Abstract

Stability and Convergence Issues in Mathematical General Relativity

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Since its conception, the FLRW cosmological solution has been used extensively to model the dynamics of our universe. With the advent of accelerated expansion of the physical universe, a cosmological constant (the simplest form of the dark energy) is often included in the model and such a model is designated as the ΛCDM model. This model is developed based on the assumption of *global* homogeneity and isotropy of the physical universe, the so-called *cosmological principle*. One important question that arises in the context of FLRW cosmology is whether these models are predicted by general relativity? To conclude that they are would seem to hinge on a proof that the purely theoretical variants of FLRW models (spatially compact models foliated by compact hyperbolic spaces) are dynamically stable. In other words, if one were to perturb these solutions in a suitable function space setting, then the natural question would be whether the perturbed spacetimes exist for all *time* as solutions of Einstein's field equations. Moreover one would like to understand whether all small data solutions (in suitable function space settings after subtracting the background solutions of the Einstein's field equations remain bounded for all time in terms of a suitable norm of the initial data and whether they exhibit asymptotic decay. There are other important physical motivations such as large-scale structure formation in the universe through cosmological perturbations. In order to understand the global evolution problem in the context of mathematical cosmology, one ought to address the interaction of the topology and geometry with the matter (or radiation) present in the spacetime. Since gravity is a highly non-linear theory, the interaction can possibly focus energy and generate singularities in finite time. Of course such singularities in the spacetime are well known in the context of black holes. However, they are hidden behind a horizon and therefore inaccessible by any observer in the domain of outer communication. In a sense these do not really imply a breakdown of the general theory of relativity. The singularities that can cause pathological breakdown of general relativity are the so called naked singularities and as such they are ruled out by Penrose's cosmic censorship conjecture. In order to even make a quantum jump towards proving such a conjecture would require a rigorous analysis of the dynamics of spacetime topology and geometry. In this thesis, we address a collection of such studies for a class of spacetimes in the regime of both 2+1 and $n+1$ ($n \geq 3$) gravity in the presence and absence of matter source. We prove a series of

stability theorems and in the process also discover several implications of non-trivial topology on the dynamics of the spacetimes.

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I Am Because We Are

Introduction

Viewed on a sufficiently coarse-grained scale the portion of our universe that is accessible to observation appears to be spatially homogeneous and isotropic. If, as is usually imagined, one should be able to extrapolate these features to (a suitably coarse-grained model of) the universe as a whole then only a handful of spatial manifolds need be considered in cosmology- the familiar Friedmann-Lematre-Robertson-Walker (FLRW) archetypes of constant positive, vanishing or negative curvature [3, 107]. These geometries consist, up to an overall, time-dependent scale factor, of the 3-sphere, $\mathbb{S}^3$, with its canonical ?round? metric, Euclidean 3-space, $\mathbb{E}^3$, hyperbolic 3-space, $\mathbb{H}^3$ and the quotient space $\mathbb{RP}(3) = \mathbb{S}^3/\mathbb{Z}_2$ obtainable from $\mathbb{S}^3$ by the identification of antipodal points [4]. Of these possibilities only the sphere and its 2-fold quotient $\mathbb{RP}^3$ are closed and thus compatible with a universe model of finite extent. It is not known of course whether the actual universe is spatially closed or not but, to simplify the present discussion, we shall limit our attention herein to models that are. More precisely we shall focus on spacetimes admitting Cauchy hypersurfaces that are each diffeomorphic to a smooth, connected 3-manifold that is compact, orientable and without boundary.

On the other hand if one takes literally the cosmological principle that only manifolds supporting a globally homogeneous and isotropic metric should be considered in models for the actual universe then, within the spatially compact setting considered here, only the 3-sphere and $\mathbb{RP}^3$ would remain. But the astronomical observations which motivate this principle are necessarily limited to a (possibly quite small) fraction of the entire universe and are compatible with models admitting metrics that are only locally, but not necessarily globally, spatially homogeneous and isotropic. As is well-known there are spatially compact variants of all of the basic Friedmann-Lematre- Robertson-Walker cosmological models, mathematically constructable (in the cases of vanishing or negative curvature) by taking suitable compact quotients of Euclidean 3-space $\mathbb{E}^3$ or of hyperbolic 3-space $\mathbb{H}^3$. One can also take infinitely many possible quotients of $\mathbb{S}^3$ to obtain the so-called spherical space forms that are locally compatible with the FLRW constant positive curvature geometry but are no longer diffeomorphic to the 3-sphere.

Still more generally though we shall find that there is a dynamical mechanism at work within the

Einstein 'flow', suitably viewed in terms of the evolution of 3-manifolds to develop 4-dimensional, globally hyperbolic spacetimes, and extended to include suitable matter sources and a positive cosmological constant Λ, that strongly suggests that even manifolds that do not admit a locally homogeneous and isotropic metric at all will nevertheless evolve in such a way as to be asymptotically compatible with the observed homogeniety and isotropy. This reflects an argument which we shall sketch that, under Einsteinian evolution, the summands making up M (in a connected sum decomposition) that do support locally homogeneous and isotropic metrics will tend to overwhelmingly dominate the spatial volume asymptotically as the universe model continues to expand and furthermore that the actual evolving (inhomogeneous, non-isotropic) metric on M will naturally tend to flow towards a homogeneous, isotropic one on each of these asymptotically volume-dominating summands.

We do not claim that this mechanism is yet so compelling, either mathematically or physically, as to convince one that the actual universe has a more exotic topology but only that such a possibility is not strictly excluded by current observations. However, it is intriguing to investigate the possibility that there may be a dynamical reason, provided by Einstein?s equations, for the observed fact that the universe seems to be at least locally homogeneous and isotropic and that this mechanism may therefore allow an attractive logical alternative to simply extrapolating observations of necessarily limited scope to the universe as a whole.

But what are the (compact, connected, orientable) 3-manifolds available for consideration? This question has been profoundly clarified in recent years by the dramatic progress on lower dimensional topology made possible through the advancements in Ricci flow [5,64]. One now knows for example that, since the Poincar conjecture has finally been proven, any such 3-manifold M that is in fact simply connected must be diffeomorphic to the ordinary 3-sphere $\mathbb{S}^3$. Setting aside this so-called ?trivial? manifold the remaining possibilities consist of an infinite list of nontrivial manifolds, each of which is diffeomorphic (designated herein by $\approx$) to a finite connected sum of the following form:

$$M \approx \tag{1}$$

$$\mathbf{S}^3/\Gamma_1\#..\#\mathbf{S}^3/\Gamma_k\#(\mathbf{S}^2\times\mathbf{S}^1)_1\#..\#(\mathbf{S}^2\times\mathbf{S}^1)_l\#\mathbf{K}(\pi,1)_1\#..\#\mathbf{K}(\pi,1)_m.$$

Here k, l and m are integers $\geq 0, k+l+m \geq 1$ and if either k, l or m is 0 then terms of that type do not occur. The connected sum $M\#N$ of two closed connected, oriented n-manifolds is constructed by removing the interiors of an embedded closed n-ball in each of M and N and then identifying the resulting $\mathbb{S}^{n-1}$ boundary components by an orientation-reversing diffeomorphism. The resulting

$n-$manifold will be smooth, connected, closed and consistently oriented with the original orientations of M and N. The above decomposition of M is only uniquely defined provided we set aside $\mathbb{S}^3$ since $M\#\mathbb{S}^3 \approx M$ for any 3-manifold M.

In the above formula if $k \geq 1$, then each Γ_i, $1 \leq i \leq k$ is a finite, nontrivial ($\Gamma_i \neq I$) subgroup of SO(4) acting freely and orthogonally on $\mathbb{S}^3$. The individual summands $\mathbb{S}^3/\Gamma_i$ are the spherical space forms alluded to previously and, by construction, each is compatible with an FLRW metric of constant positive spatial curvature (i.e., $k = +1$ models in the usual notation). The individual ?handle? summands $\mathbb{S}^2 \times \mathbb{S}^1$ admit metrics of the Kantowski-Sachs type that are homogeneous but not isotropic and so not even locally of FLRW type.

The remaining summands in the above 'prime decomposition' theorem [6] are the $K(\pi, 1)$ manifolds of Eilenberg-MacLane type wherein, by definition $\pi = \pi_1(M)$, the fundamental group of M and all of the higher homotopy groups are trivial, that is $\pi_i(M) = 0$ for $i > 1$. Equivalently, the universal covering space of M is contractible and, in this case, known to be diffeomorphic to $\mathbb{R}^3$ [6]. Since the higher homotopy groups, $\pi_i(M)$ for $i > 1$ can be interpreted as the homotopy classes of continuous maps $\mathbb{S}^i \to M$, each such map must be homotopic to a constant map. For this reason $K(\pi, 1)$ manifolds are said to be aspherical.

This general class of $K(\pi, 1)$ manifolds includes, as special cases, the 3-torus and five additional manifolds, finitely covered by the torus, that are said to be of 'flat type' since they are the only compact, connected, orientable 3-manifolds that each, individually, admits a flat metric and thus supports spatially compactified versions of the FLRW spaces of flat type (i.e., $k = 0$ models).

Other $K(\pi, 1)$ spaces include the vast set of compact hyperbolic manifolds $\mathbb{H}^3/\Gamma$, where here Γ is a discrete torsion-free (i.e., no nontrivial element has finite order) co-compact subgroup of the Lie group $\mathrm{Isom}^+(\mathbb{H}^3)$ of orientation preserving isometries of $\mathbb{H}^3$ that, in fact, is Lie-group isomorphic to the proper orthochronous Lorentz group $\mathrm{SO}^+(3, 1)$. Each of these, individually, supports spatially compactified versions of the FLRW spacetimes of constant negative (spatial) curvature (i.e., $k = -1$ models).

Additional $K(\pi, 1)$ manifolds include the trivial circle bundles over higher genus surfaces Σ_p for $p \geq 2$ (where Σ_p designates a compact, connected, orientable surface of genus p) and nontrivial circle bundles over Σ_p for $p \geq 1$. Note that the trivial circle bundles $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{T}^2 \times \mathbb{S}^1 \approx \mathbb{T}^3$ are already included among the previous prime factors discussed and that nontrivial circle bundles over $\mathbb{S}^2$ are included among the spherical space forms $\mathbb{S}^3/\Gamma$ for suitable choices of Γ. Still further examples of $K(\pi, 1)$ manifolds are compact 3-manifolds that fiber nontrivially over the circle with fiber Σ_p for $p \geq 1$. Any such manifold is obtained by identifying the boundary components of $[0, 1] \times \Sigma_p$ with a (nontrivial) orientation-reversing diffeomorphism of Σ_p.

It is known however that every prime $K(\pi, 1)$ manifold is decomposible into a (possibly trivial but always finite) collection of (complete, finite volume) hyperbolic and graph manifold components. The possibility of such a (nontrivial) decomposition arises whenever the $K(\pi, 1)$ manifold under study admits a nonempty family $\{T_i\}$ of disjoint embedded incompressible two-tori. An embedded two-torus $\mathbb{T}^2$ is said to be incompressible if every incontractible loop in the torus remains incontractible when viewed as a loop in the ambient manifold. A closed oriented 3-manifold G (possibly with boundary) is a graph manifold if there exists a finite collection $\{T_i^{'}\}$ of disjoint embedded incompressible tori $\{T_i^{'}\} \subset$G such that each component G_j of $G - \cup T_i^{'}$ is a Seifert-fibered space (A Seifert-fibered space is a 3-manifold foliated by circular fibers in such a way that each fiber has a tubular neighborhood (characterized by a pair of co-prime integers) of the special type known as a standard fibered torus). Thus a graph manifold is a union of Seifert-fibered spaces glued together by toral automorphisms along toral boundary components. The collection of tori is allowed to be empty so that, in particular, a Seifert-fibered manifold itself is a graph manifold. Decomposing a 3-manifold by cutting along essential two-spheres (to yield its prime factors) and then along incompressible tori, when present, are the basic operations that reduce a manifold to its ?geometric? constituents [6]. The Thurston conjecture that every such 3-manifold can be reduced in this way has now been established via arguments employing Ricci flow [5, 64].

It may seem entirely academic to consider such general, 'exotic' 3-manifolds as the composite (i.e., nontrivial connected sum) ones described above as arenas for general relativity when essentially all of the explicitly known solutions of Einstein's equations (in this spatially compact setting) involve only individual, 'prime factors'. As we shall see however some rather general conclusions are derivable concerning the behaviors of solutions to the field equations on such *exotic* manifolds and astronomical observations do not logically exclude the possibility that the actual universe could have such a global topological structure. It is furthermore conceivable that the validity of central open issues in general relativity like the *cosmic censorship* conjecture could depend crucially upon the spatial topology of the spacetime under study.

Chapter 1

Asymptotic behavior of a matter filled universe with exotic topology

The ADM formalism together with a constant mean curvature (CMC) temporal gauge is used to derive the monotonic decay of a weak Lyapunov function of the Einstein dynamical equations in an expanding universe with a positive cosmological constant and matter sources satisfying suitable energy conditions. While such a Lyapunov function does not, in general, represent a true Hamiltonian of the matter-coupled gravity dynamics (unlike in the vacuum case when it does), it can nevertheless be used to study the asymptotic behavior of the spacetimes. The Lyapunov function attains its infimum only in the limit that the matter sources be 'turned off' or, at least, become asymptotically negligible provided that the universe model does not re-collapse and form singularities. Later we specialize our result to the case of a perfect fluid which satisfies the desired energy conditions. However, even in this special case, we show using Shutz's velocity potential formalism cast into Hamiltonian form that unlike the vacuum spacetimes (with or without a positive cosmological constant), construction of a true Hamiltonian for the dynamics in constant mean curvature temporal gauge is difficult and therefore the Lyapunov function does not have a straightforward physical interpretation. Nevertheless, we show, for the fluid with equation of state $P = (\gamma - 1)\rho$ $(1 \leq \gamma \leq 2)$, that the general results obtained hold true and the infimum of the weak Lyapunov function can be related to the Sigma constant, a topological invariant of the manifold. Utilizing these results, some general conclusions are drawn regarding the asymptotic state of the universe and the dynamical control of the allowed spatial topologies in the cosmological models.

Construction of a complete cosmological model of the universe entails the study of the full Einstein equations with suitable matter sources. The so-called FLRW models are rather very special examples of cosmological models, where one takes the cosmological principle literally that the universe appears spatially isotropic and homogeneous if viewed on a sufficiently coarse-grained scale [46,47,48]. However, the astronomical observations that motivate the cosmological principle are necessarily limited to a fraction (possibly small) of the entire universe and such observations are compatible with spatial metrics being locally but not globally homogeneous and isotropic. Once the restriction on the 'global' topology is removed, there are compact variants of all of the basic FLRW cosmological models, mathematically constructible by taking suitable compact quotients of the basic topologies i.e., Euclidean 3-space $\mathbf{E}^3$, Hyperbolic 3-space $\mathbf{H}^3$, and 3-sphere $\mathbf{S}^3$. A natural question that then arises is whether the governing principle of general relativity does indeed support such cosmological models allowing 'exotic' topologies. Recent studies [107, 229] uncovered a surprising dynamical mechanism at work within the framework of the vacuum Einstein flow (with and without a positive cosmological constant Λ) that strongly suggests that many closed 3-manifolds that do not admit a locally homogeneous and isotropic metric at all will nevertheless evolve in such a way as to be asymptotically compatible with the observed homogeneity and isotropy of the universe. Such a striking result naturally necessitates the inclusion of suitable matter sources in the Einsteinian evolution and the deduction of whether such a matter-filled universe does exhibit the asymptotic behavior compatible with the cosmological observations. Numerous matter models such as dust [51], perfect fluid [53], and Vlasov type matter [54], have been suggested for the cosmological models. In this study, we will consider matter models in the ADM Einstein equations satisfying suitable energy conditions.

The ADM formulation of the Einstein equations, developed by Arnowitt, Deser, and Misner [55], was originally restricted to vacuum space-times. Taub [56] developed a gauge fixed Lagrangian variational principle for the relativistic perfect fluid. Later Schutz [52] used velocity potential formulation to construct a gauge free variational formalism which was still Lagrangian. Afterwards Moncrief [131] and Moncrief & Demaret [132] developed the Hamiltonian formalism for a perfect fluid and Moncrief & Demaret [132] used Taub's co-moving gauge to derive a reduced Hamiltonian (a true Hamiltonian of the dynamics) and it was applied for canonical quantization of general relativistic configurations filled with perfect fluid using Dirac's technique [59]. However, these studies do not address the topology and the future stability of the universe filled with perfect fluid, and, in particular, do not focus on constructing a suitable cosmological model. On the other hand, [107, 161, 229] constructed a reduced Hamiltonian (respecting the topological restriction) and showed its monotonic decay in the expanding direction in a suitably chosen constant mean curvature gauge. The reduced Hamiltonian (also a weak Lyapunov function), in these studies, seeks to attain its infimum and can achieve its infimum asymptotically, yielding spacetimes admitting spatial slices locally compatible with the cosmological principle. However, such studies are devoid of matter sources i.e., only vacuum spacetimes (with and without cosmological constant) are considered.

It, therefore, seems necessary to focus on the construction of a suitable Lyapunov function and study its temporal behavior in a general framework which includes a matter source along with a positive cosmological constant. We start with the ADM formalism including a general matter source and later fix a temporal gauge namely the constant mean curvature gauge. Thereafter we study several aspects of dynamical behavior associated with matter sources and assert that the obtained results hold provided that the matter satisfies physically 'reasonable' energy conditions. Later on, we specialize our results to a perfect fluid source (which satisfies the required energy conditions) and obtain the asymptotic solutions for which the Lyapunov function stays constant in the limit where the baryon density becomes negligible. Finally, we relate the asymptotic behavior of this Lyapunov function to a topological property of the manifold thereby drawing some general conclusions regarding possible 'exotic' topologies of the physical '3+1' universe compatible asymptotically with the cosmological principle.

1.1 ADM formalism with a Matter source

The ADM formalism splits the spacetime described by an 'n+1' dimensional Lorentzian manifold $\tilde{M}$ into $\mathbf{R} \times M$ with each level set $\{t\} \times M$ of the time function t being an orientable n-manifold diffeomorphic to a Cauchy hypersurface (assuming the spacetime admits a Cauchy hypersurface) and equipped with a Riemannian metric. Such a split may be executed by introducing a lapse function N and shift vector field X belonging to suitable function spaces and defined such that

$$\partial_t = N\hat{n} + X \tag{1.1}$$

with t and $\hat{n}$ being time and a hypersurface orthogonal future directed timelike unit vector i.e., $\tilde{g}(\hat{n}, \hat{n}) = -1$, respectively. The above splitting writes the spacetime metric $\tilde{g}$ in local coordinates $\{x^\alpha\}_{\alpha=0}^{n} = \{t, x^1, x^2, ..., x^n\}$ as

$$\tilde{g} = -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt) \tag{1.2}$$

and the stress-energy tensor as

$$\mathbf{T} = E\mathbf{n} \otimes \mathbf{n} + \mathbf{J} \odot \mathbf{n} + \mathbf{S}, \tag{1.3}$$

where $\mathbf{J} \in \mathfrak{X}(M)$, $\mathbf{S} \in S_0^2(M)$, and $A \odot B = \frac{1}{2}(A \otimes B + B \otimes A)$. Here, $\mathfrak{X}(M)$ and $S_0^2(M)$ are the space of vector fields and the space of symmetric covariant 2-tensors, respectively. The choice of a spatial slice in the spacetime leads to consideration of the second fundamental form k_{ij} which describes how the slice is curved in the spacetime. The trace of the second fundamental form ($tr_g k = \tau$) is the mean extrinsic curvature of the slice, which will play an important role in the analysis. Under such decomposition, the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{1.4}$$

take the form ($8\pi G = c = 1$)

$$\begin{aligned} \partial_t g_{ij} &= -2Nk_{ij} + L_X g_{ij}, \\ \partial_t k_{ij} &= -\nabla_i \nabla_j N + N\{R_{ij} + \tau k_{ij} - 2k_{ik}k_j^k \\ &\quad -\frac{1}{n-1}(2\Lambda - S + E)g_{ij} - S_{ij}\} + L_X k_{ij} \end{aligned} \tag{1.5}$$

along with the constraints (Gauss and Codazzi equations)

$$R(g) - |k|^2 + \tau^2 = 2\Lambda + 2E, \tag{1.6}$$

$$\nabla_j k_i^j - \nabla_i \tau = J_i, \tag{1.7}$$

where $S = g^{ij}S_{ij}$. The vanishing of the covariant divergence of the stress energy tensor i.e., $\nabla_\nu T^{\mu\nu} = 0$ is equivalent to the continuity equation and equations of motions of the matter

$$\begin{aligned} \frac{\partial E}{\partial t} &= L_X E + NE\tau - N\nabla_i J^i - 2J^i \nabla_i N + NS^{ij}k_{ij}, \\ \frac{\partial J^i}{\partial t} &= L_X J^i + N\tau J^i - \nabla_j(NS^{ij}) + 2Nk_j^i J^j - E\nabla^i N. \end{aligned} \tag{1.8}$$

Note that there are no evolution equations for the lapse function and the shift vector field as a consequence of gauge freedom. In a sense, the original Einstein equations (before the ADM split) consists of 6 evolution equations and 4 constraints and therefore the equations for lapse and shift should be obtained by fixing the gauge. In order to choose a time coordinate and assign uniqueness to the spatial slice, gauge fixing is required. Here we choose 'constant mean extrinsic curvature' (CMC) as the temporal gauge which yields an elliptic equation for the lapse function. Later on, we will select a suitable spatial gauge. CMC gauge reads

$$\tau = tr_g k = monotonic\ function\ of\ t\ alone \tag{1.9}$$

so τ is thus constant throughout the hypersurface and therefore can play the role of time. Using the evolution and constraint equations, one may obtain the following equation for the lapse function

$$\frac{\partial\tau}{\partial t} = \Delta_g N + \{|k|^2 + \frac{S}{n-1} + \frac{n-2}{n-1}E - \frac{2\Lambda}{n-1}\}N + L_X\tau \tag{1.10}$$

which after implementing CMC gauge ($\partial_i\tau = 0$) yields

$$\frac{\partial\tau}{\partial t} = \Delta_g N + \{|k|^2 + \frac{S}{n-1} + \frac{n-2}{n-1}E - \frac{2\Lambda}{n-1}\}N,$$

where $|k|^2 = k_{ij}k^{ij}$ and the Laplacian is defined as $\Delta_g = -\nabla[g]^i\nabla[g]_i$ and therefore has positive spectrum on compact connected manifolds.

The evolution and constraint equations derived here are purely geometric and therefore one has the freedom to choose the topology of the spatial slice M. Different choices would lead to different cosmological models. Of course, not all topologies admit metrics that satisfy the cosmological principle. Nevertheless, we will consider a general topology to start with so that the universe may not have been isotropic and homogeneous (locally) throughout its complete course of evolution. Thurston's geometrization conjecture [61, 62, 63] and its subsequent proof together with the Poincaré conjecture [61, 64, 65] and its proof allow one a complete classification of compact, orientable 3-manifolds. Leaving aside the so-called 'trivial' manifolds diffeomorphic to $\mathbf{S}^3$, the remaining possibilities consist of an infinite list of nontrivial manifolds, each of which is diffeomorphic to a finite connected sum of the following form [107, 187, 229]

$$\mathbf{S}^3/\Gamma_1\#..\#\mathbf{S}^3/\Gamma_k\#(\mathbf{S}^2\times\mathbf{S}^1)_1\#..\#(\mathbf{S}^2\times\mathbf{S}^1)_l\#\mathbf{K}(\pi,1)_1\#..\#\mathbf{K}(\pi,1)_m, \tag{1.11}$$

where k, l, and m are non-negative integers satisfying $k + l + m \geq 1$ and if either k, l or m equals to zero, then terms of that type do not occur. Here each Γ_k (for $k \geq 1$) is a nontrivial ($\neq I$) subgroup of SO(4) acting freely and properly discontinuously on $\mathbf{S}^3$ and the resulting manifold is $\mathbf{S}^3/\Gamma_k$ (see [66] and [67] for free and properly discontinuous actions). $\mathbf{S}^2\times\mathbf{S}^1$ denotes the wormhole and supports only anisotropic (even locally) homogeneous metrics and therefore is not compatible with the cosmological observations. The remaining $\mathbf{K}(\pi, 1)$ manifolds are of interest to us and defined to have non-trivial fundamental group while the rest of their higher homotopy groups vanish [67, 68]. A vast set of compact hyperbolic manifolds (and therefore supporting locally homogeneous and isotropic metrics) fall under this category. In this article, we will focus on manifolds of negative Yamabe type i.e., the scalar curvature $R(g)$ of every Riemannian metric g on M is negative somewhere (for details about negative Yamabe manifolds, the reader is referred to [107,187,229]). Note that non-flat type $\mathbf{K}(\pi,1)$ manifolds are of negative Yamabe type and presence of a single negative Yamabe type manifold in the connected sum decomposition written as (1.11) turns the whole manifold into one of negative Yamabe type. It is also known that every prime $\mathbf{K}(\pi,1)$ manifold is decomposable into a (finite) collection of (complete, finite volume) hyperbolic and graph manifolds (see introduction of [229] for the detailed description of a graph manifold). This property of $\mathbf{K}(\pi,1)$ manifold will turn out to be important later in our discussion. We will, throughout our analysis, consider an 'n+1' dimensional universe and only later specialize our results to the physical '3+1' universe.

Utilizing the 'CMC' condition, the momentum constraint reduces to

$$\nabla[g]_j k^j_i \quad = \quad J_i, \tag{1.12}$$

the solution of which may be written as

$$k^i_j = K^{tri}_j + \frac{\tau}{n}\delta^i_j \tag{1.13}$$

with K^{tr} being traceless with respect to g i.e., $K^{tr}_{ij}g^{ij} = 0$ (and so with respect to any metric conformal to g). Note that K^{tr} is obtained by solving the following equation

$$\nabla[g]_j K^{trj}_i \quad = \quad J_i. \tag{1.14}$$

The Hamiltonian constraint (4.19) may be written using the solution of momentum constraint (3.23) as follows

$$R(g) \quad = \quad |K^{tr}|^2 + 2E - \frac{n-1}{n}(\tau^2 - \frac{2n\Lambda}{n-1}).$$

Since we are primarily interested in the case $\Lambda > 0$, it might appear that $\tau^2 - \frac{2n\Lambda}{n-1}$ could be negative. But then the Hamiltonian constraint would imply that if the energy density E is non-negative, then $R(g) \geq 0$ everywhere on M which is impossible for a manifold of negative Yamabe type. Let's consider the energy condition and establish an allowed range for the constant extrinsic mean curvature τ which is now playing the role of time. The weak energy condition yields

$$\mathbf{T}(\mathbf{n},\mathbf{n}) \geq 0, \tag{1.15}$$

$$E \geq 0 \tag{1.16}$$

that is to any time-like observer the energy density is nonnegative and as such physically relevant classical matter sources are expected to satisfy this energy condition. We will only consider matter sources with point-wise nonnegative energy density throughout the spacetimes. Therefore a universe filled with matter satisfying the weak energy condition shall always have

$$\tau^2 - \frac{2n\Lambda}{n-1} > 0, \tag{1.17}$$

and for expanding models ($\frac{\partial\tau}{\partial t} > 0$),

$$-\infty < \tau < -\sqrt{\frac{2n\Lambda}{n-1}}. \tag{1.18}$$

Since we are primarily interested in the asymptotic behavior of the '**expanding**' universe model (the physically relevant case), we set the range of τ to be $(-\infty, -\sqrt{\frac{2n\Lambda}{n-1}})$ once and for all. We turn our attention to the Lapse equation in an expanding universe model by setting

$$\frac{\partial\tau}{\partial t} = \frac{n}{2(n-1)}(\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}} > 0 \tag{1.19}$$

whose solution $\tau = \tau(t)$ plays the role of time from now onwards. Note that (1.19) is a valid 'choice' of a time function since $\tau(t)$ is monotonic and chosen to be constant on a Cauchy hypersurface (CMC gauge). The lapse equation is explicitly written in this time co-ordinate as

$$\begin{aligned}\Delta_g N + (|K^{tr}|^2 + (\frac{\tau^2}{n} - \frac{2\Lambda}{n-1}) + \frac{1}{n-1}(S + (n-2)E))N \\ = \frac{n}{2(n-1)}(\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}}.\end{aligned} \tag{1.20}$$

In order for this equation to have a unique positive solution, the kernel of the elliptic operator

$$\{\Delta_g + (|K^{tr}|^2 + (\frac{\tau^2}{n} - \frac{2\Lambda}{n-1}) + \frac{1}{n-1}(S + (n-2)E))I\} \tag{1.21}$$

defined between suitable function spaces, must be trivial. Here I denotes the identity operator. If ϕ belongs to the kernel i.e., it satisfies

$$\{\Delta_g + (|K^{tr}|^2 + (\frac{\tau^2}{n} - \frac{2\Lambda}{n-1}) + \frac{1}{n-1}(S + (n-2)E))I\}\phi = 0,$$

then after multiplying both sides with ϕ, integration over the closed orientable manifold M and applying Stokes' theorem one gets

$$\int_M \{\nabla[g]_i\phi\nabla[g]^i\phi + (|K^{tr}|^2 + (\frac{\tau^2}{n} - \frac{2\Lambda}{n-1}) \\ + \frac{1}{n-1}(S + (n-2)E))\phi^2\}\mu_g = 0, \tag{1.22}$$

where $\mu_g = \sqrt{\det(g)}dx^1 \wedge dx^2 \wedge dx^3 \wedge ..... \wedge dx^n$ is the volume form on M. Existence of a trivial kernel i.e., $\phi \equiv 0$ throughout the whole of M, is implied by

$$(n-2)E + S \geq 0. \tag{1.23}$$

Therefore, for a universe filled with matter sources of everywhere non-negative energy density, an additional energy condition needs to be satisfied in order to obtain a unique solution for the lapse equation. Note that matter satisfying the strong energy condition

$$(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})n^\mu n^\nu \geq 0, \tag{1.24}$$
$$E + S \geq 0$$

falls under this category. In a sense the 'weak+strong energy condition'$\subseteq \{E \geq 0, S + (n-2)E \geq 0, n \geq 3\}$ and therefore such sources are allowed for our analysis. Of course, most physically relevant sources do satisfy the property of non-negative energy density (weak energy condition) and the attractive nature of gravity (strong energy condition). Several known sources of physical interest satisfy weak and strong energy conditions. These include for example perfect fluid and Vlasov matter (see [54, 70, 71, 72, 73] for details about Vlasov matter).

We have established the existence of a unique solution of the lapse equation provided that the matter sources in the universe satisfy a suitable energy condition. A standard maximum principle argument for the elliptic equation yields the following estimate of the lapse function

$$0 \quad < \quad \frac{\frac{n}{2(n-1)}(\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}}}{\Big((\frac{\tau^2}{n} - \frac{2\Lambda}{n-1}) + \sup(|K^{tr}|^2 + \frac{1}{n-1}(S + (n-2)E))\Big)} \leq N \leq \\ \frac{n^2}{2(n-1)}(\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}-1}. \tag{1.25}$$

The important thing here is to note that the lapse function is positive for an expanding universe model. The results obtained so far will be sufficient to study the dynamical behavior in terms of a weak Lyapunov function.

