

A geometric approach to the Yang-Mills mass gap via curvature of the orbit space

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Abstract

The orbit space, that is the space of connections of Yang-Mills theory modulo gauge transformations, is equipped with a Riemannian metric that naturally arises from the kinetic part of reduced classical action and admits a positive definite sectional curvature. The corresponding ‘quantum’ regularized *Bakry-Émery* Ricci curvature is shown to produce a positive mass gap for $2 + 1$ and $3 + 1$ dimensional Yang-Mills theory assuming the existence of a quantized Yang-Mills theory on $(\mathbb{R}^{1+2}, \eta)$ and $(\mathbb{R}^{1+3}, \eta)$, respectively. Our result on the gap calculation, described at least as a heuristic one, applies to Yang-Mills theory with any compact semi-simple Lie group (non-abelian) in the aforementioned dimensions. Due to the dimensional restriction on $3 + 1$ dimensional Yang-Mills theory, one ought to introduce a length scale to obtain an energy scale. It turns out that a certain ‘trace’ operation on the infinite dimensional geometry naturally introduces a length scale that has to be fixed by measuring the energy of the lowest glue-ball state. This introduction of a length scale also leads to a natural interpretation of linear running of the coupling constant g_{YM}^2 at the fully non-perturbative level in contrast to the logarithmic running in perturbation theory.

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1. Introduction

One of the most important questions in contemporary mathematical physics is to prove that the Hamiltonian or Schrödinger operator of non-abelian Yang-Mills fields admits a spectral

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gap. At the level of perturbative quantum field theory, two fundamental breakthroughs in the context of Yang-Mills theory (in dimensions $n + 1, n \leq 3$) are its renormalizability [1] and asymptotic freedom [2]. While the former can be interpreted in terms of suitable Sobolev embedding theorems (in the case of $3+1$ dimensions, such embedding turns out to be borderline as it fails to be compact), the latter indicates approaching a free theory at a high energy limit. At low energies where the Yang-Mills coupling is strong, the non-linearities are not *small* (in a suitable function space setting) and in such a regime of large data problems, a range of complicated processes are expected to occur that should fundamentally separate the behavior of non-abelian gauge theories from that of abelian theories such as pure QED. One such attribute of Yang-Mills theory associated with strong field processes is the expected existence of a gap in the spectrum of the Hamiltonian. Such a gap, if it exists, could represent the energy difference between the actual vacuum state and that of the lowest energy ‘glueball’ states and confirm the expectation that massless gluons cannot propagate freely as photons do. Keeping aside the perturbative treatment, little is known about the rigorous non-perturbative quantization of almost any interacting quantum field theory in $3 + 1$ dimensions. In $2 + 1$ and $1 + 1$ dimensions, the construction of quantum field theories with nonlinear interactions was made possible by the breakthrough work of Jaffe and Glimm [3, 4] among others. In 4 dimensions, through a re-normalization group argument, [18] proved the Gaussianity hence triviality of φ^4 theory.

The classical Yang-Mills theory on $\mathbb{R}^{1+n}$ is described by the extremum of the action functional $S_{YM} := -\frac{1}{4} \int_{\mathbb{R}^{1+n}} \langle F, F \rangle$, where F is the curvature associated with a principal bundle $(\mathfrak{P}, G, \mathbb{R}^{1+n})$ written in terms of the gauge covariant exterior derivative of a connection. The resulting Yang-Mills equations can be cast into a hyperbolic system (or a coupled elliptic-hyperbolic one) in a suitable choice of gauge and as such a solution can be thought of as a curve $t \mapsto (A(t), \mathcal{E}(t))$ in the reduced phase space $T^*(\mathcal{A}/\widehat{\mathcal{G}})$ ($\mathcal{A}$ is the space of *spatial* connections belonging to an appropriate Sobolev space $H^s(\mathbb{R}^n)$ and $\widehat{\mathcal{G}}$ is the group of automorphisms of the bundle $\mathfrak{P}$ after modding out the set of covariantly constant elements; note that $\mathcal{E}$ is the momentum variable associated with the connection $A \in \mathcal{A}/\widehat{\mathcal{G}}$). Since the reduced orbit space $\mathcal{A}/\widehat{\mathcal{G}}$ is an infinite dimensional manifold, one could interpret a classical solution as a particle moving in this infinite-dimensional configuration space with prescribed initial *position* $A(t = 0)$ and momentum $\mathcal{E}(t = 0)$. With this interpretation, a naive thought of writing down the Hamiltonian operator (a formal covariant Laplace-Beltrami operator defined on the configuration space $\mathcal{A}/\widehat{\mathcal{G}}$ together with a potential term) of the system and obtaining its spectrum becomes natural. Firstly however, such a covariant Laplace-Beltrami operator generates infinities while acting even on smooth functionals and therefore a suitable regularization is necessary to make sense of this operator. One may proceed to formally define a WKB ansatz for the ground state $\Phi_0 \sim N_h e^{iS[A]/\hbar}$, $N_h \in \mathbb{C}^\times$ (set of non-zero complex numbers) and arrive at a Lorentz signature Hamilton-Jacobi equation at $O(\hbar^0)$ that is not well behaved even in the case of many finite-dimensional quantum mechanical systems. One may abandon the WKB ansatz and start adopting a Euclidean signature semi-classical approach [6, 7, 8] that closely parallels the micro-local technique [9, 46] used in a finite-dimensional setting by choosing an ansatz of the type $\Phi_0 \sim N_h e^{-S[A]/\hbar}$ and expanding $S[A]$ as a formal power series in $\hbar$. The associated zero energy Euclidean signature Hamilton-Jacobi equation that is obtained at $O(\hbar^0)$ and represents the tree-level processes can be handled in a rigorous way through solving a non-linear elliptic equation (Sobolev embedding on the nonlinear terms barely fails to be compact in $3 + 1$ dimensions). The remaining task of making sense of the formal power series, solving for the excited states, and all the associated complexities would prove to be a monumental task and if completed and checked with all appropriate axioms of a quantum gauge theory (if possible at all) would correspond to rigorous quan-

tization of the Yang-Mills theory. This is however far from being realizable at the moment but this Euclidean signature semi-classical technique introduced by [8, 6, 7] seems promising and as such it keeps all the non-linearities and non-abelian gauge invariances (if present) of an interacting system fully intact at every level of the analysis. Moreover, it is proven to produce a better approximation to the wave functions in ordinary quantum mechanical systems such as non-linear harmonic oscillators than one would obtain through an application of the traditional WKB approach [5]. The *path integral* approach, if made to be rigorous, should produce the same result as that of the Hamiltonian approach adopted here.

Setting aside all technicalities for the moment, if one assumes that there exists a rigorously quantized Yang-Mills system in the context of this Euclidean-signature-semi-classical technique, then through the normalizability of the ground state one obtains a measure $e^{-2S[A]/\hbar}$ on the space $\mathcal{A}/\widehat{\mathcal{G}}$ and the problem of a spectral gap of the associated suitably regularized Hamiltonian reduces to an elliptic analysis on the infinite-dimensional weighted manifold $(\mathcal{A}/\widehat{\mathcal{G}}, \mathfrak{G}, e^{-2S[A]/\hbar})$, where $\mathfrak{G}$ is a Riemannian metric on the space $\mathcal{A}/\widehat{\mathcal{G}}$ induced by the kinetic part of the classical action. Many years ago I.M. Singer proposed an elegant, geometrical approach to the fundamental problem of the mass gap based on the fact that the classical, reduced configuration space $\mathcal{A}/\widehat{\mathcal{G}}$ for Yang-Mills dynamics has a naturally induced, curved Riemannian metric with everywhere non-negative sectional curvature [11, 12] (source of Gribov ambiguity). Now a Lichnerowicz estimate for the spectral gap of the Laplace-Beltrami operator of a compact finite-dimensional manifold admitting a positive definite Ricci curvature is well-known [13] (for a compact manifold, the spectrum of the Laplacian is discrete and a precise gap between the ground state and the first excited state is calculable employing the Ricci bound; see also [43, 44, 45] for estimates on the spectrum of a Schrodinger operator). However, in the infinite-dimensional context of the Yang-Mills reduced orbit space, the Riemann curvature is not of *trace*-class, and therefore Ricci curvature does not make sense. In order to make sense of this Ricci curvature, Singer [11] proposed using a zeta function regularization technique. Here, we adopt a point splitting regularization procedure [31, 35] which is known to be equivalent to the zeta function regularization procedure [15, 16]. Even after we regularize the Ricci curvature (in fact we shall see, if one starts with a point-splitting regularization of the functional Hamiltonian, the Ricci curvature is automatically regularized in the process of deriving the bound on the energy gap) one ought to remember that the Yang-Mills dynamics at the classical level is not merely a particle moving in the infinite-dimensional curved space $\mathcal{A}/\widehat{\mathcal{G}}$ but rather its motion is also governed by the potential term coming from the magnetic part of the action ($\approx \int_{\mathbb{R}^n} \mathcal{F}_{IJ} \cdot \mathcal{F}_{IJ}$, $A \in \mathcal{A}/\widehat{\mathcal{G}}$, note $\mathcal{F}_{IJ} \cdot \mathcal{F}_{IJ}$ is gauge-invariant and passes to the quotient). Therefore, at the level of a quantized theory, the gap of the formal Hamiltonian is expected to be influenced both by the formally defined Ricci curvature of the space and the potential term³. The potential part of the Hamiltonian manifests itself in the weight $e^{-2S[A]/\hbar}$ and as we shall see in a later section, the mass gap is determined by the so-called *Bakry-Emery* Ricci curvature (a modification of the Ricci curvature by the Hessian of the functional $S[A]$).

In the context of the weighted manifolds introduced by Lichnerowicz [25, 26], *Bakry-Emery* curvature naturally appears (see [21] for geometric properties of the Bakry-Emery Ricci tensor, Also see [27] for a recent study regarding spin geometry, ADM mass, estimates

³The potential may exhibit flat directions (Yang-Mills potential does admit flat directions when $[A, A] = 0$, polarized solutions) and whenever the potential admits flat directions, the positivity of the curvature is expected to produce a positive mass gap or the quantum fluctuation alone may generate a positive gap (in the quantum mechanical setting, there are examples of systems where potential admits flat directions but the uncertainty principle does allow for a positive mass gap [17])

on the spectrum of the Dirac operator, etc. on finite dimensional weighted manifolds). It also appears in the context of scalar-tensor gravitational theories, including Brans-Dicke theory [19], theories with Kaluza-Klein dimensional reduction [20] apart from the celebrated study by [22]. Studies by [23, 24] provide examples of the appearance of this modified Ricci curvature in the context of Lorentzian geometry. In a finite-dimensional setting with potential satisfying suitable convexity conditions, then a spectral gap estimate for the Hamiltonian operator is given by the bound on the Bakry-Emery Ricci curvature [8]. For example, for a harmonic oscillator on a flat space, the Bakry-Emery Ricci tensor produces the exact gap that is presented in every quantum mechanics textbook. Even though in such a case the ordinary Ricci curvature vanishes, the Hessian of the negative Logarithm of the ground state wave function contributes in a strictly positive manner to produce the exact gap [8]. Perhaps it would be interesting to study the spectral property of the Hamiltonian associated with the finite-dimensional setting of $2 + 1$ gravity where Teichmüller space naturally plays the role of the configuration space.

The spectral gap in the Hamiltonian of the Yang-Mills theory is absent in the perturbation theory. Recall in the perturbation theory, one splits the full gauge-invariant Lagrangian into an exactly soluble part and interactions. This procedure, however, corresponds to the breaking of the original gauge-invariance in the sense that the original gauge group $SU(N)$ undergoes a splitting $SU(N) \rightarrow \underbrace{U(1) \times U(1) \times U(1) \times U(1) \times \cdots U(1)}_{N^2-1}$. In simpler terms,

perturbation theory essentially yields several copies of photons which fail to lead to a positive mass gap. At the level of non-perturbative theory, we restore the full gauge-invariance in our analysis. However, in $3 + 1$ Yang-Mills theory, the physical constants can not produce a mass scale purely on the basis of dimensional analysis and one has to introduce a length scale that is to be fixed by measuring the mass of the lowest glue-ball state (*dimensional transmutation*). This scale is introduced through the regularization of the trace of the Riemann curvature of the infinite dimensional configuration space of the gauge theory.