[187, 229] constructed a reduced phase space as the cotangent bundle of the higher dimensional analogue of the Teichmüller space and obtained the following true Hamiltonian of the dynamics through a conformal technique

$$H_{reduced} := \frac{2(n-1)}{n}\int_M \frac{\partial\tau}{\partial t}\mu_g. \tag{1.26}$$

In that particular case of a vacuum limit, it is indeed possible to construct such a reduced Hamiltonian in 'CMC' gauge, which also acts as a weak Lyapunov function. However, as will be shown in the case of a simple matter source, the perfect fluid, the construction of a reduced Hamiltonian using a conformal diffeomorphism is restricted by certain conditions described later. However, for the purpose of the Einsteinian dynamics, we do not need to construct a reduced phase space (and associated Hamiltonian) but rather a suitable Lyapunov function and study its dynamical behavior. Let us consider the rescaled volume functional as the weak Lyapunov function and call it $L(g, k)$

$$L(g,k) = \frac{2(n-1)}{n}\int_M \frac{\partial\tau}{\partial t}\mu_g. \tag{1.27}$$

We call $L(g, k)$ a weak Lyapunov function because, following the expression of $\frac{\partial\tau}{\partial t}$, it controls the $H^1 \times L^2$ norm of the data (g,k) while the desired norm would be $H^s \times H^{s-1}$, $s > \frac{n}{2}+p$ for some $p \geq 1$. Note that we do not, at this point, have a local existence theorem of the Cauchy problem of the Einstein system in exotic spatial topologies (negative Yamabe manifolds in this case) with arbitrary matter source satisfying the desired energy conditions. For the vacuum case, Bel-Robinson energy is used by [151], which controls the $H^2 \times H^1$ norm of the data. Recently, [79] proved nonlinear stability results for small data perturbations of FLRW background cosmological model, which of course deals only with special topologies ($\mathbf{S}^3$, $\mathbf{T}^3$, and $\mathbf{H}^3$). At the moment, let us focus on the time evolution of the weak Lyapunov function namely the rescaled volume functional. Due to its weak property, we would not be able to state a theorem concerning the stability (either Lyapunov or asymptotic) of the spacetime on the basis of its time evolution. Nevertheless, we will be able to obtain important physical results related to the asymptotic behavior of the spacetime (in the expanding direction) utilizing the limiting behavior of the Lyapunov function. The time evolution of this function may be obtained as

$$
\begin{aligned}
\frac{dL(g,k)}{dt} &= \int_M \frac{\delta L(g,k)}{\delta g}\partial_t g + \int_M \frac{\delta L(g,k)}{\delta k}\partial_t k, \\
&= \frac{2(n-1)}{n}\int_M \left(\frac{\partial^2\tau}{\partial t^2} + \frac{1}{2}\frac{\partial\tau}{\partial t} g^{ij}\frac{\partial g_{ij}}{\partial t}\right)\mu_g, \\
&= \int_M \tau(\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}-1}(n\Delta_g + n|K^{tr}|^2 + (\tau^2 - \frac{2n\Lambda}{n-1}) \\
&\quad + \frac{n}{n-1}(S+(n-2)E))N + (\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}}\int_M (-N\tau + \nabla_i X^i))\mu_g, \\
&= \tau(\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}-1}\int_M (n|K^{tr}|^2 + (\tau^2 - \frac{2n\Lambda}{n-1}) \\
&\quad + \frac{n}{n-1}(S+(n-2)E) - (\tau^2 - \frac{2n\Lambda}{n-1}))N\mu_g, \\
&= n\tau(\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}-1}\int_M (|K^{tr}|^2 + \frac{1}{n-1}(S+(n-2)E))N\mu_g, \qquad (1.28)
\end{aligned}
$$

where we have used the identity

$$
\begin{aligned}
\frac{\partial^2\tau}{\partial t^2} &= \frac{n}{2(n-1)}\tau(\tau^2 - \frac{2n\Lambda}{n-1})^{\frac{n}{2}-1}(n\Delta_g + n|K^{tr}|^2 \qquad (1.29) \\
&\quad + (\tau^2 - \frac{2n\Lambda}{n-1}) + \frac{n}{n-1}(S+(n-2)E))N,
\end{aligned}
$$

obtained using the lapse equation and the CMC gauge condition ($\partial_i\tau = 0$). We have also used the Stokes' theorem to eliminate the covariant divergence terms in the integral. Along the solution curve in the expanding direction ($\frac{\partial\tau}{\partial t} > 0$ and $-\infty < \tau < -\sqrt{\frac{2n\Lambda}{n-1}}$), therefore, the Lyapunov function monotonically decays i.e.,

$$
\frac{dL(g,k)}{dt} < 0 \qquad (1.30)
$$

and only attains its infimum (i.e., $\frac{dL}{dt} = 0$) precisely when the following conditions are met

$$
\begin{aligned}
K^{tr} &= 0, \qquad (1.31) \\
S + (n-2)E &= 0. \qquad (1.32)
\end{aligned}
$$

Substitution of the first condition into the momentum constraint (3.23) immediately yields

$$
J^i \equiv 0 \qquad (1.33)
$$

as well as $Y \equiv 0$ everywhere on M. In addition to satisfying $E, S+(n-2)E \geq 0$, if the matter sources also satisfy the strong energy condition i.e., $(S+E) \geq 0$, then for $n \geq 3$ (the cases of primary interest), the second condition for the infimum of the Lyapunov function translates to

$$S + E = 0, \tag{1.34}$$

$$E = 0 \tag{1.35}$$

and therefore

$$S = 0. \tag{1.36}$$

This result, therefore, states that the weak Lyapunov function namely the rescaled volume is monotonically decaying in the direction of cosmological expansion and approaches its nfimum only in the limit that the matter sources be 'turned off' or at least become asymptotically negligible. In the limit of turned off matter, one may utilize the evolution and constraint equations by substituting $k^{TT} = \partial_t k^{TT} = 0$ to obtain the background warped product spacetimes

$$ds^2 = -\frac{n^2}{(\tau^2 - \frac{2n\Lambda}{n-1})^2} d\tau^2 + \frac{n}{(n-1)(\tau^2 - \frac{2n\Lambda}{n-1})} \gamma_{ij} dx^i dx^j, \tag{1.37}$$

with $R(\gamma) = -1$. The vital question is whether the Lyapunov function L ever attains its infimum. In the expanding universe model, if the matter sources do not re-collapse to form a singularity, then the matter density falls off and in the limiting case, may be considered to be negligible. We will indeed show such asymptotic decay of the matter density in case of a perfect fluid. Observing the monotonic decay of the Lyapunov function along a solution curve, one may be tempted to conjecture the asymptotic stability of the matter filled spacetimes. However, we remind the reader again that such property only provides a weak notion of stability if the matter sources do not develop singularities. Even a vacuum spacetime might be able to go singular that is the pure gravity can collapse to form singularity before the spatial volume of the universe reaches infinity. As of now, we do not yet have a global existence theorem for even a vacuum spacetime. Now we specialize our results to the perfect fluid matter source.

1.3 Perfect Fluid

The perfect fluid stress-energy tensor

$$T^{\mu\nu} = (P+\rho)u^{\mu}u^{\nu} + Pg^{\mu\nu} \tag{1.38}$$

yields

$$\begin{aligned} E &= T(\mathbf{n},\mathbf{n}) = (P+\rho)(\tilde{g}(u,n))^2 - P & (1.39)\\ &= (P+\rho)(Nu^0)^2 - P \geq 0 & (1.40)\end{aligned}$$

assuming the equation of state $P = (\gamma-1)\rho$, $1 \leq \gamma \leq 2$, normalization $\tilde{g}(u,u) = -1$, and a positive energy density ρ. The strong energy condition requires

$$(nP+\rho) \geq 0 \tag{1.41}$$

which is clearly satisfied as well. Therefore, satisfying both of these energy conditions, a perfect fluid falls under the category for which our analysis holds true. Through a conformal technique, we will obtain the infimum of the weak Lyapunov function described in the previous section. The conformal technique may not be applied to any matter source in a unique way in general as the conformal rescaling of at least the energy density would be different for a different choice of sources. As a consequence, the resulting nonlinear elliptic equation namely the Lichnerowicz equation arising from the rescaling of Hamiltonian constraint, which will play the central role in obtaining the infimum of the Lyapunov function, may not have a unique positive solution. Nevertheless, as we will see shortly, one may obtain a suitable conformal rescaling of matter variables in the case of a perfect fluid and therefore, the subsequent analysis will follow. Using the momentum constraint, the second fundamental form may be written as follows (1.13)

$$K^{ij} = K^{trij} + \frac{\tau}{n}g^{ij}, \tag{1.42}$$

where $g_{ij}K^{trij} = 0$, but $\nabla_j K^{trij} = J^i$ under ths CMC condition ($\partial_i \tau = 0$). We use the conformal transformation as described in $[107, 161, 229]$ only replacing the momenta conjugate to the metric g by the second fundamental form through a Legendre transformation $\pi = -\mu_g(k - (tr_g k)g)$. The conformal transformation reads

$$(g_{ij}, K^{trij}) = (\psi^{\frac{4}{n-2}}\gamma_{ij}, \psi^{-\frac{2(n+2)}{n-2}}\kappa^{trij}), \tag{1.43}$$

where γ and κ^{tr} are the scale-free fields satisfying $\gamma^{ij}\kappa^{tr}_{ij} = 0$ with $\gamma \in \mathcal{M}_{-1}$ and $\psi : M \to \mathbf{R}_{>0}$. Here $\mathcal{M}_{-1}$ is defined as $\mathcal{M}_{-1} = \{\gamma$ is a Riemannian metric on M $|R(\gamma) = -1\}$. In reality, the true dynamics assumes a metric lying in the orbit space $\mathcal{M}_{-1}/D_0$, D_0 being the group of diffeomorphisms (of M) isotopic to identity. This is a consequence of the fact that if $\gamma \in \mathcal{M}_{-1}, k, N$, and X solve the Einstein equations, so do $(\phi^{-1})^*\gamma, (\phi^{-1})^*k, (\phi^{-1})^*N = N \circ \phi^{-1}$, and ϕ_*X, where $\phi \in D_0$ and *, and $_*$ denote the pullback and push-forward operations on the cotangent and tangent bundles of M, respectively. To avoid technical complexities, the calculations may be restricted to $\mathcal{M}_{-1}$ as the entities we are interested in (such as $\mu_\gamma = \sqrt{\det \gamma_{ij}}dx^1 \wedge dx^2 ... \wedge dx^n$) are D_0 invariant yielding equivalence between $\mathcal{M}_{-1}$ and $\mathcal{M}_{-1}/D_0$ (only in this particular occasion). Note that K^{tr} may be decomposed into a transverse-traceless part (with respect to g) and a Conformal Killing tensor part

$$K^{trij} = K^{TTij} + \psi^{\frac{-2n}{n-2}}(L_Y g - \frac{2}{n}\nabla_m Y^m g)^{ij}, \tag{1.44}$$

where $Y \in \mathfrak{X}(M)$. Under this conformal transformation, the components of the conformal Killing tensor $(L_g Y - \frac{2}{n}\nabla_m Y^m g)$ transforms as follows

$$(L_g Y - \frac{2}{n}\nabla_m Y^m g)^{ij} = \psi^{\frac{-4}{n-2}}(L_\gamma Y - \frac{2}{n}\nabla_m Y^m \gamma)^{ij} \tag{1.45}$$

yielding a consistent transformation of the transverse-traceless tensor K^{TT}, that is, $K^{TTij} = \psi^{-\frac{2(n+2)}{n-2}}\kappa^{TTij}$. Therefore, we may write

$$\kappa^{trij} = \kappa^{TT} + (L_\gamma Y - \frac{2}{n}\nabla_m Y^m \gamma)^{ij} \tag{1.46}$$

The Hamiltonian constraint, under this conformal transformation, yields the Lichnerowicz equation

$$\begin{aligned} \Delta_\gamma \psi - \frac{n-2}{4(n-1)}\psi - \frac{(n-2)}{4(n-1)}\psi^{\frac{-3n+2}{n-2}}|\kappa^{TT} + (L_\gamma Y - \frac{2}{n}\nabla_m Y^m \gamma)|^2 \\ + \frac{n-2}{4n}(\tau^2 - \frac{2n\Lambda}{n-1})\psi^{\frac{n+2}{n-2}} - \frac{(n-2)E[g]}{2(n-1)}\psi^{\frac{n+2}{n-2}} = 0, \end{aligned} \tag{1.47}$$

where $E[g]$ denotes the energy density without conformal scaling. Note that we can not analyze the

Lichnerowicz equation without performing suitable scaling of $E[g]$ which we will precisely do in the case of a perfect fluid. But first, the momentum constraint after the rescaling reads

$$\nabla[\gamma]_j \kappa^{trij} = \psi^{\frac{2(n+2)}{n-2}} J[g]^i, \tag{1.48}$$

where $J[g]^i$ is the momentum density without conformal scaling. Now of course, if one chooses the following scaling for the momentum density (York scaling [223])

$$J[g]^i = \psi^{\frac{-2(n+2)}{n-2}} J[\gamma]^i, \tag{1.49}$$

the momentum constraint becomes decoupled from the Lichnerowicz equation i.e.,

$$\nabla[\gamma]_j \kappa^{trij} = J[\gamma]^i. \tag{1.50}$$

Now upon substituting κ^{tr} from equation (1.46) into equation (1.50) yields an elliptic equation for Y

$$-\Delta_\gamma Y^i + R[\gamma]^i_m Y^m + (1 - \frac{2}{n})\nabla[\gamma]^i(\nabla[\gamma]_m Y^m) \quad = \quad J[\gamma]^i. \tag{1.51}$$

The vector field Y (not a vector density) therefore soley depends on the metric γ and the matter source $J[\gamma]$, not on the conformal function ψ. Analysis completed up to now holds for any general matter source, however, in order to obtain a rescaling of the energy density, we need to know the matter type. In the case of a perfect fluid, the momentum density is expressible in terms of the basic variables P and ρ which in turn yields the energy density E. Let us split the $n+1$-velocity vector (u^μ) of the fluid into a hypersurface parallel component

$$v = u + {}^{n+1}g(u, \mathbf{n})\mathbf{n}, \tag{1.52}$$

and its orthogonal complement $-{}^{n+1}g(u, \mathbf{n})\mathbf{n}$. Here ${}^{n+1}g$ and $\mathbf{n}$ denote the spacetime metric and hypersurface orthogonal future directed unit vector, respectively. Note that $v_i = u_i$, but $v^i \neq u^i$. In the ADM language, the unscaled matter momentum density and energy density may be written as

$$J[g]^i \quad = \quad (P + \rho) N u^0 v^i, \tag{1.53}$$

$$E[g] \quad = \quad (P + \rho)(N u^0)^2 - P. \tag{1.54}$$

Now, utilizing the normalization condition $g_{\mu\nu}u^\mu u^\nu = -1$ in conjunction with the spliting (1.52), we obtain

$$-(Nu^0)^2 + g_{ij}v^i v^j = -1. \tag{1.55}$$

Since, the right hand side of this equation is a constant (and therefore conformally invariant), each term in the left hand side has to be conformally invariant leading to the following scaling of Nu^0 and v^i

$$\begin{aligned} (Nu^0)_g &= (Nu^0)_\gamma, & (1.56)\\ v^i_g &= \psi^{-\frac{2}{n-2}} v^i_\gamma, & (1.57)\end{aligned}$$

which together with the expressions of matter energy and momentum density (1.53-1.54) yields the scaling for P, ρ, and $E[g]$

$$\begin{aligned} (P+\rho)_g &= \psi^{-\frac{2(n+1)}{n-2}} (P+\rho)_\gamma, & (1.58)\\ E[g] &= \psi^{-\frac{2(n+1)}{n-2}} E[\gamma]. & (1.59)\end{aligned}$$

Rescaling of the matter energy density $E[g]$ now leads to the proper Lichnerowicz equation

$$\begin{aligned} \Delta_\gamma \psi - \frac{n-2}{4(n-1)}\psi - \frac{(n-2)}{4(n-1)}\psi^{\frac{-3n+2}{n-2}}|\kappa^{TT} + (L_Y\gamma - \frac{2}{n\mu_\gamma}\nabla_m Y^m \gamma)|^2 \\ -\frac{(n-2)E[\gamma]}{2(n-1)}\psi^{-\frac{n}{n-2}} + \frac{n-2}{4n}(\tau^2 - \frac{2n\Lambda}{n-1})\psi^{\frac{n+2}{n-2}} = 0, \end{aligned} \tag{1.60}$$

a solution of which may be obtained by the standard sub and super solution technique [223, 224]. Note that the exponent of ψ in the term $\frac{(n-2)E[\gamma]}{2(n-1)}\psi^{-\frac{n}{n-2}}$ is crucial in proving the uniqueness and existence of solutions to the Lichnerowicz equation and therefore, the rescaling of the energy density $E[g]$ is vital. Now that we have obtained a Lichnerowicz equation for the gravity coupled fluid dynamics, we may explicitly obtain the infimum of the weak Lyapunov function. A straightforward maximum principle argument, applied to the Lichnerowicz equation, shows that the unique positive solution $\psi = \psi(\tau, \gamma, \kappa^{TT}, Y)$ satisfies

$$\psi^{\frac{4}{n-2}} \geq \frac{n}{n-1}\frac{1}{(\tau^2 - \frac{2n\Lambda}{n-1})} \tag{1.61}$$

with equality holding everywhere on M iff

$$\kappa^{tr} = \kappa^{TT} + \{L_Y\gamma - \frac{2}{n}\nabla_m Y^m\gamma\} \equiv 0, \tag{1.62}$$

$$E[\gamma] \equiv 0 \tag{1.63}$$

on M. Therefore the asymptotic analysis becomes straightforward as the infimum of the Lyapunov function is attained precisely when the previous two conditions are met. In the limit of negligible matter (matter turned off condition), the Lyapunov function may be written in terms of the rescaled variables as follows

$$L(\tau, \gamma, \kappa^{tr} = 0) = \int_M (\tau^2 - \frac{2n\Lambda}{n-1})^{n/2} \psi^{2n/(n-2)}(\tau, \gamma, \kappa^{tr} = 0)\mu_\gamma. \tag{1.64}$$

The infimum of the Lyapunov function over the space $\mathcal{M}_{-1}/D_0 \times S_2^0(M)$ ($S_2^0(M)$ being the space of symmetric covariant 2-tensors) may be computed as

$$\begin{aligned} & \inf_{\mathcal{M}_{-1}/D_0 \times S_2^0(M)} L(\tau, \gamma, \kappa^{tr}) \\ = & \inf_{\mathcal{M}_{-1}/D_0 \times S_2^0(M)} \int_M (\tau^2 - \frac{2n\Lambda}{n-1})^{n/2} \psi^{2n/(n-2)}(\tau, \gamma, \kappa^{tr})\mu_\gamma, \\ & = (\frac{n}{n-1})^{n/2} \inf_{\mathcal{M}_{-1}/D_0} \int_M \mu_\gamma, \\ & = (\frac{n}{n-1})^{n/2} \{-\sigma(M)\}^{n/2}, \end{aligned} \tag{1.65}$$

where $\sigma(M)(< 0)$ is a topological invariant (higher dimensional analog of the Euler characteristics of a higher genus surfaces) of the manifold M (of negative Yamabe type considered here). For extensive details about the $\sigma-$constant (also known as Yamabe invariant) see [82, 83, 161]. For the purpose here, it suffices to know that it is a topological invariant.

The most interesting case here is the physical universe i.e., $3 + 1$ case. Utilizing Ricci-flow techniques, the σ constant (and therefore the infimum of the weak Lyapunov function) of the most general compact 3-manifolds of negative Yamabe type has been computed and as such is given by

$$|\sigma(M)| = (vol_{-1}H)^{2/3}, \tag{1.66}$$

where $vol_{-1}H$ is the volume of the hyperbolic part of M computed with respect to the hyperbolic metric normalized to have scalar curvature -1 [84,85]. Therefore, apart from the hyperbolic family of $\mathbf{K}(\pi, 1)$ manifolds, the remaining parts of M i.e., wormholes ($\mathbf{S}^2 \times \mathbf{S}^1$), spherical space forms ($\mathbf{S}^3/\Gamma_k$), and the graph manifolds (non-hyperbolic part of $\mathbf{K}(\pi, 1)$ manifold) do not contribute to the σ constant. Hence, they do not contribute to the infimum of the weak Lyapunov function as well. Since the weak Lyapunov function is geometrically the rescaled volume of M (1.27), following its monotonic decay towards an infimum dominated only by the hyperbolic component of the spatial manifold, one is led to the natural conclusion that the Einstein flow in the presence of a suitable matter sources and positive cosmological constant drives the universe towards an asymptotic state that is volume dominated by the hyperbolic components equipped with a locally homogeneous and isotropic metric. Such results while seeming innocuous may have tremendous cosmological significance. For example, let us consider the case discussed below. A well-known result in topology is that the presence of a single negative Yamabe type component in the connected sum makes the whole manifold negative Yamabe (one may find a unique solution to the Yamabe problem) and thus the asymptotic analysis described previously holds (except stand alone flat types which are zero Yamabe type). Spaces like de-sitter ($\mathbf{S}^3$ with a positive cosmological constant) or the spherical space forms ($\mathbf{S}^3/\Gamma, \Gamma \subset \mathrm{SO}(4)$) or the Schwarchild-deSitter spaces ($\mathbf{S}^2 \times \mathbf{S}^1$) may individually expand to infinity (proven results, see e.g., [88, 223]), however, while present in a connected sum with a negative Yamabe type summand, may be asymptotically volume dominated by hyperbolic parts. Let's consider the spherical space form case, for example. The spatial topology is $\mathbf{S}^3/\Gamma$ and the connected sum with a negative Yamabe manifold M is still topologically a negative Yamabe manifold. As a consequence, the individual 'expansion to infinity' property of the de-Sitter spaces (or its quotients) is sort of 'killed' by the hyperbolic parts present in M and therefore becomes asymptotically negligible. [93] showed that spherical space forms ($\mathbf{S}^3/\Gamma$) and handle ($\mathbf{S}^2 \times \mathbf{S}^1$) topologies re-collapse through formation of maximal hypersurfaces provided a number of conditions on matter sources and spatial geometry are satisfied (without a cosmological constant; see their theorem 3). Later [94] studied the closed universe re-collapse conjecture with the same spatial topologies. This provides a rough notion that these topologies may not contribute to the asymptotic state of the universe. However, in the presence of a positive cosmological constant, spherical space forms (generalized de-Sitter spaces) may avoid formation of a maximal hypersurface and continue to expand. Our result, in this particular aspect, becomes interesting in a sense that even if $\mathbf{S}^3/\Gamma$ or $\mathbf{S}^2 \times \mathbf{S}^1$ individually may expand forever (with a positive cosmological constant present), while present in a connected sum decomposition with hyperbolic manifolds, do not contribute to the asymptotic volume of the spatial universe due to the reason discussed previously. Now, compact quotients of hyperbolic space $\mathbf{H}^3$ by faithful and discrete subgroups of $\mathrm{SO}^+(3,1)$ provide an ample supply of compact hyperbolic manifolds equipped with locally homogeneous and isotropic metric. Therefore, the cosmological model is potentially enriched (and generalized in a certain sense) by including such rather exotic topologies capable of assuring that the cosmological principle holds.

Following the previous analysis, two natural questions arise. Firstly, whether, as in the vacuum case (with or without a positive cosmological constant), the Lyapunov function has a physical interpretation as a Hamiltonian of the reduced dynamics that is, can one cast the problem in such a way that the reduced Hamiltonian of the dynamics naturally turns out to be the rescaled volume functional. Secondly, how does one justify the matter being asymptotically negligible? We will try to answer both of these question by invoking the variational formulation of the gravity coupled fluid dynamics developed by [52], [132], [131]. A reduced Hamiltonian was constructed by [132] in co-moving spatial gauge and a time coordinate condition, where the lapse function and shift vector field were obtained through algebraic equations. Here, however, we explicitly worked in constant mean curvature gauge and derived the monotonic decay of the weak Lyapunov function and therefore the existence of a reduced Hamiltonian in this gauge remains to be checked. Later utilizing the CMC temporal gauge together with a co-moving gauge, we will show that indeed the matter density falls off asymptotically in an expanding universe model. We will explicitly work in a '3+1' dimensional universe.

Using Schutz's velocity potential technique [52], we will attempt to construct a reduced Hamiltonian in CMC gauge. We just provide a very brief description of the Schutz's velocity potential formulation, which is necessary for our purpose. For complete details, the reader is referred to [52].

The 4-velocity of the fluid u^{μ} may be written in terms of the five velocity potentials $\phi, \alpha, \beta, \theta$, and S as

$$u_{\mu} = \frac{1}{h}(\partial_{\mu}\phi + \alpha\partial_{\mu}\beta + \theta\partial_{\mu}S), \tag{1.67}$$

where h is the specific enthalpy of the fluid

$$h = \frac{P+\rho}{n}, \tag{1.68}$$

n being the baryon number density. From the equation of state

$$P = P(h, S), \tag{1.69}$$

we have

$$dP = ndh - nTdS, \tag{1.70}$$

T being the temperature. The equations of motions are obtained as

$$\begin{aligned}
u^\mu \partial_\mu \phi &= -h, && (1.71)\\
u^\mu \partial_\mu \alpha &= 0, && (1.72)\\
u^\mu \partial_\mu \beta &= 0, && (1.73)\\
u^\mu \partial_\mu \theta &= T, && (1.74)\\
u^\mu \partial_\mu S &= 0 && (1.75)
\end{aligned}$$

along with the continuity equation

$$\nabla_\mu (n u^\mu) = 0, \tag{1.76}$$

where the covariant derivative is with respect to the spacetime metric (e.o.m along with continuity equation are equivalent to the vanishing of the covariant divergence of the stress energy tensor).The conjugate momenta corresponding to the scalar fields $q^\eta \equiv (\phi, \alpha, \beta, \theta, S)$ are as follows [52]

$$\begin{aligned}
p^\phi &= -nNu^0, && (1.77)\\
p^\alpha &= 0, && (1.78)\\
p^\theta &= 0, && (1.79)\\
p^\beta &= \alpha p^\phi, && (1.80)\\
p^S &= \theta p^\phi. && (1.81)
\end{aligned}$$

Here, note that only α, β, and p^ϕ are independent. Using Dirac's technique (for the degenerate case as the one presented here) the fluid Hamiltonian density $\mathcal{H}_f$ is calculated, which together with the gravitational Hamiltonian density provide the total Hamiltonian density of the dynamics

$$\mathcal{H}_{total} = \mathcal{H}_{gravity} + \mu_g \mathcal{H}_f. \tag{1.82}$$

The lapse function N and shift vector field X are included in the total Hamiltonian density, which will act as Lagrange multipliers in the ADM action. Using the total Hamiltonian density, the fluid-ADM action may be written in the following form

$$S_{ADM} = \int_{I\subset\mathbf{R}} dt \int_{\mathbf{M}} (\pi^{ij}\frac{\partial g_{ij}}{\partial t} + \sum_{\eta} p^{\eta}\dot{q}_{\eta} - \mathcal{H}_{total})d^n x,$$

where π^{ij}, the gravitational momentum conjugate to g_{ij}, is obtained from the second fundamental form k^{ij} through the legendre transformation $\pi^{ij} = -\mu_g(k^{ij} - (tr_g k)g^{ij})$, and q_η and p^η are defined before. Varying the acting with respect to the Lagrange multipliers the lapse N and the shift X^i, one immediately obtains the Hamiltonian and momentum constraints of the fluid coupled gravity dynamics.

When, both of these constraints are satisfied, the term $\mathcal{H}$ totally disappears yielding the following reduced action

$$S_{reduced} = \int_{I\subset\mathbf{R}} dt \int_{\mathbf{M}} (\pi^{ij}\frac{\partial g_{ij}}{\partial t} + \sum_{\eta} p^{\eta}\dot{q}_{\eta})d^3 x.$$

Now we peform the Hamiltonian reduction via conformal transformation. Following the conventional scaling of the metric and the momenta [107, 161, 229],

$$(g, \pi^{TT}) = (\psi^4\gamma, \psi^{-4}p^{TT}), \tag{1.83}$$

the conformal Killing tensor $(L_Y g)^{ij} - \frac{2}{3}(\nabla_m Y^m)g^{ij}$ transforms as

$$\begin{aligned}(L_Y g)^{ij} - \frac{2}{3}(\nabla_m Y^m)g^{ij} = \psi^{-4}((L_Y\gamma)^{ij} \\ -\frac{2}{3}(\nabla[\gamma]_m Y^m)\gamma^{ij}).\end{aligned} \tag{1.84}$$

Using the Legendre transformation $\pi^{ij} = -\mu_g(k^{ij} - (tr_g k)g^{ij})$ and the equations (1.42-1.44), the gravitational momenta π^{ij} may be written as follows

$$\pi^{ij} = \pi^{TTij} + \frac{tr_g\pi}{3}g^{ij} - \psi^{-6}\mu_g[(L_Y g)^{ij} - \frac{2}{3}(\nabla_m Y^m)g^{ij}], \tag{1.85}$$

where π^{TT} is transverse-traceless with respect to g i.e., $\pi^{TTij}g_{ij} = 0 = \nabla[g]_j\pi^{TTij}$.

Using these conformal transformations, the reduced action becomes

$$\begin{aligned} S_{reduced} &= \int_{I\subset\mathbf{R}} dt \int_M (p^{TTij}\frac{\partial\gamma_{ij}}{\partial t} - \frac{4\mu_g}{3}\frac{\partial\tau}{\partial t} \\ &\quad -((L_Y\gamma)^{ij} - \frac{2}{3}\nabla[\gamma]_m Y^m\gamma^{ij})\frac{\partial\gamma_{ij}}{\partial t}\mu_\gamma \\ &\quad +\sum_\eta p^\eta\dot{q}_\eta)d^3x, \end{aligned} \tag{1.86}$$

where we have left the fluid variables unscaled intentionally for reasons which will be discussed now. Note that even though Schutz's formalism is only true for '3+1' spacetimes, the result we are about to state holds for any higher dimension irrespective of the fluid source. That is any non-vanishing momentum flux density J^i leads to the following problem while trying to construct an unconstrained Hamiltonian. Following the conformally scaled reduced action, one immediately observes that in order to obtain a reduced Hamiltonian, one must reduce the action to the form similar to $\int_{I\times M} dtd^3x(\pi^{TT}\partial_t\gamma + \sum_\eta p^\eta\dot{q}_\eta - \mathcal{H}_{reduced})$ and therefore, one need to somehow make the term $\int_M((L_Y\gamma)^{ij} - \frac{2}{n}\nabla[\gamma]_m Y^m\gamma^{ij})\frac{\partial\gamma_{ij}}{\partial t}\mu_\gamma$ disappear. In order to do so, we must work in a slice of the D_0 action on $\mathcal{M}_{-1}$ i.e., $\frac{\partial\gamma_{ij}}{\partial t} = \frac{\partial\gamma_{ij}}{\partial r^a}\dot{r} = l^{TT}_{ij}\dot{r}$, for some $\gamma-$transverse-traceless covariant 2-tensor l and co-ordinates $\{r^a\}$ on some local chart of $\mathcal{M}_{-1}/D_0$. However, for dimension $n > 2$, such procedure is restricted as a consequence of the absence of a globally integrable distribution i.e., a slice of the D_0 action is not a submanifold of $\mathcal{M}_{-1}$ in general (and therefore is not integrable). For 2 dimensions, the Teichmüller space has a nice global manifold structure (as a submanifold of $\mathcal{M}_{-1}$) and thus is globally integrable. Of course, in the absence of a matter source (with or without cosmological constant), the vector field Y, the generator of the diffeomorphism of M vanishes due to the momentum constraint and the weak Lyapunov function $\frac{4}{3}\int_M \frac{\partial\tau}{\partial t}\mu_g$ becomes a reduced Hamiltonian.

Even though we could not obtain a reduced Hamiltonian in CMC gauge, we may show using CMC time gauge and a co-moving gauge, that the matter sources (perfect fluid), in an expanding universe, becomes asymptotically negligible. Let us write the time and spatial slice gauge explicitly first. In '3+1' CMC gauge, a time coordinate is chosen as (consistent throughout the study)

$$\frac{\partial\tau}{\partial t} = \frac{3}{4}(\tau^2 - 3\Lambda)^{\frac{3}{2}} \tag{1.87}$$

along with the co-moving spatial gauge condition

$$u^i = 0, \tag{1.88}$$

which through the normalization condition $g_{\mu\nu}u^{\mu}u^{\nu} = -1$ yields

$$u^0 = \frac{1}{\sqrt{N^2 - X_i X^i}} \tag{1.89}$$

with $N^2 > X_i X^i$. Noting that both E and S are nonnegative for perfect fluids, an estimate of N follows immediately from (1.25)

$$0 < \frac{\frac{9}{4}(\tau^2 - 3\Lambda)^{\frac{1}{2}}}{\left(1 + \frac{1}{\tau^2 - 3\Lambda}\sup(|K^{tr}|^2 + \frac{1}{2}(S + 2E))\right)} \leq N \leq \frac{9}{4}(\tau^2 - 3\Lambda)^{\frac{1}{2}}.$$

Since the condition $N^2 > X_i X^i$ has to be satisfied point-wise through the spacetime, $X_i X^i$ may be written in terms of N^2 as CN^2 for a suitable $1 > C > 0$. Finally, the continuity equation (1.76) may be written in the chosen gauge as

$$\frac{\partial}{\partial t}(\sqrt{N\mu_g} n u^0) = 0, \tag{1.90}$$

which upon substituting u^0 and $X_i X^i$ yields

$$n \approx C_1 \sqrt{\frac{N}{\mu_g}}, \tag{1.91}$$

where $C_1 > 0$ is a suitable constant. In an expanding universe (if initially expanding, it will only continue to expand, because, the formation of a maximal hypersurface and subsequent re-collapse are strictly prohibited; τ can never vanish by virtue of the Hamiltonian constraint) , the spatial volume form μ_g satisfies the estimate $\mu_g \approx (\tau^2 - 3\Lambda)^{-2\alpha}$, for some $\alpha > 0$ (may be obtained by time differentiating $\int_M \mu_g$ in conjunction with the use of field equations and estimate of lapse function), which together with the estimate of N yields the decay of the baryon density

$$n \approx C_2(\tau^2 - 3\Lambda)^{\frac{1}{4}+\alpha}, \tag{1.92}$$

for some $C_2 > 0$. In the asymptotic limit $\tau \to -\sqrt{3\Lambda}$, the baryon density becomes negligible if the matter source does not already collapse to form a singularity. Therefore, the previous asymptotic analysis holds true and the infimum of the Lyapunov function is attained in this limit allowing the persistence of topological consequences discussed previously. However, an important thing to note here is that the analysis presented here does not address the global existence by any means (despite the fact that the Lyapunov function provides a weak notion). Even in the absence of matter sources, pure gravity could 'blow up' before the volume tends to infinity i.e., gravitational singularities could prevent global existence. Such global existence is addressed by [206] in the special case of small data limit and vacuum spacetimes (without cosmological constant) of arbitrary dimension. In case of vacuum spacetimes with a positive cosmological constant, proof of the global existence of the sufficiently small perturbations about background conformal spacetimes (eq. 1.37; through studying the asymptotic stability of such spacetimes) is under preparation by the author [90]. [103] studied the global existence of small data perturbations in case of vacuum spacetimes with a cosmological constant by establishing the Lyapunov stability of the background solutions. In the presence of matter sources, global existence, even in the small data limit, is an open problem.

Developing a cosmological model is of fundamental importance in general relativity. The present so-called FLRW models are developed based on astronomical observations of local homogeneity and isotropy (since the observations are only limited to a possibly quite small fraction of the universe) that is the procedure resembles 'model fitting' in certain sense. Extrapolating the local observations to a global scale thereby allows only special simple topologies and leads to the reduction of Einstein equations to ordinary differential equation for one unknown namely the scale parameter. This is, in a sense, against the whole spirit of general relativity where one should include more general topologies and study the Einsteinian dynamics in a rigorous way. The resulting cosmological model should then be compared locally with the observations. For example, for a compact spatial topology, within the injectivity radius around a point, space seems locally flat despite the possibility that the manifold may originally be globally exotic (with strictly nonvanishing curvature). In addition, manifolds may also admit locally homogeneous and isotropic metrics with no such global extension. In a sense, the role of spatial topologies in the dynamics of general relativity is somewhat underestimated and we precisely address possibly a small fraction of such roles here. We show by invoking a weak Lyapunov function that while the initially expanding universe in the presence of suitable matter sources (satisfying suitable energy conditions) and a positive cosmological constant, may not be isotropic and homogeneous (not even locally), it nevertheless will evolve to support a homogeneous and isotropic metric if the evolution continues long enough without going singular. Therefore, one does not in general need to start the evolution with an isotropic and homogeneous metric at all. Einsteinian evolution automatically leads to the spatially homogeneous and isotropic universe in the asymptotic limit where the baryon number density becomes negligible. Our analysis of course only addresses the issue of asymptotic negligibility of the matter in the case of a rather special perfect fluid source through invoking Schutz's velocity potential formalism in a constant mean curvature co-moving gauge. In the limiting case, as shown previously, the asymptotic vanishing of the matter density simplifies the associated conformal equation (Lichnerowicz equation) leading us to the main result. Of course, during the course of evolution, matter or gravitational radiation may collapse to form a singularity and a rough notion of such singularity formation (or at least geodesic incompleteness via formation of caustics) is stated by the singularity theorems for matter sources satisfying the strong energy condition. In our case while a positive cosmological constant is included, even the formation of caustics is excluded in an expanding universe. In order to establish an absolute claim that 'true' singularities are avoided, one must investigate the dynamics of Einstein equations in the presence of exotic topologies numerically (which is currently only limited to very special manifolds with several symmetries). Nevertheless, it is realized through our analysis that the universe may be locally isotropic and homogeneous, but does not have to be globally so. Such topological non-triviality demonstrated here, may be only the beginning of a bigger picture and may potentially be able to answer the fundamental questions related to issues such as the flatness problem, horizon problem or even have consequences for the **Cosmic Censorship** conjecture.

Another interesting fact revealed through our analysis is that while the associated Lyapunov function has the physical meaning of being a reduced Hamiltonian ('true Hamiltonian of the dynamics) only in case of a vacuum universe (with or without a positive cosmological constant), and therefore may be used for the purpose of quantization, matter sources (even the special perfect fluid case) rule out such significance in CMC gauge. Nevertheless, we are able to draw general conclusions about the asymptotic behavior of the spacetimes using the Lyapunov function alone. However, the weak nature (in terms of norm control) as discussed previously, does not allow a global existence theorem and for the time being we are only limited to the asymptotic results. Nevertheless, our analysis opens up several interesting questions which as described before may potentially address several existing problems. Are the Cauchy hypersurfaces always asymptotically volume-dominated by their hyperbolic components with the rescaled metrics on these components asymptotically approaching homogeneity and isotropy? Do wormholes ($\mathbf{S}^2 \times \mathbf{S}^1$) , spherical space forms ($\mathbf{S}^3/\Gamma, \Gamma \in$SO(4)), and graph manifolds always asymptotically pinch off from the universe? How is the fundamental question of 'Cosmic Censorship' influenced by answers to these questions? what happens if one includes matter sources not satisfying either or both of the energy conditions mentioned? Upon further development of Einsteinian dynamics with matter sources, these questions may be answered yielding a self consistent cosmological model.

Chapter 2

Attractors of the ‘n+1’ dimensional Einstein-Λ flow

Here we prove a global existence theorem for sufficiently small however fully nonlinear perturbations of a family of background solutions of the ‘$n+1$’ dimensional vacuum Einstein equations in the presence of a positive cosmological constant Λ. The future stability of vacuum solutions in the small data and zero cosmological constant limit has been studied previously for both ‘$3+1$’ and higher dimensional spacetimes. However, with the advent of dark energy driven accelerated expansion of the universe, it is of fundamental importance in mathematical cosmology to include a positive cosmological constant, the simplest form of the dark energy for the vacuum Einstein equations. Such Einsteinian evolution is here designated as the ‘Einstein-Λ’ flow. We study the background solutions of this ‘Einstein-Λ’ flow in ‘$n+1$’ dimensional spacetimes in constant mean curvature spatial harmonic gauge, $n \geq 3$ and establish both linear and non-linear stability of such solutions. In the cases of number of spatial dimensions being strictly greater than 3, the finite dimensional Einstein moduli spaces form the center manifolds of the dynamics. A suitable shadow gauge condition [206] is implemented in order to treat these cases. In addition, the autonomous character of the suitably re-scaled Einstein flow breaks down as a consequence of including $\Lambda(>0)$. We construct a Lyapunov function (controlling a suitable norm of the small data) similar to a wave equation type energy for the non-linear non-autonomous evolution of the small data and prove its decay in the direction of cosmological expansion utilizing the structure of the non-autonomous terms and smallness assumption on the data. Our results demonstrate the future stability and geodesic completeness of the perturbed spacetimes, and show that the scale-free geometry converges to an element of the space of constant negative scalar curvature metrics sufficiently close to and containing the Einstein moduli space (a point for $n=3$ and a finite dimensional space for $n>3$), which has significant consequences for the cosmic topology while restricting to the case of $n=3$.

Since the Einstein equations formulated as a Cauchy problem leave the spatial topology of the universe unrestricted, a natural question arises whether one may constrain the topology through studying the dynamics of the Einsteinian evolution, while also satisfying the cosmological principle. Recent articles [107, 229] unfolded such a possibility i.e., a dynamical mechanism at work within the Einstein flow (both with and without Λ) which strongly suggests that many closed 3-manifolds that do not admit a locally homogeneous and isotropic metric at all (and thus are incompatible with the cosmological principle) will nevertheless evolve under Einsteinian evolution to be asymptotically compatible with the observed, approximate, spatial homogeneity and isotropy of the universe. Results of these studies, based on the monotonic decay of a weak Lyapunov function namely the reduced Hamiltonian, suggested that a 3-manifold of negative Yamabe type, if it contains parts supporting hyperbolic metrics in its connected sum decomposition, will be volume dominated by these hyperbolic components asymptotically by the Einstein flow (Einstein-Λ flow for $\Lambda \neq 0$). On the other hand, if one takes the cosmological principle literally, the so-called FLRW model restricts the choice of global spatial topology of the universe to a small set consisting of negatively curved hyperbolic space $\mathbf{H}^3$, flat Euclidean space $\mathbf{E}^3$, positively curved 3-sphere $\mathbf{S}^3$ with its canonical round metric and its two fold quotient $\mathbf{PR}^3 = \mathbf{S}^3/Z_2$. However, the astronomical observations that motivate the cosmological principle are necessarily limited to a fraction (possibly small) of the entire universe and such observations are compatible with spatial metrics being locally but not globally homogeneous and isotropic. Once the restriction on the topology (a global property) is removed, numerous closed manifolds may be constructed as the quotients of $\mathbf{H}^3, \mathbf{E}^3$, and $\mathbf{S}^3$ by discrete, proper, and torsion free subgroups of their respective isometry groups, with each satisfying the local homogeneity and isotropy criteria but no longer being globally homogeneous or isotropic. In order for these topologically rich spatially compact spacetimes to be possible candidates for the cosmological models, it is crucial to study the asymptotic behaviour of the fully nonlinear perturbations of these models. Non-linear stability of these spacetimes clearly opens up the possibility for the universe to have an exotic spatial topology. In this paper, we study the linear (which acts as a motivation towards studying non-linear stability) and non-linear stability of the spacetimes with spatial part being negative Einstein for the cases of $n \geq 3$ (hyperbolic for $n = 3$). However, we will assume certain smallness condition on the fully nonlinear perturbations.

The family of background solutions (a class of fixed points of the Einstein-Λ flow) designated as *'conformal spacetimes'* (due to the fact that each of these spacetimes admits a timelike conformal Killing field) with the spacetime topology $\mathbf{R} \times M$ (M being the spatial manifold) in constant mean curvature spatial harmonic gauge (CMCSH), may be written in the following warped product form

$$\hat{g} = -\frac{n^2}{(\tau^2 - \frac{2n\Lambda}{n-1})^2} d\tau \otimes d\tau + \frac{1}{(\tau^2 - \frac{2n\Lambda}{n-1})} \gamma_{ij} dx^i \otimes dx^j, \tag{2.1}$$

where γ is an Einstein metric satisfying $R_{ij}(\gamma) = -\frac{n-1}{n^2}\gamma_{ij}$ and $\tau \in (-\infty, -\sqrt{\frac{2n\Lambda}{n-1}})$ is the mean extrinsic curvature of M in the globally hyperbolic spacetime on $\mathbf{R} \times M$. For the special case of $n = 3$, the negative Einstein spaces are hyperbolic i.e., the Einstein moduli space reduces to a point. In the limit of $\Lambda = 0$, [206] calls these spacetimes the Lorentz cone space times (these are constructed by taking quotient of the interior of the future light cone in Minkowski space by $SO^+(3,1)$ in case of $n = 3$). Stability of these '3+1' Lorentz cone spacetimes was proven by [151] utilizing the Bel-Robinson energy. In the more general setting of $n > 3$, a finite dimensional space of Einstein metrics provides the 'center manifold' towards which the re-scaled spatial metric is flowing in the limit of infinite cosmological expansion. [206] proved the stability of these background solutions by invoking a so called shadow gauge condition and later utilized a wave equation type of energy for the sufficiently small however fully nonlinear perturbations. This energy acts like a Lyapunov function for these perturbations (and, in particular vanishes at fixed points), which is defined to control the desired norm of perturbations. In the case of the '2+1' Einstein flow on $\mathbf{R} \times \Sigma_{genus}$ with $genus > 1$, Teichmüller space plays the role of the Einstein moduli space and special techniques [197, 198] were used to study the global existence (by utilizing the properness of the Dirichlet energy functional defined on the Teichmüller space). Recently [103] studied the Lyapunov stability of these background solutions including a positive cosmological constant. However, in order to establish the 'attractor' property of certain solutions, it is necessary to prove their asymptotic stability. [161,229] constructed a reduced phase space as the cotangent bundle of the higher dimensional analogue of the Teichmüller space and obtained the following true Hamiltonian of the dynamics while expressed as a functional of the reduced phase space variables through a conformal technique

$$H_{reduced} \quad := \quad \frac{2(n-1)}{n} \int_M \frac{\partial \tau}{\partial t} \mu_g. \tag{2.2}$$

They have shown that the reduced Hamiltonian acting as a Lyapunov function decays along certain solutions of the Einstein equations and achieves its infimum precisely for the background Lorentz cone spacetimes (conformal spacetimes for $\Lambda \neq 0$). Such a property provides a notion of the stability of these background solutions for arbitrarily large perturbations. However, the Lyapunov function only controls the $H^1 \times L^2$ norm of the reduced data and therefore, such a notion of stability is weak. Motivated by these results, we intend to study the stability of these background solutions for sufficiently small fully nonlinear perturbations in the case when a positive consmological constant is included in the Einstein equations. A subtlety is that the properties of the wave equation type energy exploit the information about the lowest eigenvalue of the Lichnerowicz type Laplacian (acting on the space of symmetric (0,2) tensors on M, that is, $S_2^0(M)$) which enters into the evolution equation. In this paper, we consider the complete generality of the problem in a framework of sufficiently small however fully nonlinear perturbations of the background solutions. However, the inclusion of a positive cosmological constant introduces several seemingly restrictive features of the field equations. A few examples may be seen as follows. In the CMCSH gauge, the vacuum Einstein equations with $\Lambda = 0$ are non-autonomous due to the fact that the mean extrinsic curvature acting as time explicitly appears in the equations. However, after a suitable re-scaling, the equations can be made to be autonomous. In the presence of a nonzero Λ, such a property is lost. This is one of the major differences from the case of $\Lambda = 0$. Nevertheless, one may still obtain estimates necessary to prove the decay property of a suitably defined Lyapunov function by introducing a Newtonian like time co-ordinate. In particular, we take the advantage of the structure of the Einstein's equations. The explicitly time dependent terms appear in the field equation in such a way as to help drive the flow towards a class of fixed points. Roughly speaking, the potentially problematic terms in the expression for the time derivative of the energy either gets cancelled point-wise or are multiplied by asymptotically decaying factor e^{-T} ($T \in (-\infty, \infty)$). Therefore, one needs to keep track of the explicitly time dependent terms accurately to reach the conclusion of energy decay. This subtlety does not arise in the case of vanishing cosmological constant and one may easily obtain the decay of the suitably defined Lyapunov function by introducing a correction factor (see [206] for detail). In the process of controlling the energy decay, invoking the shadow gauge in order to handle the nontrivial moduli space (such moduli space is assumed to have a smooth structure and be stable) becomes necessary. In the $\Lambda = 0$ case, the scale-free geometry converges to the Einstein structure and therefore the attractors of the Einstein flow is identified to be the Einstein spaces. However, in the presence of $\Lambda > 0$, the scale-free geometry exponentially converges to space of metrics with constant negative scalar curvature sufficiently close to and containing the Einstein structure. In other words, if we start the evolution with an arbitrary metric g sufficiently close to the Einstein moduli space such that the difference is L^2 orthogonal to the moduli space then the conformal geometry converges in infinite time to an element of the space of metrics with constant negative scalar curvature sufficiently close to and containing the Einstein structure. On the other hand perturbations tangential to the Einstein moduli space exponentially decays such that the metric converges to another element of the moduli space, that is, tangential perturbations exhibit trivial asymptotic stability. Therefore, the attractor of the Einstein-Λ flow is identified to be the space of constant negative scalar curvature metrics lying sufficiently close to and containing the Einstein moduli space. In addition, the only requirement for the asymptotic stability of these spacetimes is the stability of the negative Einstein structure i.e., the non-negativity of the eigen spectrum of the associated Lichnerowicz type Laplacian operator. These are some major differences between the $\Lambda = 0$ and $\Lambda > 0$ cases. The asymptoically stable physical conformal spacetimes are given by the following warp product metric

$$\hat{g} = -\frac{n^2}{(\tau^2 - \frac{2n\Lambda}{n-1})^2} d\tau \otimes d\tau + \frac{1}{(\tau^2 - \frac{2n\Lambda}{n-1})} \gamma^{\dagger}_{ij} dx^i \otimes dx^j, \tag{2.3}$$

where $R(\gamma^{\dagger}) = -\frac{n-1}{n}$. Our strategy would be to perturb the fixed point solutions described by the negative Einstein spaces and employ the shadow gauge to treat the evolution of the perturbations orthogonal and tangential to the Einstein moduli space. Using necessary estimates, we then prove that the spatial metric exponentially converges (in appropriate norm topology) to a point of the space of constant negative scalar curvature metrics sufficiently close to and containing the moduli space. In summary, the structure of the paper is the following. We start with the gauge fixed Einstein-Λ equations and state the necessary theorems ensuring local well-posedness of the Cauchy problem. Next, we move on to computing background solutions and study their linear stability. After obtaining a series of estimates utilizing the elliptic equations arising as a result of gauge fixing and imposing the shadow gauge condition, we construct a Lyapunov function for the small data. In the last part, utilizing the obtained estimates, we prove the decay of the constructed energy functional (which vanishes only for the spacetimes of type (3.1) and remains constant for the spacetimes of type (2.3)) thereby establishing the stability of the background solutions.

2.1 Notations and facts

We denote the '$n+1$' dimensional ($n \geq 3$) spacetime manifold by $\tilde{M}$ with its topology being $\mathbf{R} \times M$, M being the n-dimensional spatial slice diffeomorphic to a Cauchy hypersurface. The space of Riemannian metrics on M is denoted by $\mathcal{M}$. $\mathcal{M}_{-\frac{n-1}{n}}$ is defined as follows

$$\mathcal{M}_{-\frac{n-1}{n}} := \{g \in \mathcal{M} | R(g) = -\frac{n-1}{n}\},$$

with $R(g)$ being the scalar curvature associated with g. $R_{ijkl}[g]$ and $\Gamma[g]^i_{jk}$ denote the Riemann curvature and connection coefficients with respect to the metric g, respectively. In terms of the function space of fields (metric, second fundamental form etc.), we work in the L^2 (with respect to a given metric) Sobolev space $W^{s,2}$ for $s > \frac{n}{2} + 2$, also denoted by H^s. We denote the L^2 inner product between two 2-tensors on M with respect to a background metric γ as

$$< u|v >_{L^2} = \int_M u_{ij} v_{kl} \gamma^{ik} \gamma^{jl} \mu_g$$

and the inner product on derivatives as

$$< \nabla[\gamma]u|\nabla[\gamma]v >_{L^2} \quad = \quad \int_M \nabla[\gamma]_m u_{ij} \nabla[\gamma]_n v_{kl} g^{mn} \gamma^{ik} \gamma^{jl} \mu_g,$$

where μ_g is the volume form associated with $g \in \mathcal{M}$

$$\mu_g = \sqrt{\det(g_{ij})} dx^1 \wedge dx^2 \wedge dx^3 \wedge .... \wedge dx^n.$$

Abusing notation, we use μ_g to denote both the volume form as well as $\sqrt{\det(g_{ij})}$. The rough Laplacian $\Delta_{g,\gamma}$ acting on a vector bundle (symmetric covariant 2-tensors are sections of this bundle) over (M, g) is defined as

$$\Delta_{g,\gamma} h_{ij} := -\frac{1}{\mu_g} \nabla[\gamma]_m (g^{mn} \mu_g \nabla[\gamma]_n h_{ij}). \tag{2.4}$$

This rough Laplacian is self-adjoint with respect to the L^2 inner product on covariant 2-tensors. Using the rough Laplacian, a self-adjoint Lichnerowicz type Laplacian which will be crucial later is defined as follows

$$\mathcal{L}_{g,\gamma} h_{ij} := \Delta_{g,\gamma} h_{ij} - 2R[\gamma]_i{}^k{}_j{}^l h_{kl}. \tag{2.5}$$

We may sometimes drop the Sobolev index to simplify the notation. The reader is expected to assume the function space for (g, k, N, X) to be $H^s \times H^{s-1} \times H^{s+1} \times H^{s+1}$ with $s > \frac{n}{2} + 2$. The Laplacian Δ_g is defined so as to have a non-negative spectrum i.e.,

$$\Delta_g \equiv -g^{ij} \nabla_i \nabla_j. \tag{2.6}$$

For $a, b \in R_{>0}$, $a \lesssim b$ is defined to be $a \leq Cb$ for some constant $0 < C < \infty$. The spaces of symmetric covariant 2-tensors and vector fields on M are denoted by $S^0_2(M)$ and $\mathfrak{X}(M)$, respectively.

2.2 Field equations and gauge fixing

The ADM formalism splits the spacetime described by an 'n+1' dimensional Lorentzian manifold $\tilde{M}$ into $\mathbf{R} \times M$ with each level set $\{t\} \times M$ of the time function t being an orientable n-manifold diffeomorphic to a Cauchy hypersurface (assuming the spacetime to be globally hyperbolic) and equipped with a Riemannian metric. Such a split may be implemented by introducing a lapse function N and shift vector field X belonging to suitable function spaces and defined such that

$$\partial_t \quad = \quad N\hat{n} + X \tag{2.7}$$

with t and $\hat{n}$ being time and a hypersurface orthogonal future directed timelike unit vector i.e., $\hat{g}(\hat{n}, \hat{n}) = -1$, respectively. The above splitting puts the spacetime metric $\hat{g}$ in local coordinates $\{x^\alpha\}_{\alpha=0}^n = \{t, x^1, x^2, ..., x^n\}$ into the form

$$\hat{g} \quad = \quad -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt) \tag{2.8}$$

where $g_{ij} dx^i \otimes dx^j$ is the induced Riemannian metric on M. In order to describe the embedding of the Cauchy hypersurface M into the spacetime $\tilde{M}$, one needs the information about how the hypersurface is curved in the ambient spacetime. Thus, one needs the second fundamental form k defined as

$$k_{ij} \quad = -\tfrac{1}{2N}(\partial_t g_{ij} - (L_X g)_{ij}), \tag{2.9}$$

the trace of which ($\tau = g^{ij} k_{ij}$, $g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} := g^{-1}$) is the mean extrinsic curvature of M in $\tilde{M}$. Here L denotes the Lie derivative operator. The vacuum Einstein equations with a cosmological constant Λ

$$R_{\mu\nu}(\hat{g}) - \frac{1}{2} R(\hat{g}) \hat{g}_{\mu\nu} + \Lambda \hat{g}_{\mu\nu} = 0 \tag{2.10}$$

may now be expressed as the evolution and Gauss and Codazzi constraint equations of g and k

$$\partial_t g_{ij} = -2Nk_{ij} + (L_X g)_{ij}, \tag{2.11}$$

$$\partial_t k_{ij} = -\nabla_i \nabla_j N + N(R_{ij} + \tau k_{ij} - 2k_i^k k_{jk} - \frac{2\Lambda}{n-1} g_{ij}) + (L_X k)_{ij}, \tag{2.12}$$

$$2\Lambda = R(g) - |k|_g^2 + (tr_g k)^2, \tag{2.13}$$

$$0 = \nabla^i k_{ij} - \nabla_j \tau, \tag{2.14}$$

where $\tau = tr_g k$. A solution to the Einstein evolution and constraint equations is a curve $t \mapsto (g(t), k(t), N(t), X(t))$ in $H^s \times H^{s-1} \times H^{s+1} \times H^{s+1}$ (at least in our case where the local existence theorem holds in this function space) satisfying equations (4.17)-(4.19). The spacetime metric $\tilde{g}$ given in terms of (g, N, X) by (4.14) solves the Einstein equation (4.16) if and only if (g, k, N, X) solves the evolution and constraint equations (4.17)-(4.19). However, the system (4.17)-(4.19) is not hyperbolic. We may reduce the system to a canonical hyperbolic evolution equation for g by fixing gauge. The physical concept of gauge fixing (spatial and temporal) may be described as follows. Let us first consider the spatial gauge. With the spacetime topology of $\mathbf{R} \times M$, one has the freedom to choose the spatial slice as long as it is diffeomorphic to a Cauchy hypersurface. Let M be a Cauchy hypersurface with an induced metric g which together with (k, N, X) satisfies the Einstein evolution and constraint equations (4.17-4.19). Now let ϕ ($t-$independent) be an element of the identity component of the diffeomorphism group ($\mathcal{D}_0$) of M. Then $(\phi^{-1})^* g, (\phi^{-1})^* k, (\phi^{-1})^* N = N \circ \phi^{-1}$, and $\phi_* X$ solves the Einstein equations as well, where *, and $_*$ denote the pullback and push-forward operations on the cotangent and tangent bundles of M, respectively. This is obvious due to the spatial covariant nature of the evolution and constraint equations. More generally, let the identity component of the diffeomorphism group act on M by a time dependent element ϕ_t. The evolution equation (4.17) under the action of ϕ_t reads

$$\begin{aligned}
\partial_t((\phi_t^{-1})^* g)_{ij} &= -2(\phi_t^{-1})^*(Nk)_{ij} + (L_{\phi_{t*}X}(\phi_t^{-1})^* g)_{ij}, \qquad (2.15)\\
((\phi_t^{-1})^* \partial_t g)_{ij} + (\partial_t(\phi_t^{-1})^* g)_{ij} &= -2(\phi_t^{-1})^*(Nk)_{ij} \\
&\quad + \frac{\partial}{\partial s}((\phi_t^{-1} \Psi_s^X \phi_t)^* (\phi_t^{-1})^* g)_{ij}|_{s=0}, \\
((\phi_t^{-1})^* \partial_t g)_{ij} + (\partial_s(\phi_{t+s}^{-1})^* g)_{ij}|_{s=0} &= -2(\phi_t^{-1})^*(Nk)_{ij} + (\phi_t^{-1})^*(L_X g)_{ij}, \\
(\phi_t^{-1})^* \partial_t g_{ij} + (\phi_t^{-1})^*(L_Y g)_{ij} &= -2(\phi_t^{-1})^*(Nk)_{ij} + (\phi_t^{-1})^*(L_X g)_{ij}, \\
(\phi_t^{-1})^* \left\{ \partial_t g_{ij} = -2Nk_{ij} + (L_{X-Y} g)_{ij} \right\}.
\end{aligned}$$

Here Y is the vector field associated with the flow ϕ_t and Ψ_s^X is the flow of the shift vector field X. A similar calculation for the evolution equation for the second fundamental form shows that if we make a trasformation $X \mapsto X+Y$, the Einstein evolution and constraint (due to their natural spatial covariance nature) equations are satisfied by the transformed fields. The choice of spatial hypersurface is fixed by choosing constant mean extrinsic curvature spatial harmonic gauge. Constant mean extrinsic curvature gauge defines a time function and therefore it is the temporal gauge choice. We briefly describe the gauge fixing below starting with spatial harmonic gauge. Let $\phi : (M, g) \rightarrow (M, \gamma)$ be a harmonic map. Clearly it satisfies the Euler-Lagrange equations arising from criticality of the associated Dirichlet energy $\frac{1}{2}\int_M g^{ij}\frac{\partial\phi^k}{\partial x^i}\frac{\partial\phi^l}{\partial x^j}\gamma_{kl}\mu_g$ i.e.,

$$g^{ij}\left(\frac{\partial^2\phi^k}{\partial x^i\partial x^j} - \Gamma[g]^l_{ij}\frac{\partial\phi^k}{\partial x^l} + \Gamma[\gamma]^k_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\right) = 0. \tag{2.16}$$

Now, we fix the gauge by imposing the condition that $\phi = id$, which leads to the following equation

$$-g^{ij}(\Gamma[g]^k_{ij} - \hat{\Gamma}[\gamma]^k_{ij}) \quad = \quad 0. \tag{2.17}$$

where $\hat{\Gamma}[\gamma]^k_{ij}$ is the connection with respect to some arbitrary background Riemannian metric γ. Choice of this spatial harmonic gauge yields an elliptic equation for the shift vector field X after time differentiating equation (3.29). The spatial harmonic slice is chosen to have uniform mean extrinsic curvature i.e.,

$$\partial_i\tau \quad = \quad 0, \tag{2.18}$$

and thus τ may play the role of time i.e.,

$$t = monotonic\ function\ of\ \tau, \tag{2.19}$$

and in this case, we choose $t = \tau$. Choice of the Constant mean extrinsic curvature gauge (CMC) yields an elliptic equation for the lapse N. Note that we do not have evolution equations for the lapse and shift. However, they are constrained by the elliptic equations obtained through gauge fixing which together with the evolution equations for g and k comprises the full 'Einstein-Λ' system

$$
\begin{aligned}
\partial_t g_{ij} &= -2Nk_{ij} + (L_X g)_{ij}, & (2.20)\\
\partial_t k_{ij} &= -\nabla_i\nabla_j N + N(R_{ij} + \tau k_{ij} - 2k_i^k k_{jk} - \frac{2\Lambda}{n-1} g_{ij} - \alpha_{ij}) & (2.21)\\
&\quad + (L_X k)_{ij}, &\\
\frac{\partial\tau}{\partial t} &= \Delta_g N + (k_{ij}k^{ij} - \frac{2\Lambda}{n-1})N, & (2.22)\\
\Delta_g X^i - R^i_j X^j + L_X V^i &= (\nabla^i N)\tau - 2\nabla^j N k^i_j + (2Nk^{jk} & (2.23)\\
&\quad -2\nabla^j X^k)(\Gamma[g]^i_{jk} - \hat{\Gamma}[\gamma]^i_{jk}) - g^{jk}\partial_t\hat{\Gamma}[\gamma]^i_{jk}, &
\end{aligned}
$$

where $\alpha_{ij} = \frac{1}{2}(\nabla_i V_j + \nabla_j V_i)$ and $-V^i$ is the tension field defined as

$$V^i = g^{jk}(\Gamma[g]^i_{jk} - \hat{\Gamma}[\gamma]^i_{jk}). \tag{2.24}$$

In addition, we also have the constraints

$$
\begin{aligned}
2\Lambda &= R(g) - |k|^2_g + (tr_g k)^2, & (2.25)\\
0 &= \nabla^i k_{ij}, & (2.26)
\end{aligned}
$$

which are conserved throughout the term of evolution as a consequence of the Bianchi identity. $V^i = 0$ essentially corresponds to the spatial harmonic gauge. This Cauchy problem with constant mean extrinsic curvature and spatially harmonic gauge is referred to as '**CMCSH Cauchy**' problem.

2.2.1 local well-posedness and gauge conservation

[150] proved a well-posedness theorem for the Cauchy problem for a family of elliptic-hyperbolic systems that included the '$n+1$' dimensional vacuum Einstein equations in CMCSH gauge. [229] sketched how to apply the theorem of [150] to a gauge fixed system of 'Einstein-Λ' field equations. Since the 'Einstein-Λ' field equations only differ from the vacuum equations by the addition of some rather innocuous linear terms, most of the technicalities of this extended application of their theorem are straightforward to verify. There are however a couple of subtle points involving the elliptic equations for the lapse function and the shift vector field. Firstly, inclusion of $\Lambda > 0$ seemingly creates an obstruction to achieving a trivial kernel for the lapse equation (2.22). However, note that we are primarily interested in negative Yamabe manifolds (see [107] and [229] for the relevant definitions) and therefore the scalar curvature $R(g)$ can never be positive everywhere on M yielding

the following range of allowed mean extrinsic curvature (for cosmologically expanding solutions) by virtue of the Hamiltonian constraint

$$-\infty < \tau < -\sqrt{\frac{2n\Lambda}{n-1}}. \tag{2.27}$$

This condition indeed guarantees a unique positive solution of the lapse equation (2.22). Secondly, allowing for the time dependent behavior of the background metric (a negative Einstein metric in our case) introduces the extra term '$-g^{jk}\partial_t\hat{\Gamma}[\gamma]^i_{jk}$' in the elliptic equation for the shift vector field. However, our primary concern is the small perturbations about the background and the term '$-g^{jk}\partial_t\hat{\Gamma}[\gamma]^i_{jk}$' acts as small perturbation (see lemma (3) and (4) for the relevant estimates). Therefore, the extra term in the shift equation due to time dependence of the background spatial metric does not affect the existence and uniqueness results. In a sense, these previous studies together complete the desired local well-posedness for the 'Einstein-Λ' system. For this reason we shall mostly refer the reader to the relevant sections of [150], [206], and [229] rather than reiterate the detailed arguments herein. The most important point to note is that the local existence theorem provides the time of existence in terms of the size of the initial data. Therefore, in the global existence argument, if the appropriate norm of the perturbation is bounded, one may immediately use to local existence to obtain the desired result.

In addition to proving the local well-posedness of the 'Einstein-Λ' quasi hyperbolic evolution equations, we also need to ensure the conservation of gauges and constraints i.e., whenever (g, k, N, X) solve the 'Einstein-Λ' equations (2.20-2.23), the following entities, if vanishing initially, are zero along the solution curve

$$A = \tau - t, \tag{2.28}$$

$$V^i = g^{jk}(\Gamma[g]^i_{jk} - \hat{\Gamma}[\gamma]^i_{jk}), \tag{2.29}$$

$$F = R(g) + \tau^2 - |k|^2 - \nabla_i V^i - 2\Lambda, \tag{2.30}$$

$$D_i = \nabla_i \tau - 2\nabla^k k_{ki}. \tag{2.31}$$

One may show by direct calculation using the modified evolution equations (2.20-2.21) that the set of constraint and gauge entities (A, V^i, F, D_i) satisfy exactly the same induced evolution equations as those given in equations (4.4a-d) in [150]. Thus the energy argument in section 4 of this reference goes through unchanged and shows that if $(A, F, V^i, D_i) = 0$ for the initial data $(g(t_0), k(t_0))$, then $(A, F, V^i, D_i) \equiv 0$ along the solution curve $(g(t), k(t), N(t), X(t))$. This completes the analysis of the desired local well-posedness and gauge conservation criteria.