The introduction of a length scale has an important consequence on the (squared) coupling constant g_{YM}^2 . The infinite dimensional curvature involves a factor of g_{YM}^2 and essentially describes the strength of interaction. Now the expectation of the regularized Ricci curvature of the infinite dimensional geometry with respect to the functional measure $e^{-2S/\hbar}$ generates the mass gap of the theory where one needs the introduction of a length scale to respect the dimensional consistency. The length scale is introduced naturally through the regularization of the trace geometry and appears as g_{YM}^2/L (L being the introduced length scale, ultraviolet cut-off). Now in order for this entity to make sense g_{YM}^2 must run as L in the limit $L \rightarrow 0$ (high energy limit). In contrast, g_{YM}^2 runs logarithmically with scale in the perturbation theory. However, the expectation value of the Ricci containing g_{YM}^2 that appears in the gap formula is essentially an integral over the infinite dimensional orbit space. If calculated order by order (and possibly after a re-summation is performed to make the formal series convergent), it contains every quantum correction and therefore one would naturally expect that the logarithmic running at the perturbation theory is promoted to a linear running at the full non-perturbative theory.

Progress has been made in the context of the $2 + 1$ dimensional Yang-Mills theory. In a series of articles, [29, 30, 31, 32] carried out the Hamiltonian analysis of the $2 + 1$ Yang-Mills theory using a gauge invariant matrix parametrization of fields. The point-splitting regularization is utilized in order to make sense of the Laplace-Beltrami operator defined on the configuration space. These studies have explicitly computed the lowest eigenstates of this regularized operator and established the existence of a positive gap in its spectrum. The potential part, however, was not included. The nature of the expected correction to

the mass gap due to the presence of the potential term is discussed perturbatively. Our approach on the other hand, as mentioned earlier, incorporates the potential energy suitably and manifests itself as the Bakry-Emery correction to the ordinary regularized Ricci curvature of the configuration space to produce a gap. The point-splitting regularization is essentially adopted from the aforementioned study of $2 + 1$ dimensional Yang-Mills theory that was extended to the $3 + 1$ dimensional case by [37, 34, 35]. We expect the Ricci curvature of the infinite dimensional geometry to appear at the level of loop corrections to the semi-classical amplitudes. In fact a natural conjecture would be that the renormalization group flow for the metric of the Yang-Mills theory would be an infinite dimensional forced Ricci flow. A tremendous amount of work and new ideas are essentially required to rigorously quantize the Yang-Mills theory. But we suspect that these ideas related to the infinite dimensional geometry may have substantial contribution.

2. Geometry of the orbit space $\mathcal{A}/\widehat{\mathcal{G}}$

We denote by $\mathfrak{P}$ a C^∞ principal bundle with base an $n+1$ dimensional Lorentzian manifold M and a Lie group G . We assume that G is compact (for physical purposes) and therefore admits a positive definite non-degenerate bi-invariant metric. Its Lie algebra $\mathfrak{g}$ by construction admits an adjoint invariant, positive definite scalar product denoted by $\langle \cdot, \cdot \rangle$ which enjoys the property: for $A, B, C \in \mathfrak{g}$,

$$\langle [A, B], C \rangle = \langle A, [B, C] \rangle. \quad (2.1)$$

as a consequence of adjoint invariance. A Yang-Mills connection is defined as a 1-form ω on $\mathfrak{P}$ with values in $\mathfrak{g}$ endowed with compatible properties. It's representative in a local trivialization of $\mathfrak{P}$ over $U \subset M$

$$\varphi : p \mapsto (x, a), \quad p \in \mathfrak{P}, \quad x \in U, \quad a \in G \quad (2.2)$$

is the 1-form $s^*\omega$ on U , where s is the local section of $\mathfrak{P}$ corresponding canonically to the local trivialization $s(x) = \varphi^{-1}(x, e)$, called a *gauge*. Let A_1 and A_2 be representatives of ω in gauges s_1 and s_2 over $U_1 \subset M$ and $U_2 \subset M$. In $U_1 \cap U_2$, one has

$$A_1 = Ad(u_{12}^{-1})A_2 + u_{12}\Theta_{MC}, \quad (2.3)$$

where Θ_{MC} is the Maurer-Cartan form on G , (or $A_1 \mapsto u_{12}^{-1}A_1u_{12} + u_{12}du_{12}^{-1}$) and $u_{12} : U_1 \cap U_2 \rightarrow G$ generates the transformation between the two local trivializations:

$$s_1 = R_{u_{12}}s_2, \quad (2.4)$$

$R_{u_{12}}$ is the right translation on $\mathfrak{P}$ by u_{12} . Given the principal bundle $\mathfrak{P} \rightarrow \mathbb{R}^{1+n}$, a Yang-Mills potential A on M is a section of the fibered tensor product $T^*M \otimes_M \mathfrak{P}_{Affine, \mathfrak{g}}$ where $\mathfrak{P}_{Affine, \mathfrak{g}}$ is the affine bundle with base M and typical fibre $\mathfrak{g}$ associated to $\mathfrak{P}$ via relation (2.3) (in other words, the connection does not transform as a tensor under a gauge transformation). If $\widehat{A}$ is another Yang-Mills potential on M , then $A - \widehat{A}$ is a section of the tensor product of vector bundles $T^*M \otimes_M \mathfrak{P}_{Ad, \mathfrak{g}}$, where $\mathfrak{P}_{Ad, \mathfrak{g}} := \mathfrak{P} \times_{Ad} \mathfrak{g}$ is the vector bundle associated to $\mathfrak{P}$ by the adjoint representation of G on $\mathfrak{g}$ (the difference of two connections does transform as a tensor under a gauge transformation). There is an inner product in the fibres of $\mathfrak{P}_{Ad, \mathfrak{g}}$, deduced from that on $\mathfrak{g}$. The curvature Ω of the connection ω considered as a 1-form on $\mathfrak{P}$ is a $\mathfrak{g}$ -valued 2-form on $\mathfrak{P}$. Its representative in a gauge where ω is represented by A is given by

$$F := dA + [A, A], \quad (2.5)$$

and the relation between two representatives F_1 and F_2 on $U_1 \cap U_2$ is $F_1 = Ad(u_{12}^{-1})F_2$ and therefore F is a section of the vector bundle $\Lambda^2 T^*M \otimes_M \mathfrak{P}_{Ad, \mathfrak{g}}$. For a section $\mathfrak{D}$ of the vector bundle $\otimes^k T^*M \otimes_M \mathfrak{P}_{Ad, \mathfrak{g}}$, a natural covariant derivative is defined as follows

$$\widehat{\nabla} \mathfrak{D} := \nabla \mathfrak{D} + [A, \mathfrak{D}], \quad (2.6)$$

where ∇ is the usual covariant derivative induced by the Lorentzian structure of M and by construction $\widehat{\nabla} \mathfrak{D}$ is a section of the vector bundle $\otimes^{k+1} T^*M \otimes_M \mathfrak{P}_{Ad, \mathfrak{g}}$. The associated exterior derivative is denoted by $d^{\widehat{\nabla}}$. The Yang-Mills coupling constant g_{YM} is kept hidden within the structure constants of the commutators.

The classical Yang-Mills equations (in the absence of sources) correspond to setting the natural (spacetime and gauge as defined in 2.6) covariant divergence of this curvature two-form F to zero. By virtue of its definition in terms of the connection, this curvature also satisfies the Bianchi identity that asserts the vanishing of its gauge covariant exterior derivative. Taken together these equations provide a geometrically natural nonlinear generalization of Maxwell's equations (when the latter are written in terms of a 'vector potential') and of course play a fundamental role in modern elementary particle physics. If nontrivial bundles are considered or nontrivial spacetime topologies are involved, then the foregoing so-called 'local trivializations' of the bundles in question must be patched together to give global descriptions but, by virtue of the covariance of the formalism, there is a natural way of carrying out this patching procedure at least over those regions of spacetime where the connections are well-defined. From now on, we set $M = \mathbb{R}^{1+n}$, $n = 2, 3$ equipped with the Minkowski metric η . In addition, in a chosen Lie algebra basis we write the commutation $[\cdot, \cdot]$ on $\mathfrak{g}$ explicitly in terms of the structure constants i.e., $[A_i, A_j]^P = f^{PQR} A_i^Q A_j^R$. We denote the space of connections in a particular Sobolev class H^s by $\mathcal{A}$. The following lemma yields a local expression for the metric on the orbit space $\mathcal{A}/\mathcal{G}$ (note that [11, 12] also obtained metrics on the orbit space). We provide an explicit expression for the metric for the convenience of forthcoming calculations.

Lemma 2.1. *Let $F = d^{\widehat{\nabla}} A$ be the curvature of the principle G -bundle $\mathfrak{P}$ over $\mathbb{R}^{1+n}$. The associated Yang-Mills action functional is defined as $S = -\frac{1}{4} \int_{\mathbb{R}^{1+n}} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle$. The metric induced by the action functional S on the orbit space $\mathcal{A}/\mathcal{G}$ verifies the following expression in the local Coulomb coordinates (i.e., connection verifies $\nabla^i A_i = 0$ in a small enough open neighborhood of the flat connection $A_i = 0$)*

$$\mathfrak{G}[A]_{A_i^P(x) A_j^Q(x')} = \delta_{ij} \delta_{PQ} \delta(x - x') + f^{PRV} A_i^V(x) \Delta_A^{-1}(x, x') f^{RUQ} A_j^U(x'),$$

where f^{PQR} are the structure constants defined via $[A_i, A_j]^P = f^{PQR} A_i^Q A_j^R$ in a chosen Lie algebra basis. Here $\mathcal{G}$ is the group of automorphisms of the bundle $\mathfrak{P}$ i.e., the group of gauge transformations (under a gauge transformation $\varphi(x)$, a connection $A \in \mathcal{A}$ transforms as $A \mapsto \varphi A \varphi^{-1} + \varphi d \varphi^{-1}$).

Remark 1. *Note that the space $\mathcal{A}/\mathcal{G}$ is in general not a manifold since the group action $\mathcal{G}$ on $\mathcal{A}$ is not free due to the potential presence of gauge symmetry i.e., the gauge transformations that leave a connection A invariant or equivalently solutions of the equation $\varphi A \varphi^{-1} + \varphi d \varphi^{-1} = A$ or $d \varphi^{-1} + [A, \varphi^{-1}] = 0$ i.e., the elements of the bundle automorphism group that are covariantly constant. However, we can work with the space of irreducible connections i.e., $\mathcal{A}/\widehat{\mathcal{G}}$ where $\widehat{\mathcal{G}}$ is obtained by modding out the covariantly constant elements of $\mathcal{G}$. $\mathcal{A}/\widehat{\mathcal{G}}$ is an infinite dimensional manifold. This property is important as we shall see at the later sections. From now on $\mathcal{A}/\widehat{\mathcal{G}}$ is to be understood as the space of connections belonging to an appropriate Sobolev space $H^s(\mathbb{R}^n)$.*

Proof. The Gauss Law constraint

$$\widehat{\nabla}_\nu F^{0\nu} = 0 \quad (2.7)$$

yields

$$\widehat{\nabla}_i \widehat{\nabla}_i A_0 = \nabla_i \partial_0 A_i + [A_i, \partial_0 A_i] \quad (2.8)$$

which after an application of the Coulomb coordinate condition $\partial_i A_i = 0$ yields

$$\widehat{\nabla}_i \widehat{\nabla}_i A_0 = [A_i, \partial_0 A_i] \quad (2.9)$$

and therefore A_0 may be obtained by formally inverting the elliptic operator $\widehat{\nabla}_i \widehat{\nabla}_i = \Delta_A$

$$A_0 = \Delta_A^{-1}([A_i, \partial_t A_i]). \quad (2.10)$$

Now write the usual commutation for the elements of the Lie algebra $\mathfrak{g}$

$$[\chi_i, \chi_j]^P = f^{ABC} \chi_i^B \chi_j^C \text{ i.e., } [A_i, \partial_t A_i]^P = f^{ABC} A_i^B \partial_t A_i^C. \quad (2.11)$$

We may obtain A_0^P by formally inverting Δ_A

$$A_0^P = \Delta_A^{-1}(f^{PQR} A_i^Q \partial_t A_i^R). \quad (2.12)$$

In the Coulomb coordinate, the action functional $S = \int_{\mathbb{R}^{1,n}} \left(\frac{1}{2} F^P{}_{0i} F^P{}_{0i} - \frac{1}{4} F^P{}_{ij} F^P{}_{ij} \right) d^{n+1}x$ takes the following form

$$\begin{aligned} S &= \int_{\mathbb{R}^{1,n}} \left(\frac{1}{2} \partial_t A_i^P \partial_t A_i^P - \partial_t A_i \partial_i A_0^P + \frac{1}{2} \partial_i A_0^P \partial_i A_0^P + \partial_t A_i^P [A_0, A_i]^P \right. \\ &\quad \left. - \partial_i A_0^P [A_0, A_i^T]^P + \frac{1}{2} [A_0, A_i]^P [A_0, A_i]^P - \frac{1}{4} F^P{}_{ij} F^P{}_{ij} \right) d^{n+1}x \\ &= \int_{\mathbb{R}^{1,n}} \left(\frac{1}{2} \partial_t A_i^P \partial_t A_i^P - \frac{1}{2} A_0^P \Delta A_0^P + \partial_t A_i^P [A_0, A_i]^P - \partial_i A_0^P [A_0, A_i]^P + \right. \\ &\quad \left. \frac{1}{2} [A_0, A_i]^P [A_0, A_i]^P - \frac{1}{4} F^P{}_{ij} F^P{}_{ij} \right) d^{n+1}x - \int_{\partial \mathbb{R}^{1,n}} (\partial_t A_i^P A_0^P - \frac{1}{2} A_0^P \partial_i A_0^P). \end{aligned} \quad (2.13)$$