2.3 Re-scaled equations

In this section, we convert the evolution and constraint equations to scale free equations after rescaling the dimensionful entities by suitable powers of the conformal factor $\phi^2 = \tau^2 - \frac{2n\Lambda}{n-1}$ (which is strictly positive according to the condition (2.27)). Before rescaling, we observe that the solution of the momentum constraint

$$\nabla^j k_{ij} = 0, \tag{2.32}$$

may be written as

$$k = K^{TT} + \frac{\tau}{n} g, \tag{2.33}$$

where K^{TT} is traceless with respect to g. We will obtain equations in terms of K^{TT}. We denote the dimensional entities by a ˜ sign, while dimensionless entities are written simply without ˜ sign for convenience. The re-scaled entities are given as follows

$$\tilde{g}_{ij} = \frac{1}{\phi^2} g_{ij},\ \tilde{N} = \frac{1}{\phi^2} N,\ \tilde{X}^i = \frac{1}{\phi} X^i,\ \tilde{K}^{TT}_{ij} = \frac{1}{\phi} K^{TT}_{ij}, \tag{2.34}$$

where $\phi = -\sqrt{\tau^2 - \frac{2n\Lambda}{n-1}}$ such that $\frac{\phi}{\tau} > 0$. In CMCSH gauge, the re-scaled evolution and constraint equations may be written as

$$\partial_T \tau = -\frac{\phi^2}{\tau}, \tag{2.35}$$

$$\partial_T g_{ij} = \frac{2\phi(\tau)}{\tau} N K^{TT}_{ij} - 2(1 - \frac{N}{n}) g_{ij} - \frac{\phi(\tau)}{\tau} (L_X g)_{ij}, \tag{2.36}$$

$$\begin{aligned} \partial_T K^{TT}_{ij} &= -(n-1) K^{TT}_{ij} - \frac{\phi(\tau)}{\tau} N (R_{ij} + \frac{n-1}{n^2} g_{ij} - \alpha_{ij}) \\ &\quad + \frac{\phi(\tau)}{\tau} \nabla_i \nabla_j N + \frac{2\phi(\tau)}{\tau} N K^{TT}_{im} K^{Tm}_{j} \\ &\quad - \frac{\phi(\tau)}{n\tau} (\frac{N}{n} - 1) g_{ij} - (n-2)(\frac{N}{n} - 1) K^{TT}_{ij} - \frac{\phi(\tau)}{\tau} (L_X K^{TT})_{ij}, \end{aligned} \tag{2.37}$$

$$R + \frac{n-1}{n} - |K^{TT}|^2 = 0, \tag{2.38}$$

$$\nabla_j K^{TTij} = 0. \tag{2.39}$$

Here the new time coordinate is defined as

$$\partial_T = -\frac{\phi^2}{\tau}\partial_\tau \tag{2.40}$$

which may be integrated explicitly to yield

$$\phi(\tau) = -e^{-T}, \tag{2.41}$$
$$\tau = -\sqrt{e^{-2T} + \frac{2n\Lambda}{n-1}} \tag{2.42}$$

with T being Newtonian like i.e., $-\infty < T < \infty$. Note an important fact that $0 < \frac{\phi(\tau)}{\tau} < 1$ and $\frac{\phi(\tau)}{\tau} \approx \sqrt{\frac{n-1}{2n\Lambda}}e^{-T}$ as $T \to \infty$. The re-scaled elliptic equations for the lapse function and the shift vector field may be expressed as follows

$$\Delta_g N + (|K^{TT}|^2 + \frac{1}{n})N = 1, \tag{2.43}$$
$$\begin{aligned}\frac{\phi(\tau)}{\tau}(\Delta_g X^i - R^i_j X^j + L_X V^i) &= \frac{\phi(\tau)}{\tau}(2NK^{Tjk} - 2\nabla^j X^k)(\Gamma[g]^i_{jk} \\ - \Gamma[\gamma]^i_{jk}) - (2-n)\nabla^i(\frac{N}{n} - 1) &- \frac{2\phi(\tau)}{\tau}\nabla^j N K^{Ti}_j + g^{jk}\partial_T\Gamma[\gamma]^i_{jk}.\end{aligned}$$

2.3.1 Background solutions: conformal spacetimes

The fixed point solutions are computed as the solutions of the following set of equations in CMCSH gauge i.e., by setting $V^i = 0$ and taking τ =a monotonic function of t ($t = \tau$ in this case)

$$0 = \frac{2\phi(\tau)}{\tau}NK^{TT}_{ij} - 2(1 - \frac{N}{n})g_{ij} - \frac{\phi(\tau)}{\tau}(L_X g)_{ij}, \tag{2.44}$$
$$\begin{aligned}0 = -(n-1)K^{TT}_{ij} - \frac{\phi(\tau)}{\tau}N(R_{ij} + \frac{n-1}{n^2}g_{ij}) + \frac{\phi(\tau)}{\tau}\nabla_i\nabla_j N \\ + \frac{2\phi(\tau)}{\tau}NK^{TT}_{im}K^{Tm}_j - \frac{\phi(\tau)}{n\tau}(\frac{N}{n} - 1)g_{ij} - (n-2)(\frac{N}{n} - 1)K^{TT}_{ij} \\ - \frac{\phi(\tau)}{\tau}(L_X K^{TT})_{ij},\end{aligned} \tag{2.45}$$
$$1 = \Delta_g N + (|K^{TT}|^2 + \frac{1}{n})N, \tag{2.46}$$
$$\begin{aligned}\frac{\phi(\tau)}{\tau}(\Delta_g X^i - R^i_j X^j) &= \frac{\phi(\tau)}{\tau}(2NK^{Tjk} - 2\nabla^j X^k)(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) \\ &-(2-n)\nabla^i(\frac{N}{n} - 1) - \frac{2\phi(\tau)}{\tau}\nabla^j N K^{Ti}_j + g^{jk}\partial_T\Gamma[\gamma]^i_{jk}.\end{aligned} \tag{2.47}$$

Contracting equation (2.44) with K^{TT}, using the momentum constraint $\nabla_j K^{Tij} = 0$, and integrating over M, we obtain

$$\int_M \frac{2\phi(\tau)}{\tau} N|K^{TT}|_g^2 \mu_g \quad = \quad 0. \tag{2.48}$$

Standard maximum principle arguments for the elliptic equation (2.43), yield estimates for the re-scaled lapse

$$0 < \frac{1}{\sup(|K^{TT}|^2) + \frac{1}{n}} \leq N \leq n \tag{2.49}$$

which, together with equation (2.48) implies

$$K^{TT} \equiv 0 \tag{2.50}$$

on M. From the lapse equation (2.43), we immediately obtain

$$N = n \tag{2.51}$$

on M. Substituting equations (2.50) and (2.51) into equation (2.44) leads to

$$L_X g|_{K^{TT}=0} \quad = \quad 0, \tag{2.52}$$

which implies that the shift vector field is a generator of the isometry group of $\mathbf{M}$. After substituting the available variables into the fixed point equation of the transverse traceless second fundamental form (2.45), we obtain the re-scaled metric to be a negative Einstein metric

$$R_{ij}(g) = -\frac{n-1}{n^2} g_{ij}. \tag{2.53}$$

Now, the isometry group of compact manifold $\mathbf{M}$ with negative Ricci curvature is discrete. Therefore, the existing Killing fields are only trivial i.e., $X = 0$. A sketch of the proof is as follows. The divergence of the Killing equation along with commutation of covariant derivative yields

$$-\Delta_g X^i + R^i_j X^j + \nabla^i \nabla_j X^j = 0. \tag{2.54}$$

The trace of the Killing equation provides $\nabla_i X^i = 0$ and therefore, after multiplying both sides of equation (2.54) with X_i and integrating over M, the following expression is obtained

$$\int_M [\nabla_j X_i \nabla^j X^i + \frac{n-1}{n^2} g_{ij} X^i X^j] \mu_g = 0 \tag{2.55}$$

which implies

$$X \equiv 0 \tag{2.56}$$

everywhere on M. One observes that the unknowns obtained from the momentum constraint and the dynamical equations satisfy the Hamiltonian constraint. Therefore, we have proved the following theorem.

Theorem 1. *Let* $\mathbf{M}$ *be a closed (compact without boundary) connected orientable n-manifold,* $n \geq 3$*, of negative Yamabe type. Then the fixed point solutions of the re-scaled 'Einstein-*Λ*' flow (2.44-4.22) on* $(T_-, T_+) \times M$*,* $-\infty \leq T_- < T_+ \leq \infty$*, have the Cauchy data* (g, K^{TT}, N, X)*=*$(g_0, K_0^{TT}, N_0, X_0)$ *which satisfy the following equations:*
$R_{ij}(g_0) = -\frac{n-1}{n^2} g_0, \quad K^{TT} = 0, \quad N_0 = n, \quad X_0 = 0.$
For convenience, we may replace g_0 by γ with

$$R_{ij}(\gamma) = -\frac{n-1}{n^2}\gamma_{ij}, \tag{2.57}$$

The physical variables are then given by

$$\tilde{g}_{ij} = \frac{1}{\phi^2} g_{ij} = \frac{1}{\tau^2 - \frac{2n\Lambda}{n-1}}\gamma_{ij}, \tag{2.58}$$

$$\tilde{N} = \frac{N}{\tau^2 - \frac{2n\Lambda}{n-1}} = \frac{n}{\tau^2 - \frac{2n\Lambda}{n-1}}, \tag{2.59}$$

$$\tilde{X}^i = \frac{X^i}{\sqrt{\tau^2 - \frac{2n\Lambda}{n-1}}} = 0. \tag{2.60}$$

If M admits an Einstein metric γ, then the corresponding re-scaled variables $(\gamma, 0, n, 0)$ provide constant mean extrinsic curvature Cauchy data (g, K, N, X) through equations (50-53) for a vacuum spacetime with a positive cosmological constant on $(-\infty, -\sqrt{\frac{2n\Lambda}{n-1}}) \times \mathbf{M}$, locally expressible as

$$ds^2 = -\frac{n^2}{(\tau^2 - \frac{2n\Lambda}{n-1})^2} d\tau^2 + \frac{1}{(\tau^2 - \frac{2n\Lambda}{n-1})}\gamma_{ij} dx^i dx^j. \tag{2.61}$$

This so called 'trivial' evolution exists for $n = 3$ if and only if the spatial manifold $\mathbf{M}$ is hyperbolizable (by the Mostow rigidity theorem). For $n > 3$, the existence of a negative Einstein space is sufficient to guarantee the existence of this Cauchy data. This is the isolated fixed point for $n = 3$ and is in general non-isolated (by virtue of non-trivial Einstein moduli spaces) for $n > 3$. The spacetime (2.61) admits a globally defined time-like conformal Killing field $Y = \sqrt{\tau^2 - \frac{2n\Lambda}{n-1}}\partial_\tau$ i.e.,

$$L_Y g^{n+1} = -\frac{2\tau}{\sqrt{\tau^2 - \frac{2n\Lambda}{n-1}}} g^{n+1}. \tag{2.62}$$

We therefore designate these spacetimes as '*conformal spacetimes*'. A summary of the results obtained so far yields the following theorem

Theorem 2. *Let M be a closed connected oriented n-manifold of negative Yamabe type. The fixed points of the non-autonomous re-scaled 'Einstein-Λ' evolution and constraint equations on $(-\infty, -\sqrt{\frac{2n\Lambda}{n-1}}) \times M$ with the gauge condition $t = \tau$ and spatial harmonic slice gauge condition are the 'trivial' spacetimes given by (2.61). Such spacetimes admit $Y = \sqrt{\tau^2 - \frac{2n\Lambda}{n-1}}\partial_\tau$ as a globally defined time-like conformal Killing vector field.*

A second class of solutions of the re-scaled equations lying sufficiently close to and containing the one described by the negative Einstein metrics will be of particular importance to us. These solutions are described by

$$R(g) = -\frac{n-1}{n}, X = 0, N = 0, K^{TT} = 0 \tag{2.63}$$

provided that $||g - \gamma||_{H^s} < \epsilon, \gamma \in Ein_{-\frac{n-1}{n^2}}, s > \frac{n}{2} + 2$ and $\epsilon << ||\gamma||_{H^s}$. Small calculation similar to the previous case together with lemma 5 shows that the space described by the set $\{R(g) = -\frac{n-1}{n}, X = 0, N = 0, K^{TT} = 0\}$ satisfies the scale-free Einstein's equations (2.35-2.39) in CMCSH gauge in the limit $\frac{\phi}{\tau} \to 0$. Clearly the centre manifold described by the negative Einstein spaces in theorem 1 is a subset of the solutions $\{R(g) = -\frac{n-1}{n}, X = 0, N = 0, K^{TT} = 0\}$ since $Ein_{-\frac{n-1}{n^2}} \subset \mathcal{M}_{-\frac{n-1}{n}}$. We will designate the space of solutions $\{R(g) = -\frac{n-1}{n}\}$ lying sufficiently close to the Einstein structure with metric g satisfying the spatial harmonic gauge condition the '**extended centre manifold**' and denote it by $\mathcal{M}^{\epsilon}_{-\frac{n-1}{n}} \cap \mathcal{S}_\gamma$ (the reason for such a notation will be clear in section 4.1). Our intention is to prove that this extended centre manifold is indeed the attractor of the Einstein-Λ flow.

2.3.2 Linear stability of the conformal spacetimes

[229] constructed the following reduced Hamiltonian (while expressed as a functional of the reduced phase space variables)

$$H_{reduced} = \frac{2(n-1)}{n}\int_M \frac{\partial\tau}{\partial t}\mu_g > 0 \tag{2.64}$$

of the dynamics which was shown to decay monotonically along the solution curves and achieve its infimum precisely for the conformal spacetimes described by equation (2.61). Such a reduced Hamiltonian plays the role of a weak Lyapunov function of the reduced dynamics, which indicates that these background solutions may be stable against perturbations. Motivated by this notion we conduct a linear stability analysis of the re-scaled equations about the background solutions. Let the perturbation be ($h_{ij} = \delta g_{ij}, \delta K^{TT}_{ij} = K^{TT}_{ij}, \delta N, \delta X^i$). However, δN and δX^i satisfy elliptic equations from which we prove that they vanish if the background metric is negative Einstein (which is the case here). We state and prove the following lemma regarding the vanishing of the perturbations to the lapse function and the shift vector field.

Lemma 1: *Let M be a closed connected oriented n-manifold of negative Yamabe type. The fixed points of the non-autonomous re-scaled 'Einstein-Λ' evolution and constraint equations on $(-\infty, -\sqrt{\frac{2n\Lambda}{n-1}})\times M$ with the gauge condition $t=\tau$ and spatial harmonic slice gauge condition are the 'trivial' spacetimes given by (2.61). Let the perturbations about these background solutions be $(h_{ij}, \delta\pi^{Tij}, \delta N, \delta X^i)$. Then $\delta N \equiv 0$ and $\delta X^i \equiv 0$ everywhere on M.*

Proof: Perturbation of the lapse equation leads to

$$\Delta_\gamma \delta N + \frac{1}{n}\delta N = 0. \tag{2.65}$$

Application of the standard maximum principle immediately yields $\delta N \equiv 0$ everywhere on M, which completes the proof of the first part. Perturbation of the shift equation yields

$$\Delta_\gamma \delta X_i - R[\gamma]_{ij}\delta X^j = 0. \tag{2.66}$$

Now, multiplying both sides with δX^i and integrating over closed M, we obtain

$$\int_M \left(\delta X^i \Delta_\gamma \delta X_i - \delta X^i R[\gamma]_{ij}\delta X^j\right)\mu_\gamma = 0, \tag{2.67}$$

$$\int_M \left(\nabla[\gamma]^i \delta X^j \nabla[\gamma]_i \delta X_j - R[\gamma]_{ij}\delta X^i \delta X^j\right)\mu_\gamma = 0. \tag{2.68}$$

Substituting $R_{ij}[\gamma] = -\frac{n-1}{n^2}\gamma_{ij}$ yields

$$\int_M \left(\nabla[\gamma]^i \delta X^j \nabla[\gamma]_i \delta X_j + \frac{n-1}{n^2} g_{ij} \delta X^i \delta X^j \right) \mu_\gamma \quad = \quad 0, \tag{2.69}$$

which implies $\delta X^i \equiv 0$ everywhere on M, completing the second part of the proof. With the results $\delta N = 0 = \delta X^i$, the spacetime perturbations (at the linear level) reduce to $(h_{ij}, K_{ij}^{TT}, 0, 0)$. In the next lemma, we prove that utilizing the constraint and gauges, the perturbations h_{ij} to the spatial metric may be reduced to pure transverse-traceless form with respect to the background metric.

Lemma 2: *Let the perturbations describing the dynamics be (h_{ij}, K_{ij}^{TT}). Full-filling the Hamiltonian constraint by the perturbed data $(g_{ij} = \gamma_{ij} + h_{ij}, K_{ij}^{TT} = 0 + K_{ij}^{TT})$ through the constant mean extrinsic curvature spatial harmonic gauge implies the transverse-traceless property of the metric perturbations.*

Proof: Variation of Hamiltonian constraint (2.38) reads

$$DR \cdot h = 0, \tag{2.70}$$

which upon using the expression of the Frechet derivative of R yields

$$\Delta_\gamma tr_\gamma h + \nabla^i \nabla^j h_{ij} - R[\gamma]_{ij} h^{ij} = 0. \tag{2.71}$$

Following the Hodge-like decomposition, we may write the symmetric 2-tensor h_{ij} as

$$h_{ij} \quad = \quad h_{ij}^{TT} + f\gamma_{ij} + (L_W \gamma)_{ij}, \tag{2.72}$$

where h^{TT} is a symmetric transverse-traceless (w.r.t the background metric γ_{ij}) 2 tensor and f and W are a function and vector field lying in suitable function spaces. Upon substituting decomposition (2.72) into the variation of the Hamiltonian constraint and noticing $\Delta_\gamma tr_\gamma (L_W \gamma) + \nabla^i \nabla^j (L_W \gamma)_{ij} - R[\gamma]_{ij} (L_W \gamma)^{ij} \equiv 0$ one arrives at

$$\begin{aligned} 3\Delta_\gamma f + \gamma^{ij} \nabla_i \nabla_j f - R[\gamma] f = 0, \\ 2\Delta_\gamma f - R[\gamma] f = 0, \end{aligned} \tag{2.73}$$

where $R[\gamma] = -\frac{n-1}{n}$. Noting that the operator $(2\Delta_\gamma + \frac{n-1}{n})$ is invertible on compact M, we immediately obtain

$$f \equiv 0 \tag{2.74}$$

throughout M. The vector field W may be obtained in terms of f through the following elliptic equation, which follows from the variation of the spatial harmonic gauge condition (i.e., variation of the tension field). At the linear level, the gauge condition '$id : (M, \gamma + h) \to (M, \gamma)$ is harmonic' yields

$$\gamma^{jk}\left(\Gamma[\gamma + h]^i_{jk} - \Gamma[\gamma]^i_{jk}\right) = 0 \tag{2.75}$$

$$2\nabla_j h^{ij} - \nabla^i tr_\gamma h = 0$$

$$-\nabla^i f - 2\Delta_\gamma W^i + 2R[\gamma]^i_j W^j = 0$$

$$\Delta_\gamma W^i - R[\gamma]^i_j W^j = -\frac{1}{2}\nabla^i f. \tag{2.76}$$

Now substituting $f = 0$ and $R[\gamma]_{ij} = -\frac{n-1}{n^2}\gamma_{ij}$, yields

$$\Delta_\gamma W^i + \frac{n-1}{n^2}W^i \quad = \quad 0. \tag{2.77}$$

Again, invertibility of the operator $(\Delta_\gamma + \frac{n-1}{n^2})$ in compact M implies $W \equiv 0$ throughout M. Therefore, $h_{ij} = h^{TT}_{ij}$, which concludes the proof.

Following the previous lemma, we need only consider the perturbations of the transverse-traceless type and no information is lost doing so. We will denote h^{TT}_{ij} simply by h_{ij} if there is no confusion. Now the linearized equations of motion around the background solution take the following forms

$$\partial_T h_{ij} \quad = \quad \frac{2n\phi(\tau)}{\tau}K^{TT}_{ij}, \tag{2.78}$$

$$\partial_T K^{TT}_{ij} \quad = \quad -(n-1)K^{TT}_{ij} - \frac{n\phi(\tau)}{\tau}\left(\delta R_{ij} + \frac{n-1}{n^2}h_{ij}\right), \tag{2.79}$$

where we have dropped δN and δX^i following lemma 1. We may now obtain the wave equation for the metric perturbation as follows. Using the perturbation to the Ricci tensor

$$\delta R_{ij} = \frac{1}{2}[-\frac{2(n-1)}{n^2}h_{ij} + (R[\gamma]_{kijm} + R[\gamma]_{kjim})h^{km} \tag{2.80}$$

$$-\nabla[\gamma]^m\nabla[\gamma]_m h_{ij}]$$

along with $\gamma^{ij}h_{ij} = 0$, $\nabla[\gamma]_m h^{ml} = 0$, and $h^{ij} = \gamma^{ik}\gamma^{jl}h_{kl} \neq \delta g^{ij}$, the wave equation for the metric perturbation takes the form

$$\begin{aligned} \frac{\partial^2 h_{ij}}{\partial T^2} + \left[(n-1) + \frac{2n\Lambda}{(n-1)\tau^2}\right]\frac{\partial h_{ij}}{\partial T} & \\ + \frac{\phi^2(T)}{\tau^2(T)} n^2 (\Delta_\gamma h_{ij} + (R[\gamma]_{kijm} + R[\gamma]_{kjim})h^{km}) &= 0. \end{aligned} \tag{2.81}$$

Let the Laplacian be defined as $\Delta_\gamma = -\nabla[\gamma]^i\nabla[\gamma]_i$. Utilizing the eigenvalue equation of the linear differential operator on the right hand side of the previous equation i.e.,

$$\Delta_\gamma h_{ij} + (R[\gamma]_{kijm} + R[\gamma]_{kjim})h^{km} = \lambda h_{ij}, \tag{2.82}$$

the wave equations for the transverse-traceless metric perturbations reduce to the following set of ordinary differential equations

$$\frac{\partial^2 h_{ij}}{\partial T^2} + \left[(n-1) + \frac{2n\Lambda}{(n-1)\tau^2}\right]\frac{\partial h_{ij}}{\partial T} + \frac{\phi^2(T)}{\tau^2(T)}\lambda n^2 h_{ij} = 0. \tag{2.83}$$

Here, we assume that the negative Einstein spaces are stable, that is, $\lambda \geq 0$. No example of an unstable, compact, negative Einstein space is known [206]. Therefore we will assume $\lambda \geq 0$. This leads to the result of [161] in the limit of $\Lambda = 0$. The time coordinate T satisfies the following from equation (4.48)

$$-\frac{\phi^2}{\tau}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T}, \tag{2.84}$$

which yields the range of T as $(-\infty, \infty)$ for $\tau \in (-\infty, -\sqrt{\frac{2n\Lambda}{n-1}})$. Note that for h in the kernel of the differential operator i.e., for $-\Delta_\gamma h_{ij} - (R[\gamma]_{kijm} + R[\gamma]_{kjim})h^{km} = 0$, (i.e., $\lambda = 0$) the equation reduces trivially to

$$\frac{\partial^2 h_{ij}}{\partial T^2} + \left[(n-1) + \frac{2n\Lambda}{(n-1)\tau^2}\right]\frac{\partial h_{ij}}{\partial T} = 0, \tag{2.85}$$

which yields asymptotic stability following the fact that the damping coefficient is strictly positive. Asymptotic stability of perturbations lying in the space orthogonal to the kernel of $\mathcal{L}$ may be shown by explicitly constructing a Lyapunov function and deriving its monotonic decay property in the time future direction. Let us convert the second order equation (2.83) to a system of first order equations by substituting $h_{ij} = u$ and $\frac{\partial h_{ij}}{\partial T} = v$

$$\partial_T u = \frac{\phi}{\tau} v, \tag{2.86}$$

$$\partial_T v = -(n-1)v - \frac{\phi(\tau)}{\tau} n^2 \lambda u, \tag{2.87}$$

Let the Lyapunov function be defined as

$$E = \frac{1}{2}v^2 + \frac{1}{2}n^2\lambda u^2, \tag{2.88}$$

the time derivative of which reads

$$\begin{aligned} \frac{dE}{dT} &= -(n-1)v^2 \\ &< 0 \end{aligned} \tag{2.89}$$

i.e., energy decays monotonically and $\frac{dE}{dT} = 0$ only when $v \equiv 0$. The equations (2.86) and (2.87) together with the decay of the Lyapunov function yield $\partial_T u \lesssim e^{-2T}, v \lesssim e^{-T}$ as $T \to \infty$ and therefore u remains bounded and limits to $u^* < \infty$. The asymptotic solution is as follows

$$|u - u^*| \lesssim e^{-2T}, \tag{2.90}$$

$$|v| \lesssim e^{-T}. \tag{2.91}$$

We may prove the future completeness of these perturbed spacetimes at the linear level as follows. In order to establish the future completeness of the spacetime, we need to show that the length of a timelike geodesic goes to infinity. For a family (homotopy class) of rectifiable timelike curves $c : [a, b] \to \mathbf{R} \times M$, the length of its geodesic representative is computed as follows

$$\sup_c \int_a^b \sqrt{(-\hat{g}(\dot{c}, \dot{c}))}dt = d_{\hat{g}}(a, b) = l_{\hat{g}}(\mathcal{C}), \tag{2.92}$$

where $\mathcal{C}$ is the geodesic representative of the family c and $\hat{g}$ is the spacetime metric. Note that the problem arises due to the fact that these family of curves are not necessarily uniformly timelike. Here we will use a theorem proved in [223], which serves as sufficient condition for the geodesic completeness. Such condition is satisfied in this particular case and therefore future completeness holds.

Theorem [223] Sufficient conditions for future timelike and null geodesic completeness of a regularly sliced spacetime are that

1. $|\nabla N|_g$ is bounded by a function of t which is integrable on $[a, \infty)$

2. $|K|_g$ is bounded by a function of t which is integrable on $[a, \infty)$ for some $a < \infty$. Note that both of these conditions are trivially satisfied in our case of linearized perturbations. We will obtain a separate proof of future completeness in the fully nonlinear case. **Theorem 3:** *Let M be a closed connected oriented n-manifold of negative Yamabe type. The background solutions (2.61) of the Einstein-Λ evolution and constraint equations on $(-\infty, -\sqrt{\frac{2n\Lambda}{n-1}}) \times M$ with the gauge condition $t = \tau$ and spatial harmonic slice gauge condition are stable and future complete against linear perturbations.*

2.4 Fully nonlinear perturbations

Theorem 3 provides us with a notion of stability of the background solutions. However, it leaves out the fully non-linear evolution of the small perturbations. The reduced Hamiltonian described in equation (2.64) is always at hand and may be used to study fully non-linear and arbitrarily large perturbations. But, it seems to control only the $H^1 \times L^2$ norm of the reduced phase space data (see [161] and [229] for a definition of reduced phase space). However, following the local existence theorem developed here, we need to control the $H^s \times H^{s-1}$ norm with $s > \frac{n}{2} + 2$, $n \geq 3$. Therefore we construct a Lyapunov function of the dynamics which indeed controls the required Sobolev norm (L^2 norm of the $s > \frac{n}{2} + 2$ spatial derivatives) of the fully non-linear small perturbations. We show that it decays along the solution curve if we start sufficiently close to the background spacetime. In addition, as mentioned previously, finite dimensionality of the Einstein moduli space in the case of higher dimensions ($n > 3$) has to be addressed carefully in order to show the attractor property of the centre manifold. However, before doing so, a few important geometric notions are to be discussed which have substantial impact on the stability analysis in the case of the spatial dimension $n > 3$.

2.4.1 **Deformation space**

Here we provide a brief description of the deformation space of the Einstein structure necessary for the nonlinear analysis. Details can be found in several studies [98, 206]. The background solutions of the Einstein-Λ flow in CMCSH gauge are the conformal spacetimes as described in the previous section. The spatial metric component of these conformal spacetimes is a negative Einstein metric i.e., the spatial metric satisfies

$$R_{ij}(\gamma) = -\frac{n-1}{n^2}\gamma_{ij}. \tag{2.93}$$

Let us denote the space of metrics satisfying equation (4.16) by $\mathcal{E}in_{-\frac{n-1}{n^2}}$ and consider the following map

$$\begin{aligned} F : \mathcal{M} &\rightarrow \mathcal{M}^{'}, \\ \gamma &\mapsto R_{ij}(\gamma) + \frac{n-1}{n^2}\gamma_{ij}, \end{aligned} \tag{2.94}$$

where $\mathcal{M}^{'}$ is a subspace of the space of symmetric $(0,2)$ tensors. $\mathcal{E}in_{-\frac{n-1}{n^2}} = F^{-1}(0)$ will be a submanifold of $\mathcal{M}$ provided that the $T_{F^{-1}(0)}F$ is surjective (i.e., F is a submersion). The differential $D(Ric+\frac{n-1}{n^2}\gamma)\cdot h$ may be reduced to a second order elliptic differential operator acting on the variation h, whose adjoint is injective and thus $T_{F^{-1}(0)}F$ is surjective. The tangent space of $\mathcal{E}in_{-\frac{n-1}{n^2}}$ may be calculated as the kernel of $T_\gamma F$. The operator $T_\gamma F = D(Ric + \frac{n-1}{n^2}\gamma)$ may be decomposed in terms of the Lichnerowicz type Laplacian $\mathcal{L}$, via

$$T_\gamma F \cdot h = \mathcal{L}h - 2\delta^*\delta h - \nabla[\gamma]d(trh), \tag{2.95}$$

where $\mathcal{L}h_{ij} = \Delta_\gamma h_{ij} - 2R[\gamma]_{ikjl}h^{kl}$, $(\delta h)_i = \nabla[\gamma]^j h_{ij}$, and $(\delta^* Y)_{ij} = -\frac{1}{2}(L_Y\gamma)_{ij}$. The space of symmetric 2 tensors S^2M may be decomposed as

$$S^2M = C^{TT}(S^2M) \oplus (\mathcal{F}(M) \otimes \gamma) \oplus \mathcal{IM}(L), \tag{2.96}$$

where $C^{TT}(S^2M)$, $\mathcal{F}(M)$, and $\mathcal{IM}(L)$ are the space of symmetric transverse-traceless 2-tensors, the space of functions on M, and the image of the Lie derivative L acting on γ with respect to a vector field $Y \in TM$ (a section of TM to be precise), respectively. In local co-ordinates, this decomposition may be represented as

$$h_{ij} = h_{ij}^{TT} + f\gamma_{ij} + (L_Y\gamma)_{ij}. \tag{2.97}$$

The kernel $\mathcal{K}(T_\gamma F)$ at $\gamma \in F^{-1}(0)$ may be obtained through the solution of the following equation

$$\mathcal{L}(h^{TT} + f\gamma + L_Y\gamma) - 2\delta^*\delta(h^{TT} + f\gamma + L_Y\gamma) \\ -\nabla d(tr(h^{TT} + f\gamma + L_Y\gamma)) = 0, \tag{2.98}$$

where we notice that $\mathcal{L}(L_Y\gamma) - 2\delta^*\delta(L_Y\gamma) - \nabla d(tr(L_Y\gamma)) \equiv 0$ and thus $L_Y\gamma \in \mathcal{K}(T_{\gamma\in F^{-1}(0)}F)$. The remaining terms lead to

$$(\mathcal{L}h^{TT})_{ij} + (\Delta_\gamma f)\gamma_{ij} - 2fR_{ij} + \nabla_i\nabla_j f + \nabla_j\nabla_i f - n\nabla_i\nabla_j f \quad = \quad 0, \tag{2.99}$$

which upon taking the trace yields

$$2(n-1)\Delta_\gamma f - 2R(\gamma)f \quad = \quad 0. \tag{2.100}$$

For $\gamma \in F^{-1}(0)$, the scalar curvature $R(\gamma) = -\frac{n-1}{n}$ and therefore $f = 0$ everywhere on M. The resulting tangent space of $\mathcal{E}in_{-\frac{n-1}{n^2}} = F^{-1}(0)$ consists of the kernel of the operator $\mathcal{L}$ (a subspace of the space of transverse-traceless symmetric 2-tensors) and the image of the Lie derivative operator with respect to a vector field

$$T_\gamma\mathcal{E}in_{-\frac{n-1}{n^2}} = \ker(\mathcal{L}) \oplus \mathcal{IM}(L). \tag{2.101}$$

Let $\gamma_0 \in \mathcal{E}in_{-\frac{n-1}{n^2}}$ and $\mathcal{V}$ be its connected component. Also consider $\mathcal{S}_\gamma$ to be the harmonic slice of the identity diffeomorphism i.e., the set of $\gamma \in \mathcal{E}in_{-\frac{n-1}{n^2}}$ for which the identity map $Id : (M,\gamma) \to (M,\gamma_0)$ is harmonic. This condition is equivalent to the vanishing of the tension field $-V^k$ that is

$$-V^k = -\gamma^{ij}(\Gamma[\gamma]^k_{ij} - \Gamma[\gamma_0]^k_{ij}) = 0. \tag{2.102}$$

For $\gamma \in \mathcal{E}in_{-\frac{n-1}{n^2}}$, $\mathcal{S}_\gamma$ is a submanifold of $\mathcal{M}$ for γ sufficiently close to γ_0 [206,229]. The deformation space $\mathcal{N}$ of $\gamma_0 \in \mathcal{E}in_{-\frac{n-1}{n^2}}$ is defined as the intersection of the γ_0-connected component $\mathcal{V} \subset \mathcal{E}in_{-\frac{n-1}{n^2}}$ and the harmonic slice $\mathcal{S}_\gamma$ i.e.,

$$\mathcal{N} = \mathcal{V} \cap \mathcal{S}_\gamma. \tag{2.103}$$

$\mathcal{N}$ is assumed to be smooth. In the case of $n = 3$, the negative Einstein structure is rigid which follows from the Mostow rigidity theorem [96, 121] and this structure corresponds to the hyperbolic structure up to isometry. For higher genus surfaces Σ_{genus} ($genus > 1$) in two dimensions, the deformation space modulo isotopy diffeomorphisms is the Teichmüller space diffeomorphic to $\mathbf{R}^{6genus-6}$. For $n > 3$, it is a finite dimensional submanifold of $\mathcal{M}$. Examples of higher dimensional ($n > 3$) negative Einstein spaces with non-trivial deformation spaces include Kahler-Einstein manifolds [178]. [206] provides the details of constructing numerous negative Einstein spaces through a product operation. Therefore, we do not repeat the same here. Readers are referred to the aforementioned paper.

Following equations (2.101), (3.73), and (3.74), the tangent space $T_\gamma\mathcal{N}$ in local coordinates is represented as

$$\frac{\partial\gamma}{\partial q^a} = h^{TT||} + L_{Y^{||}}\gamma, \tag{2.104}$$

where $h^{TT||} \in \ker(\mathcal{L}) = C^{TT||}(S^2M) \subset C^{TT}(S^2M)$, $Y \in \mathfrak{X}(M)$ satisfy

$$-[\nabla[\gamma]^m\nabla[\gamma]_mY^i + R[\gamma]^i_mY^m] + (h^{TT||} + L_{Y^{||}}\gamma)^{mn}(\Gamma[\gamma]^i_{mn} - \Gamma[\gamma_0]^i_{mn}) = 0,$$

and, $\{q^a\}_{a=1}^{dim(\mathcal{N})}$ is a local chart on $\mathcal{N}$, $\mathfrak{X}(M)$ is the space of vector fields on M (in a suitable function space setting). Also note that the space of transverse-traceless tensors may be decomposed as follows

$$C^{TT}(S^2M) = C^{TT||}(S^2M) \oplus C^{TT\perp}(S^2M), \tag{2.105}$$

where $C^{TT\perp}$ is the orthogonal complement of $\ker(\mathcal{L})$ in $C^{TT}(S^2M)$. An important thing to note is that all known examples of closed negative Einstein spaces have integrable deformation spaces. Note that the extended centre manifold is correctly defined as the intersection of the $\epsilon-$neighbourhood of deformation space $\mathcal{N}$ in $\mathcal{M}_{-\frac{n-1}{n}} \cap \mathcal{S}_\gamma$, that is, $\mathcal{M}^\epsilon_{-\frac{n-1}{n}} \cap \mathcal{S}_\gamma, \gamma \in Ein_{-\frac{n-1}{n^2}}$ (since the spatial harmonic gauge is imposed, the metric must lie in the submanifold $\mathcal{S}_\gamma$ as well). Obviously, $\mathcal{N} \subset \mathcal{M}^\epsilon_{-\frac{n-1}{n}} \cap \mathcal{S}_\gamma$ and $T\mathcal{N} \subset T(\mathcal{M}^\epsilon_{-\frac{n-1}{n}} \cap \mathcal{S}_\gamma)$. This concludes the business of describing the centre (and extended centre) manifold of the dynamics.