Notice that there are problematic terms such as $\int_{\mathbb{R}^{1,n}} \partial_i A_0^P [A_0, A_i^T]^P$. However this term is cancelled in a point-wise manner after expanding ΔA_0^P using equation (2.9)

$$\begin{aligned} -\frac{1}{2} A_0^P \Delta A_0^P - \partial_i A_0^P [A_0, A_i]^P &= A_0^P [A_i, \partial_i A_0]^P + \frac{1}{2} A_0^P [A_i, [A_i, A_0]]^P \\ &\quad - \frac{1}{2} A_0^P [A_i, \partial_t A_i]^P - \partial_i A_0^P [A_0, A_i]^P. \end{aligned}$$

Now $A_0^P [A_i, \partial_i A_0]^P - \partial_i A_0^P [A_0, A_i]^P$ vanishes due to the property (2.1). Therefore ignoring the boundary terms (assuming fields belong to the Schwartz space), the action reads

$$S = \int_{\mathbb{R}^n} \left(\frac{1}{2} \partial_t A_i^P \partial_t A_i^P + \frac{1}{2} A_0^P [A_i, \partial_t A_i]^P - \frac{1}{4} F^P{}_{ij} F^P{}_{ij} \right) d^{n+1}x.$$

Now after an explicit calculation using the Lie-algebra commutation relation, one writes the action in the usual form, that is, as the difference between the kinetic and potential terms

$$L = \int_{(\mathbb{R}^n)^2} \left(\frac{1}{2} \partial_t A_i^P(x) \partial_t A_i^P(x') \delta(x - x') + \frac{1}{2} f^{PQR} A_i^Q(x) \partial_t A_i^R(x) \Delta_A^{-1}(x, x') f^{PUV} A_k^U(x') \partial_t A_k^V(x') \right)$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1}{2} \mathfrak{G}[A]_{A_i^P(x) A_j^Q(x')} \partial_t A_i^P(x) \partial_t A_j^Q(x') - \frac{1}{4} \int_{\mathbb{R}^n} \mathcal{F}^P{}_{ij} \mathcal{F}^P{}_{ij},$$

where

$$\mathfrak{G}[A]_{A_i^P(x) A_j^Q(x')} = \delta_{ij} \delta_{PQ} \delta(x - x') + f^{PRV} A_i^V(x) \Delta_A^{-1}(x, x') f^{RUQ} A_j^U(x').$$

This concludes the proof of the lemma. Note that $\Delta^{-1}(x, x') := \frac{1}{4\pi} \frac{-1}{|x - x'|}$ for $n = 3$ and $\Delta^{-1}(x, x') := \frac{1}{2} \ln(|x - x'|/a)$ for $n = 2$, a some fixed constant with dimension of length. This metric was obtained by [11, 12] by a different method (mention that). $\square$

Lemma 2.2. $\mathfrak{G}$ is a Riemannian metric.

Proof. Follows from the positive definiteness of the Kinetic energy (a consequence of the compactness of the gauge group). $\square$

The Riemannian metric induced by the action functional on the configuration space $\mathcal{A}/\widehat{\mathcal{G}}$ is in general curved. As such one may compute the Riemann curvature of this metric $\mathfrak{G}[A]$ at any point $\widehat{A}$ of $\mathcal{A}/\widehat{\mathcal{G}}$ by explicit calculations or by expanding it in the normal coordinate around $\widehat{A}$. We compute the Riemann curvature at $\widehat{A} = 0$ in the following lemma. Note that [11, 12] performed similar calculations as well.

We define the formal single trace operation on sections of suitable bundles on the infinite dimensional manifold $\mathcal{A}/\widehat{\mathcal{G}}$ as follows

$$\begin{aligned} & (\text{tr}\Phi)_{A_{I_1}^{P_1}(x_1) A_{I_2}^{P_2}(x_2) A_{I_3}^{P_3}(x_3) \dots \widehat{A}_{I_i}^{P_i}(x_i) \dots \widehat{A}_{I_j}^{P_j}(x_j) \dots A_{I_n}^{P_n}(x_n)} \\ &:= \int_{x_i, x_j} \mathfrak{G}_{A_{I_i}^{P_i}(x_i) A_{I_j}^{P_j}(x_j)} \Phi_{A_{I_1}^{P_1}(x_1) A_{I_2}^{P_2}(x_2) A_{I_3}^{P_3}(x_3) \dots \widehat{A}_{I_i}^{P_i}(x_i) \dots \widehat{A}_{I_j}^{P_j}(x_j) \dots A_{I_n}^{P_n}(x_n)}, \end{aligned} \quad (2.14)$$

where the hat symbol implies the deletion of the respective indices. For example, if we consider Riemann curvature i.e., $\Phi := \mathcal{R}_{A_{I_1}^{P_1}(x_1) A_{I_2}^{P_2}(x_2) A_{I_3}^{P_3}(x_3) A_{I_4}^{P_4}(x_4)}$, then the formal Ricci curvature would be defined as follows

$$\text{Ric}_{A_{I_2}^{P_2}(x_2) A_{I_4}^{P_4}(x_4)} := \int_{x_1, x_3} \mathfrak{G}_{A_{I_1}^{P_1}(x_1) A_{I_3}^{P_3}(x_3)} \mathcal{R}_{A_{I_1}^{P_1}(x_1) A_{I_2}^{P_2}(x_2) A_{I_3}^{P_3}(x_3) A_{I_4}^{P_4}(x_4)}. \quad (2.15)$$

Here $\mathfrak{G}_{A_{I_1}^{P_1}(x_1) A_{I_3}^{P_3}(x_3)} := (\mathfrak{G}^{-1})_{A_{I_1}^{P_1}(x_1) A_{I_3}^{P_3}(x_3)}$.

Lemma 2.3. The formal Ricci curvature of the metric $\mathfrak{G}$ in local Coulomb coordinates around $A = 0$ satisfies

$$\text{Ric}(X, Y) = 3(f^{VPR} X_i^R(x) \text{tr} \Delta^{-1}(x, x') f^{VQU} Y_j^U(x')). \quad (2.16)$$

where $\Delta^{-1} : L^2(\mathbb{R}^n) \rightarrow H^1(\mathbb{R}^n)$ is the inverse of the Laplacian $\Delta := \eta^{ij} \nabla_i \nabla_j$ and tr denotes the formal trace operation defined by multiplication of δ^{PQ} and the distribution $\delta(x - x')$ to yield the coincident limit at $A = 0$.

Proof. First recall the definition of the Covariant derivative $\mathfrak{D}$

$$2\mathfrak{G}(Z, \mathfrak{D}_X Y) = X \cdot \mathfrak{G}(Z, Y) + Y \cdot \mathfrak{G}(Z, X) - Z \cdot \mathfrak{G}(X, Y), \quad (2.17)$$

and that of the Riemann curvature

$$\mathcal{R}(X, Y)Z := \mathfrak{D}_X \mathfrak{D}_Y Z - \mathfrak{D}_Y \mathfrak{D}_X Z - \mathfrak{D}_{[X, Y]}Z, \quad (2.18)$$

for $X, Y, Z, W \in \mathfrak{H}_A$. Using the expression, one may explicitly compute at $A = 0$

$$\mathcal{R}(W, Z, X, Y) = -2\langle [Y_j, W_j], \Delta^{-1}[X_i, Z_i] \rangle - \langle [Z_j, W_j], \Delta^{-1}[X_i, Y_i] \rangle + \langle [X_j, W_j], \Delta^{-1}[Z_i, Y_i] \rangle.$$

The quadratic form associated with the Ricci curvature may be computed by taking the formal *trace* (infinite dimensional) of the Riemann curvature

$$\text{Ric}(X, Y) = 3\text{tr}(\langle [X, \cdot], \Delta^{-1}[Y, \cdot] \rangle) \quad (2.19)$$

Expanding the bracket in terms of structure constants yields the result.

Notice the following point of view that is different from direct calculations. Remarkably, the Coulomb coordinate chart based at $A = 0$ is naturally a geodesic normal chart (based at $A = 0$) since $\mathfrak{G}_{\dot{A}_i^P(x)\dot{A}_j^Q(x')}|_{A=0} = \delta_{ij}\delta_{PQ}\delta(x - x')$. In addition connections $\Gamma_{A_j^Q A_k^R}^{A_i^P}|_{A=0} = 0$ since $\frac{\delta \mathfrak{G}}{\delta A}|_{A=0} = 0$ due to the non-constant terms in the metric being at least quadratic in A . Now recall the expression of the metric derived in the previous lemma and compare with the expression of the metric in a normal neighbourhood based at the flat connection $A = 0$

$$\mathfrak{G}_{\dot{A}_i^P(x)\dot{A}_j^Q(x')} = \delta_{ij}\delta_{PQ}\delta(x - x') - \frac{1}{3}\mathcal{R}_{A_i^P(x)A_k^R(x_1)A_j^Q(x_2)A_l^U(x')}A_k^R(x_1)A_l^U(x_2) + O(|A|^3) \quad (2.20)$$

to yield

$$\mathcal{R}_{A_i^P(x)A_k^R(x_1)A_j^Q(x')A_l^U(x_2)}A_k^R(x_1)A_l^U(x_2) = 3f^{VPR}A_i^R(x)\Delta^{-1}(x, x')f^{VQU}A_j^U(x'). \quad (2.21)$$

The invariant quadratic form for the Ricci tensor is then obtained by taking *formal* trace of the Riemann tensor i.e.,

$$\text{Ric}(X, Y) = 3(f^{VPR}X_i^R(x)\text{tr}\Delta^{-1}(x, x')f^{VQU}Y_j^U(x')). \quad (2.22)$$

This concludes the lemma. $\square$

Remark 2. It is not difficult to see that at an arbitrary point $\hat{A} \in \mathcal{A}/\mathcal{G}$, the formal Ricci quadratic form is simply $\text{Ric}(X, Y) = 3(f^{VPR}X_i^R(x)\text{tr}\Delta_{\hat{A}}^{-1}(x, x')f^{VQU}Y_j^U(x'))$, where $\Delta_{\hat{A}} := \eta^{ij}\widehat{\nabla}_i^{\hat{A}}\widehat{\nabla}_j^{\hat{A}}$ is the gauge covariant Laplacian. The sectional curvature $\mathcal{K}_{X, Y} := \langle \mathcal{R}(X, Y)Y, X \rangle$ of a 2-plane spanned by the orthonormal vectors $X, Y \in \mathfrak{H}_A$ is then

$$\mathcal{K}_{X, Y} = 3\langle [X, Y], \Delta_{\hat{A}}^{-1}[X, Y] \rangle. \quad (2.23)$$

This can be achieved by choosing the generalized Coulomb coordinate chart based at $\hat{A}$ defined by $\eta^{ij}\widehat{\nabla}_i^{\hat{A}}(A - \hat{A})_j = 0$ and obtaining an expression of the metric $\mathfrak{G}$ in this chart. $\mathfrak{H}_A$ is tangent at A to the horizontal subspace of the bundle $\mathcal{A} \rightarrow \mathcal{A}/\hat{\mathcal{G}}$.

3. Lichnerowicz type estimate for the Hamiltonian operator: Regularization

In the finite dimensional setting, a lower bound on the Ricci curvature and compactness yields a lower bound on the first eigenvalue of the Laplace-Beltrami operator due to Lichnerowicz [13]. In the presence of a potential, a Bakry-Emery correction to the ordinary Ricci curvature is required to estimate a precise gap in the spectrum (there are several studies on estimating the gap of a Schrodinger operator in finite dimensions using direct analysis [43, 44, 45]). In an infinite dimensional setting, a straightforward generalization does not work. Note in particular that the Riemann tensor of $\mathcal{A}/\widehat{\mathcal{G}}$ is not of trace class. Recall the definition of the trace. At the flat connection $A = 0$, the trace would correspond to contraction with respect to the flat metric and therefore to setting $P = Q$ and $x = x'$ in the expression (2.22). This would correspond to the evaluation of the coincidence limit of $\Delta^{-1}(x, x')$. However in 2 dimensions $\Delta^{-1}(x, x') = \frac{1}{2} \ln |x - x'|$ and in 3 + 1 dimensions, $\Delta^{-1}(x, x') = -\frac{1}{4\pi} \frac{1}{|x - x'|}$, whose coincident limits of course do not exist (or in the QFT terminology, one has occurrence of ultraviolet divergences). In order to make sense of the Ricci tensor, one needs to invoke a regularization scheme. In the regularization scheme that we adopt, we split the points by approximating Dirac's distribution and taking a suitable limit.

Definition 1. *Let us endow the local coordinates $\{x^i\}$ of a smooth n -manifold with the dimension of length while the metric (co-variant) coefficients are left dimensionless. The point-splitting of Dirac's distribution associated with the usual Dirac's distribution $\delta(x, x_0) = \frac{\delta(x - x_0)}{\mu_g(x)} = \frac{\prod_{i=1}^n \delta(x^i - x_0^i)}{\mu_g(x)}$ on a Riemannian n -manifold (M, g) , $x, x_0 \in M$, is defined as follows*

$$\delta_\chi(x, x^0) := \frac{\prod_{i=1}^n \frac{\chi}{\pi} e^{-(x^i - x_0^i)^2 \chi^2}}{\mu_g(x)}. \quad (3.1)$$

The usual distribution is recovered after letting $\chi \rightarrow \infty$.

A rigorous quantum Yang-Mills theory if it exists should consist of a separable Hilbert space $\mathcal{H}$, a unitary representation of the Poincaré group in $\mathcal{H}$, an operator-valued gauged distribution A on $\mathcal{S}(\mathbb{R}^n)$ and a dense subspace $\mathcal{D} \subset \mathcal{H}$ such that appropriate axioms of quantum gauge theory hold. As we have mentioned in the introduction this is a monumental task even for non-gauge interacting field theories. Putting aside these issues we assume a rigorous quantum field theory exists. In other words, we dodge the hardest question and study its consequences for the mass gap. The functional Hamiltonian operator defined on the orbit space $\mathcal{A}/\widehat{\mathcal{G}}$ needs regularization since even while acting on a smooth functional, it generates infinities. The formal Schrödinger operator for a Yang-Mills field in $n + 1$ dimensions, of the type that we shall consider, is given by ⁴

$$H = \int_{\mathbb{R}^n} \left(-\frac{\hbar^2}{2} \int_{\mathbb{R}^n} \mathfrak{G}^{A_I^P(x) A_J^Q(y)} \frac{\mathfrak{D}}{\mathfrak{D} A_I^P(x)} \frac{\mathfrak{D}}{\mathfrak{D} A_J^Q(x)} + \frac{1}{4} \mathcal{F}_{IJ} \cdot \mathcal{F}_{IJ} \right) d^n x, \quad (3.2)$$

where $\frac{\mathfrak{D}}{\mathfrak{D} A_I^P}$ is the covariant derivative on the Riemannian manifold $(\mathcal{A}/\widehat{\mathcal{G}}, \mathfrak{G})$ and $\mathcal{F}$ denotes the Yang-Mills field strength after descending to the orbit space ⁵. The delta distribution in $\mathfrak{G}$ is replaced by the point-split distribution defined in (3.1). Note that contrary to the

⁴In co-variant quantization, if one uses a formal path integral with a Fadeev-Popov gauge fixing term, one may need to address a slightly modified Hamiltonian

⁵Notice that the potential is gauge invariant and therefore naturally descends to the quotient i.e., the orbit space

Laplacian, the Hessian $\frac{\mathfrak{D}}{\mathfrak{D}A_I^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_J^Q(y)}$ is well-defined on a smooth functional. The flat part of the covariant functional Laplacian is ill defined. Utilizing the point-splitting of Dirac's distribution introduced previously in (3.1) we define the regularization of the flat Laplacian as follows

$$\int_x \frac{\delta}{\delta A_I^P(x)} \frac{\delta}{\delta A_I^P(x)} \mapsto \int_{x,y} \delta_\chi(x,y) \frac{\delta}{\delta A_I^P(x)} \Theta_{PQ}(x,y) \frac{\delta}{\delta A_I^Q(y)}, \quad (3.3)$$

where $\Theta_{AB}(x,y)$ is a parallel propagator between x and y and defined as a solution of the parallel propagation equation, $\Theta_{PQ}(x,y) := (\mathcal{P}e^{-\int_y^x A_i dz^i})_{PQ}$, $\mathcal{P}$ denotes the path ordering of the exponential. This is inserted in order to preserve the gauge invariance (note $\Theta_{AB}(x,y)$ transforms under a gauge transformation $\varphi \in \mathcal{G}$ as $\Theta_{PQ}(x,y) \mapsto (\varphi(x)\Theta(x,y)\varphi^{-1}(y))_{PQ}$). The result would not depend on the choice of the path from x to y in the limit $\chi \rightarrow \infty$, which we are interested in after subtracting possible infinities. Naturally, this regularization descends to the orbit space $\mathcal{A}/\widehat{\mathcal{G}}$ due to its gauge invariance. In addition, the parallel propagator is chosen to be such that it is symmetric under the transformation $A \rightarrow B$, $x \rightarrow y$ (see [30, 31, 32, 35] for detail). Since

$$\int_{x,y} \delta_\chi(x,y) \left(\frac{\delta}{\delta A_I^P(x)} \Theta_{PB}(x,y) \right) \frac{\delta}{\delta A_I^B(y)} = 0, \quad (3.4)$$

we may write the regularization (see [30, 31, 32, 35] for 2+1 dimensions and [33, 36, 37] (also see the thesis [34]) for 3 + 1 dimensions) as

$$\int_x \frac{\delta}{\delta A_I^A(x)} \frac{\delta}{\delta A_I^A(x)} \mapsto \int_{x,y} \delta_\chi(x,y) \Theta_{AB}(x,y) \frac{\delta}{\delta A_I^A(x)} \frac{\delta}{\delta A_I^B(y)}, \quad (3.5)$$

where note that we recover the usual flat functional Laplacian in the limit $\chi \rightarrow \infty$. We will proceed with this regularization scheme. Therefore we write the regularized Hamiltonian that we shall work with as follows

$$\widehat{H} := -\frac{\hbar^2}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \Theta^{PQ}(x,y) (\mathfrak{G}_{\delta_\chi}^{-1})^{A_I^P(x) A_J^Q(y)} \frac{\mathfrak{D}}{\mathfrak{D}A_I^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_J^Q(y)} + \int_{\mathbb{R}^n} \frac{1}{4} \mathcal{F}_{IJ} \cdot \mathcal{F}_{IJ} d^3x, \quad (3.6)$$

where we have point-split the Dirac's distribution appearing in the metric $\mathfrak{G}$ i.e.,

$$(\mathfrak{G}_{\delta_\chi})_{A_I^P(x) A_J^Q(x')} := \delta_{ij} \delta_{PQ} \delta_\chi(x, x') + f^{PRV} A_i^V(x) \Delta_A^{-1}(x, x') f^{RUQ} A_j^U(x').$$

In order to estimate the gap in the spectrum of the Hamiltonian, we must perform a Bochner-type analysis on the gauge covariant Hamiltonian acting on wave functionals. To this end, we will adapt the Euclidean signature semi-classical (ESSC) introduced by [6, 7, 8] for renormalizable interacting Bosonic field theories (borderline Sobolev embedding for 3 + 1 dimensional Yang-Mills theory). This technique is in a similar spirit to the microlocal method (see [46] for a comprehensive review) used for the analysis of Schrödinger eigenvalue problems even though the latter has not previously been applicable to field theoretic problems due to technical reasons. Suppose we fix an open set $U \subset \mathcal{A}/\widehat{\mathcal{G}}$, $A \in U$ and substitute the following node-less formal expression for the semi-classical expansion of the ground state wave functional⁶

$$\Psi[A]_g = N_\hbar e^{-\frac{S_\hbar[A]}{\hbar}}, \quad S_\hbar[A] \simeq S_0[A] + \hbar S_1[A] + \frac{\hbar^2}{2!} S_2[A] + \cdots \frac{\hbar^k}{k!} S_k[A] + \cdots, \quad (3.7)$$

⁶Contrary to the microlocal approach, if one assumes a WKB ansatz, then the tree level processes are governed by a Lorentz signature Hamilton-Jacobi equation that yields finite time blow up even in finite dimensional problems due to the presence of caustics in the configuration space

$$E_h^0 \simeq \hbar \left(E_0 + \hbar E_1 + \hbar^2 E_2 + \dots \hbar^k E_k + \dots \right) \quad (3.8)$$

into the Schrödinger equation

$$\hat{H}\Psi[A] = E_h^0 \Psi[A] \quad (3.9)$$

and impose equality order by order in the Planck constant to conclude that S_0 satisfies the following functional Hamilton-Jacobi equation

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1}{2} \mathfrak{G}^{A_i^P(x_1) A_j^Q(x_2)} \frac{\delta S_0}{\delta A_i^P(x_1)} \frac{\delta S_0}{\delta A_j^Q(x_2)} - \int_{\mathbb{R}^n} \frac{1}{4} \mathcal{F}_{jk} \cdot \mathcal{F}_{jk} = 0. \quad (3.10)$$

Now notice that $\frac{\delta S_0}{\delta A(x)}$ is well defined (no need for regularization at this tree level) and S_0 can be obtained as Hamilton's principal function for the Euclidean signature Yang-Mills action functional i.e.,

$$S_0 := \inf_{\mathcal{A} \in H^1(\mathbb{R}^{n+1})} \mathcal{I}_{es}[\mathcal{A}], \quad (3.11)$$

where $\mathcal{I}_{es}[\mathcal{A}] := \frac{1}{2} \int_{\mathbb{R}^- \times \mathbb{R}^n} \left(\sum_{\mu, \nu=0}^n \mathcal{F}[\mathcal{A}]_{\mu\nu}^I \mathcal{F}[\mathcal{A}]_{\mu\nu}^I \right) d^{n+1}x$. The minimization procedure may be described as follows. Given A as the boundary condition for $\mathcal{A}$ on $\{0\} \times \mathbb{R}^n$ in the respective Sobolev trace space, one wants to minimize the Euclidean signature action functional in $\mathbb{R}^- \times \mathbb{R}^n$ with $\mathcal{A}$ approaching the flat connection on $\{-\infty\} \times \mathbb{R}^n$. This minimization procedure is essentially solving a semi-linear elliptic equation with a prescribed Dirichlet boundary value in a suitable choice of gauge (generalized Coulomb or Hodge gauge is one such choice). However, the non-linearity is critical for $n+1=4$ dimensions in the sense that the Sobolev embedding $H^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ is continuous but just fails to be compact and therefore a straightforward application of variational techniques on $\mathcal{I}_{es}[\mathcal{A}]$ having proved its convexity, coercivity, and lower semi-continuity does not work. This can be handled by means of refined elliptic estimates. Another vital problem that appears is the presence of self-dual solutions that are absolute minimizers of the Euclidean signature Yang-Mills action functional in 4 dimensions and constitute a finite-dimensional moduli space (if the action functional is same in the upper and lower half-spaces for two different self-dual solutions, then the minimization is no longer unique causing trouble). These could in turn prove to be an obstruction to the uniqueness of the minimizer S_0 leading to its not everywhere differentiability property. This, however, does not seem to cause a substantial problem at the *tree* level (semi-classical) but rather causes complications when one attempts to compute the quantum loop corrections to the S_0 functional and obtain the $S_h[A]$ functional which is what one ultimately wants. This is due to the fact that in order to compute the quantum loop corrections to the S_0 functional, one ought to solve a sequence of transport equations that are sourced by the differentiated S_0 functional that is obtained by the minimization procedure. For example, at the level of 1 loop (i.e., $O(\hbar)$), S_1 is obtained by solving the following transport equation

$$- \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{G}^{A_i^P(x) A_j^Q(x')} \frac{\delta S_0}{\delta A_i^P(x)} \frac{\delta S_1}{\delta A_j^Q(x')} + \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{G}^{A_i^P(x) A_j^Q(x')} \frac{\mathfrak{D}}{\mathfrak{D} A_i^P(x)} \frac{\delta S_0}{\delta A_j^Q(x')} = \mathcal{E}_0 \quad (3.12)$$

But, since S_0 appears in a differentiated manner, the transport equation does not seem to make sense at all if S_0 is not differentiable at least almost everywhere in the orbit space. Secondly, the S_0 functional appearing as a source term for the transport equation is acted on by the functional covariant Laplacian. This problem can however be circumvented by employing the gauge-invariant point-splitting regularization procedure mentioned in (3.5).