2.4.2 Perturbations and shadow gauge

The previous section describes the non-trivial deformation space of a negative Einstein structure. The presence of a non-trivial deformation space necessitates the consideration of perturbations L^2 orthogonal to the deformation space of the Einstein structures forming the center manifold of the re-scaled dynamics. Perturbations tangent to the deformation space are trivially stable at the linear level, which was shown earlier and will be repeated later as well. For the treatment of the orthogonal perturbations, we invoke the shadow gauge introduced by [206] in addition to the constant mean curvature spatial harmonic gauge (CMCSH). Let $\gamma \in \mathcal{N}$ and consider metric $g \in \mathcal{M}$ close to γ_0 such that $||g-\gamma_0||_{H^s} < \delta$, and $||\gamma-\gamma_0||_{H^s} < \delta$ imply that $||g-\gamma||_{H^s} < 2\delta$ through the triangle inequality. The shadow gauge is defined by requiring that the perturbation $(g-\gamma)$ be L^2 orthogonal to the deformation space $\mathcal{N}$. Adopting the local coordinate system $\{q^a\}_{a=1}^N$ on $\mathcal{N}$ with $N = dim(\mathcal{N}) < \infty$, the local basis for the tangent space of $\mathcal{N}$ may be written as $\frac{\partial \gamma_{ij}}{\partial q^a} = h^{TT||} + L_{Y^{||}}\gamma$. The shadow gauge condition is equivalent to

$$
\begin{aligned}
< (g-\gamma), \frac{\partial \gamma}{\partial q^a} >_{L^2} &= 0, \\
\int_{\mathbf{M}} (g_{ij}-\gamma_{ij}) \frac{\partial \gamma^{ij}}{\partial q^a} \mu_\gamma &= 0,
\end{aligned}
\tag{2.106}
$$

where $\frac{\partial \gamma^{ij}}{\partial q^a} = -\gamma^{im}\gamma^{jn}\frac{\partial \gamma_{mn}}{\partial q^a}$. Technically, this shadow condition is equivalent to γ being a projection of g onto $\mathcal{N}$. In other words, there is a projection map $\mathcal{P} : \mathcal{M} \to \mathcal{N}$ such that

$$\gamma = \mathcal{P}[g]. \tag{2.107}$$

Noting that the space $\mathcal{N}$ is assumed to have smooth structure, this projection, in a sense, is a smoothing operation. We call γ the shadow of g. The time derivative of γ in $\mathcal{N}$ may be obtained as

$$\frac{\partial \gamma}{\partial T} = D\mathcal{P}[g] \cdot \frac{\partial g}{\partial T}. \tag{2.108}$$

The expression for $D\mathcal{P}[g]$ may be obtained by time differentiating the shadow metric condition i.e.,

$$\frac{d}{dT}\int_{\mathbf{M}} (g_{ij}-\gamma_{ij}) \frac{\partial \gamma^{ij}}{\partial q^a} \mu_\gamma = 0, \tag{2.109}$$

$$
\begin{aligned}
&\int_M \partial_T g_{ij} \frac{\partial \gamma^{ij}}{\partial q^a}\mu_\gamma + \int_M (g_{ij}-\gamma_{ij})\frac{\partial}{\partial q^b}(\frac{\partial \gamma^{ij}}{\partial q^a})\frac{\partial q^b}{\partial T}\mu_\gamma \\
&+\frac{1}{2}\int_M (g_{ij}-\gamma_{ij})\frac{\partial \gamma^{ij}}{\partial q^a}\gamma^{lm}\frac{\partial \gamma_{lm}}{\partial q^b}\frac{\partial q^b}{\partial T}\mu_\gamma - \int_M \frac{\partial \gamma_{ij}}{\partial q^b}\frac{\partial \gamma^{ij}}{\partial q^a}\frac{\partial q^b}{\partial T}\mu_\gamma = 0,
\end{aligned}
\tag{2.110}
$$

where the matrix $-\int_M \frac{\partial\gamma_{ij}}{\partial q^b}\frac{\partial\gamma^{ij}}{\partial q^a}\mu_\gamma = \int_M \frac{\partial\gamma_{ij}}{\partial q^b}\gamma^{im}\gamma^{jn}\frac{\partial\gamma_{mn}}{\partial q^a}\mu_\gamma$ is invertible due to $\frac{\partial\gamma_{ij}}{\partial q^b}$ being a basis for $T_\gamma\mathcal{N}$. For the small data limit i.e., $(g-\gamma) < 2\delta, \delta > 0$, the combined matrix

$$\begin{aligned}\mathcal{B} = \int_M (g_{ij}-\gamma_{ij})\left(\frac{\partial}{\partial q^b}(\frac{\partial\gamma^{ij}}{\partial q^a}) + \frac{\partial\gamma^{ij}}{\partial q^a}\gamma^{lm}\frac{\partial\gamma_{lm}}{\partial q^b}\right)\mu_\gamma \\ + \int_M \mu_\gamma\frac{\partial\gamma_{ij}}{\partial q^b}\gamma^{im}\gamma^{jn}\frac{\partial\gamma_{mn}}{\partial q^a}\mu_\gamma\end{aligned} \tag{2.111}$$

is invertible as well yielding the following time evolution of the shadow metric γ in the deformation space

$$\begin{aligned}\frac{\partial q^b}{\partial T} &= -(\mathcal{B}^{-1})^{ba}\int_M \partial_T g_{ij}\frac{\partial\gamma^{ij}}{\partial q^a}\mu_\gamma, \\ \frac{\partial\gamma_{ij}}{\partial T} &= -\frac{\partial\gamma_{ij}}{\partial q^b}(\mathcal{B}^{-1})^{ba}\int_M \partial_T g_{mn}\frac{\partial\gamma^{mn}}{\partial q^a},\end{aligned} \tag{2.112}$$

where $\frac{\partial g_{mn}}{\partial T}$ may be obtained from the re-scaled field equation (2.36). This is again equivalent to the following projection operation $D\mathcal{P} : T_g\mathcal{M} \to T_\gamma\mathcal{N}$

$$\frac{\partial\gamma}{\partial T} = D\mathcal{P}[g]\cdot\frac{\partial g}{\partial T} \tag{2.113}$$

In a sense, the following estimate holds

$$||\frac{\partial\gamma}{\partial T}||_{H^s} \leq C||\frac{\partial g}{\partial T}||_{H^{s-1}}, \tag{2.114}$$

for some constant $C = C(\delta) > 0$. More generally, one may have the following estimate for $z \in T_g\mathcal{U}$ ($g \in \mathcal{U} \subset \mathcal{M}$) while considering the projection operation $\mathcal{P} : T_g\mathcal{U} \to T_\gamma\mathcal{N}$

$$||D\mathcal{P}\cdot z||_{H^s} \leq C||z||_{H^{s-1}}. \tag{2.115}$$

In a sense, the projection is a smoothing operation. We are primarily interested in studying the evolution of sufficiently small however fully nonlinear perturbations under Einstein-Λ flow. In order to do so, we need to first define the small data. The background solutions (conformal space-times (2.61)) may be expressed in terms of the re-scaled variables $(\gamma, N = n, X^i = 0)$ after dimensionalization by suitable factors of $\phi^2 = (\tau^2 - \frac{2n\Lambda}{n-1}) > 0$. In addition to these three entities, we also have the corresponding background re-scaled transverse-traceless second fundamental form $K^{TT} = 0$. Therefore, the complete set of small data is defined as the difference between the background and perturbed solutions i.e., $(g-\gamma, K^{TT}, \frac{N}{n}-1, X)$ whose norm is sufficiently small in the suitable function space.

Now we state and prove a series of important lemmas which shall be required later to obtain several important estimates.

Lemma 3: *Let* $s > \frac{n}{2}+2$, $\gamma_0 \in \mathcal{E}in_{-\frac{n-1}{n^2}}$ *and* $\mathcal{N}$ *be its integrable deformation space and* $\gamma \in \mathcal{N}$. *Also assume* $g \in \mathcal{U} \subset \mathcal{M}$ *and* $\mathcal{U}$ *is a neighborhood of* $\mathcal{N}$ *in* $\mathcal{M}$ *and* $g \in \mathcal{U}$ *satisfies* $||g-\gamma_0||_{H^s} < \delta > 0$. *Let* $z \in T_g\mathcal{U}$. *Then*

$$\gamma^{mn} D\Gamma^i_{mn}[\gamma] D\mathcal{P} \cdot z : H^{s-1} \to H^s \tag{2.116}$$

satisfies the following estimate

$$||\gamma^{mn} D\Gamma^i_{mn}[\gamma] D\mathcal{P} \cdot z||_{H^s} \leq C(\delta)||\gamma-\gamma_0||_{H^s}||z||_{H^{s-1}}. \tag{2.117}$$

with $C(\delta) > 0$.

Proof: Following equation (3.75), any element belonging to $T_\gamma\mathcal{N}$ may be written as

$$h = h^{TT||} + L_{Y^{||}}\gamma, \tag{2.118}$$

with $Y^{||}$ satisfying

$$\begin{aligned} -[\nabla[\gamma]^m \nabla[\gamma]_m Y^{||i} + R[\gamma]^{||i}_m Y^{||m}] + (h^{TT||} + L_{Y^{||}}\gamma)^{mn}(\Gamma[\gamma]^i_{mn} \\ -\Gamma[\gamma_0]^i_{mn}) = 0. \end{aligned} \tag{2.119}$$

$\gamma^{mn} D\Gamma^i_{mn}[\gamma]h$ for $h \in T_\gamma\mathcal{N}$ may be written as

$$\begin{aligned} 2\gamma^{mn} D\Gamma[\gamma]^i_{mn} h &= \gamma^{mn}\gamma^{ik}\left(\nabla[\gamma]_m h_{kn} + \nabla[\gamma]_n h_{mk} - \nabla[\gamma]_k h_{mn}\right) \\ &= 2\gamma^{ik}\nabla[\gamma]^n h_{kn} - \nabla[\gamma]^i (tr_\gamma h), \end{aligned} \tag{2.120}$$

from which $\gamma^{mn}D\Gamma^i_{mn}[\gamma]h^{TT||}=0$ follows immediately and the remaining terms lead to the following equation

$$
\begin{aligned}
2\gamma^{mn}D\Gamma[\gamma]^i_{mn}h &= 2\gamma^{ik}\nabla[\gamma]^n\left(\nabla[\gamma]_kY_n+\nabla[\gamma]_nY_k\right)-2\nabla[\gamma]^i(\nabla[\gamma]_mY^m)\\
&= 2\gamma^{ik}(\nabla[\gamma]_k\nabla[\gamma]_nY^n+R[\gamma]_{mk}Y^m)+2\nabla[\gamma]^n\nabla[\gamma]_nY^i-2\nabla[\gamma]^i(\nabla[\gamma]_mY^m)\\
&= 2(\nabla[\gamma]^n\nabla[\gamma]_nY^i+R[\gamma]^i_mY^m)\\
&= 2(L_{Y^{||}}\gamma+h^{TT})^{mn}(\Gamma[\gamma]^i_{mn}-\Gamma[\gamma_0]^i_{mn})\\
&= 2h^{mn}(\Gamma[\gamma]^i_{mn}-\Gamma[\gamma_0]^i_{mn}).
\end{aligned}
$$

Now, using the previous expression, for γ close to γ_0, we have the following estimate

$$||\gamma^{mn}D\Gamma[\gamma]^i_{mn}h||_{H^s}\leq C||\gamma-\gamma_0||_{H^s}||h||_{H^{s-1}} \tag{2.121}$$

which upon substituting $T_\gamma\mathcal{N}\ni h=D\mathcal{P}\cdot z$ together with (2.115) yields the required estimate

$$||\gamma^{mn}D\Gamma^i_{mn}[\gamma]D\mathcal{P}\cdot z||_{H^s}\leq C||\gamma-\gamma_0||_{H^s}||z||_{H^{s-1}}. \tag{2.122}$$

We have therefore proved the lemma.

Lemma 4: *let $s>\frac{n}{2}+2,\gamma_0\in\mathcal{E}in_{-\frac{n-1}{n^2}}$, and $\mathcal{N}$ be the integrable deformation space of γ_0, and let $\gamma\in\mathcal{N}$ be the shadow of g* i.e., $\mathcal{P}[g]=\gamma$, $g\in\mathcal{U}\subset\mathcal{M}$ with $\mathcal{U}$ being a neighborhood of $\mathcal{N}$ in $\mathcal{M}$ and $g\in\mathcal{U}$ satisfying $||g-\gamma_0||_{H^s}<\delta$ for some $\delta>0$, then

$$||g^{mn}D\Gamma^i_{mn}[\gamma]D\mathcal{P}|_gz||_{H^s}\leq C(\delta)\left(||g-\gamma||_{H^s}+||\gamma-\gamma_0||_{H^s}\right)||z||_{H^{s-1}}, \tag{2.123}$$

for $z\in T_g\mathcal{U}$.

Proof:

$$
\begin{aligned}
g^{mn}D\Gamma^i_{mn}[\gamma]D\mathcal{P}|_gz &= (g^{mn}-\gamma^{mn})D\Gamma^i_{mn}|_\gamma\cdot D\mathcal{P}|_gz+\gamma^{mn}D\Gamma^i_{mn}|_\gamma\cdot D\mathcal{P}|_gz,\\
||g^{mn}D\Gamma^i_{mn}[\gamma]D\mathcal{P}|_gz||_{H^s} &= ||(g^{mn}-\gamma^{mn})D\Gamma^i_{mn}|_\gamma\cdot D\mathcal{P}|_gz\\
&\quad+\gamma^{mn}D\Gamma^i_{mn}|_\gamma\cdot D\mathcal{P}|_gz||_{H^s}\\
&\leq ||(g^{mn}-\gamma^{mn})D\Gamma^i_{mn}|_\gamma\cdot D\mathcal{P}|_gz||_{H^s}+||\gamma^{mn}D\Gamma^i_{mn}|_\gamma\cdot D\mathcal{P}|_gz||_{H^s}.
\end{aligned}
$$

Application of lemma 1 in conjunction with (2.122) yields the desired estimate

$$||g^{mn} D\Gamma^i_{mn}[\gamma] D\mathcal{P}|_g \cdot z||_{H^s} \leq C\left(||g-\gamma||_{H^s} + ||\gamma-\gamma_0||_{H^s}\right)||z||_{H^{s-1}}. \tag{2.124}$$

Lemma 5: *let $s > \frac{n}{2}+2, \gamma_0 \in \mathcal{E}in_{-\frac{n-1}{n^2}}$, and $\mathcal{N}$ be the integrable deformation space of γ_0, and let $\gamma \in \mathcal{N}$ be the shadow of g i.e., $\mathcal{P}[g] = \gamma$, $g \in \mathcal{U} \subset \mathcal{M}$ with $\mathcal{U}$ being a neighborhood of $\mathcal{N}$ in $\mathcal{M}$ and $g \in \mathcal{U}$ satisfying $||g-\gamma_0||_{H^s} < \delta$ for some $\delta > 0$, then the following map*

$$\begin{aligned} P: H^{s+1}(\mathfrak{X}(M)) &\to H^{s-1}(\mathfrak{X}(M)), && (2.125)\\ X &\mapsto \Delta_g X^i - R[g]^i_j X^j + (2\nabla^j X^k)(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) && (2.126)\end{aligned}$$

is an isomorphism.

Proof: Let us say ψ_s is the flow of the shift vector field X and thus a one parameter group of diffeomorphism of M. Therefore, by ψ_s at a fixed time, we may push forward and pull back the sections of the tangent and the co-tangent bundles, respectively. The negative of the tension vector field is defined as a section of the tangent bundle TM and locally expressible as

$$V^i = g^{jk}(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}). \tag{2.127}$$

The co-vector counterpart of V may be pulled back and dualized as

$$\begin{aligned} (\psi_s^* V)^i &= (\psi_s^* g)^{jk} \psi_s^*((\Gamma[g] - \Gamma[\gamma])^i_{jk}, && (2.128)\\ &= (\psi_s^* g)^{jk}\left(\Gamma[\psi_s^* g]^i_{jk} - \Gamma[\psi_s^*\gamma]^i_{jk}\right). \end{aligned}$$

The right hand side follows as a consequence of the tensor transformation property of the difference of connection coefficients (Γ^i_{jk}). Differentiation with respect to s at $s=0$ yields

$$\begin{aligned} (\frac{d}{ds}(\psi_s^* V)^i)|_{s=0} &= \frac{d}{ds}((\psi_s^* g)^{jk}\left(\Gamma[\psi_s^* g]^i_{jk} - \Gamma[\psi_s^*\gamma]^i_{jk}\right))|_{s=0}, && (2.129)\\ L_X V^i &= \frac{d}{ds}((\psi_s^* g)^{jk})|_{s=0}\left(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}\right) \\ &\quad + g^{jk}\frac{d}{ds}\left(\Gamma[\psi^* g]^i_{jk} - \Gamma[\psi^*\gamma]^i_{jk}\right)|_{s=0}, \\ L_X V^i &= L_X g^{jk}\left(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}\right) + g^{jk}\frac{d}{ds}\left(\Gamma[\psi^* g]^i_{jk} - \Gamma[\psi^*\gamma]^i_{jk}\right)|_{s=0}. \end{aligned}$$

Using the formula for the Fréchet derivative of the connection coefficients, we may obtain

$$\begin{aligned}
\frac{d}{ds}\Gamma^i_{jk}[\psi_s^* g]|_{s=0} &= \frac{1}{2}g^{im}[\nabla_j(\frac{d}{ds}((\psi_s^* g)_{mk})|_{s=0}) + \nabla_k(\frac{d}{ds}((\psi_s^* g)_{jm})|_{s=0}) \\
&\quad -\nabla_m(\frac{d}{ds}((\psi_s^* g)_{jk})|_{s=0})] \\
&= \frac{1}{2}g^{im}\left(\nabla_j(L_X g_{mk}) + \nabla_k(L_X g_{jm}) - \nabla_m(L_X g_{jk})\right) \\
&= \frac{1}{2}g^{im}(\nabla_j(\nabla_m X_k + \nabla_k X_m) + \nabla_k(\nabla_m X_j + \nabla_j X_m) \\
&\quad -\nabla_m(\nabla_k X_j + \nabla_j X_k)) \\
&= \frac{1}{2}\left(\nabla_{(j}\nabla_{k)}X^i + g^{im}(R_{knjm} + R_{jnkm})X^n\right) \\
&= \frac{1}{2}\left(\nabla[g]_{(j}\nabla[g]_{k)}X^i + (R[g]^i_{jnk} + R[g]^i_{knj})X^n\right),
\end{aligned}$$

where $\nabla_{(i}X_{j)}$ is the symmetrization of $\nabla_i X_j$ i..e, $\nabla_{(i}X_{j)} := \nabla_i X_j + \nabla_j X_i$. Similarly the following holds

$$\frac{d}{ds}\Gamma^i_{jk}[\psi_s^* \gamma]|_{s=0} = \frac{1}{2}\left(\nabla[\gamma]_{(j}\nabla[\gamma]_{k)}X^i + (R[\gamma]^i_{jnk} + R[\gamma]^i_{knj})X^n\right). \tag{2.130}$$

The previous expressions altogether yield

$$\begin{aligned}
L_X V^i &= (-2\nabla[g]^j X^k)\left(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}\right) + g^{jk}[\nabla[g]_j\nabla[g]_k X^i + R[g]^i_{knj}X^n \\
&\quad -\nabla[\gamma]_j\nabla[\gamma]_k X^i - R[\gamma]^i_{knj}X^n]
\end{aligned}$$

which leads to

$$\begin{aligned}
&\Delta_g X^i - R[g]^i_j X^j + (2\nabla^j X^k)(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) + L_X V^i \\
&\qquad = -g^{jk}(\nabla[\gamma]_j\nabla[\gamma]_k X^i + R[\gamma]^i_{knj}X^n),
\end{aligned} \tag{2.131}$$

or

$$\begin{aligned}
&\Delta_g X^i - R[g]^i_j X^j + (2\nabla^j X^k)(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) \\
&\qquad = -g^{jk}(\nabla[\gamma]_j\nabla[\gamma]_k X^i + R[\gamma]^i_{knj}X^n).
\end{aligned} \tag{2.132}$$

upon imposing the spatial harmonic gauge condition $V^i = 0$. Given that we've established the relation (2.132), it is sufficient to show the injectivity of the map $X \mapsto -g^{jk}(\nabla[\gamma]_j\nabla[\gamma]_k X^i + R[\gamma]^i_{knj}X^n)$ (a map from $H^{s+1}(\mathfrak{X}(M))$ to $H^{s-1}(\mathfrak{X}(M))$) in order to prove the isomorphism property (then surjectivity will follow from self-adjointness) of the map P. Let $Z \in \ker(X \mapsto -g^{jk}(\nabla[\gamma]_j\nabla[\gamma]_k X^i + R[\gamma]^i_{knj}X^n))$ i.e.,

$$-g^{jk}(\nabla[\gamma]_j\nabla[\gamma]_k Z^i + R[\gamma]^i_{knj}Z^n) = 0, \tag{2.133}$$

which upon multiplying both sides by Z_i and integrating over M after imposing $V^i = 0$ yields

$$\int_M (-g^{jk}\nabla[\gamma]_j Z_i \nabla_k Z^i + g^{jk}R[\gamma]_{iknj}Z^i Z^n)\mu_g = 0. \tag{2.134}$$

Now g is sufficiently close to γ and $Ric_\gamma(Z, Z) = -\frac{n-1}{n^2}\gamma(Z, Z) < 0$. Therefore $g^{jk}R[\gamma]_{iknj}Z^i Z^n \leq 0$ is satisfied leading to

$$Z = 0, \tag{2.135}$$

and therefore

$$\ker(X \mapsto -g^{jk}(\nabla[\gamma]_j\nabla[\gamma]_k X^i + R[\gamma]^i_{knj}X^n)) = \{0\}.$$

This concludes the proof that P is an isomorphism between H^{s+1} and H^{s-1}.

Using the previous lemmas, we will prove three additional lemmas which will be crucial towards proving the stability results.

Lemma 6: *Let $s > \frac{n}{2} + 2$. Let $B_{s,\delta}(\gamma_0, 0) \subset H^s \times H^{s-1}$ be a ball of radius δ centered at $(\gamma_0, 0)$ and $(g, K^{TT}) \in B_{s,\delta}(\gamma_0, 0)$. Let $(\tau^2 - \frac{2n\Lambda}{n-1}) > 0$, $\partial_T = -\frac{\tau^2 - \frac{2n\Lambda}{n-1}}{\tau}\partial_\tau = -\frac{\phi^2}{\tau}\partial_\tau$, and assume that the CMCSH gauge condition is satisfied. Then there exists a constant $C = C(\delta) > 0$ such that the following inequality holds for any T satisfying $-\infty < T_1 < T < T_2 < \infty$*

$$||\frac{N}{n} - 1||_{H^{s+1}} \leq C||K^{TT}||^2_{H^{s-1}}. \tag{2.136}$$

Proof: Let's consider the re-scaled Lapse equation

$$\Delta_g N + (|K^{TT}|^2 + \frac{1}{n})N = 1 \tag{2.137}$$

and substitute $Q = \frac{N}{n} - 1$ i.e., $N = n(1 + Q)$. We obtain

$$\Delta_g Q + (|K^{TT}|^2 + \frac{1}{n})Q = -|K^{TT}|^2. \tag{2.138}$$

Clearly, as $(|K^{TT}|^2+\frac{1}{n}) > 0$, "$\Delta_g+(|K^{TT}|^2+\frac{1}{n})id$" is an isomorphism of H^{s+1} onto H^{s-1}. Therefore, from the elliptic regularity of "$\Delta_g + (|K^{TT}|^2 + \frac{1}{n})id$", we may write the following inequality [104]

$$||Q||_{H^{s+1}} \leq C||(\Delta_g + (|K^{TT}|^2 + \frac{1}{n})id)Q||_{H^{s-1}}, \tag{2.139}$$

and using equation (2.138), we may immediately write

$$||Q||_{H^{s+1}} \leq C||K^{TT}||^2_{H^{s-1}} \tag{2.140}$$

or

$$||\frac{N}{n} - 1||_{H^{s+1}} \leq C||K^{TT}||^2_{H^{s-1}}. \tag{2.141}$$

We have thus proved the lemma.

Lemma 7: *Let $s > \frac{n}{2} + 2$. Let $B_{s,\delta}(\gamma_0, 0) \subset H^s \times H^{s-1}$ be a ball of radius δ centered at $(\gamma_0, 0)$ and $(g, K^{TT}) \in B_{s,\delta}(\gamma_0, 0)$. Let $(\tau^2 - \frac{2n\Lambda}{n-1}) > 0$ and $\partial_T = -\frac{\tau^2 - \frac{2n\Lambda}{n-1}}{\tau}\partial_\tau = -\frac{\phi^2}{\tau}\partial_\tau$, $0 < \frac{\phi(\tau)}{\tau} < 1$, and assume that the CMCSH gauge condition is satisfied. Then there exists a constant $C = C(\delta) > 0$ such that the following inequality holds for any T satisfying $-\infty < T_1 < T < T_2 < \infty$*

$$||X||_{H^{s+1}} \quad \leq \quad C(||K^{TT}||_{H^{s-1}} + \frac{\tau}{\phi}||K^{TT}||^2_{H^{s-1}}). \tag{2.142}$$

Proof: The elliptic equation (4.22) for the shift X reads

$$\begin{aligned}\Delta_g(\frac{\phi(\tau)}{\tau}X^i) - R[g]^i_j(\frac{\phi(\tau)}{\tau}X^j) + (2\nabla^j(\frac{\phi(\tau)}{\tau}X^k))(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) = \\ \frac{\phi(\tau)}{\tau}(2NK^{Tjk})(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) - (2-n)\nabla[g]^i(\frac{N}{n} - 1) \\ -\frac{2\phi(\tau)}{\tau}\nabla[g]^j NK^{Ti}_j + g^{jk}\partial_T\Gamma[\gamma]^i_{jk},\end{aligned}$$

and we have proved in lemma (5) that the operator $P : H^{s+1} \to H^{s-1}$ i.e.,

$$X^i \quad \mapsto \quad \Delta_g X^i - R[g]^i_j X^j + 2\nabla^j X^k(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk})$$

is an isomorphism and thus the following estimate holds

$$||X||_{H^{s+1}} \quad \leq \quad C||\Delta_g X^i - R[g]^i_j X^j + 2\nabla^j X^k(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk})||_{H^{s-1}}.$$

Therefore, use of the shift equation yields

$$||\frac{\phi(\tau)}{\tau}X||_{H^{s+1}} \leq ||g^{mn}\partial_T\Gamma[\gamma]^i_{mn}||_{H^s}. \tag{2.143}$$

Note that every term on the right hand side of the equation (2.143) is of second or higher order except the last term $g^{jk}\partial_T\Gamma[\gamma]^i_{jk}$. Using the estimate (2.123), and the re-scaled field equation (2.36), and $0 < \frac{\phi(\tau)}{\tau} < 1$, we obtain

$$\begin{aligned}||g^{mn}\partial_T\Gamma[\gamma]^i_{mn}||_{H^s} \leq C(||g-\gamma||_{H^s} + ||\gamma-\gamma_0||_{H^s})(||\frac{\phi(\tau)}{\tau(T)}X||_{H^{s+1}} \\ +||\frac{\phi(\tau)}{\tau}K^{TT}||_{H^{s-1}} + ||K^{TT}||^2_{H^{s-1}}) \\ \leq C\frac{\phi(\tau)}{\tau}(||g-\gamma||_{H^s} + ||\gamma-\gamma_0||_{H^s})(||X||_{H^{s+1}} + ||K^{TT}||_{H^{s-1}} + \frac{\tau}{\phi}||K^{TT}||^2_{H^{s-1}}),\end{aligned}$$

which upon substituting in (4.47) leads to the desired estimate

$$\begin{aligned}||X||_{H^{s+1}} \quad \leq \quad & C(||g-\gamma||_{H^s} + ||\gamma-\gamma_0||_{H^s})(||X||_{H^{s+1}} + ||K^{TT}||_{H^{s-1}} \\ & +\frac{\tau}{\phi}||K^{TT}||^2_{H^{s-1}}).\end{aligned} \tag{2.144}$$

Obviously, we can find a constant $C = C(\delta)$, such that the following holds for sufficiently small $(||g-\gamma||_{H^s} + ||\gamma-\gamma_0||_{H^s})$

$$C(||g-\gamma||_{H^s} + ||\gamma-\gamma_0||_{H^s}) \leq 1. \tag{2.145}$$

Therefore, we have the following estimate for X

$$||X||_{H^{s+1}} \quad \leq \quad C(||K^{TT}||_{H^{s-1}} + \frac{\tau}{\phi}||K^{TT}||^2_{H^{s-1}}), \tag{2.146}$$

and thus we have proved the lemma. Notice an important fact that the potentially dangerous term $\frac{\tau}{\phi}$ which satisfies e^T growth as $T \to \infty$ appears in the lemma. However, as we shall see, this dangerous factor cancels with its inverse in the energy inequalities which will be derived later.

Finally, we obtain an estimate on the term $\frac{\phi(\tau)}{\tau}X + Y^{||}$, which is stated in the next lemma.

Lemma 8: *Let $s > \frac{n}{2}+2$. Let $B_{s,\delta}(\gamma_0, 0) \subset H^s \times H^{s-1}$ be a ball of radius δ centered at $(\gamma_0, 0)$ and $(g, K^{TT}) \in B_{s,\delta}(\gamma_0, 0)$. Let $(\tau^2 - \frac{2n\Lambda}{n-1}) > 0$ and $\partial_T = -\frac{\tau^2-\frac{2n\Lambda}{n-1}}{\tau}\partial_\tau = -\frac{\phi^2}{\tau}\partial_\tau$, and assume that the CMCSH gauge condition is satisfied. Then there exists a constant $C = C(\delta) > 0$ such that the* *following inequality holds for any T satisfying $-\infty < T_1 < T < T_2 < \infty$*

$$||\frac{\phi(\tau)}{\tau}X + Y^{||}||_{H^{s+1}} \quad \leq \quad C(\frac{\phi(\tau)}{\tau}||g-\gamma||_{H^s}||K^{TT}||_{H^{s-1}} + ||K^{TT}||^2_{H^{s-1}}), \tag{2.147}$$

Proof: Now let's consider $\gamma \in \mathcal{N}$. Thus, $T_\gamma \mathcal{N} \ni \partial_T \gamma$ may be written as

$$\partial_T \gamma \quad = \quad h^{TT} + L_{Y^{||}}\gamma, \tag{2.148}$$

where h^{TT} is a transverse-traceless tensor and $Y^{||}$ solves the following equation

$$-[\nabla[\gamma]^m \nabla[\gamma]_m Y^i + R[\gamma]^i_m Y^m] + (h^{TT||} + L_{Y^{||}}\gamma)^{mn}(\Gamma[\gamma]^k_{mn} - \Gamma[\gamma_0]^k_{mn}) = 0.$$

We have already shown (2.120) that the following equation holds

$$\gamma^{mn}\partial_T \Gamma[\gamma]^i_{mn} = \gamma^{mn} D\Gamma[\gamma]^i_{mn}\partial_T\gamma = \gamma^{mn} D\Gamma[\gamma]^i_{mn}(h^{TT} + L_{Y^{||}}\gamma) \tag{2.149}$$

$$= \gamma^{mn} D\Gamma[\gamma]^i_{mn} L_{Y^{||}}\gamma = (\nabla[\gamma]^n \nabla[\gamma]_n Y^{||i} + R[\gamma]^i_m Y^{||m}) \tag{2.150}$$

Now adding $(-\nabla[\gamma]^n\nabla[\gamma]_n((\frac{\phi(\tau)}{\tau}X^i) - R[\gamma]^i_m(\frac{\phi(\tau)}{\tau}X^m))$ to both sides of equation (2.143), we obtain

$$\begin{aligned}
&\Delta_g(\frac{\phi(\tau)}{\tau}X^i) - R[g]^i_j(\frac{\phi(\tau)}{\tau}X^j) + (2\nabla^j(\frac{\phi(\tau)}{\tau}X^k))(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk})\\
&+(-\nabla[\gamma]^n\nabla[\gamma]_n((\frac{\phi(\tau)}{\tau}X^i) - R[\gamma]^i_m(\frac{\phi(\tau)}{\tau}X^m)) = \frac{\phi(\tau)}{\tau}(2NK^{Tjk})(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk})\\
&-(2-n)\nabla[g]^i(\frac{N}{n}-1) - \frac{2\phi(\tau)}{\tau}\nabla[g]^j N K^{Ti}_j\\
&+g^{jk}\partial_T\Gamma[\gamma]^i_{jk} + (-\nabla[\gamma]^n\nabla[\gamma]_n((\frac{\phi(\tau)}{\tau}X^i) - R[\gamma]^i_m(\frac{\phi(\tau)}{\tau}X^m))
\end{aligned}$$

$$\begin{aligned}
&-g^{jk}(\nabla[\gamma]_j\nabla[\gamma]_k(\frac{\phi(\tau)}{\tau}X^i) + R[\gamma]^i_{knj}(\frac{\phi(\tau)}{\tau}X^n)) + (-\nabla[\gamma]^n\nabla[\gamma]_n(\frac{\phi(\tau)}{\tau}X^i)\\
&-R[\gamma]^i_m(\frac{\phi(\tau)}{\tau}X^m)) = \frac{\phi(\tau)}{\tau}(2NK^{Tjk})(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) - (2-n)\nabla[g]^i(\frac{N}{n}-1)\\
&-\frac{2\phi(\tau)}{\tau}\nabla[g]^j N K^{Ti}_j + g^{jk}\partial_T\Gamma[\gamma]^i_{jk} + (-\nabla[\gamma]^n\nabla[\gamma]_n((\frac{\phi(\tau)}{\tau}X^i) - R[\gamma]^i_m(\frac{\phi(\tau)}{\tau}X^m))
\end{aligned}$$

$$(-\nabla[\gamma]^n\nabla[\gamma]_n(\frac{\phi(\tau)}{\tau}X^i) - R[\gamma]^i_m(\frac{\phi(\tau)}{\tau}X^m)) = \frac{\phi(\tau)}{\tau}(2NK^{Tjk})(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk})$$
$$-(2-n)\nabla[g]^i(\frac{N}{n}-1) - \frac{2\phi(\tau)}{\tau}\nabla[g]^j NK^{Ti}_j + g^{jk}\partial_T\Gamma[\gamma]^i_{jk}$$
$$+(\gamma^{mn} - g^{mn})(-\nabla[\gamma]_m\nabla[\gamma]_n((\frac{\phi(\tau)}{\tau}X^i) - R[\gamma]^i_{mjn}(\frac{\phi(\tau)}{\tau}X^j)).$$

Now adding $-(\nabla[\gamma]^n\nabla[\gamma]_n Y^{||i} + R[\gamma]^i_m Y^{||m}) = -\gamma^{mn}\partial_T\Gamma[\gamma]^i_{mn}$ to the both sides of previous equation, we obtain

$$(-\nabla[\gamma]^n\nabla[\gamma]_n(\frac{\phi(\tau)}{\tau}X^i + Y^{||i}) - R[\gamma]^i_m(\frac{\phi(\tau)}{\tau}X^m + Y^{||m})) \tag{2.151}$$
$$= \frac{\phi(\tau)}{\tau}(2NK^{Tjk})(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) - (2-n)\nabla[g]^i(\frac{N}{n}-1) - \frac{2\phi(\tau)}{\tau}\nabla[g]^j NK^{Ti}_j$$
$$+g^{mn}\partial_T\Gamma[\gamma]^i_{mn} + (\gamma^{mn} - g^{mn})(-\nabla[\gamma]_m\nabla[\gamma]_n((\frac{\phi(\tau)}{\tau}X^i)$$
$$-R[\gamma]^i_{mjn}(\frac{\phi(\tau)}{\tau}X^j)) - \gamma^{mn}\partial_T\Gamma[\gamma]^i_{mn}$$

$$=> (-\nabla[\gamma]^n\nabla[\gamma]_n(\frac{\phi(\tau)}{\tau}X^i + Y^{||i}) - R[\gamma]^i_m(\frac{\phi(\tau)}{\tau}X^m + Y^{||m}))$$
$$= \frac{\phi(\tau)}{\tau}(2NK^{Tjk})(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) - (2-n)\nabla[g]^i(\frac{N}{n}-1) - \frac{2\phi(\tau)}{\tau}\nabla[g]^j NK^{Ti}_j$$
$$+(\gamma^{mn} - g^{mn})(-\nabla[\gamma]_m\nabla[\gamma]_n((\frac{\phi(\tau)}{\tau}X^i) - R[\gamma]^i_{mjn}(\frac{\phi(\tau)}{\tau}X^j))$$
$$+(g^{mn} - \gamma^{mn})\partial_T\Gamma[\gamma]^i_{mn}.$$

Now after applying estimate of $\frac{N}{n} - 1$, X, smoothing operation by shadow gauge $||\frac{\partial\gamma}{\partial T}||_{C^\infty} \lesssim ||\frac{\partial g}{\partial T}||_{H^{s-1}}$, the elliptic regularity of the operator P, and the algebra property of the space H^s for $s > \frac{n}{2}$, we note that every term in the right hand side of the previous equation contributes at least to a second order. Trivial power counting of $\frac{\phi}{\tau}$ yields

$$||\frac{\phi(\tau)}{\tau}X + Y^{||}||_{H^{s+1}} \leq C(\frac{\phi(\tau)}{\tau}||g-\gamma||_{H^s}||K^{TT}||_{H^{s-1}} + ||K^{TT}||^2_{H^{s-1}}), \tag{2.152}$$

for some $C = C(\delta) > 0$. This concludes the proof of the lemma.

In order to construct a Lyapunov function and to establish its decay property, we need the evolution equations for the small data $(g-\gamma, K^{TT})$. Through the following lemmas and utilizing equations (2.36)-(2.37), we arrive at the final set of evolution equations required to define an energy functional (Lyapunov function for small data) and obtain its estimate.

Lemma 9: *Let $(g_0, K_0^{TT}, N, X) = (\gamma, 0, n, 0)$ be a fixed point solution of the re-scaled 'Einstein-Λ' equations, where $R(\gamma) = -\frac{n-1}{n^2}\gamma$. Define $u = g - \gamma, v = 2nK^{TT}$, and $w = \frac{N}{n}$. The 'Einstein-Λ' evolution equations are equivalent to the following system*

$$\begin{aligned}
\partial_T u_{ij} &= \frac{\phi}{\tau} w v_{ij} - \frac{\phi}{\tau} X^m \nabla[\gamma]_m u_{ij} - h_{ij}^{TT||} + 2(w-1)(u_{ij} + \gamma_{ij}) \\
&\quad -(L_{\frac{\phi}{\tau}X+Y^{||}}\gamma)_{ij} - \frac{\phi}{\tau}(u_{im}\nabla[\gamma]_j X^m + u_{mj}\nabla[\gamma]_i X^m), \\
\partial_T v_{ij} &= -(n-1)v_{ij} - \frac{\phi}{\tau} n^2 w \mathcal{L}_{g,\gamma} u_{ij} - \frac{\phi}{\tau} X^m \nabla[\gamma]_m v_{ij} - 2\frac{\phi}{\tau} n^2 w (R[g]_{ij} \\
&\quad + \frac{n-1}{n^2} g_{ij} - \alpha_{ij}) + \frac{2\phi}{\tau} n^2 \nabla_i \nabla_j w + \frac{\phi}{\tau} w v_{im} v_j^m - 2\frac{\phi}{\tau}(w-1)(u_{ij} + \gamma_{ij}) \\
&\quad -(n-2)(w-1)v_{ij} - \frac{\phi}{\tau}(v_{im}\nabla[\gamma]_j X^m + v_{mj}\nabla[\gamma]_i X^m) + 8\frac{\phi(\tau)}{\tau} n^3 w v_{im} v_j^m,
\end{aligned} \tag{2.153}$$

where $\phi^2 = \tau^2 - \frac{2n\Lambda}{n-1} > 0$, $v_j^m = g^{ml} v_{lj}$, and $\partial_T = -\frac{\phi^2}{\tau}\partial_\tau$.

Proof: A direct calculation after substituting the transformed variables $u = g - \gamma$, $v = 2nK^{TT}$, and $N = nw$ along with the fact that $0 \neq \frac{\partial \gamma}{\partial T} \in T_\gamma \mathcal{N}$ and thus $\frac{\partial \gamma}{\partial T} = h^{TT||} + L_{Y^{||}}\gamma$, we thereby obtain the evolution equation for u.

Now, we need the following lemma.

Lemma 10 [206]: *The following holds for $R_{ij}[g]$*

$$R_{ij}[g] - \alpha_{ij} + \frac{n-1}{n^2} g_{ij} \quad = \quad \frac{1}{2}\mathcal{L}_{g,\gamma}(g-\gamma)_{ij} + J_{ij}, \tag{2.154}$$

where $\alpha_{ij} = \frac{1}{2}(L_V g)_{ij}$, $\mathcal{L}_{g,\gamma} h_{ij} = \Delta_{g,\gamma} h_{ij} - 2R[\gamma]_{ikjl} h^{kl}$, $\Delta_{g,\gamma} h_{ij} = -\frac{1}{\mu_g}\nabla[\gamma]_m (g^{mn}\mu_g \nabla[\gamma]_n h_{ij})$, and J_{ij} satisfies the following estimate

$$||J||_{H^{s-2}} \quad \leq \quad C||g-\gamma||_{H^s}^2. \tag{2.155}$$

Proof: A direct calculation using the definitions of $\mathcal{L}_{g,\gamma}$ and $\Delta_{g,\gamma}$ yields the result. Note that $\mathcal{L}_{\gamma,\gamma}$ is just $\mathcal{L}$ defined in section (4.1).

Using this lemma (10), the evolution equations follow from a direct calculation.

Lemma 11: *The evolution equations for u and v are equivalent to the following system*

$$\partial_T u \quad = \quad \frac{\phi}{\tau} w v - \frac{\phi}{\tau} X^m \nabla[\gamma]_m u - h^{TT||} + 2(w-1)\gamma + \mathcal{F}_u, \tag{2.156}$$

$$\partial_T v \quad = \quad -(n-1)v - \frac{\phi}{\tau} n^2 w \mathcal{L}_{g,\gamma} u - \frac{\phi}{\tau} X^m \nabla[\gamma]_m v + \mathcal{F}_v, \tag{2.157}$$

where $(\mathcal{F}_u)_{ij} = 2(w-1)u_{ij} - (L_{\frac{\phi}{\tau}X+Y^{||}}\gamma)_{ij} - \frac{\phi}{\tau}(u_{im}\nabla[\gamma]_j X^m + u_{mj}\nabla[\gamma]_i X^m)$ and $(\mathcal{F}_v)_{ij} = -2\frac{\phi}{\tau} n^2 w J_{ij} +$

$\frac{2\phi}{\tau}n^2\nabla_i\nabla_j w+\frac{\phi}{\tau}wv_{im}v^m_j-2\frac{\phi}{\tau}(w-1)(u_{ij}+\gamma_{ij})-(n-2)(w-1)v_{ij}-\frac{\phi}{\tau}(v_{im}\nabla[\gamma]_jX^m+v_{mj}\nabla[\gamma]_i)+8\frac{\phi(\tau)}{\tau}n^3wv_{im}v^m_j$, *and they satisfy the following estimates*

Proof: Trivial.

2.4.3 Linearization

Even though we have already established the linearized stability, we may quickly reprove the result using the dynamical equations obtained for the perturbations. Here, we construct an energy functional (Lyapunov function) for the linearized equations, which will motivate the construction of the energy functional for the fully nonlinear stability problem. At this point, we have dynamical equations for perturbations both parallel and perpendicular to $\mathcal{N}$. However, we will see shortly that the parallel component of the perturbation is trivially stable. Once again the fixed points satisfy

$$w_0=\frac{N}{n}=1, X^i_0=0, u_0=\gamma, R(\gamma)=-\frac{n-1}{n^2}\gamma, v_0=2nK^{TT}_0=0. \tag{2.158}$$

Linearization about these fixed points preserving the gauges and constraints i.e.,

$$\delta u=u^{TT},\delta v=v^{TT},\delta w=0,\delta X=0,\delta(\frac{\phi}{\tau}X+Y^{||})=0,h^{TT||}=\frac{\phi}{\tau}v^{||} \tag{2.159}$$

together with the field equations yield

$$\begin{aligned}
\partial_T u^{\perp} &= \frac{\phi}{\tau}v^{\perp}, && (2.160)\\
\partial_T v^{||} &= -(n-1)v^{||}, && (2.161)\\
\partial_T v^{\perp} &= -(n-1)v^{\perp}-\frac{\phi(\tau)}{\tau}n^2\mathcal{L}_{\gamma,\gamma}u^{\perp}, && (2.162)
\end{aligned}$$

where we have used the L^2 orthogonal decomposition $u^{TT}=u^{TT||}+u^{TT\perp}, v^{TT}=v^{TT||}+v^{TT\perp}$, and $u^{TT||}=0$ (at the linear level, u is L^2 orthogonal to $\mathcal{N}$). We immediately obtain as $T\to\infty$

$$v^{||}(T)=e^{-(n-1)(T-T_0)}v^{||}(T_0) \tag{2.163}$$

The linearized equation for the L^2 orthogonal component satisfies the following pdes (let's write $u^{\perp} = u$ and $v^{'\perp} = v$ for simplicity)

$$\partial_T u = \frac{\phi}{\tau} v, \tag{2.164}$$

$$\partial_T v = -(n-1)v - \frac{\phi(\tau)}{\tau} n^2 \mathcal{L}_{\gamma,\gamma} u, \tag{2.165}$$

where the operator $\mathcal{L}_{\gamma,\gamma}$ satisfies the eigenvalue equation

$$\mathcal{L}_{\gamma,\gamma}\mathcal{X} = \lambda\mathcal{X} \tag{2.166}$$

with

$$\lambda \geq 0. \tag{2.167}$$

Note that the eigentensor corresponding to $\lambda = 0$ is tangent to the centre manifold $\mathcal{N}$. Such perturbations are trivially stable as evident from equation (164). Since, on the compact manifold, the spectrum of the second order elliptic operator is essentially discrete, we need to focus on the minimum positive eigenvalue of $\mathcal{L}_{\gamma,\gamma}$. Let the positive minimum of the spectrum of $\mathcal{L}_{\gamma,\gamma}$ be $\lambda_0 > 0$ i.e., $\lambda > \lambda_0 > 0 \ \ \forall\ \lambda \in Spec(\mathcal{L})$. Clearly, the coupled pde system can be reduced to the following pair of odes

$$\partial_T u = \frac{\phi}{\tau} v, \tag{2.168}$$

$$\partial_T v = -(n-1)v - \frac{\phi(\tau)}{\tau} n^2 \lambda u. \tag{2.169}$$

In the linearized analysis section, we have already constructed a Lyapunov function for this system. The most natural energy (Lyapunov function) may be defined as follows

$$\mathcal{E} = \frac{1}{2}v^2 + \frac{n^2\lambda}{2}u^2. \tag{2.170}$$

The energy $\mathcal{E}$ is positive semi-definite and vanishes precisely when $(u, v) \equiv 0$, that is, at the fixed points. The time derivative of the energy reads

$$d_T\mathcal{E} = -(n-1)v^2 < 0 \tag{2.171}$$

Therefore, we observe that the energy $\mathcal{E}$ is monotonically decaying. Utilizing well known theorem of dynamical system, strictly monotonic decay of Lyapunov function implies asymptotic stability. As $T \to \infty$, we have the following decay

$$v(T) \lesssim e^{-T}, \partial_T u \lesssim e^{-2T}, |u - u^*| \lesssim e^{-2T}. \tag{2.172}$$

Here we have utilized the fact that $\frac{\phi(\tau)}{\tau} \sim e^{-T}$ as $T \to \infty$. Using the evolution equation, we observe that the re-scaled metric converges to a limit metric (in the space of metrics with constant scalar curvature $-\frac{n-1}{n}$) as $T \to \infty$.

2.4.4 **Non-linear perturbations**

From here onward, we will focus on fully non-linear perturbations to the background solutions. Let us fix a background metric $\gamma_0 \in \mathcal{E}in_{-\frac{n-1}{n^2}}$. Let $\mathcal{N}$ be the deformation space with respect to γ_0 and assume γ is close to γ_0. There exists a harmonic slice $S_\gamma \subset \mathcal{M}$ as the solution of the following equation satisfied by the tension field equation i.e.,

$$V^i = g^{jk} \left(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk} \right) = 0. \tag{2.173}$$

We want to consider $(g \in \mathcal{M}, K^{TT})$ which satisfies the constraint equations (2.38-2.39) as well as the harmonicity condition that the idenitity map '$id : (M, g) \to (M, \gamma)$' is harmonic. Let us denote this constraint slice by $\mathcal{S}_{c,\gamma}$ corresponding to $\mathcal{S}_\gamma$. Following the analysis of [151] (see lemma 2.3), we may represent the constraint slice $\mathcal{S}_{c,\gamma}$ as a graph over its tangent space, that is, we may write $(g, K^{TT}) \in \mathcal{S}_{c,\gamma}$ in the following form

$$g - \gamma = u^{TT} + z, \tag{2.174}$$

$$2nK^{TT} = v^{TT} + r, \tag{2.175}$$

where u^{TT} and v^{TT} are transverse-traceless with respect to γ with $< z|u^{TT} >_{L^2}= 0$, $< v^{TT}|r >_{L^2}= 0$, and $||z||_{H^s} \leq C(||u^{TT}||^2_{H^s} + ||v^{TT}||^2_{H^{s-1}})$, and $||r||_{H^{s-1}} \leq C(||u^{TT}||^2_{H^s} + ||v^{TT}||^2_{H^{s-1}})$ for $C > 0$. From here onward, we will write u and v (resp. v) for $u^{TT} + z$ and $v^{TT} + r$, respectively for simplicity.

2.5 Constructing the Lyapunov functional: definition of Energy

The spectrum of the self-adjoint operator $\mathcal{L}_{g,\gamma}$ will play a vital role in the definition of the energy. In general for a closed manifold, the spectrum of $\mathcal{L}_{g,\gamma}$ is non-negative (because, we have assumed that the compact negative Einstein spaces are stable) i.e., λ satisfying

$$\mathcal{L}_{g,\gamma}\mathcal{X} = \lambda\mathcal{X} \tag{2.176}$$

also satisfies

$$\lambda \geq 0. \tag{2.177}$$

We will observe later that $\lambda = 0$ case is trivially stable provided the smallness condition ($H^s \times H^{s-1}$ norm) on the initial data is met. Let us re-write down the fully non-linear evolution equations for $(u, v) \in B_{s,\delta}(0, 0)$,

$$\partial_T u = \frac{\phi}{\tau}wv - \frac{\phi}{\tau}X^m\nabla[\gamma]_m u - h^{TT||} + 2(w-1)\gamma_{ij} + \mathcal{F}_u, \tag{2.178}$$

$$\partial_T v = -(n-1)v - \frac{\phi}{\tau}n^2 w\mathcal{L}_{g,\gamma}u - \frac{\phi}{\tau}X^m\nabla[\gamma]_m v + \mathcal{F}_v, \tag{2.179}$$

where $(\mathcal{F}_u)_{ij} = 2(w-1)u_{ij} - (L_{\frac{\phi}{\tau}X+Y^{||}}\gamma)_{ij} - \frac{\phi}{\tau}(u_{im}\nabla[\gamma]_j X^m + u_{mj}\nabla[\gamma]_i X^m)$ and $(\mathcal{F}_v)_{ij} = -2\frac{\phi}{\tau}n^2 wJ_{ij} + \frac{2\phi}{\tau}n^2\nabla_i\nabla_j w + \frac{\phi}{\tau}wv_{im}v^m_j - 2\frac{\phi}{\tau}(w-1)(u_{ij}+\gamma_{ij}) - (n-2)(w-1)v_{ij} - \frac{\phi}{\tau}(v_{im}\nabla[\gamma]_j X^m + v_{mj}\nabla[\gamma]_i) + 8\frac{\phi(\tau)}{\tau}n^3 wv_{im}v^m_j$, and they roughly satisfy a third order estimates. The exact estimates will be derived later (when necessary).

Motivated by the energy associated with the linear stability analysis, we define a natural wave equation type of energy (can be read off from the evolution equations) as follows

$$\begin{aligned}\mathcal{E}_i &= \frac{1}{2} < v|\mathcal{L}^{i-1}_{g,\gamma}v >_{L^2} + \frac{n^2}{2} < u|\mathcal{L}^i_{g,\gamma}u >_{L^2} \\ &= \frac{1}{2}\int_M (v_{ij}\mathcal{L}^{i-1}_{g\gamma}v_{kl})\gamma^{ik}\gamma^{jl}\mu_g + \frac{n^2}{2}\int_M (u_{ij}\mathcal{L}^i_{g\gamma}u_{kl})\gamma^{ik}\gamma^{jl}\mu_g.\end{aligned} \tag{2.180}$$

The lowest order term $\mathcal{E}_1$ may be explicitly calculated as follows

$$\begin{aligned}\mathcal{E}_1 = \frac{1}{2}\int_M v_{ij}v_{kl}\gamma^{ik}\gamma^{jl}\mu_g + \frac{n^2}{2}\int_M (&\nabla[\gamma]_m u_{ij}\nabla[\gamma]_n u_{kl}g^{mn}\gamma^{ik}\gamma^{jl} \\ &-2R[\gamma]_i{}^m{}_j{}^n u_{mn}u_{kl}\gamma^{ik}\gamma^{jl})\mu_g.\end{aligned} \tag{2.181}$$

The total energy may be defined by summing all order energies up to s

$$\mathcal{E}_s = \sum_{i=1}^{s} \mathcal{E}_i. \tag{2.182}$$

This energy is positive semi-definite and it vanishes only when $(u,v) \equiv 0$, that is, on the centre manifold. We will now check the non-negative definiteness of the hessian of the energy functional, which will be used to obtain several useful estimates. The first variation of the energy with $\delta u = h$, and $\delta v = k$ at $(0,0)$ vanishes

$$D\mathcal{E}_s(h,k) \quad = \quad 0, \tag{2.183}$$

i.e., $(u,v) = (0,0)$ is a critical point of $\mathcal{E}_s$. The second variation about the critical point yields

$$D^2\mathcal{E}_s((h,k),(h,k)) = \sum_{i=1}^{s} < k|\mathcal{L}_{\gamma,\gamma}^{i-1} k >_{L^2} + n^2 \sum_{i=1}^{s} < h|\mathcal{L}_{\gamma,\gamma}^{i} h >_{L^2}$$

and we immediately obtain the positive semi-definiteness of the hessian of energy using the spectrum of $\mathcal{L}_{\gamma,\gamma}$

$$D^2\mathcal{E}_s((h,k),(h,k)) \geq 0 \tag{2.184}$$

with equality holding if and only if $h = h^{TT||}$ and $k = 0$. Therefore $(0,0)$ is a non-degenerate critical point of $\mathcal{E}_s$. Once we have established the positive semi-definiteness of the hessian of the energy functional, we use this property to obtain a control of the desired $H^s \times H^{s-1}$ norm of the data (u,v) in terms of the energy. The following two lemmas will in fact provide such control of the desired norm.

Lemma 12: *Let (γ, g, K^{TT}) be such that $(g-\gamma)$ satisfies the shadow gauge and $g - \gamma = u$, $2nK^{TT} = v$. There exists a constant $\delta > 0$ sufficiently small, and a constant $C = C(\delta) > 0$ such that if $(u,v) \in \mathcal{B}_\delta(0,0) \in H^s \times H^{s-1}$, the following estimate holds*

$$||u^{||}||_{H^s} \leq C\left(||u^{\perp}||^2_{H^s} + ||v||^2_{H^{s-1}}\right). \tag{2.185}$$

Proof: Following the shadow gauge (u is L^2-orthogonal to $\mathcal{N}$), we may write

$$\begin{aligned} <(g-\gamma), h^{TT||} + L_{Y^{||}}\gamma>_{L^2} &= 0 \\ <u^{||}+u^{\perp}+z, h^{TT||}+L_{Y^{||}}\gamma>_{L^2} &= 0 \\ <u^{||}, h^{TT||}>_{L^2} + <z, h^{TT||}+L_{Y^{||}}\gamma>_{L^2} &= 0 \end{aligned} \tag{2.186}$$

where we have used the facts that $<u^{\perp}, h^{TT||}>_{L^2}=0$ and $<u^{TT}, L_{Y^{||}}\gamma>_{L^2}=0$. Using the relation obtained, we may say that $u^{||}$ is a smooth function of z which satisfies

$$||z||_{H^s} \leq C(||u^{TT}||^2_{H^s} + ||v^{TT}||^2_{H^{s-1}}), \tag{2.187}$$

and therefore, $u^{||}$ satisfies the following estimate

$$||u^{||}||_{H^s} \leq C\left(||u^{\perp}||^2_{H^s} + ||v||^2_{H^{s-1}}\right). \tag{2.188}$$

Using the above definition of the energy together with the positive definiteness of its hessian at $(0,0)$, we have the following crucial lemma which together with the lemma (12) will yield a control of the $H^s \times H^{s-1}$ norm of the data (u,v) in terms of the energy.

Lemma 13: *Let $s > \frac{n}{2}+2$, $\gamma \in \mathcal{E}in_{-\frac{n-1}{n^2}}$, and $\mathcal{E}_s$ be the the total energy defined in (2.182). Then $\exists\ \delta>0, C=C(\delta)>0$, such that $\forall (u,v) \in \mathcal{B}_{\delta}(0,0) \in H^s \times H^{s-1}$, the following estimate holds*

$$||u^{\perp}||^2_{H^s} + ||v||^2_{H^{s-1}} \leq C\mathcal{E}_s. \tag{2.189}$$

Proof: We have observed the positive semi-definiteness of the hessian of the energy functional while restricted to the subspace $\mathcal{H}^s_{TT} = H^s_{TT\perp} \times H^{s-1}_{TT}$ (in local co-ordinates $(u^{\perp}, v)$). Thus, $D^2\mathcal{E}_s : \mathcal{H}^s_{TT} \rightarrow Image(D^2\mathcal{E}_s)$ is an isomorphism leading to the following inequality

$$||u^{\perp}||^2_{H^s} + ||v||^2_{H^{s-1}} \leq CD^2\mathcal{E}_s \cdot ((h,k),(h,k)), \tag{2.190}$$

for some finite $C>0$. Now, using a version of the Morse lemma (Hilbert space version) on the non-degenerate critical point $(0,0)$, we obtain that there exists a $\delta>0$ such that for variations lying within $B_{\delta}(0,0)$ and restricted to $\mathcal{H}^s_{TT}$, the following holds up to a possibly non-linear diffeomorphism $\mathcal{E}_s = \mathcal{E}_s(0,0) + D^2\mathcal{E}_s \cdot ((h,k),(h,k)) = D^2\mathcal{E}_s \cdot ((h,k),(h,k))$ (notice that $\mathcal{E}_s(0,0)=0$). Therefore, we prove the lemma

$$||u^{\perp}||^2_{H^s} + ||v||^2_{H^{s-1}} \leq C\mathcal{E}_s. \tag{2.191}$$

Using the previous two lemmas (12 and 13), we immediately obtain the following crucial result

$$||u||^2_{H^s} + ||v||^2_{H^{s-1}} \leq C\mathcal{E}_s, \tag{2.192}$$

which clearly shows that the energy controls the desired norm of the data (u, v).

2.6 Decay of the energy (or the Lyapunov function)

In this section we study the time evolution of the total energy functional. In order to obtain the decay property of the energy, we state several necessary lemmas. Utilizing these lemmas, we compute the time evolution for the lowest order energy and following analogous calculations, the time evolution of higher order energies are obtained. Since $||A||_{H^{s_1}} \lesssim ||A||_{H^{s_2}}$ for $s_1 < s_2$, we will bound every Sobolev norm of a tensor or a vector field by its maximum available Sobolev norm. Occasionally we will use the Sobolev embedding $||A||_{L^\infty} \lesssim ||A||_{H^a}$ for $a > \frac{n}{2}$ and the following product estimates

$$||AB||_{H^s} \lesssim (||A||_{L^\infty}||B||_{H^s} + ||A||_{H^s}||B||_{L^\infty}), s > 0, \tag{2.193}$$

$$||AB||_{H^s} \lesssim ||A||_{H^s}||B||_{H^s}, s > \frac{n}{2}, \tag{2.194}$$

$$||[P, A]B||_{H^a} \lesssim (||\nabla A||_{L^\infty}||B||_{H^{s+a-1}} + ||A||_{H^{s+a}}||B||_{L^\infty}), P \in \mathcal{OP}^s, s > 0, a \geq 0,$$

where $\mathcal{OP}^s$ denotes the pseudo-differential operators with symbol in the Hormander class $S^s_{1,0}$ (see [175] for details). The first and second inequalities essentially emphasize the algebra property of $H^s \cap L^\infty$ for $s > 0$ and of H^s for $s > \frac{n}{2}$, respectively. In addition, we of course use integration by parts, Holder's and Minkowski inequality whenever necessary.

Lemma 14: *Let $s > \frac{n}{2} + 2$, $\gamma \in \mathcal{E}in_{-\frac{n-1}{n^2}}$ be the shadow of $g \in \mathcal{M}$, $g - \gamma = u$, $2nK^{TT} = v$, and assume there exists a $\delta > 0$ such that $(u, v) \in \mathcal{B}_\delta(0, 0) \subset H^s \times H^{s-1}$, then the following estimates* *hold*

$$(1)|\int_M < \mathcal{L}_{g,\gamma}u, h^{TT||} > \mu_g| \leq C\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}}, \tag{2.195}$$

$$(2)\int_M < u, \partial_T\mathcal{L}_{g,\gamma}u > \mu_g = \int_M < \partial_T u|\mathcal{L}_{g,\gamma}u > \mu_g + \mathcal{R}, |\mathcal{R}| \leq C||u||^2_{H^s}||v||_{H^{s-1}},$$

$$(3)\int_M < v, v > g^{ij}\partial_T g_{ij}\mu_g \leq C(\frac{\phi}{\tau}||v||^3_{H^{s-1}} + ||v||^4_{H^{s-1}}),$$

$$(4)\int_M < u, \mathcal{L}_{g,\gamma}u > g^{ij}\partial_T g_{ij}\mu_g) \leq C(\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}} + ||u||^2_{H^s}||v||^2_{H^{s-1}}),$$

$$(5)|\int_M \frac{\phi}{\tau} < v, X^m\nabla[\gamma]_m v > \mu_g| \leq C(\frac{\phi}{\tau}||v||^3_{H^{s-1}} + ||v||^4_{H^{s-1}}),$$

$$(6)|\int_M \frac{\phi}{\tau} < \mathcal{L}_{g,\gamma}u, X^m\nabla[\gamma]_m u > \mu_g| \leq C(\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}} + ||u||^2_{H^s}||v||^2_{H^{s-1}}),$$

and $C = C(\delta) > 0$.