One could compare this with the formal path integral method to quantize the gauge theory. The formal partition function in the non-covariant Coulomb gauge would read

$$\mathcal{Z} = \int_{\mathcal{A}/\widehat{\mathcal{G}}} [dA] \det(\widehat{\nabla}\partial)(\det \widehat{\nabla}^2)^{-\frac{1}{2}} \delta(\partial^i A_i^P) e^{-\frac{1}{2} \int_I dt \int_{\mathbb{R}^3} A_i^P (\partial_t^2 - \partial^j \partial_j) A_i^P + \mathcal{S}'}, \quad (3.13)$$

where

$$\begin{aligned} \mathcal{S}' := & \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} f^{PRV} \partial_t A_i^P(x) A_i^V(x) \Delta_A^{-1}(x, x') f^{RUQ} A_j^U(x') \partial_t A_i^Q(x') \\ & + \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} f^{NV R} \Delta_A^{-1}(x'', x') f^{VMP} A_j^M(x') A_i^R(x''). \end{aligned} \quad (3.14)$$

One could proceed to compute the tree-level processes and ideally address the convergence of the associated formal series. This complete task, however, can be handled in the Euclidean signature semi-classical or microlocal approach by means of the analysis of the zero energy Hamilton-Jacobi equation (3.10). Secondly the loop divergences are essentially at the similar level as the divergence issue associated with (3.12). In addition, as we shall observe in the later section, the same regularization is required for the Ricci curvature as well necessarily promoting it to a ‘quantum’ object. A natural question would be if one could relate the Ricci curvature to loop amplitudes. In fact, one would wonder is one could deduce the S -matrix elements of Yang-Mills theory by means of geometric information of the orbit space $\mathcal{A}/\widehat{\mathfrak{G}}$ and the gauge invariant potential. This issue is under investigation.

Making sense of the formal series (3.7) is the most difficult part. Even if one regularizes the Hamiltonian, extends it to a self-adjoint operator, and proves the almost everywhere differentiability property of the S_0 functional (since the self-dual solutions form a finite-dimensional space, one would expect an almost everywhere differentiability property), even then the formal series (3.7) never converges. This problem is tied to the whole issue of rigorous quantization of Yang-Mills theory itself which remains open today. Here we assume that there exists a rigorous quantization. In other words, appropriate axioms of the quantum gauge theory are satisfied. In particular, a unique ground state exists that is Póincare invariant and this state has zero energy. This ground state is an element of a separable *Hilbert space* of the theory. Our goal is to present some geometrical arguments that suggest if there is a rigorous quantization of the Yang-Mills fields, then the associated Hamiltonian (suitably regularized) exhibits a positive mass gap. Under such a bold assumption, the ground state wave functional is normalizable

$$\int_{\mathcal{A}/\widehat{\mathcal{G}}} \Psi[A]_g^\dagger \Psi[A]_g \mu_{\mathfrak{G}} = |N_h|^2 \int_{\mathcal{A}/\widehat{\mathcal{G}}} e^{-2S[A]/\hbar} \mu_{\mathfrak{G}} = 1 \quad (3.15)$$

for $N_h \in \mathbb{C} - \{0\}$ and with corresponding eigenvalue E_h^0 . Note that in order to respect the boost-invariance $E_h^0 \equiv 0$ (in fact the whole energy-momentum vector of the ground state must vanish). The formal naive measure $\mu_{\mathfrak{G}} = [DA] \sqrt{\det(\mathfrak{G})}$ does not make sense. However, due to (3.15), we can use $e^{-2S[A]/\hbar} \mu_{\mathfrak{G}}$ as a measure on the orbit space $\mathcal{A}/\widehat{\mathcal{G}}$. Once again, we stress the fact that all of these hold under the assumption that we have a rigorous quantum Yang-Mills theory. The first excited state wave functional may be written as

$$\Psi^*[A] = \varphi[A] e^{-S[A]/\hbar} \quad (3.16)$$

with $\varphi : \mathcal{A}/\widehat{\mathcal{G}} \rightarrow \mathbb{C}$ and energy E_h^* . We are interested in estimating $E_h^* - E_h^0$. The following lemma establishes a gap estimate by means of the lower bound on the *Bakry-Emery* Ricci curvature of the moduli space $\mathcal{A}/\widehat{\mathcal{G}}$.

Remark 3. Normalizability of the ground state yields a measure $e^{-2S[A]/\hbar}\mu_{\mathfrak{G}}$ on $\mathcal{A}/\widehat{\mathcal{G}}$. Having constructed the complete S functional, ideally one should be able to prove an estimate of the type $|S[A]| \geq \|A\|_{H^s(\mathbb{R}^n)}^k$ for an appropriate $k \geq 2$, $s \geq \frac{1}{2}$ and therefore a rapid decay of $e^{-2S[A]/\hbar}$ at infinity (of $\mathcal{A}/\widehat{\mathcal{G}}$). If $e^{-2S[A]/\hbar}\mu_{\mathfrak{G}}$ is rigorously proven to satisfy all the criteria to be a legitimate measure on the orbit space, then one may proceed to perform analysis on $\mathcal{A}/\widehat{\mathcal{G}}$. In particular, with respect to this measure, one could integrate the total divergence term to yield zero i.e.,

$$\int_{\mathcal{A}/\widehat{\mathcal{G}}} \int_{x^1, x^2} \frac{\mathfrak{D}}{\mathfrak{D}A_I^P(x^1)} (\mathfrak{G}^{A_I^P(x^1)A_J^Q(x^2)} \frac{\mathfrak{D}}{\mathfrak{D}A_J^Q(x^2)} \mathfrak{F}[A] e^{-2S[A]/\hbar}) \mu_{\mathfrak{G}} = 0 \quad (3.17)$$

Remark 4. Note that since we are interested in the energy difference $E_h^* - E_h^0$, we do not need to normal order the Hamiltonian in an appropriate way.

Lemma 3.1. Under the assumption of a rigorous existence of the S functional (possibly after a re-summation procedure is applied), the self-adjoint property of the regularized Hamiltonian, and an uniform lower bound of the Bakry-Emery Ricci curvature $\mathcal{R}^{B.E}$

$$\mathcal{R}^{B.E}(\alpha[A]_{\epsilon}, \alpha[A]_{\epsilon}) \geq \Delta \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{G}^{A_I^P(x)A_J^Q(y)} \frac{\mathfrak{D}\varphi_{\epsilon}[A]^{\dagger}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi_{\epsilon}[A]}{\mathfrak{D}A_j^Q(y)}, \quad (3.18)$$

where $\Delta > 0$ and

$$\begin{aligned} \mathcal{R}^{B.E}(\alpha[A]_{\epsilon}, \alpha[A]_{\epsilon}) := & \int_{\mathfrak{K}} \delta_{\chi}(y, z) \Theta^{LM}(y, z) \int_{\mathfrak{K}} \left(\mathfrak{G}^{A_k^M(z)A_k^L(y)} \mathfrak{R}_{A_k^M(z)A_n^N(x'')A_k^L(y)A_i^P(x)} \frac{\mathfrak{D}\varphi_{\epsilon}^{\dagger}}{\mathfrak{D}A_n^N(x'')} \frac{\mathfrak{D}\varphi_{\epsilon}}{\mathfrak{D}A_i^P(x)} \right) \\ & + \frac{2}{\hbar} \mathfrak{G}^{A_I^P(x)A_J^Q(x')} \mathfrak{G}^{A_K^M(y)A_L^N(y')} \frac{\mathfrak{D}}{\mathfrak{D}A_I^P(x)} \frac{\mathfrak{D}S}{\mathfrak{D}A_K^M(y)} \frac{\mathfrak{D}\varphi_{\epsilon}^{\dagger}}{\mathfrak{D}A_J^Q(x')} \frac{\mathfrak{D}\varphi_{\epsilon}}{\mathfrak{D}A_L^N(y')} \end{aligned}$$

the spectral gap $E_h^* - E_h^0$ verifies the bound

$$E_h^* - E_h^0 \geq \frac{\hbar^2 \Delta}{2}. \quad (3.19)$$

Proof. Let us define the following entity

$$\mathcal{Q} := \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{G}_{\chi}^{A_i^P(x)A_j^Q(x')} \Theta^{PQ}(x, x') \left(\frac{\mathfrak{D}\varphi[A]^{\dagger}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi[A]}{\mathfrak{D}A_j^Q(x')} e^{-2S[A]/\hbar} \right) d^n x d^n x' \quad (3.20)$$

and apply the regularized covariant functional Laplacian to yield (denote $\mathbb{R}^n \times \mathbb{R}^n$ by $\mathfrak{K}$)

$$\int_{\mathfrak{K}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}}{\mathfrak{D}A_l^M(z)} \mathcal{Q} \quad (3.21)$$

$$\begin{aligned}
&= \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}}{\mathfrak{D}A_l^M(z)} \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_i^P(x)A_j^Q(x')} \Theta^{PQ}(x, x') \left(\frac{\mathfrak{D}\varphi[A]^{\dagger}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi[A]}{\mathfrak{D}A_j^Q(x')} e^{-2S[A]/\hbar} \right) \\
&= \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_i^P(x)A_j^Q(x')} \Theta^{PQ}(x, x') \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \left(\frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right. \\
&\quad \left. + \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} - \frac{2}{\hbar} \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \right) \\
&= \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_i^P(x)A_j^Q(x')} \Theta^{PQ}(x, x') \left\{ \left(\frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_l^M(z)} \right. \right. \\
&\quad \left. \left. + \Re_{A_l^M(z)A_n^N(x'')A_k^L(y)A_i^P(x)} \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_n^N(x'')} \right) \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} + \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right. \\
&\quad \left. + \left(\frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} + \Re_{A_l^M(z)A_n^N(x'')A_k^L(y)A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_n^N(x'')} \right) \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right. \\
&\quad \left. + \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} - \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_k^L(y)} e^{-2S/\hbar} \right. \\
&\quad \left. - \frac{2}{\hbar} \frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}S}{\mathfrak{D}A_k^L(y)} e^{-2S/\hbar} - \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \left(\frac{\mathfrak{D}\varphi^{\dagger}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \right) \right\}.
\end{aligned}$$

We have utilized the fact that the functional covariant derivative commutes with the parallel propagator (Wilson line). Now notice that the Riemann curvature of the space $\mathcal{A}/\widehat{\mathcal{G}}$ appears in the previous expression, which without regularization would lead to the formal Ricci curvature which would not make sense as a trace of a non-trace class operator. In order to relate this expression to the spectral gap of the regularized Hamiltonian operator $\widehat{H}$, first, recall the following identity

$$\begin{aligned}
\int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \left(\frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} \right) e^{-S/\hbar} d^n y d^n z &= -\frac{2}{\hbar^2} (\widehat{H} - E_h^0) (\varphi e^{-S/\hbar}) \quad (3.23) \\
&+ \frac{2}{\hbar} \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \frac{\mathfrak{D}S}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} e^{-S/\hbar} d^n x.
\end{aligned}$$

Now in order to obtain a lower bound for the spectrum of $\widehat{H} - E_h^0$, we need to manipulate the expression for the entity $\int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}}{\mathfrak{D}A_l^M(z)} \mathcal{Q}$. Under the assumption of the existence of a rigorous quantum field theory, we may take S and φ to be smooth functionals of A . Therefore, under this bold assumption, all the integrals supposedly yield finite values rendering an application of Fubini's theorem to interchange the integrals over $\mathfrak{R}$ whenever necessary. In addition, having assumed the existence of a quantized theory, the regularized operator $\widehat{H} - E_h^0$ can be assumed to be self-adjoint and as a consequence, we may discard the boundary terms that arise in the process. For now let us evaluate $\int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}}{\mathfrak{D}A_l^M(z)} \mathcal{Q}$

$$\int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}}{\mathfrak{D}A_l^M(z)} \mathcal{Q} \quad (3.24)$$

$$\begin{aligned}
&= \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_i^P(x)A_j^Q(x')} \Theta^{PQ}(x, x') \left(-2 \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right. \\
&\quad + \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_i^P(x)} e^{-2S/\hbar} + \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_i^P(x)} e^{-2S/\hbar} \\
&\quad + \Re_{A_l^M(z)A_n^N(x'')A_k^L(y)A_i^P(x)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_n^N(x'')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} + \Re_{A_l^M(z)A_n^N(x'')A_k^L(y)A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_n^N(x'')} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \\
&\quad + \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} + \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \\
&\quad - \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_k^L(y)} e^{-2S/\hbar} - \frac{2}{\hbar} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}S}{\mathfrak{D}A_k^L(y)} e^{-2S/\hbar} \\
&\quad + \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \left(\frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right) + \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \left(\frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right) \\
&\quad \left. - \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \left(\frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \right) \right).
\end{aligned}$$