Proof: (1) Using the self-adjoint property of $\mathcal{L}_{g,\gamma}$, we may write

$$\int_M < \mathcal{L}_{g,\gamma}u, h^{TT||} > \mu_g = \int_M < u, \mathcal{L}_{g,\gamma}h^{TT||} > \mu_g. \tag{2.196}$$

We have

$$(\Delta_{g,\gamma}h)_{ij} = -\frac{1}{\mu_g}\nabla[\gamma]_m\left(g^{mn}\mu_g(\nabla[\gamma]_n h_{ij})\right) \tag{2.197}$$

and the definition of $\mathcal{L}_{g,\gamma}$

$$\begin{aligned}
\mathcal{L}_{g,\gamma}h_{ij} &= \Delta_{g,\gamma}h_{ij} - 2R[\gamma]_{ikjl}h^{kl}, \\
&= -g^{mn}(\nabla[\gamma]_m\nabla[\gamma]_n h_{ij}) - V^m\nabla[\gamma]_m h_{ij} - 2R[\gamma]_{ikjl}h^{kl}, \\
&= -(g^{mn} - \gamma^{mn})(\nabla[\gamma]_m\nabla[\gamma]_n h_{ij}) - \gamma^{mn}\nabla[\gamma]_m\nabla[\gamma]_n h_{ij} \\
&\quad -2R[\gamma]_{ikjl}h^{kl}, \\
&= -(g^{mn} - \gamma^{mn})(\nabla[\gamma]_m\nabla[\gamma]_n h_{ij}) + \mathcal{L}_{\gamma,\gamma}h_{ij},
\end{aligned} \tag{2.198}$$

where we have used the identity $\nabla[\gamma]_m(\mu_g g^{-1})^{mn} = -V^n\mu_g$, and set $V^m = 0$. Replacing h by $h^{TT||}$

$$\mathcal{L}_{g,\gamma}h^{TT||}_{ij} = -(g^{mn} - \gamma^{mn})(\nabla[\gamma]_m\nabla[\gamma]_n h^{TT||}_{ij}) \tag{2.199}$$

as a consequence of $\mathcal{L}_{\gamma,\gamma} h^{TT||} = 0$. Now we will exploit the shadow gauge condition to obtain an estimate of $h^{TT||}$. The shadow gauge reads

$$\begin{aligned} < g - \gamma | \frac{\partial \gamma}{\partial q^\alpha} >_{L^2} &= 0, & (2.200) \\ < u | h^{TT||\alpha} + L_{Y^{||\alpha}} \gamma >_{L^2} &= 0 & (2.201) \end{aligned}$$

which upon time differentiation becomes

$$\begin{aligned} &< \partial_T u | h^{TT||\alpha} + L_{Y^{||\alpha}} \gamma >_{L^2} + \textit{second order terms} = 0, \\ &< \frac{\phi}{\tau} wv - \frac{\phi(\tau)}{\tau} X^m \nabla[\gamma]_m u - h^{TT||} + \mathcal{F}_u | h^{TT||\alpha} + L_{Y^{||\alpha}} \gamma >_{L^2} \\ &\qquad + \textit{second order terms} = 0. \end{aligned}$$

Now, using the estimates on $(w-1)$, X, and $\mathcal{F}_u$, and the identity $< A^{TT} | L_Z \gamma >_{L^2} = 0$ for any transverse-traceless tensor A^{TT} and vector field $Z \in \mathfrak{X}(M)$, we immediately obtain

$$< \frac{\phi}{\tau} v - h^{TT||} | h^{TT||\alpha} >_{L^2} + \textit{second order terms} = 0 \tag{2.202}$$

which leads to

$$h^{TT||} = \frac{\phi}{\tau} v^{||} + \textit{second order terms}, \tag{2.203}$$

where $v^{||}$ is the projection of v onto the subspace of TT tensors belonging to the kernel of $\mathcal{L}_{g,\gamma}$. Now using the equation (2.199), we observe that every term of $< u, \mathcal{L}_{g,\gamma} h^{TT||} >_{L^2}$ is of at least third order and the following claim follows

$$| \int_M < \mathcal{L}_{g,\gamma} u, h^{TT||} >_{L^2} \mu_g | \leq C \frac{\phi}{\tau} ||u||^2_{H^s} ||v||_{H^{s-1}}. \tag{2.204}$$

(2) We need the estimate for the term $< u | \partial_T \mathcal{L}_{g\gamma} u >_{L^2}$. Using the explicit expression for $\mathcal{L}_{g,\gamma}$, we may write

$$\begin{aligned} \partial_T \mathcal{L}_{g,\gamma} u_{ij} &= \partial_T \left(\Delta_{g,\gamma} u_{ij} - 2R[\gamma]_{ikjl} u^{kl} \right), & (2.205) \\ &= \partial_T \left(-g^{mn} (\nabla[\gamma]_m \nabla[\gamma]_n u_{ij}) - V^m \nabla[\gamma]_m u_{ij} - 2R[\gamma]_{ikjl} u^{kl} \right) \end{aligned}$$

and imposing spatial harmonic gauge $V^i = 0$

$$\begin{aligned}
\partial_T \mathcal{L}_{g,\gamma} u_{ij} &= \partial_T \left(-g^{mn}(\nabla[\gamma]_m \nabla[\gamma]_n u_{ij}) - 2R[\gamma]_{ikjl} u^{kl}\right), \\
&= \partial_T(-g^{mn}(\nabla[\gamma]_m \nabla[\gamma]_n u_{ij}) - 2(DR[\gamma]_{ikjl} \cdot \partial_T \gamma) u^{kl} \\
&\quad -2R[\gamma]_{ikjl} \partial_T u^{kl}.
\end{aligned} \tag{2.206}$$

Let the operator $g^{mn}\nabla[\gamma]_m \nabla[\gamma]_n$ be denoted as $\mathcal{D}$, then we write

$$\partial_T \mathcal{D}[g,\gamma] u_{ij} = (\frac{\partial \mathcal{D}[g,\gamma]}{\partial g} \cdot \partial_T g + \frac{\partial \mathcal{D}[g,\gamma]}{\partial \gamma} \cdot \partial_T \gamma) u_{ij} + \mathcal{D}[g,\gamma] \partial_T u_{ij} \tag{2.207}$$

which yields

$$\begin{aligned}
\partial_T \mathcal{L}_{g,\gamma} u_{ij} = \mathcal{L}_{g,\gamma} \partial_T u_{ij} - (\frac{\partial \mathcal{D}[g,\gamma]}{\partial g} \cdot \partial_T g + \frac{\partial \mathcal{D}[g,\gamma]}{\partial \gamma} \cdot \partial_T \gamma) u_{ij} \\
-2(DR[\gamma]_{ikjl} \cdot \partial_T \gamma) u^{kl}.
\end{aligned} \tag{2.208}$$

Since the metric γ is now time dependent, we need to control the terms involving $\partial_T \gamma$. Fortunately, we do have the shadow gauge at our disposal. Utilizing the smoothing operation via shadow gauge one readily bounds $\frac{\partial \gamma}{\partial T}$ in terms of $\frac{\partial g}{\partial T}$

$$||\partial_T \gamma||_{H^s} \leq C ||\partial_T g||_{H^{s-1}}. \tag{2.209}$$

On the other hand, we of course have the estimate for $\frac{\partial g}{\partial T}$ from the evolution equation

$$||\frac{\partial g}{\partial T}||_{H^{s-1}} \lesssim (\frac{\phi}{\tau}||K^{TT}||_{H^{s-1}} + \frac{\phi}{\tau}||X||_{H^{s+1}} + ||K^{TT}||^2_{H^{s-1}}) \tag{2.210}$$

which yields through $||X||_{H^{s+1}} \lesssim (||K^{TT}||_{H^{s-1}} + \frac{\tau}{\phi}||K^{TT}||^2_{H^{s-1}})$

$$||\partial_T \gamma||_{H^s} \lesssim ||K^{TT}||_{H^{s-1}}. \tag{2.211}$$

In fact, following the C^∞ topology of the deformation space $\mathcal{N}$, one may write

$$||\partial_T \gamma||_{C^\infty} \lesssim ||K^{TT}||_{H^{s-1}}. \tag{2.212}$$

Therefore, one is led to the conclusion that the term $-(\frac{\partial \mathcal{D}[g,\gamma]}{\partial g} \cdot \partial_T g + \frac{\partial \mathcal{D}[g,\gamma]}{\partial \gamma} \cdot \partial_T \gamma) u_{ij} - 2(DR[\gamma]_{ikjl} \cdot$

$\partial_T\gamma)u^{kl}$ satisfies a second order estimate. This estimates together yields the desired result

$$< u|\partial_T\mathcal{L}_{g\gamma}u >_{L^2} \quad = \quad < \partial_T u|\mathcal{L}_{g,\gamma}u >_{L^2} + C||u||^2_{H^s}||v||_{H^{s-1}}, \tag{2.213}$$

which concludes the proof of the second part. (3,4) Let us explicitly compute $g^{ij}\frac{\partial g_{ij}}{\partial T}$ using the evolution equation

$$g^{ij}\frac{\partial g_{ij}}{\partial T} = -2n(1-\frac{N}{n}) - \frac{\phi}{\tau}(\nabla[g]_i X^i). \tag{2.214}$$

Therefore, we obtain the following expressions

$$\int_M < v,v > g^{ij}\partial_T g_{ij}\mu_g = -2n\int_M < v,v > \left((1-\frac{N}{n}) + \frac{\phi}{\tau}\nabla[g]_i X^i\right)\mu_g,$$
$$\int_M < u,\mathcal{L}_{g,\gamma}u > g^{ij}\partial_T g_{ij}\mu_g = -2n\int_M < u,\mathcal{L}_{g,\gamma}u > \left((1-\frac{N}{n}) + \frac{\phi}{\tau}\nabla[g]_i X^i\right)\mu_g.$$

Now we invoke the elliptic equation for the lapse N

$$\Delta_g N + (|K^{TT}|^2 + \frac{1}{n})N = 1. \tag{2.215}$$

An straightforward maximum principle yields the following estimate for N

$$N \leq n \tag{2.216}$$

throughout M yielding $1-\frac{N}{n} \geq 0$ and since, $< v,v >$ and $< u,\mathcal{L}_{g,\gamma}u >$ are non-negative definite, we may immediately write

$$\int_M < v,v > g^{ij}\partial_T g_{ij}\mu_g \leq -2n\frac{\phi}{\tau}\int_M < v,v > \nabla[g]_i X^i\mu_g, \tag{2.217}$$
$$\int_M < u,\mathcal{L}_{g,\gamma}u > g^{ij}\partial_T g_{ij}\mu_g \leq -2n\frac{\phi}{\tau}\int_M < u,\mathcal{L}_{g,\gamma}u > \nabla[g]_i X^i\mu_g$$

yielding the desired estimates

$$\int_M < v,v > g^{ij}\partial_T g_{ij}\mu_g \leq C(\frac{\phi}{\tau}||v||^3_{H^{s-1}} + ||v||^4_{H^{s-1}}), \tag{2.218}$$
$$\int_M < u,\mathcal{L}_{g,\gamma}u > g^{ij}\partial_T g_{ij}\mu_g \leq C(\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}} + ||u||^2_{H^s}||v||^2_{H^{s-1}})$$

upon utilizing the estimate of X from lemma 7. (4) and (5) are straightforward to obtain using the estimates of X, N in terms of v. Importantly, it is clear that each one satisfies a third order estimate. The following lemma characterizes the temporal evolution of the lowest order energy. The higher order energy behaviour may be computed in a similar way.

Lemma 15: *Let $s > \frac{n}{2} + 2$, $\gamma \in \mathcal{E}in_{-\frac{n-1}{n^2}}$ be the shadow of $g \in \mathcal{M}$, and assume there exists a $\delta > 0$ such that $(u, v) \in \mathcal{B}_\delta(0,0) \subset H^s \times H^{s-1}$, , then the following holds*

$$\partial_T \mathcal{E}_1 = -(n-1) < v|v >_{L^2} + \mathcal{A}_1, \tag{2.219}$$

with $\mathcal{A}_1$ satisfying

$$\begin{aligned}\mathcal{A}_1 \leq C(||u||_{H^s}||v||^2_{H^{s-1}} + ||u||^2_{H^s}||v||^2_{H^s} + ||v||^4_{H^{s-1}} + \frac{\phi}{\tau}||u||_{H^s}||v||^2_{H^{s-1}} \\ + \frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}} + \frac{\phi}{\tau}||v||^3_{H^{s-1}}),\end{aligned} \tag{2.220}$$

and $C = C(\delta) > 0$.

Proof: A direct calculation using equations (2.156)-(2.179) yields

$$\begin{aligned}\partial_T \mathcal{E}_1 =& < v|\partial_T v >_{L^2} + \frac{n^2}{2} < \partial_T u|\mathcal{L}_{g,\gamma} u >_{L^2} + \frac{n^2}{2} < u|\partial_T \mathcal{L}_{g,\gamma} u >_{L^2} \\ &+ \frac{1}{4}\int_M ((v_{ij}v_{kl})\gamma^{ik}\gamma^{jl} + n^2(u_{ij}\mathcal{L}_{g\gamma}u_{kl})\gamma^{ik}\gamma^{jl})g^{mn}\frac{\partial g_{mn}}{\partial T}\mu_g \\ &- \int_M ((v_{ij}v_{kl})\mu_g + n^2(u_{ij}\mathcal{L}_{g\gamma}u_{kl}))\gamma^{im}\gamma^{kn}\gamma^{jl}\frac{\partial \gamma_{mn}}{\partial T}\mu_g \\ =& < v| - (n-1)v - \frac{\phi}{\tau}n^2 w\mathcal{L}_{g,\gamma}u - \frac{\phi}{\tau}X^m\nabla[\gamma]_m v + F_v >_{L^2} \\ &+ \frac{n^2}{2} < \frac{\phi}{\tau}wv - \frac{\phi}{\tau}X^m\nabla[\gamma]_m u - h^{TT||} + 2(w-1)\gamma + \mathcal{F}_u|\mathcal{L}_{g,\gamma}u >_{L^2} \\ &+ \frac{n^2}{2} < u|\partial_T\mathcal{L}_{g,\gamma}u >_{L^2} + \mathcal{B} \\ =& -(n-1) < v|v >_{L^2} - \frac{\phi}{\tau} < v|X^m\nabla[\gamma]_m v >_{L^2} - \frac{n^2\phi}{2\tau} < X^m\nabla[\gamma]_m u|\mathcal{L}_{g,\gamma}u >_{L^2} \\ &- \frac{n^2}{2} < h^{TT||}|\mathcal{L}_{g,\gamma} >_{L^2} + n^2 < (w-1)\gamma|\mathcal{L}_{g,\gamma}u >_{L^2} - \frac{n^2\phi}{2\tau} < wv|\mathcal{L}_{g,\gamma}u >_{L^2} \\ &+ \frac{n^2}{2} < u|\partial_T\mathcal{L}_{g,\gamma}u >_{L^2} + < v|\mathcal{F}_v >_{L^2} + \frac{n^2}{2} < \mathcal{F}_u|\mathcal{L}_{g,\gamma}u >_{L^2} + \mathcal{B}.\end{aligned}$$

Now utilizing point (2) of lemma (14), we may write

$$\int_M < u, \partial_T\mathcal{L}_{g,\gamma}u > \mu_g = \int_M < \partial_T u|\mathcal{L}_{g,\gamma}u > \mu_g + \mathcal{R}, |\mathcal{R}| \leq C||u||^2_{H^s}||v||_{H^{s-1}},$$

and therefore the time derivative of first order energy (note that $||\frac{\partial\gamma}{\partial T}||_{H^s} \lesssim ||\frac{\partial g}{\partial T}||_{H^s}$ from shadow gauge) becomes

$$\begin{aligned}\partial_T \mathcal{E}_1 = &-(n-1) < v|v >_{L^2} - \frac{\phi}{\tau} < v|X^m \nabla[\gamma]_m v >_{L^2} - \frac{n^2\phi}{\tau} < X^m \nabla[\gamma]_m u|\mathcal{L}_{g,\gamma} u >_{L^2} \\ &-n^2 < h^{TT||}|\mathcal{L}_{g,\gamma} >_{L^2} + 2n^2 < (w-1)\gamma|\mathcal{L}_{g,\gamma} u >_{L^2} - \frac{n^2\phi}{2\tau} < wv|\mathcal{L}_{g,\gamma} u >_{L^2} \\ &+ \frac{n^2\phi}{2\tau} < wv|\mathcal{L}_{g,\gamma} u >_{L^2} + < v|\mathcal{F}_v >_{L^2} + n^2 < \mathcal{F}_u|\mathcal{L}_{g,\gamma} u >_{L^2} + \mathcal{B} \\ = &-(n-1) < v|v >_{L^2} - \frac{\phi}{\tau} < v|X^m \nabla[\gamma]_m v >_{L^2} - \frac{n^2\phi}{\tau} < X^m \nabla[\gamma]_m u|\mathcal{L}_{g,\gamma} u >_{L^2} \\ &-n^2 < h^{TT||}|\mathcal{L}_{g,\gamma} >_{L^2} + 2n^2 < (w-1)\gamma|\mathcal{L}_{g,\gamma} u >_{L^2} + < v|\mathcal{F}_v >_{L^2} \\ &+ n^2 < \mathcal{F}_u|\mathcal{L}_{g,\gamma} u >_{L^2} + \mathcal{B}.\end{aligned}$$

We note that the potentially problematic term $\frac{n^2\phi}{2\tau} < wv|\mathcal{L}_{g,\gamma} u >_{L^2}$ gets cancelled in the previous expression. Now we will estimate each term in the energy expression utilizing lemma (14) and the basic inequalities stated at the beginning of this section. Once again, we will bound every term by their maximum available Sobolev norm (since $||A||_{H^{s_1}} \lesssim ||A||_{H^{s_2}}$ for $s_1 < s_2$). The following holds

$$\begin{aligned}&| < v|X^m \nabla[\gamma]_m v >_{L^2} | \lesssim ||v||^3_{H^{s-1}} + \frac{\tau}{\phi} ||v||^4_{H^{s-1}}, | < X^m \nabla[\gamma]_m u|\mathcal{L}_{g,\gamma} u >_{L^2} | \\ &\lesssim ||u||^2_{H^s} ||v||_{H^{s-1}} + \frac{\tau}{\phi} ||u||^2_{H^s} ||v||^2_{H^{s-1}}, | < h^{TT||}|\mathcal{L}_{g,\gamma} >_{L^2} | \lesssim \frac{\phi}{\tau} ||u||^2_{H^s} ||v||_{H^s}, \\ &| < (w-1)\gamma|\mathcal{L}_{g,\gamma} u >_{L^2} | \lesssim ||u||_{H^s} ||v||^2_{H^{s-1}}, \mathcal{B} \lesssim \frac{\phi}{\tau} ||v||^3_{H^{s-1}} + ||v||^4_{H^{s-1}} \\ &+ \frac{\phi}{\tau} ||u||^2_{H^s} ||v||_{H^{s-1}} + ||u||^2_{H^s} ||v||^2_{H^{s-1}}.\end{aligned}$$

Now, let us compute the product $< v|\mathcal{F}_v >_{L^2}$ and $< \mathcal{F}_u|\mathcal{L}_{g,\gamma} u >_{L^2}$

$$\begin{aligned}< v|\mathcal{F}_v >_{L^2} = &-2n^2 \frac{\phi}{\tau} < v|wJ >_{L^2} + 2n^2 \frac{\phi}{\tau} < v|\nabla \otimes \nabla w >_{L^2} + \frac{\phi}{\tau} < v|wv \cdot v >_{L^2} \\ &- \frac{2\phi}{\tau} < v|(w-1)(u+\gamma) >_{L^2} - (n-2) < v|(w-1)v >_{L^2} + 8n^3 \frac{\phi}{\tau} < v|wv \cdot v >_{L^2} \\ &- \frac{\phi}{\tau} < v|v \circ \nabla X >_{L^2}, \\ < \mathcal{F}_u|\mathcal{L}_{g,\gamma} u >_{L^2} = &\, 2n^2 < (w-1)u|\mathcal{L}_{g,\gamma} u >_{L^2} - n^2 < L_{\frac{\phi}{\tau} X + Y^{||}} \gamma|\mathcal{L}_{g,\gamma} u >_{L^2} \\ &- n^2 \frac{\phi}{\tau} < u \circ \nabla X|\mathcal{L}_{g,\gamma} u >_{L^2},\end{aligned}$$

where $(\nabla \otimes \nabla w)_{ij} = \nabla_i \nabla_j w, (v \cdot v)_{ij} = v_{im} v^m_j$, and $(v \circ \nabla X)_{ij} = v_{im} \nabla_j X^m + v_{jm} \nabla_i X^m$. Once again utilizing lemma (14), basic inequalities, and the elliptic estimates for the lapse function and the shift vector field, each term of the above expressions may be estimated as follows

$$
\begin{aligned}
|<v|wJ>_{L^2}| \lesssim ||u||^2_{H^s}||v||_{H^{s-1}}, |<v|\nabla\otimes\nabla w>_{L^2}| \lesssim ||v||^3_{H^{s-1}}, \\
|<v|wv\cdot v>_{L^2}| \lesssim ||v||^3_{H^{s-1}}, |<v|(w-1)(u+\gamma)>_{L^2}| \lesssim ||u||_{H^s}||v||^3_{H^{s-1}}, \\
|<v|(w-1)v>_{L^2}| \lesssim ||v||^4_{H^{s-1}}, |<v|wv\cdot v>_{L^2}| \lesssim ||v||^3_{H^{s-1}}, \\
|<v|v\circ\nabla X>_{L^2}| \lesssim ||v||^3_{H^{s-1}} + \frac{\tau}{\phi}||v||^4_{H^{s-1}}, |<(w-1)u|\mathcal{L}_{g,\gamma}u>_{L^2}| \\
\lesssim ||u||^2_{H^s}||v||^2_{H^{s-1}}, |<L_{\frac{\phi}{\tau}X+Y^{||}}\gamma|\mathcal{L}_{g,\gamma}u>_{L^2}| \lesssim \frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}} + \\
||u||^2_{H^s}||v||^2_{H^{s-1}}, |<u\circ\nabla X|\mathcal{L}_{g,\gamma}u>_{L^2}| \lesssim ||u||^2_{H^s}||v||_{H^{s-1}} + \frac{\tau}{\phi}||u||^2_{H^s}||v||^2_{H^{s-1}}.
\end{aligned}
\tag{2.221}
$$

Note that the dangerous factor $\frac{\tau}{\phi}$ cancel out in the time derivative of the energy by its inverse $\frac{\phi}{\tau}$. Utilizing these estimates and carefully counting powers of $\frac{\phi}{\tau}$, we obtain the desired differential equation for the first order energy $\mathcal{E}_1$

$$\partial_T\mathcal{E}_1 = -(n-1) <v|v>_{L^2} + \mathcal{A}_1, \tag{2.222}$$

with $\mathcal{A}_1$ satisfying

$$
\begin{aligned}
|\mathcal{A}_1| \leq C(||u||_{H^s}||v||^2_{H^{s-1}} + ||u||^2_{H^s}||v||^2_{H^s} + ||v||^4_{H^{s-1}} \\
+\frac{\phi}{\tau}||u||_{H^s}||v||^2_{H^{s-1}} + \frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}} + \frac{\phi}{\tau}||v||^3_{H^{s-1}})
\end{aligned}
\tag{2.223}
$$

This proves the lemma.

Now we need to derive the time derivative of the higher order energies in order to obtain a time evolution of the total energy. In order to do so, we need a few additional estimates. The next lemma provides the required estimates.

Lemma 16: *Let $s > \frac{n}{2}+2$, $\gamma \in \mathcal{E}in_{-\frac{n-1}{n^2}}$ be the shadow of $g \in \mathcal{M}$, and assume there exists a $\delta > 0$*

such that $(u,v)\in\mathcal{B}_\delta(0,0)\subset H^s\times H^{s-1}$*, then the following estimates hold*

$$(1)|\int_M <\mathcal{L}^i_{g,\gamma}u, h^{TT||}>\mu_g|\leq C\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}}, \qquad (2.224)$$
$$(2)\int_M <u,\partial_T\mathcal{L}^i_{g,\gamma}u>\mu_g=\int_M <\partial_T u|\mathcal{L}^i_{g,\gamma}u>\mu_g+\mathcal{R}^{'}, |\mathcal{R}^{'}|\leq C||u||^2_{H^s}||v||_{H^{s-1}},$$
$$(3)\int_M <v,\mathcal{L}^{i-1}_{g,\gamma}v>g^{kl}\partial_T g_{kl}\mu_g\leq C(\frac{\phi}{\tau}||v||^3_{H^{s-1}}+||v||^4_{H^{s-1}}),$$
$$(4)\int_M <u,\mathcal{L}^i_{g,\gamma}u>g^{kl}\partial_T g_{kl}\mu_g)\leq C(\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}}+||u||^2_{H^s}||v||^2_{H^{s-1}}),$$
$$(5)|\int_M\frac{\phi}{\tau}<\mathcal{L}^{i-1}_{g,\gamma}v,X^m\nabla[\gamma]_m v>\mu_g|\leq C(\frac{\phi}{\tau}||v||^3_{H^{s-1}}+||v||^4_{H^{s-1}}),$$
$$(6)|\int_M\frac{\phi}{\tau}<\mathcal{L}^i_{g,\gamma}u,X^m\nabla[\gamma]_m u>\mu_g|\leq C(\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}}+||u||^2_{H^s}||v||^2_{H^{s-1}}),$$

for $2\leq i\leq s$ *and* $C=C(\delta)>0$.

Proof: Following a calculation analogous to that of lemma (14) and using the formula for the higher order estimates provided in the beginning of this section (i.e., the product estimates and the commutator estimates) for $s>\frac{n}{2}+2$, each of the claims follows.

Now that we have the necessary estimates, we may obtain the time derivative of the higher order energies. The following lemma states the time derivative of the higher order energies.

Lemma 17: *Let* $s>\frac{n}{2}+2$, $\gamma\in\mathcal{E}in_{-\frac{n-1}{n^2}}$ *be the shadow of* $g\in\mathcal{M}$*, and assume there exists a* $\delta>0$ *such that* $(u,v)\in\mathcal{B}_\delta(0,0)\subset H^s\times H^{s-1}$*, then the following holds for* $1<i\leq s$

$$\partial_T\mathcal{E}_i=-(n-1)<v|\mathcal{L}^{i-1}_{g,\gamma}v>_{L^2}+\mathcal{A}_i, \qquad (2.225)$$

with $\mathcal{A}_i$ *satisfying*

$$|\mathcal{A}_i|\leq C(||u||_{H^s}||v||^2_{H^{s-1}}+||u||^2_{H^s}||v||^2_{H^s}+||v||^4_{H^{s-1}}$$
$$+\frac{\phi}{\tau}||u||_{H^s}||v||^2_{H^{s-1}}+\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}}+\frac{\phi}{\tau}||v||^3_{H^{s-1}}), \qquad (2.226)$$

for $C=C(\delta)>0$.

Proof: A calculation analogous to that of lemma (15) and the higher order estimates from lemma (16) directly yield the desired result.

Now that we have concluded with the proofs of the important lemmas, we will study the time evolution of the total energy. The time derivative of the total energy may be written using the lemma (15) and (17) as follows

$$\frac{\partial \mathcal{E}}{\partial T}=-(n-1)\sum_{i=1}^{s}< v|\mathcal{L}_{g,\gamma}^{i-1}v>_{L^2}+\sum_{i=1}^{s}\mathcal{A}_i, \tag{2.227}$$

where each $\mathcal{A}_i$ satisfies higher order estimate

$$\begin{aligned}\mathcal{A}_i &\leq C(||u||_{H^s}||v||^2_{H^{s-1}}+||u||^2_{H^s}||v||^2_{H^s}+||v||^4_{H^{s-1}} \\ &+\frac{\phi}{\tau}||u||_{H^s}||v||^2_{H^{s-1}}+\frac{\phi}{\tau}||u||^2_{H^s}||v||_{H^{s-1}}+\frac{\phi}{\tau}||v||^3_{H^{s-1}})\end{aligned} \tag{2.228}$$

The above expression may be rewritten as follows

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial T} \leq -\left((n-1)-C(||u||_{H^s}+||u||^2_{H^s}+||v||^2_{H^s})\right)\sum_{i=1}^{s}< v|\mathcal{L}_{g,\gamma}^{i-1}v>_{L^2} \\ +C\frac{\phi}{\tau}||v||_{H^{s-1}}\mathcal{E},\end{aligned} \tag{2.229}$$

using the fact that $||v||^2_{H^{s-1}} \lesssim \sum_{i=1}^{s} < v|\mathcal{L}_{g,\gamma}^{i-1}v >_{L^2}, ||u||^2_{H^s}+||v||^2_{H^{s-1}} \lesssim \mathcal{E}$. Also note that $||u||_{H^s}, ||v||_{H^{s-1}} \lesssim \mathcal{E}^{1/2}$. Now notice that zero eigenvalues of $\mathcal{L}_{g,\gamma}$ would correspond to the trivially stable case for small data. If $(u,v)\in \ker(\mathcal{L}_{g,\gamma})$, then

$$\frac{\partial \mathcal{E}}{\partial T} \leq -2(n-1)\mathcal{E}+C\frac{\phi}{\tau}\mathcal{E}^{3/2} \tag{2.230}$$

and if the initial data is sufficiently small in $H^s\times H^{s-1}$, then $\mathcal{E}(T)\lesssim e^{-2(n-1)(T-T_0)}$ as $T\to\infty$. Therefore, we will focus on perturbations (u,v), satisfying $< v|\mathcal{L}_{g,\gamma}^{i-1}v >_{L^2}> 0\ \forall 1\leq i\leq s$. Note an extremely important fact that the possible feedback term (which might lead to energy growth) $||v||_{H^{s-1}}\mathcal{E}$ is multiplied by a factor of $\frac{\phi}{\tau}$. Noting that $\frac{\phi}{\tau}\sim e^{-T}$ as $T\to\infty$, one immediate guess would be that the energy at least remains bounded if the initial energy is chosen to be sufficiently small. Now the time derivative of the energy may be further reduced to the following form

$$\frac{\partial \mathcal{E}}{\partial T} \leq -\left((n-1)-C\sqrt{\mathcal{E}}\right)\sum_{i=1}^{s}< v|\mathcal{L}_{g,\gamma}^{i-1}v>_{L^2} \tag{2.231}$$

$$+C\frac{\phi}{\tau}||v||_{H^{s-1}}\mathcal{E}. \tag{2.232}$$

Now, if the energy is sufficiently small (i.e., $(u,v)\in\mathcal{B}_\delta(0,0)\in H^s\times H^{s-1}$ such that $\mathcal{E}<(\frac{n-1}{2C})^2$ for example), then the first term contributes a negative factor and for such small data limit, the linear terms $||v||_{H^{s-1}}$ are absorbed in the constant and the energy satisfies

$$\frac{\partial \mathcal{E}}{\partial T} \leq C\frac{\phi}{\tau}\mathcal{E}, \tag{2.233}$$

integration of which yields

$$\mathcal{E}(T) \leq \mathcal{E}(T_0) e^{C \ln \left| \frac{e^{-T_0}+\sqrt{e^{-2T_0}+\frac{2n\Lambda}{n-1}}}{e^{-T}+\sqrt{e^{-2T}+\frac{2n\Lambda}{n-1}}} \right|} = \mathcal{E}(T_0) \left(\frac{e^{-T_0}+\sqrt{e^{-2T_0}+\frac{2n\Lambda}{n-1}}}{e^{-T}+\sqrt{e^{-2T}+\frac{2n\Lambda}{n-1}}} \right)^C. \tag{2.234}$$

Therefore, we have the boundedness of energy at the limit $T \to \infty$ ($\lim_{T\to\infty}(e^{-T}+\sqrt{e^{-2T}+\frac{2n\Lambda}{n-1}}) = \sqrt{\frac{2n\Lambda}{n-1}}$)

$$\lim_{T\to\infty} \mathcal{E}(T) \leq C^{'}\mathcal{E}(T_0). \tag{2.235}$$

In order to close the argument, the following must be satisfied

$$\sqrt{\mathcal{E}(T)} < \frac{n-1}{2C} \tag{2.236}$$

to ensure that the first term in the inequality (2.231) contributes a negative term for all time (which we assumed at the beginning). We enforce the following condition which will impose necessary smallness condition on the initial energy $\mathcal{E}(T_0)$

$$\lim_{T\to\infty} \mathcal{E}(T) \leq C^{'}\mathcal{E}(T_0) < \frac{(n-1)^2}{4C^2}. \tag{2.237}$$

It yields the following smallness of the initial data

$$\mathcal{E}(T_0) < \frac{(n-1)^2}{4C^2C'}. \tag{2.238}$$

Therefore, by ensuring that the initial data is small enough such that (2.237) holds, we ensure that the first term in the differential inequality (2.231) always contributes a negative term. Therefore, we close the argument and obtain that the suitable norm of the data remains bounded by the initial data as $T \to \infty$ if the initial data is chosen small enough i.e.,

$$\mathcal{E}(T) \lesssim \mathcal{E}(T_0), \tag{2.239}$$

or

$$||u(T)||^2_{H^s} + ||v(T)||^2_{H^{s-1}} \lesssim ||u(T_0)||^2_{H^s} + ||v(T_0)||^2_{H^{s-1}}. \tag{2.240}$$

Now, since the time interval of existence depends on the size of $||u||_{H^s}$ and $||v||_{H^{s-1}}$ from the local existence theorem, we obtain the global existence.

Even though we proved the boundedness of the suitable norm of the dynamical variables yielding global existence, this is not satisfactory. We want to prove the attractor property of the Einstein-Λ flow and as such we need to establish decay property of suitable norms. [206] showed the attractor property of the Einstein flow with $\Lambda = 0$. Now, with $\Lambda > 0$ included, one would naturally expect that the accelerated expansion should kill away the perturbations and drive the flow towards the centre manifold (extended center manifold to be precise). However, we only seem to obtain a boundedness of the energy without any decay. This is mainly due to the fact that we have underestimated the large damping term $-(n-1)\sum_{i=1}^{s} < v|\mathcal{L}^{i-1}_{g,\gamma} v >_{L^2}$. Using an iterated scheme, we will now achieve the sharp decay which does agree with the linear analysis as $T \to \infty$ (as it should).