Now utilizing the identity (3.23), we may write the following

$$\begin{aligned}
&\int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_i^P(x)A_j^Q(x')} \Theta^{PQ}(x, x') \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \\
&= \frac{4}{\hbar^4} \left\{ (\hat{H} - E_h^0)(\varphi e^{-S/\hbar}) \right\} \left\{ (\hat{H} - E_h^0)(\varphi^\dagger e^{-S/\hbar}) \right\} + \frac{2}{\hbar} \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \int_{\mathfrak{R}} \mathfrak{G}_{\chi}^{A_i^P(x)A_j^Q(x')} \\
&\Theta^{PQ}(x, x') \left(\frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}S}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} + \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \right. \\
&\quad \left. - \frac{4}{\hbar^2} \frac{\mathfrak{D}S}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \right)
\end{aligned}$$

substitution of which in the expression for $\int_{\mathfrak{R}} \delta_{\chi}(y, z) \Theta^{LM}(y, z) \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}}{\mathfrak{D}A_k^M(z)} \mathcal{Q}$ yields

$$\int_{\mathfrak{R}} \Theta^{LM}(y, z) \mathfrak{G}_{\chi}^{A_I^L(y)A_J^M(z)} \frac{\mathfrak{D}}{\mathfrak{D}A_I^L(y)} \frac{\mathfrak{D}}{\mathfrak{D}A_J^M(z)} \mathcal{Q} \quad (3.27)$$

$$\begin{aligned}
&= -\frac{8}{\hbar^4} \left\{ (\hat{H} - E_h^0)(\varphi e^{-S/\hbar}) \right\} \left\{ (\hat{H} - E_h^0)(\varphi^\dagger e^{-S/\hbar}) \right\} + \int_{\mathfrak{R}} \mathfrak{G}_\chi^{A_k^L(y)A_l^M(z)} \Theta^{LM}(y, z) \int_{\mathfrak{R}} \mathfrak{G}_\chi^{A_i^P(x)A_j^Q(x')} \Theta^{PQ}(x, x') \\
&\quad \left(\Re_{A_l^M(z)A_n^N(x'')A_k^L(y)A_i^P(x)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_n^N(x'')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} + \Re_{A_l^M(z)A_n^N(x'')A_k^L(y)A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_n^N(x'')} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right. \\
&\quad + \frac{4}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}S}{\mathfrak{D}A_k^M(z)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_k^L(z)} e^{-2S/\hbar} + 2 \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \\
&\quad + \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \left(\frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right) + \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \left(\frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_l^M(z)} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} e^{-2S/\hbar} \right) \\
&\quad - \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_k^L(y)} \left(\frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_i^P(x)} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}S}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \right) - \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \left(\frac{\mathfrak{D}\varphi}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}S}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \right) \\
&\quad \left. - \frac{2}{\hbar} \frac{\mathfrak{D}}{\mathfrak{D}A_i^P(x)} \left(\frac{\mathfrak{D}\varphi^\dagger}{\mathfrak{D}A_j^Q(x')} \frac{\mathfrak{D}\varphi}{\mathfrak{D}A_k^L(y)} \frac{\mathfrak{D}S}{\mathfrak{D}A_l^M(z)} e^{-2S/\hbar} \right) \right)
\end{aligned}$$

Assuming the existence of the $S[A]$ functional and the rapid decay of $e^{-2S[A]/\hbar}$ at infinity, we may neglect the boundary terms while integrating over the reduced configuration space $\mathcal{A}/\widehat{\mathcal{G}}$. In field theories, one would expect the existence of a continuous spectrum and in the current case the spectrum would have a finite gap the bottom. The first excited state is orthogonal to the ground state formally satisfying

$$\int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi_\epsilon[A]^\dagger e^{-S[A]/\hbar} (\hat{H} - E_h^*) \varphi_\epsilon[A] e^{-S[A]/\hbar} \mu_{\mathfrak{G}} \geq 0 \quad (3.29)$$

$$\int_{\mathcal{A}/\widehat{\mathcal{G}}} \left\{ (\hat{H} - E_h^*) \varphi_\epsilon[A] e^{-S[A]/\hbar} \right\}^\dagger (\hat{H} - E_h^*) \varphi_\epsilon[A] e^{-S[A]/\hbar} \mu_{\mathfrak{G}} \leq \epsilon^2 \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi_\epsilon^\dagger \varphi_\epsilon[A] e^{-2S[A]/\hbar} \mu_{\mathfrak{G}} \quad (3.30)$$

$$\int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi_\epsilon[A]^\dagger e^{-2S[A]/\hbar} \mu_{\mathfrak{G}} = 0$$

for any $\epsilon > 0$. Notice that the last condition is essential for the validity of the first condition. Otherwise one could take $\varphi_\epsilon \rightarrow 1$ and yield a contradiction. Application of the Cauchy-Schwartz with respect to the measure $e^{-2S[A]/\hbar} \mu_{\mathfrak{G}}$, using the property (3.29), and expanding $\hat{H} - E_h^*$ formally yields

$$0 \leq \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi_\epsilon[A]^\dagger e^{-S[A]/\hbar} (\hat{H} - E_h^*) \varphi_\epsilon[A] e^{-S[A]/\hbar} \mu_{\mathfrak{G}} \leq \epsilon \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi_\epsilon[A]^\dagger \varphi_\epsilon[A] e^{-2S[A]/\hbar} \mu_{\mathfrak{G}}.$$

Similarly, we may write the following

$$\begin{aligned}
&\int_{\mathcal{A}/\widehat{\mathcal{G}}} \left\{ (\hat{H} - E_h^0) \varphi_\epsilon[A] e^{-S[A]/\hbar} \right\}^\dagger (\hat{H} - E_h^0) \varphi_\epsilon[A] e^{-S[A]/\hbar} \mu_{\mathfrak{G}} \\
&= \int_{\mathcal{A}/\widehat{\mathcal{G}}} \left\{ (\hat{H} - E_h^*) \varphi_\epsilon[A] e^{-S[A]/\hbar} \right\}^\dagger (\hat{H} - E_h^*) \varphi_\epsilon[A] e^{-S[A]/\hbar} \mu_{\mathfrak{G}} \\
&+ \int_{\mathcal{A}/\widehat{\mathcal{G}}} \left((E_h^* - E_h^0)^2 \varphi[A]_\epsilon^* \varphi[A]_\epsilon e^{-2S[A]/\hbar} + 2(E_h^* - E_h^0) \varphi[A]_\epsilon^\dagger e^{-S[A]/\hbar} (\hat{H} - E_h^*) \varphi[A]_\epsilon e^{-S[A]/\hbar} \right) \mu_{\mathfrak{G}} \\
&\leq (E_h^* - E_h^0 + \epsilon)^2 \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi[A]_\epsilon^\dagger \varphi[A]_\epsilon e^{-2S[A]/\hbar} \mu_{\mathfrak{G}}
\end{aligned}$$

where we have utilized (3.31) and (3.30). Now performing an exact similar calculation as (3.27) but with $\varphi_\epsilon e^{-S[A]/\hbar}$, we obtain

$$(E_h^* - E_h^0 + \epsilon)^2 \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi[A]_\epsilon^\dagger \varphi[A]_\epsilon e^{-2S[A]/\hbar} \mu_{\mathfrak{G}}$$

$$\begin{aligned}
&\geq \int_{\mathcal{A}/\widehat{\mathcal{G}}} \left\{ (\widehat{H} - E_h^*) \varphi_\epsilon[A] e^{-S[A]/h} \right\}^\dagger (\widehat{H} - E_h^*) \varphi_\epsilon[A] e^{-S[A]/h} \mu_{\mathfrak{G}} \\
&\geq \frac{\hbar^4}{4} \left[\int_{\mathcal{A}/\widehat{\mathcal{G}}} \mathfrak{Ricci}(\alpha_\epsilon[A], \alpha_\epsilon[A]) \mu_{\mathfrak{G}} e^{-2S/h} \right. \\
&\quad \left. + \frac{2}{\hbar} \int_{\mathcal{A}/\widehat{\mathcal{G}}} \left\{ \int_{\mathfrak{K}} \int_{\mathfrak{K}} \mathfrak{G}^{A_I^P(x) A_J^Q(x')} \mathfrak{G}^{A_K^M(y) A_L^N(y')} \frac{\mathfrak{D}}{\mathfrak{D} A_I^P(x)} \frac{\mathfrak{D} S}{\mathfrak{D} A_K^M(y)} \frac{\mathfrak{D} \varphi_\epsilon^\dagger}{\mathfrak{D} A_J^Q(x')} \frac{\mathfrak{D} \varphi_\epsilon}{\mathfrak{D} A_L^N(y')} \right\} \mu_{\mathfrak{G}} e^{-2S/h} \right],
\end{aligned}$$

where

$$\mathfrak{Ricci}(\alpha_\epsilon[A], \alpha_\epsilon[A]) := \int_{\mathfrak{K}} \delta_\chi(y, z) \Theta^{LM}(y, z) \int_{\mathfrak{K}} \left(\mathfrak{G}^{A_k^M(z) A_k^L(y)} \mathfrak{R}_{A_k^M(z) A_n^N(x'') A_k^L(y) A_i^P(x)} \frac{\mathfrak{D} \varphi_\epsilon^\dagger}{\mathfrak{D} A_n^N(x'')} \frac{\mathfrak{D} \varphi_\epsilon}{\mathfrak{D} A_i^P(x)} \right).$$

Now expanding $\widehat{H} - E_h^0$, consider the following identity

$$\frac{2}{\hbar^2} \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi_\epsilon[A]^\dagger e^{-S/h} (\widehat{H} - E_h^0) \varphi_\epsilon[A] e^{-S/h} = \int_{\mathcal{A}/\widehat{\mathcal{G}}} \left(\int_{\mathbb{R}^n} \frac{\mathfrak{D} \varphi_\epsilon[A]^\dagger}{\mathfrak{D} A_i^P(x)} \frac{\mathfrak{D} \varphi_\epsilon[A]}{\mathfrak{D} A_i^P(x)} \right) e^{-2S/h} \quad (3.31)$$

and assume that The Bakry-Emery Ricci curvature verifies the point-wise bound

$$\begin{aligned}
\mathfrak{Ricci}(\alpha_\epsilon[A], \alpha_\epsilon[A]) + \frac{2}{\hbar} \mathfrak{G}^{A_I^P(x) A_J^Q(x')} \mathfrak{G}^{A_K^M(y) A_L^N(y')} \frac{\mathfrak{D}}{\mathfrak{D} A_I^P(x)} \frac{\mathfrak{D} S}{\mathfrak{D} A_K^M(y)} \frac{\mathfrak{D} \varphi_\epsilon^\dagger}{\mathfrak{D} A_J^Q(x')} \frac{\mathfrak{D} \varphi_\epsilon}{\mathfrak{D} A_L^N(y')} \quad (3.32) \\
\geq \Delta \int_{\mathbb{R}^n} \frac{\mathfrak{D} \varphi_\epsilon[A]^\dagger}{\mathfrak{D} A_i^P(x)} \frac{\mathfrak{D} \varphi_\epsilon[A]}{\mathfrak{D} A_i^P(x)}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&(E_h^* - E_h^0 + \epsilon)^2 \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi[A]_\epsilon^\dagger \varphi[A]_\epsilon e^{-2S[A]/h} \mu_{\mathfrak{G}} \\
&\geq \frac{\hbar^4}{4} \Delta \int_{\mathcal{A}/\widehat{\mathcal{G}}} \left(\int_{\mathbb{R}^n} \frac{\mathfrak{D} \varphi_\epsilon[A]^\dagger}{\mathfrak{D} A_i^P(x)} \frac{\mathfrak{D} \varphi_\epsilon[A]}{\mathfrak{D} A_i^P(x)} \right) e^{-2S/h} = \Delta \frac{\hbar^2}{2} \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi_\epsilon[A]^\dagger e^{-S/h} (\widehat{H} - E_h^0) \varphi_\epsilon[A] e^{-S/h} \\
&= \Delta \frac{\hbar^2}{2} \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi_\epsilon[A]^\dagger e^{-S/h} (\widehat{H} - E_h^* + E_h^* - E_h^0) \varphi_\epsilon[A] e^{-S/h} \\
&\geq (E_h^* - E_h^0) \frac{\hbar^2}{2} \Delta \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi[A]_\epsilon^\dagger \varphi[A]_\epsilon e^{-2S[A]/h} \mu_{\mathfrak{G}}
\end{aligned}$$

or

$$(E_h^* - E_h^0) + \frac{\epsilon^2}{E_h^* - E_h^0} + 2\epsilon \geq \frac{\hbar^2 \Delta}{2} \quad \forall \epsilon > 0 \quad (3.33)$$

yielding

$$E_h^* - E_h^0 \geq \frac{\hbar^2 \Delta}{2}. \quad (3.34)$$

Notice that $E_h^* - E_h^0$ can not be zero since that would imply $\frac{\hbar^4}{4} \Delta \int_{\mathcal{A}/\widehat{\mathcal{G}}} \left(\int_{\mathbb{R}^n} \frac{\mathfrak{D} \varphi_\epsilon[A]^\dagger}{\mathfrak{D} A_i^P(x)} \frac{\mathfrak{D} \varphi_\epsilon[A]}{\mathfrak{D} A_i^P(x)} \right) e^{-2S/h} \leq \epsilon^2 \int_{\mathcal{A}/\widehat{\mathcal{G}}} \varphi[A]_\epsilon^\dagger \varphi[A]_\epsilon e^{-2S[A]/h} \mu_{\mathfrak{G}} \quad \forall \epsilon > 0$ yielding a contradiction to non-constancy of $\varphi[A]$ (and hence its orthogonality to the ground state). This concludes the proof of the lemma. $\square$

Remark 5. Note that the term $\int_{\mathfrak{R}} \delta_{\chi}(y, z) \Theta^{LM}(y, z) \int_{\mathfrak{R}} \left(\mathfrak{G}^{A_k^M(z) A_l^L(y)} \mathfrak{R}_{A_k^M(z) A_n^N(x'') A_l^L(y) A_i^P(x)} \right) + \frac{2}{\hbar} \int_{\mathfrak{R}} \frac{\mathfrak{D}}{\mathfrak{D} A_i^P(x)} \frac{\mathfrak{D} S}{\mathfrak{D} A_k^M(y)}$ is nothing but the Bakry-Emery Ricci curvature of the configuration space $\mathcal{A}/\widehat{\mathcal{G}}$.

Remark 6. Notice that Ricci curvature always requires regularization indicating its certain “quantum” nature. In the perturbation theory calculations, one can show that Ricci curvature shows up in the expression of the loop amplitudes. In fact a catuarl conjecture would be that the re-normalization group flow of the metric on the moduli space is a type of infinite dimensional Ricci flow.

Notice an important point: The regularized Ricci curvature together with the Hessian of the functional S constitute the so-called functional *Bakry-Emery* Ricci tensor. In the finite-dimensional setting, it appears as the ordinary *Bakry-Emery* Ricci tensor. We note studies of this Ricci tensor that naturally appear in the study of weighted manifolds by [21, 27]. Our setting could formally be an infinite dimensional version of a weighted manifold of the type $(\mathcal{A}/\widehat{\mathcal{G}}, \mathfrak{G}, e^{-2S/\hbar})$. This microlocal or Euclidean signature semi-classical technique can be used to study the quantum mechanical systems satisfying suitable conditions (see [5] for the study of the nonlinear anharmonic oscillators).

4. Explicit calculations for the gap in 2 and 3 dimensions

Since the Ricci curvature appears in a regularized way, we can explicitly compute it and later take the limit $\chi \rightarrow \infty$. However, doing so would inevitably introduce infinities (an ultraviolet divergence; since χ has a dimension of inverse length). The regular value of the Ricci curvature is then obtained by subtracting these infinities. Recall at the level of perturbation theory, one would remove the infinities starting at the level of the action by adding counter terms such that those counter terms generate the exact infinities (at the loop level) with opposite signs and therefore a cancellation occurs (in the process one obtains scaling differential equations for the coupling constants). Renormalizability of the Yang-Mills theory ([1]) suggests that one only requires a finite number of counterterms to cancel out the infinities.

Lemma 4.1. *The formal Ricci curvature satisfies the following expression at the flat connection $\widehat{A} = 0$ in terms of the cut-off parameter χ in 2 and 3 spatial dimensions⁷*

$$\begin{aligned} \mathfrak{Ric}_{\chi}(\alpha, \alpha) &= \frac{3}{4\pi} \delta^{BP} f^{ABC} f^{APQ} \int_x \left(2e^{-\chi^2|x|^2} \ln\left(\frac{|x|}{|x_0|}\right) - Ei(-\chi^2|x|^2) \right) \alpha^C(x) \alpha^Q(x) d^2x \\ &= \frac{3C_2(G)g_{YM}^2}{4\pi} \int_x \left(Ei(-\chi^2|x|^2) - 2e^{-\chi^2|x|^2} \ln\left(\frac{|x|}{|x_0|}\right) \right) \alpha^P(x) \alpha^P(x) d^2x, \quad n=2, \\ \mathfrak{Ric}_{\chi}(\alpha, \alpha) &= \frac{3\chi C_2(G)g_{YM}^2}{2\pi^3} \int_x e^{-\chi^2|x|^2} \alpha^P(x) \alpha^P(x) d^3x, \quad n=3, \end{aligned} \quad (4.2)$$

where $C_2(G)$ is the Casimir invariant for the adjoint representation of the compact gauge group $G = SU(N)$. Here $|x_0|$ is a reference constant with dimension of length.

Proof. At the flat connection $\widehat{A} = 0$, the operator $\Delta_{\widehat{A}}^{-1}$ reduces to the ordinary inverse Laplacian Δ^{-1} on $\mathbb{R}^n$. Now recalling $\Delta^{-1}(x, x') = \frac{1}{2} \ln \left| \frac{x-x'}{|x_0|} \right|$ for $n=2$, $\Delta^{-1}(x, x') =$

⁷ $Ei(x)$ is the exponential integral function $Ei(x) := \int_{-\infty}^x e^t t^{-1} dt$

$-\frac{1}{4\pi} \frac{1}{|x-x'|}$ for $n = 3$, we write the coincident limit by means of the point-splitting delta distribution δ_χ as appears in the mass gap integral of lemma (3.1) to yield

$$\begin{aligned} & \Re ic_\chi(\alpha, \alpha) \quad (4.3) \\ &= \frac{3\chi^2}{8\pi^3} \delta^{BP} f^{ABC} f^{APQ} \int_x \alpha^C(x) \alpha^Q(x) \left(\int_{|x|}^\infty r \ln(r/|x_0|) e^{-\chi^2 r^2} dr \right) d^2x, \quad n = 2 \end{aligned}$$

$$\begin{aligned} & \Re ic_\chi(\alpha, \alpha) \quad (4.4) \\ &= -\frac{12\chi^3}{\pi^2} \delta^{BP} f^{ABC} f^{APQ} \int_x \alpha^C(x) \alpha^Q(x) \left(\int_{|x|}^\infty r e^{-\chi^2 r^2} dr \right) d^3x, \quad n = 3. \end{aligned}$$

Explicit integration and recalling $f^{ABC} f^{ABQ} = -C_2(G) g_{YM}^2 \delta^{CQ}$ (note that our definition of the commutator is $[X, Y]^A = f^{ABC} X^B Y^C$ i.e., a factor $i = \sqrt{-1}$ is absorbed in the structure constants and that g_{YM} is the Yang-Mills coupling constant), we obtain the result. $\square$

Now we perform an elementary dimensional analysis and argue that for $3 + 1$ dimensional Yang-Mills theory, one needs to introduce a length scale L in order to obtain a mass gap. We set the light speed c equals to 1. With this convention, we have $[t] = [x] = L$. The classical action $\int_{\mathbb{R}^{1,3}} \langle F, F \rangle dt d^3x$ has the dimension of $\hbar$. Therefore A has the dimension of $\frac{\hbar^{\frac{1}{2}}}{L}$ and g_{YM}^2 has the dimension of $\frac{1}{\hbar}$. Now according to (4.2), the dimension of $\frac{\Re ic_\chi(\alpha, \alpha)}{\int_{\mathbb{R}^{1,3}} |\alpha|^2}$ is $\frac{1}{\hbar L}$ since χ has dimension $\frac{1}{L}$. Therefore Δ in (3.19) of lemma 3.1 has dimension $\frac{1}{\hbar L}$ yielding the dimension of $E_h^* - E_h^0$ to be $\frac{\hbar}{L}$ which is the correct dimension of energy. Therefore, the introduction of a finite χ (inverse length) is absolutely necessary to generate a energy gap in the quantum Yang-Mills theory in $3 + 1$ dimensions.

Contrary to $3 + 1$ dimensions, in the chosen convention $c = 1$, the Yang-Mills coupling constant has an appropriate dimension. In other words, g_{YM}^2 has the dimension of $\frac{1}{\hbar L}$ that yields a dimension of $\frac{1}{\hbar L}$ for the Bakry-emery bound Δ from (4.1). This in turn implies that the gap $E_h^* - E_h^0$ has the correct dimension of $\frac{\hbar}{L}$. Therefore, we do not need to introduce an additional length scale for the energy gap in $2 + 1$ dimensional quantum Yang-Mills theory.

Remark 7. Now note an extremely important fact that the function $-2e^{-\chi^2|x|^2} \ln(\frac{|x|}{|x_0|}) + Ei(-\chi^2|x|^2)$ that appears in the case of $2 + 1$ Yang-Mills theory behaves near $x = 0$ as $\gamma + \ln \chi |x_0|$ (γ is Euler's constant and x_0 is an arbitrary reference length scale multiplied to make the argument of the Logarithm dimensionless) and therefore exhibits divergence as $\chi \rightarrow \infty$ (length $\rightarrow 0$) as expected. This function exhibits rapid decay away from 0 and after subtracting the infinite part $\ln \chi$ and setting $\chi \rightarrow \infty$, one obtains a finite positive value $\approx \gamma$.

Remark 8. In $3 + 1$ Yang-Mills theory, the scenario is substantially different since g_{YM} is dimensionless and thus one can not create a mass out of the occurring constants (i.e., $g_{YM}, \hbar, c = 1$) (purely on dimensional grounds). In order to generate the dimension of mass, one must introduce an additional length scale as we have demonstrated previously. However, this naturally appears through the regularization process as one can not simply eliminate χ . Therefore in the $3 + 1$ case, one may not take the limit $\chi \rightarrow \infty$ (or length approaching zero) but set it to $\frac{1}{L}$, $L > 0$. This L is then to be fixed possibly by measuring the mass of the lowest glue-ball state. Roughly the Ricci curvature is proportional to $\frac{3C_2(G)g_{YM}^2}{2\pi^3 L}$. Notice that $1/L$ may be absorbed in the coupling constant, g_{YM} to force it to vary with distance i.e., g_{YM} becomes a running coupling constant $g_{YM} = \sqrt{L} g'_{YM}$, where g'_{YM} has dimension $[\text{length}]^{-1/2}$ to make g_{YM} dimensionless. In order for g'_{YM} to be finite as $L \rightarrow 0$, $g_{YM} \rightarrow 0$ as $L \rightarrow 0$ indicating a free theory at low length or high energy scale.

Remark 9. Now one would wonder the physical significance of the scaling of the coupling $g_{YM}^2 \sim L$. In other words how is absorbing L into the bare coupling g^{YM} is justified? Notice that in the gap estimate one has the pure curvature contribution $E_h^* - E_h^0 \geq \frac{\int_{A/\widehat{G}} \text{Ricci}_{reg}(\alpha[A], \alpha[A]) e^{-2S/h}}{\int_{A/\widehat{G}} \mathfrak{G}(\alpha[A], \alpha[A]) e^{-2S/h}}$. This is essentially the expectation value of the regularized Ricci curvature Ricci_{reg} with respect the measure $e^{-2S/h} \mu_{\mathfrak{G}}$. Now if one did a perturbative calculation to compute this expectation value of the regularized Ricci, then the quantum corrections that are logarithms but in the fully non-perturbative regime, collection of all the terms and possible re-summation would yield a linear ($\sim L$) behaviour. At least from a physical perspective, this is extremely interesting since the tracing operation on infinite dimensional geometry can potentially explain the non-perturbative running of the coupling constant.

Remark 10. In $3 + 1$ dimensions, the introduction of a length scale L , introduces another scale the mass m_0 of the lowest glue-ball state. Essentially the ratio of the two scales $m_0 L$ is the meaningful entity.

In the previous section, we observed that at the flat connection $A = 0$, the regularized Ricci tensor enjoys a positive definite property. In this particular case the elliptic operator that appears is simply the inverse Laplacian which made the explicit calculations possible. However, away from the flat connection $A = 0$, one ought to regularize the trace of the inverse gauge-covariant Laplacian $\Delta_{\widehat{A}}^{-1}$. While the spectrum is still positive, it is difficult to perform explicit calculations. Nevertheless, we may still prove that the trace of the regularized operator has a strictly positive lower bound. Recall the identity

$$\lambda^{-s} = \frac{1}{\Gamma[s]} \int_0^\infty t^{s-1} e^{-t\lambda} dt, \quad (4.5)$$

where $\Gamma[s]$ is the gamma function $\int_0^\infty t^{s-1} e^{-t} dt$ that has discrete poles for negative s . The above formula is valid for any $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > 0$. Now recall the definition of the heat kernel associated with the positive elliptic operator $\Delta_{\widehat{A}}$

$$e^{t\Delta_{\widehat{A}}} := \int_{\text{Spec}(\Delta_{\widehat{A}})} e^{-t\lambda} dE_\lambda, \quad (4.6)$$

where E_λ is the spectral resolution of $\Delta_{\widehat{A}}|_{H^2}$ in L^2 . We have the following proposition for a L^2 section of the bundle $\mathfrak{P}_{Ad, \mathfrak{g}}$ (assuming that the kernel of the gauge-covariant derivative $\widehat{\nabla}$ is trivial which is the case for irreducible connections; irreducible connections are generic).

Proposition 4.1. *The heat kernel $e^{t\Delta_{\widehat{A}}}$ is smoothing on $L^2(\mathbb{R}^n)$, more precisely*

$$\|e^{t\Delta_{\widehat{A}}} f\|_{H^{2k}(\mathbb{R}^n)} \lesssim (1 + t^{-k}) \|f\|_{L^2(\mathbb{R}^n)} \quad \forall k \in \mathbb{Z}^+. \quad (4.7)$$

Proof. For a section of the bundle $\mathfrak{P}_{AD, \mathfrak{g}}$, the natural gauge invariant Sobolev norm of order $2k$ is defined by means of the positive operator $(-\Delta_{\widehat{A}})^k$ i.e., for a compactly supported section h of the bundle $\mathfrak{P}_{Ad, \mathfrak{g}}$,

$$\|h\|_{H^k}^2 := \sum_{I=0}^k \int_{\mathbb{R}^n} \langle h, (-\Delta_{\widehat{A}})^I h \rangle. \quad (4.8)$$

Now

$$\|(-\Delta_{\widehat{A}})^I e^{t\Delta_{\widehat{A}}} f\|_{L^2} = \left(\int_0^\infty (\lambda^I e^{-t\lambda})^2 d\|E_\lambda f\|_{L^2}^2 \right)^{\frac{1}{2}} \leq \sup_{\lambda \in (0, \infty)} (\lambda^I e^{-t\lambda}) \|f\|_{L^2} \leq (I/t)^I e^{-I} \|f\|_{L^2}$$

and therefore

$$\|e^{t\Delta_{\hat{A}}}f\|_{H^{2k}} \leq C \left(1 + \sum_{I=0}^k \left(\frac{I}{t}\right)^I e^{-I}\right) \|f\|_{L^2} \lesssim (1 + t^{-k})\|f\|_{L^2}. \quad (4.9)$$

□

Using this heat kernel, we may therefore formally write the following

$$\begin{aligned} (-\Delta_{\hat{A}})^{-s}(x, x')f(x') &:= \int_{\text{Spec}(\Delta_{\hat{A}})} \lambda^{-s} dE_{\lambda}f(x) = \frac{1}{\Gamma[s]} \int_{\text{Spec}(\Delta_{\hat{A}})} \int_0^{\infty} t^{s-1} e^{-t\lambda} dt dE_{\lambda}f(x) \\ &= \frac{1}{\Gamma[s]} \int_0^{\infty} t^{s-1} \left(\int_{\text{Spec}(-\Delta_{\hat{A}})} e^{-t\lambda} dE_{\lambda}f(x) \right) dt = \frac{1}{\Gamma[s]} \int_0^{\infty} t^{s-1} \left(e^{t\Delta_{\hat{A}}}(x, x')f(x') \right) dt \end{aligned} \quad (4.10)$$

where we have used the boundedness of the inner integral $\int_0^{\infty} t^{s-1} e^{-t\lambda} dt$ for $\lambda > 0$ to interchange the order of the integrals. This integral can have the problem of producing infinities near $t = 0$ and $t = \infty$. The later happens if the $\text{Spec}(-\Delta_{\hat{A}})$ contains zero or negative numbers. This is the so called *infrared* divergence issue while divergence at $t = 0$ is essentially the *ultraviolet* divergence issue. Denoting $e^{t\Delta_{\hat{A}}}(x, y)$ as $K^{\hat{A}}(t; x, y)$ the previous expression may also be expressed as follows

$$(-\Delta_{\hat{A}})^{-s}(x, x')f(x') = \frac{1}{\Gamma[s]} \int_0^{\infty} t^{s-1} \left(\int_{x'} K^{\hat{A}}(x, x')f(x') dx' \right) dt. \quad (4.11)$$

Now let us write down a formal power series expansion of $K^{\hat{A}}(t; x, y)$ as $t \rightarrow 0$

$$K^{\hat{A}}(t; x, y) = K(t; x, y)(1 + ta_1(x, y) + t^2 a_2(x, y) + \dots), \quad (4.12)$$

where $K(t; x, y) = \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}}$ is the usual heat Kernel on $\mathbb{R}^n$. The coincident limits $\{a_k(x, x)\}$ are local invariants (invariant polynomials of curvature) given in terms of the curvature of the connection $\hat{A}$. On $\mathbb{R}^n$ equipped with the flat metric, one may find through explicit calculations that $a_1(x, x) = 0$, $a_2(x, x) = \frac{1}{12} F^P[\hat{A}]_{ij} F^P[\hat{A}]^{ij}$ i.e,

$$K^{\hat{A}}(t; x, x) = K(t; x, x) \left(1 + \frac{t^2}{12} F^P[\hat{A}]_{ij} F^P[\hat{A}]^{ij} + O(t^3) \right). \quad (4.13)$$

Setting Λ to be a small but fixed positive number, we write the trace integral as follows

$$\begin{aligned} I_{\epsilon} &= \int_{\mathbb{R}^n} \alpha(x) (-\Delta_{\hat{A}})^{-s}(x, x) \alpha(x) d^n x \\ &= \frac{1}{\Gamma[s]} \int_{\mathbb{R}^n} \alpha(x) \left(\int_{\epsilon}^{\Lambda} t^{s-1} K(t; x, x) \left(1 + \frac{t^2}{12} F^P[\hat{A}]_{ij} F^P[\hat{A}]^{ij} + O(t^3) \right) \right. \\ &\quad \left. + \int_{\Lambda}^{\infty} t^{s-1} K^{\hat{A}}(t; x, x) dt \right) \alpha(x) d^n x. \end{aligned} \quad (4.14)$$

We recover the original integral after taking the limit $\epsilon \rightarrow 0$ in a suitable way. Note that the infrared divergence is absent since the spectrum of $\Delta_{\hat{A}}$ does not contain zero or negative modes (generic connections are considered) yielding a finite positive contribution from the integral

$\frac{1}{\Gamma[s]} \int_{\mathbb{R}^n} \alpha(x) \left(\int_{\Lambda}^{\infty} t^{s-1} K^{\hat{A}}(t; x, x) dt \right) \alpha(x) d^n x$. The problem of ultraviolet divergence occurs at the flat value necessarily since $K(t; x, x) = \frac{1}{(4\pi t)^{\frac{n}{2}}}$ and therefore $\int_{\epsilon}^{\Lambda} t^{s-1-\frac{n}{2}} dt$ yields a $\log \epsilon$ divergence for $n = 2$ and $\epsilon^{-\frac{1}{2}}$ divergence for $n = 3$ at $s = 1$ as expected from the previous lemma concerning the trace of Δ^{-1} . This is natural since $t \approx [\text{length}]^2$, Δ^{-1} in 2 and 3 dim essentially behaves like $\ln[\text{Length}/\text{Length}_0]$ (for some arbitrary reference length Length_0) and $1/[\text{Length}]$, respectively. This ultraviolet divergence is then regularized by means of the previous lemma 4.1. The following lemma yields an estimate for the finite part of the Ricci curvature away from the flat connection $\hat{A} = 0$

Lemma 4.2. *The regularized Ricci quadratic form $\int_{\mathbb{R}^n} \alpha(x)(-\Delta)_{\hat{A}}^{-s}(x, x)\alpha(x)d^n x$ satisfies*

$$F.P \left\{ \int_{\mathbb{R}^n} \alpha(x)(-\Delta)_{\hat{A}}^{-1}(x, x)\alpha(x)d^n x \right\} > F.P \left\{ \left(\int_{\mathbb{R}^n} \alpha(x)(-\Delta)_{\hat{A}}^{-1}(x, x)\alpha(x)d^n x \right)_{\hat{A}=0} \right\} + O(\Lambda^{4-\frac{n}{2}}),$$

$n = 2, 3$, where $F.P$ denotes the finite part.

Proof. Note the fact that $t^{s-1+k-\frac{n}{2}}$ is integrable at zero with $s = 1$ for $k > \frac{n}{2} - 1$. Therefore recalling the divergences that occur near $t = 0$, we obtain

$$\begin{aligned} F.P \left\{ \int_{\mathbb{R}^n} \alpha(x)(-\Delta)_{\hat{A}}^{-1}(x, x)\alpha(x)d^n x \right\} &\geq F.P \left\{ \left(\int_{\mathbb{R}^n} \alpha(x)(-\Delta)_{\hat{A}}^{-1}(x, x)\alpha(x)d^n x \right)_{\hat{A}=0} \right\} \\ &\quad + \frac{\Lambda^{3-\frac{n}{2}}}{12(3-\frac{n}{2})} \int_{\mathbb{R}^n} \alpha(x) F[\hat{A}(x)]_{ij} F[\hat{A}(x)]^{ij} \alpha(x) d^n x + O(\Lambda^{4-\frac{n}{2}}) \\ &> F.P \left\{ \left(\int_{\mathbb{R}^n} \alpha(x)(-\Delta)_{\hat{A}}^{-1}(x, x)\alpha(x)d^n x \right)_{\hat{A}=0} \right\} + O(\Lambda^{4-\frac{n}{2}}). \end{aligned} \tag{4.15}$$

□

Remark 11. *Note that away from the flat connection, the finite part of the Ricci quadratic form is modified by a strictly positive entity at the leading order.*

5. Concluding Remarks

Topology and geometry of the configuration space of the classical gauge theory have been studied previously [11, 12, 39]. At the classical level, the geometry of the configuration space is not known to play a vital role in the sense that one does not require the geometric information of the configuration space (its curvature, etc) while studying local Cauchy problems and even in understanding the long *time* dynamics. Classical Yang-Mills fields are globally well-posed on both $\mathbb{R}^{1+2}$ and $\mathbb{R}^{1+3}$ and in the proof of such global well-posedness [40, 41, 42] nowhere does the geometry of the configuration space enter crucially. It is suspected however that the geometry of the configuration space has an important role to play at the level of quantum field theory. While very little effort is paid towards understanding the role of the geometry of the classical configuration space in quantized field theory in contemporary high energy physics, it is certainly worth the attention. In a finite-dimensional setting, sharp estimates on the spectrum of the Hamiltonian operator of a quantum theory are obtainable through Lichnerowicz-type estimates on a constructed weighted manifold under a suitable convexity assumption on the potential [8]. In addition, several results on the estimates of the spectrum of the Schrodinger operator are available [43, 44, 45]. At the level of field theory,

this is much more delicate since the operators do not make sense without appropriate regularization. Even after one performs such regularization, making sense of a rigorous quantum theory requires new novel ideas that are yet to be thought of.

The loop corrections to the tree solution (semi-classical approximation) obtained by solving the functional Hamilton-Jacobi equation require regularization. This is because the tree contribution appears as a source term acted upon by the functional Laplacian in the transport equations for the quantum corrections. At the semi-classical level, no such regularization is required since the functional Hamilton-Jacobi equation does not involve singular operators. This singular nature of the operators appearing at the loop level is essentially related to the divergences associated with the quantum field theory. Similarly, notice that the Riemann curvature of the configuration space is a purely classical object. However, to define the Ricci curvature, we had to invoke the same regularization scheme, and as such the formal non-regularized Ricci curvature contains ultraviolet divergence terms. This hints towards a conclusion that the quantum field theory is affected by the geometry at the level of Ricci curvature (Bakry-Emery Ricci where the potential contribution is considered). Appropriate invariants of the Riemann curvature should show up at the tree-level scattering amplitudes. It is almost certainly expected that the Ricci curvature would inevitably show up when one tries to compute the loop amplitudes indicating a *quantum*-nature of the Ricci curvature of this infinite-dimensional configuration space. A natural conjecture would be that the renormalization group flow for the metric corresponds to a forced (due to the presence of the potential term) infinite dimensional Ricci flow. This should result in a flat metric at the high energy limit indicating the asymptotic freedom.

Another example of the orbit space geometry playing an important role is the case of scalar electrodynamics where photons remain gap-less due to vanishing Riemann curvature of the orbit space of the $U(1)$ connections while moduli of charged scalar fields are shown to have a non-vanishing curvature [8]. It should be interesting to consider the large N limit of $SU(N)$ non-abelian gauge theories since at such a limit at least the perturbative theory is simplified. In the finite-dimensional setting, it would be worth investigating $2 + 1$ gravity from this perspective since the geometry of Teichmüller space (the configuration space of the classical theory) is well understood. These issues are under intense investigation.

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