The following analysis holds in the limit $T \to \infty$. We will simply compute the time derivative of $\mathcal{E}_v := \sum_{i=1}^{s-1} < v|\mathcal{L}^{i-1}_{g,\gamma} v >_{L^2}, s > \frac{n}{2} + 2$

$$\begin{aligned} \frac{d\mathcal{E}_v}{dT} &\leq -(n-1)\sum_{i=1}^{s-1} < v|\mathcal{L}^{i-1}_{g,\gamma} v >_{L^2} + C||v||^4_{H^{s-2}} \\ &+\frac{\phi}{\tau}(||v||^3_{H^{s-2}} + ||u||_{H^{s-1}}||v||^3_{H^{s-2}} + ||u||_{H^s}||v||_{H^{s-2}}). \end{aligned} \tag{2.241}$$

Note an important fact that we loose one degree of regularity because, in the computation of the time derivative of $\mathcal{E}_v := \frac{1}{2}\sum_{i=1}^{s} < v|\mathcal{L}^{i-1}_{g,\gamma} v >_{L^2}$ alone, the dangerous term $\sum_{i=1}^{s} < \mathcal{L}_{g,\gamma} u|\mathcal{L}^{i-1}_{g,\gamma} v >_{L^2}$ does not get cancelled unlike the case of the time derivative of the total energy $\mathcal{E}$. Therefore, we loose one order of regularity. However, since we have $s > \frac{n}{2} + 2$, we will still be able to obtain the desired pointwise decay estimate. The previous inequality may also be expressed as follows

$$\begin{aligned} \frac{d\mathcal{E}_v}{dT} &\leq -((n-1) - C||v||^2_{H^{s-2}})\sum_{i=1}^{s-1} < v|\mathcal{L}^{i-1}_{g,\gamma} v >_{L^2} \\ &+C\frac{\phi}{\tau}(||v||^3_{H^{s-2}} + ||u||_{H^{s-1}}||v||^3_{H^{s-2}} + ||u||_{H^s}||v||_{H^{s-2}}) \end{aligned} \tag{2.242}$$

Recalling $||v||^2_{H^{s-2}} \lesssim \delta^2 < \frac{n-1}{2C}$ and $||u||_{H^s} \lesssim \delta$, we obtain

$$\frac{d\mathcal{E}_v}{dT} \leq -\frac{n-1}{2}\sum_{i=1}^{s-1} < v|\mathcal{L}^{i-1}_{g,\gamma}v >_{L^2} + C\frac{\phi}{\tau}||v||^2_{H^{s-2}}(1+||u||_{H^{s-1}}) + C\frac{\phi}{\tau}||u||_{H^s}||v||_{H^{s-2}} \tag{2.243}$$

$$= -(n-1)\mathcal{E}_v + C\frac{\phi}{\tau}||v||^2_{H^{s-2}}(1+||u||_{H^{s-1}}) + C\frac{\phi}{\tau}||u||_{H^s}||v||_{H^{s-2}}$$

integration of which yields ($n \geq 3$)

$$\mathcal{E}_v(T) \lesssim e^{-(n-1)(T-T_0)}\mathcal{E}(T_0) + Ce^{-T}, \tag{2.244}$$

i.e.,

$$\mathcal{E}_v \lesssim e^{-T} \; or \; ||v||_{H^{s-2}} \lesssim e^{-T/2}. \tag{2.245}$$

Using $s > \frac{n}{2} + 2$, we obtain applying Sobolev embedding on compact domain

$$||v||_{L^\infty} \lesssim e^{-T/2}. \tag{2.246}$$

Now, note that this decay is not optimal. An iteration scheme would yield a better decay rate. We go back to the expression of $\frac{d\mathcal{E}_v}{dT}$ and observe

$$\frac{d\mathcal{E}_v}{dT} \leq -(n-1)\sum_{i=1}^{s-1} < v|\mathcal{L}^{i-1}_{g,\gamma}v >_{L^2} + C||v||^4_{H^{s-2}} + C\frac{\phi}{\tau}(||v||^3_{H^{s-2}} + ||u||_{H^{s-1}}||v||^3_{H^{s-2}} + ||u||_{H^s}||v||_{H^{s-2}}) \tag{2.247}$$

to obtain

$$\frac{d\mathcal{E}_v}{dT} \leq -2(n-1)\mathcal{E}_v + Ce^{-(1+1/2)T} + Ce^{-2T} \tag{2.248}$$

since $||v||^4_{H^{s-2}} \lesssim e^{-2T}$, $\frac{\phi}{\tau}(||v||^3_{H^{s-2}} + ||u||_{H^{s-1}}||v||^3_{H^{s-2}} + ||u||_{H^s}||v||_{H^{s-2}}) \lesssim e^{-(1+1/2)T/2}$ using the previous estimate. Therefore, the previous differential inequality yields

$$\mathcal{E}_v \lesssim e^{-(1+1/2)T} \; or \; ||v||_{H^{s-2}} \lesssim e^{-\frac{(1+1/2)T}{2}}. \tag{2.249}$$

Notice that we have gained a factor of 1/4 in the decay estimate. Using this estimate, in the next iteration, we obtain

$$\frac{d\mathcal{E}_v}{dT} \leq -2(n-1)\mathcal{E}_v + Ce^{-\{1+1/2(1+1/2)\}T} \tag{2.250}$$

and subsequently

$$\mathcal{E}_v \lesssim e^{-\{1+1/2(1+1/2)\}T}, \ or \ ||v||_{H^{s-2}} \lesssim e^{-\frac{1}{2}\{1+1/2(1+1/2)\}T}. \tag{2.251}$$

If we continue to iterate, we obtain the decay rate to be the sum of an infinite series, that is, the final decay rate is computed to be

$$1+\frac{1}{2}(1+\frac{1}{2}(1+\frac{1}{2}(1+\frac{1}{2}(....... = \sum_{k=0}^{\infty}\frac{1}{2^k} = \frac{1}{1-1/2} = 2. \tag{2.252}$$

Therefore, the final estimate reads

$$\mathcal{E}_v \lesssim e^{-2T} \ or \ ||v||_{H^{s-2}} \lesssim e^{-T}. \tag{2.253}$$

Note that this is optimal in a sense that substituting this decay back into the equation (2.247) returns the same decay. Now we compute the time derivative of $\mathcal{E}_s := \frac{n^2}{2}\sum_{i=1}^{s-1} < u|\mathcal{L}^i_{g,\gamma}u >_{L^2}$ $+\frac{1}{2}\sum_{i=1}^{s-1} < v|\mathcal{L}^{i-1}_{g,\gamma}v >_{L^2}$ which yields

$$\begin{aligned}\frac{d\mathcal{E}_s}{dT} \leq -(n-1)\sum_{i=1}^{s-1} < v|\mathcal{L}^{i-1}_{g,\gamma}v >_{L^2} &+C(||u||_{H^{s-1}} + ||u||^2_{H^{s-1}} \\ &+||v||^2_{H^{s-2}})||v||^2_{H^{s-2}} + C\frac{\phi}{\tau}||v||_{H^{s-2}}\mathcal{E}_s.\end{aligned} \tag{2.254}$$

Utilizing the estimate $||v||_{s-2} \lesssim e^{-T}$ and $||u||_{H^{s-1}} < \delta$, we observe that every term decays exponentially as $T \to \infty$ i.e., $C(||u||_{H^{s-1}} + ||u||^2_{H^{s-1}} + ||v||^2_{H^{s-2}})||v||^2_{H^{s-2}} \lesssim e^{-2T}$, $\frac{\phi}{\tau}||v||_{H^{s-2}}\mathcal{E}_s \lesssim e^{-2T}$ and $\sum_{i=1}^{s-1} < v|\mathcal{L}^{i-1}_{g,\gamma}v >_{L^2} \lesssim e^{-2T}$ yielding

$$\frac{d\mathcal{E}_s}{dT} \lesssim e^{-2T}, \tag{2.255}$$

i.e., $\frac{d\mathcal{E}_s}{dT}$ decays as$T \to \infty$. In fact, we may show that $\mathcal{E}_s$ actually decays as $T \to \infty$. Using the decay estimate $||v||_{H^s} \lesssim e^{-T}$ and noting the order of each term

$$\begin{aligned}\frac{d\mathcal{E}_s}{dT} &\leq -(n-1)\delta^2 e^{-2T} + C\delta^3 e^{-2T} + C\delta^3 e^{-2T} \\ &= -\left((n-1) - 2C\delta\right)\delta^2 e^{-2T}.\end{aligned} \tag{2.256}$$

Now of course there exists a $\delta > 0$ such that $\delta < \frac{n-1}{2C}$ yielding

$$\frac{d\mathcal{E}_s}{dT} < 0 \tag{2.257}$$

and $\frac{d\mathcal{E}_s}{dT} = 0$ if and only if $||v||_{H^{s-2}} \equiv 0$ since every term in the right hand side is multiplied by a factor of $||v||_{H^{s-2}}$. This indicates that the re-scaled metric converges to a limit metric. Now, we go back to the evolution equation of the metric since, we have not yet established a sharp decay of the metric. The evolution equation for the metric reads

$$\partial_T g_{ij} = \frac{2\phi(\tau)}{\tau} N K^{TT}_{ij} - 2(1 - \frac{N}{n})g_{ij} - \frac{\phi(\tau)}{\tau}(L_X g)_{ij}. \tag{2.258}$$

Now, we have estimated $||v||_{H^{s-2}} = 2n||K^{TT}||_{H^{s-2}} \lesssim e^{-T}$ which yields through the elliptic estimate (lemma 6 and 7)

$$(||\frac{N}{n} - 1)||_{H^s} \lesssim e^{-2T}, ||X||_{H^s} \lesssim e^{-T}. \tag{2.259}$$

Therefore, utilizing these estimates ($s > \frac{n}{2} + 2$) we obtain

$$||\partial_T g_{ij}||_{H^{s-2}} \lesssim e^{-2T} \ or \ ||\partial_T g_{ij}||_{L^\infty} \lesssim e^{-2T} \tag{2.260}$$

which implies that the re-scaled metric g decays to a limit metric γ i.e., utilizing the evolution equation $||g - \gamma^{\dagger}||_{H^{s-1}} \lesssim e^{-2T}$ or equivalently $||g - \gamma^{\dagger}||_{L^\infty} \lesssim e^{-2T}$ from Sobolev embedding on compact domain. The question that remains is what are these limit metrics? Invoking the Hamiltonian constraint

$$R + \frac{n-1}{n} = |K^{TT}|^2, \tag{2.261}$$

one obtains

$$\lim_{T\to\infty}(R+\frac{n-1}{n})=\lim_{T\to\infty}|K^{TT}|^2=\lim_{T\to\infty}e^{-2T}=0. \tag{2.262}$$

This precisely implies that the re-scaled metric g_{ij} converges to an element of the space $\mathcal{M}^{\epsilon}_{-\frac{n-1}{n}}\cap\mathcal{S}_{\gamma}$ i.e., the extended centre manifold. The decay estimate is as follows

$$||g-\gamma^{\dagger}||_{H^{s-1}}\lesssim e^{-2T}, \tag{2.263}$$

where $\gamma^{\dagger}\in\mathcal{M}^{\epsilon}_{-\frac{n-1}{n}}\cap\mathcal{S}_{\gamma}$. In summary, the metric g along with (K^{TT},X,N) satisfies

$$\begin{aligned}\lim_{T\to\infty}||\frac{\partial g_{ij}}{\partial T}||_{L^\infty}=0, \lim_{T\to\infty}||K^{TT}||_{L^\infty}=0,\\ \lim_{T\to\infty}||N||_{L^\infty}=n, \lim_{T\to\infty}||X||_{L^\infty}=0, \lim_{T\to\infty}R[g]=-\frac{n-1}{n},\end{aligned} \tag{2.264}$$

that is

$$\lim_{T\to\infty}g(T)=\gamma^{\dagger}\ (\textit{in }H^{s-1}\textit{ topology}) \tag{2.265}$$

with $\gamma^{\dagger}\in\mathcal{M}^{\epsilon}_{-\frac{n-1}{n}}\cap\mathcal{S}_{\gamma}$. This implies that metrics lying in $\mathcal{M}^{\epsilon}_{-\frac{n-1}{n}}\cap\mathcal{S}_{\gamma}$ simply evolves to another point in $\mathcal{M}^{\epsilon}_{-\frac{n-1}{n}}\cap\mathcal{S}_{\gamma}$ and metrics lying sufficiently close to $\mathcal{N}$ yet not in $\mathcal{M}^{\epsilon}_{-\frac{n-1}{n}}\cap\mathcal{S}_{\gamma}$ exponentially converges to $\mathcal{M}^{\epsilon}_{-\frac{n-1}{n}}\cap\mathcal{S}_{\gamma}$ in infinite time. This completes the proof of the attractor property of the centre manifold (extended to be precise) under the Einstein flow. Theorem 5 formally summarizes the result.

Once we have obtained the estimates for the perturbations to the primary dynamical variables (u,v), the estimates for the lapse function and the shift vector field follow from lemma (6) and (7), respectively

$$||\frac{N}{n}-1||_{H^s}\lesssim e^{-2T}, \tag{2.266}$$

$$||X||_{H^s}\lesssim e^{-T}. \tag{2.267}$$

and thus

$$||\frac{N}{n}-1||_{L^\infty}\lesssim e^{-2T}, \tag{2.268}$$

$$||X||_{L^\infty}\lesssim e^{-T} \tag{2.269}$$

following Sobolev embedding theorem on compact domain with $s > \frac{n}{2}+2$. We proved earlier that $h^{TT||} = \frac{\phi}{\tau}v^{||}$ up to a second order correction and $Y^{||}$ is estimated by lemma (8). Therefore, the following holds for $h^{TT||}$ and $Y^{||}$

$$||h^{TT}||_{H^{s-1}} \lesssim e^{-2T}, \ or \ ||h^{TT}||_{L^\infty} \lesssim e^{-2T} \tag{2.270}$$

$$||Y^{||}||_{H^s} \lesssim e^{-2T} \ or \ ||Y^{||}||_{L^\infty} \lesssim e^{-2T}. \tag{2.271}$$

Therefore, we summarize the following decay estimates for the perturbations L^2 orthogonal to the deformation space $\mathcal{N}$

$$\begin{aligned} ||\frac{\partial g}{\partial T}||_{L^\infty} \lesssim e^{-2T}, ||g-\gamma^\dagger||_{L^\infty} \lesssim e^{-2T}, ||K^{TT}||_{L^\infty} \lesssim e^{-T}, \\ ||\frac{N}{n}-1||_{L^\infty} \lesssim e^{-2T}, ||X||_{L^\infty} \lesssim e^{-T}, ||h^{TT}||_{L^\infty} \lesssim e^{-2T}, ||Y^{||}||_{L^\infty} \lesssim e^{-2T}. \end{aligned} \tag{2.272}$$

For purely tangential perturbations (tangential to the deformation space $\mathcal{N}$), the following decay estimates hold using (2.230), the elliptic estimates (lemma 6, 7, and 8), and the evolution equations

$$\begin{aligned} ||\frac{\partial g}{\partial T}||_{L^\infty} \lesssim e^{-nT}, ||g-\gamma^\dagger||_{L^\infty} \lesssim e^{-nT}, ||K^{TT}||_{L^\infty} \lesssim e^{-(n-1)T}, \\ ||\frac{N}{n}-1||_{L^\infty} \lesssim e^{-2(n-1)T}, ||X||_{L^\infty} \lesssim e^{-(n-1)T}, ||h^{TT}||_{L^\infty} \lesssim e^{-nT}, \\ ||Y^{||}||_{L^\infty} \lesssim e^{-2nT}. \end{aligned} \tag{2.273}$$

Note an important fact that the asymptotic decay estimates of $\frac{\partial g}{\partial T}$, $(g-\gamma^\dagger)$, and K^{TT} match with the linear decay estimates as expected. Utilizing these asymptotic decay estimates for the relevant fields, we therefore obtain the following theorem regarding the attractor property of the Einstein-Λ flow

Theorem 5: *Let $(g_0, K_0^{TT}) \in B_\delta(\gamma_0, 0) \subset H^{s-1} \times H^{s-2}, s > \frac{n}{2}+2$ with $\gamma_0 \in \mathcal{N}$ and assume the triple $(\gamma_0, g_0, K_0^{TT})$ satisfies the shadow gauge condition. The Newtonian like time $-\infty < T < \infty$ is defined as the solution of the equation $\partial_T = -\frac{\tau^2 - \frac{2n\Lambda}{n-1}}{\tau}\partial_\tau$, with $\tau \in (-\infty, -\sqrt{\frac{2n\Lambda}{n-1}})$ and $\Lambda > 0$ being the mean extrinsic curvature (constant) of the Cauchy hypersurface M and the cosmological constant, respectively. Let $T \mapsto (\gamma(T), g(T), K^{TT}(T))$ be the maximal development of the Cauchy problem for the system (2.35, 2.36, 2.37, 2.38, 2.39, 2.43,2.44) with shadow gauge condition imposed and initial data $(\gamma_0, g_0, K_0^{TT})$. Then there exists a $\gamma^* \in \mathcal{N}$ and a $\gamma^\dagger \in \mathcal{M}^\epsilon_{-\frac{n-1}{n}} \cap \mathcal{S}_\gamma$ such that the* *triple* (γ, g, K^{TT})*) flows toward* $(\gamma^*, \gamma^\dagger, 0)$ *in the limit of infinite time that is*

$$\lim_{T\to\infty}(\gamma(T), g(T), K^{TT}(T)) = (\gamma^*, \gamma^\dagger, 0). \tag{2.274}$$

and moreover either $\gamma^\dagger = \gamma^*$ *or* $\mathcal{P}\gamma^\dagger = \gamma^*$*. Here* $\mathcal{P}$ *is the projection operator defined in (2.107).*

In the limit of infinite time (infinite expansion of the physical metric), the complete solution satisfies

$$\lim_{T\to\infty}(\gamma(T), g(T), K^{TT}(T), N(T), X(T)) = (\gamma^*, \gamma^\dagger, 0, n, 0). \tag{2.275}$$

In order to establish the future completeness of the spacetime, we need to show that the length of a timelike geodesic goes to infinity. In other words, the solution of the geodesic equation must exist for an infinite interval of the affine parameter. Let's designate the timelike geodesic by $\mathcal{C}$. The tangent vector $\alpha = \frac{d\mathcal{C}}{d\lambda} = \alpha^\mu \partial_\mu$ to $\mathcal{C}$ for the affine parameter λ satisfies $\hat{g}(\alpha, \alpha) = -1$, where $\hat{g}$ is the spacetime metric. As $\mathcal{C}$ is causal, we may parametrize it as $(T, \mathcal{C}^i)$, $i = 1, 2, 3$. We must show that $\lim_{T\to\infty}\lambda(T) = +\infty$, that is,

$$\lim_{T\to\infty}\int_{T_0}^{T}\frac{d\lambda}{dT'}dT^{'} = +\infty. \tag{2.276}$$

Noting that $\alpha^0 = \frac{dT}{d\lambda}$, we must show

$$\lim_{T\to\infty}\int_{T_0}^{T}\frac{1}{\alpha^0}dT^{'} = +\infty. \tag{2.277}$$

We follow the method of [151] to achieve this. Showing that $|\bar{N}\alpha^0|$ is bounded and therefore $\lim_{T\to\infty}\int_{T_0}^{T}\bar{N}dT^{'} = +\infty$ is enough to ensure the geodesic completeness. We first show that $|\bar{N}\alpha^0|$ is bounded. Let's consider a co-vector field $Z_\mu = \bar{N}\delta_\mu^0$ in local coordinates using the $n+1$ decomposition, where δ_μ^ν is the Kronecker delta. This shows that α may be expressed as

$$\alpha = \bar{N}\alpha^0 Z + W, \tag{2.278}$$

where $W \in \mathfrak{X}(M)$. Noting $\hat{g}(\alpha, \alpha) = -1$, we obtain

$$|W|_{\bar{g}}^2 = \bar{N}^2(\alpha^0)^2 - 1. \tag{2.279}$$

Here $\tilde{g}$ is the induced Riemannian metric on M. Clearly we have

$$|W|^2_{\tilde{g}} < \bar{N}^2(\alpha^0)^2. \tag{2.280}$$

Let us compute the entity $\frac{d(\bar{N}^2(\alpha^0)^2)}{dT}$ as follows

$$\frac{d(\bar{N}^2(\alpha^0)^2)}{dT} = \frac{d}{dT}(\hat{g}(\alpha, Z))^2 = \frac{2}{\alpha^0}\hat{g}(\alpha, Z)\hat{g}(\alpha, \nabla[\hat{g}]_\alpha Z), \tag{2.281}$$

where, we have used the fact that for $\mathcal{C}$ being a geodesic, $\nabla[\hat{g}]_\alpha \alpha = 0$ and appealed to the Koszul formula for the derivative. Using $\hat{g}(\alpha, Z) = -\bar{N}\alpha^0$ and writing the covariant derivative of the spacetime metric as a direct sum of its projection onto the tangent space of M and the second fundamental form of M, we obtain

$$\frac{d(\bar{N}^2(\alpha^0)^2)}{dT} = -2\bar{N}(\alpha^0\nabla_W\bar{N} - \tilde{K}_{ij}W^iW^j). \tag{2.282}$$

Decomposing $\tilde{K} = \tilde{K}^{TT} + \frac{\tau}{n}\tilde{g}$ and noting that $\tau < 0$, we obtain

$$|\frac{d}{dT}(\ln(\bar{N}^2(\alpha^0)^2))| \leq ||\nabla\bar{N}||_{L^\infty;\tilde{g}} + ||\bar{N}\tilde{K}^{TT}||_{L^\infty;\tilde{g}}. \tag{2.283}$$

Now, in the time coordinate $\frac{d\tau}{dt} = 1$ and $\frac{-\phi^2}{\tau}\frac{d}{d\tau} = \frac{d}{dT}$, the spacetime metric reads

$$\hat{g} = -\tilde{N}^2 dt \otimes dt + \tilde{g}_{ij}(dx^i + \tilde{X}^i dt) \otimes (dx^j + \tilde{X}^j dt) \tag{2.284}$$

$$= -\frac{\tilde{N}^2\phi^4}{\tau^2}dT \otimes dT + \tilde{g}_{ij}(dx^i - \frac{X^i\phi^2}{\tau}dT)(dx^j - \frac{X^j\phi^2}{\tau}dT), \tag{2.285}$$

and therefore $\bar{N} = \frac{\tilde{N}^2\phi^4}{\tau^2}$. Now, we utilize the estimate obtained for the lapse function and the transverse-traceless second fundamental form. Note that these fields are not dimensionless and therefore we need to multiply them with suitable powers of $\frac{1}{\phi} \sim e^T$(2.34) to extract the dimensionless part. We obtain, $\bar{N} = \frac{\tilde{N}\phi^2}{|\tau|} = \frac{N}{|\tau|}$ and $\tilde{K}^{TT} = \frac{1}{\phi}K^{TT}$, where N and K^{TT} are dimensionless. Utilizing the estimates $||\frac{N}{n} - 1||_{H^s} \lesssim e^{-2T}$ and $||K^{TT}||_{H^{s-2}} \lesssim e^{-T}$ (for $s > \frac{n}{2} + 2$, from Sobolev embedding on a compact domain, bounded H^s (resp. H^{s-2}) norm of N (resp. K^{TT}) implies bounded L^∞ norm, $||\nabla\bar{N}||_{L^\infty;\tilde{g}} := \sup_M \sqrt{\tilde{g}^{ij}\nabla_i\bar{N}\nabla_j\bar{N}}$, $||K^{TT}||_{L^\infty;\tilde{g}} := \sup_M \sqrt{\tilde{g}^{ij}\tilde{g}^{kl}K^{TT}_{ik}K^{TT}_{jl}}$), we observe that the following holds

$$|\frac{d}{dT}(\ln(\bar{N}^2(\alpha^0)^2))| \lesssim e^{-2T} \tag{2.286}$$

as $T \to \infty$ and therefore $\bar{N}^2(\alpha^0)^2$ is bounded, i.e.,

$$\bar{N}^2(\alpha^0)^2(T) \leq C \tag{2.287}$$

for some $C < \infty$. Therefore, we need to show that $\lim_{T\to\infty}\int_{T_0}^{T}\bar{N}dT^{'} = +\infty$ in order to finish the proof of timelike geodesic completeness. Once again using $\bar{N} = \frac{N}{|\tau|}$, the estimate (2.49)

$$0 < \frac{1}{\sup(|K^{TT}|^2) + \frac{1}{n}} \leq N \leq n, \tag{2.288}$$

and $||K^{TT}||_{H^{s-1}} \lesssim e^{-T}$ as $T \to \infty$, we clearly see that N is bounded from below by a strictly positive number and therefore

$$\lim_{T\to\infty}\int_{T_0}^{T}\bar{N}dT^{'} = +\infty. \tag{2.289}$$

Therefore, the solution of the geodesic equation must exist for a semi-infinite interval of the affine parameter. This completes the proof timelike geodesic completeness. The case of null geodesics can be handled exactly the same way.

This proves the future completeness of this family of spacetimes. Previous analysis together with the attractor property stated in theorem 5 yields the following global existence theorem.

Theorem 6: *Let $\mathcal{N}$ be the integrable deformation space of γ_0. Then $\exists\delta > 0$ such that for any $(g(T_0), K^{TT}(T_0)) \in B_\delta(\gamma_0, 0) \subset H^s \times H^{s-1}$, with the triple $(\gamma_0, g(T_0), K^{TT}(T_0))$ satisfying the shadow gauge condition, the Cauchy problem for the re-scaled Einstein-Λ system with constant mean extrinsic curvature (CMC) and spatial harmonic (SH) gauge is globally well posed to the future and the spacetime is future complete.*

One might recall for $\Lambda = 0$ case that in order to obtain a sharp decay of the energy, [206] added a correction term to the ordinary wave equation type energy. One could naturally ask whether introduction of such a correction term is necessary, that is, can one just initially bound the energy and later use the iteration scheme to yield the improved decay. However, there is a major difference between the $\Lambda = 0$ and $\Lambda > 0$ case. In $\Lambda > 0$ case, we have the 'good' term $\frac{\phi}{\tau}$ (which decays as e^{-T} as $T \to \infty$) which plays an extremely important role in obtaining the decay estimate. Roughly, one sees in the second iteration from equation (2.247) that given the boundedness of $||u||_{H^{s-1}}$, and $||v||_{H^{s-2}} \lesssim e^{-T/2}$, the terms multiplied by $\frac{\phi}{\tau}$ adds an extra factor of 1 in the decay estimate of $||v||^2_{H^{s-2}}$. In case of $\Lambda = 0$, the absence of $\frac{\phi}{\tau}$ would lead to a circular argument in the first step and prohibit one to obtain a decay estimate. Therefore, introduction of a corrected energy becomes essential. Physically, this behaviour is expected since addition of a positive cosmological constant yields an accelerated expansion which is expected to destroy the perturbations. Sufficiently small data and the positivity of the spectrum of the operator $\mathcal{L}_{g,\gamma}$ are sufficient to establish the asymptotic stability.

2.7 Concluding remarks

We have proved a global existence theorem for sufficiently small however fully nonlinear perturbations of a family of background solutions (conformal spacetimes) derived from Einstein's equations in the presence of a positive cosmological constant. However, our current result assumes the initial data to be within a small neighbourhood (in the proper function space setting) of the background solutions, and as such is very far from stating a global result for arbitrarily large data. Nevertheless, several interesting physically relevant features are revealed through the analysis. Firstly, consider the case of $n = 3$ i.e., the physical spacetimes. Following Mostow rigidity, the Einstein moduli space consists of a single point that corresponds to the hyperbolic geometry. Therefore, the background spacetime is essentially foliated by compact hyperbolic manifolds which are locally homogeneous and isotropic thereby conforming to the cosmological principle (astronomical observations that motivate the cosmological principle are local). Notice that if we start with an inhomogeneous anisotropic initial spatial metric sufficiently close to the background (in suitable function space settings), this metric does not only remain within a bounded neighbourhood of the background, it actually approaches the space of metrics with constant scalar curvature sufficiently close and containing the Einstein structure. In addition, recent astronomical observations support the claim that the spatial slice of the physical universe is indeed negatively curved (slightly). This notion together with our result opens up the possibility of a rather exotic spatial topology of the universe (hyperbolic 3-manifolds are topologically rich). Of course, our result can only provide an indication of such a claim being true. A complete analysis would entail inclusion of suitable matter sources on the one hand and treating arbitrarily large data perturbations on the other. While [166] proved the non-linear stability of the small perturbations to the FLRW background solutions in the presence of irrotational perfect fluid and a positive cosmological constant on $T^3 \times R$ (and therefore flat spatial topology), such non-linear stability of spacetimes foliated by compact hyperbolic manifolds (which is of physical interest) is still open and the flat model is not likely a viable candidate for the physical universe. A linear stability of spacetimes foliated by compact hyperbolic spatial slices in the presence of a perfect fluid and $\Lambda > 0$ (compact variants of the $k = -1$ FLRW model) is under preparation by the current author. [165] has recently studied the asymptotic behavior of a universe filled with matter sources satisfying suitable energy conditions and a positive cosmological constant based on a monotonic decay property of a suitably constructed Lyapunov function. While such a Lyapunov function can treat arbitrarily large data, it can only control the lowest order norm ($H^1 \times L^2$) of the data and therefore, is unable to state a result regarding global existence (at the physically interesting classical solutions we are interested). In a sense, any definite result regarding the evolution of the physical universe requires global existence (or blow up in finite time) for large data perturbations.

This question of global existence is far from obvious and an extremely important (and difficult) open problem in classical general relativity. Global existence is known to be violated for some known examples of spacetime via formation of black holes. These include Schwarzschild, Reissner-Nordström, Kerr spacetimes, where a true curvature singularity occurs within the event horizon of the black hole. Even in the vacuum case, pure gravity could 'blow up' (curvature concentration) i.e., gravitational singularities could prevent global existence or the spacetime could simply lose the global hyperbolicity through formation of Cauchy horizons (Taub-NUT spacetimes for example). Of course, such issues lead to the fundamental question of the **Cosmic Censorship** conjecture [111], which still remains open. Available results related to the global existence address rather special cases such as spacetimes with non-trivial symmetry groups [112, 114, 191] (and which therefore are not generic) or where a certain smallness condition on the initial data is assumed [151, 206]. There is however a rather ambitious program under development by Moncrief to control the pointwise (L^∞ norm) behaviour of the spacetime curvature through the use of *light cone estimates* [117]. This method is recently applied to establish the global existence of Yang-Mills and Klein-Gordon fields in curved spacetimes by the current author and Moncrief. Application of this light cone estimates to establish the small data global existence for certain background solutions is currently under investigation by the current author.

An interesting question which arises through our result is what role can these non-trivial exotic topologies ($H^3/\Gamma,\ \Gamma \in SO^+(3,1)$ proper, discrete, and torsion free) play in answering the question of global existence or finite time blow up. In a sense, can the topological properties of these interesting manifolds have any control on the fundamental question of large data global existence (or finite time blow up)? Can the interrelation between the dynamics and topology provide crucial information to handle the issue of global existence or finite time blow up, at least in $n = 3$ dimensions?

Chapter 3

The linear stability of the n+1 dimensional FLRW spacetimes

Here we prove the linear stability of a family of '$n+1$'-dimensional cosmological models of general relativity locally isometric to the Friedmann Lemaître Robertson Walker (FLRW) spacetimes including a positive cosmological constant. We show that the solutions to the linearized Einstein-Euler field equations around a class of FLRW metrics with compact spatial topology (negative Einstein spaces and in particular hyperbolic for $n = 3$) arising from regular initial data remain uniformly bounded and decay to a family of metrics with constant negative spatial scalar curvature. To accomplish the result, we express the Einstein-Euler system in constant mean extrinsic curvature spatial harmonic gauge and linearize about the chosen FLRW background. Utilizing a Hodge decomposition of the fluid's $n-$velocity 1-form, the linearized system becomes elliptic-hyperbolic (and non-autonomous) in the CMCSH gauge facilitating an application of an energy type argument. Utilizing the estimates derived from the associated elliptic equations, we first prove the uniform boundedness of a energy functional (controlling an appropriate norm of the data) in the expanding direction. Utilizing the uniform boundedness, we later obtain a sharp decay estimate which suggests accelerated expansion (for $\Lambda > 0$) of this particular universe model may be sufficient to control the non-linearities (including possible shock formation) of the Einstein-Euler system in a potential future proof of the fully non-linear stability. In addition, the rotational and harmonic parts of the fluid's $n-$velocity field only couple to the remaining degrees of freedom in higher orders, which once again indicates a straightforward extension of current analysis to the fully non-linear setting in the sufficiently small data limit. In addition, our results require a certain integrability condition on the expansion factor and a suitable range of the adiabatic index γ_{ad} ($(1, \frac{n+1}{n})$ i.e., $(1, \frac{4}{3})$ in the physically relevant '$3+1$' universe) if the barotropic equation of state $p = (\gamma_{ad} - 1)\rho$ is chosen.

Recent astronomical observations [119, 120] indicate a possibility that the spatial universe may have slightly negative curvature. This leads to the consideration of the well-known FLRW spacetime with negative spatial curvature as a possible cosmological model. Of course, the spatial manifold of this model is $\mathbb{H}^3$and it satisfies the global homogeneity and isotropy criteria as dictated by the cosmological principle. However, the astronomical observations that motivate the cosmological principle are necessarily limited to a fraction (possibly small) of the entire universe and such observations are compatible with spatial metrics being locally but not globally homogeneous and isotropic. Once the restriction on the topology (a global property) is removed, numerous closed manifolds may be constructed as the quotients of $\mathbf{H}^3$ by discrete, proper, and torsion-free subgroups of $SO^+(1,3)$, with each satisfying the local homogeneity and isotropy criteria but no longer being globally homogeneous or isotropic. In other words, the indication of the spatial universe to have a constant negative curvature opens up the possibility of quite 'exotic' topologies. In fact, there are infinitely many closed (compact without boundary) hyperbolic $3-$manifolds topologically distinct from each other. Each such manifold may serve as a possible candidate model for the spatial universe. These spatially compact model spacetimes may be written as the following warped product form in their '$n+1$' dimensional generalization

$$^{n+1}g = -dt \otimes dt + a(t)^2 \gamma_{ij} dx^i \otimes dx^j, \tag{3.1}$$

where $R[\gamma]_{ij} = -\frac{1}{n}\gamma_{ij}$, $t \in (0,\infty)$, $a(t)$ is the scale factor satisfying $\partial_t a > 0$ for this expanding universe model, and $R[\gamma]_{ij}$ is the Ricci tensor of the metric γ. Restricting to $n = 3$ yields the compact hyperbolic manifolds by Mostow rigidity [121]. However, an important question remains to be answered. Are the FLRW models (variants of FLRW to be precise since spatial manifolds of these models are not globally isotropic and homogeneous only locally so) predicted by general relativity? To conclude that they are may seem to hinge on proof that the purely theoretical FLRW models with compact hyperbolic spatial parts are dynamically stable. A natural first step towards such proof would be to establish stability at the level of linear perturbation theory.

The theory of cosmological perturbations and linearization stability is not new. A substantial amount of work has been done on the topic by several authors (e.g., [123, 124, 125, 126]) since the first study by [122]. Despite the tremendous amount of existing work in the literature, these are not the 'true' linear stability problem rather 'formal' mode stability results where the relevant fields are decomposed into modes via eigenfunction (of Laplacian) decomposition and the problem is subsequently reduced to ordinary differential equations in time. However, *the true problem of linear stability concerns general solutions to the linearized Einstein-Euler system about (3.1) arising from regular initial data, not simply the fixed modes.* Consider the following two linear stability statements [129]: (a) whether all solutions of the linearized Einstein-Euler system about [3.1] remain bounded for all times in terms of a suitable norm of their initial data, (b) the asymptotic stability i.e., whether all solutions to the linearized equations asymptotically decay. The mode analysis yields necessary but not sufficient conditions for either (a) or (b) of 'true' linear stability. In addition, the mode analysis does not work at the non-linear level due to 'all to all' coupling (of modes) and therefore a straightforward extension of the linearized mode analysis to study the fully non-linear problem (irrespective of the size of the data) becomes tremendously difficult if not impossible. In particular, we want to extend our analysis in the future to study the fully non-linear problem in our choice of gauge (CMCSH-to be described later) on spacetimes of topological type $M \times \mathbb{R}$ (where M is compact negative Einstein for $n > 3$ and in particular compact hyperbolic for $n = 3$).

In the regime of fully non-linear stability of Einstein's equations (vacuum or coupled to suitable matter sources), there has been considerable progress in the last three decades. Global nonlinear stability proof of de-Sitter spacetime by [147] marked the initiation of this progress. This was followed by the proof of the stability of the Minkowski space by [148]. Later [149] provided with a simplified proof of the same in spacetime harmonic or wave gauge. Andersson and Moncrief [151] proved the global existence of '$3+1$' dimensional vacuum Einstein's equations for sufficiently small however fully nonlinear perturbations of spacetimes of the type $\mathbb{R} \times M_t$ in constant mean curvature spatial harmonic (CMCSH) gauge, where M_t is a compact hyperbolic manifold. Following these fundamental studies, numerous studies have been performed regarding the small data global existence issues associated with Einstein's equations including various sources [153,155,156,157,159,160]. Two of the studies that will be most relevant to the current study are that by Rodnianski and Speck [166] and Speck [176]. Rodnianski and Speck [166] studied the small data perturbations of the $\mathbb{R} \times \mathbb{T}^3$ type FLRW model with a positive cosmological constant in spacetime harmonic gauge, where an irrotational perfect fluid model was assumed. Later Speck [176] extended this analysis on the same topology (and same gauge) to include non-zero vorticity utilizing suitably defined energy currents. Commonly, the perfect fluid matter model is considered to be 'bad' because of its finite time shock singularity formation property without even coupling to gravity (e.g., [154, 163]). One would surely not hope that while coupled to gravity, this property may be 'suppressed' by gravity since gravity itself may yield finite-time singularity. However, [158, 166, 176] have utilized the property that the presence of a positive cosmological constant induces an accelerated expansion of the physical universe, which avoids shocks in the regime of a small data limit (rapid expansion obstructs energy concentration by non-linearities). With an accelerated expansion, there are universally decaying 'good' terms in the evolution equations that dominate the nonlinearities and drive the universe towards a stable configuration. Recently, [183] proved a nonlinear stability result for Milne universe including dust fluid source. This model is devoid of a positive cosmological constant and therefore slowly expanding.

Our analysis indicates that if the expansion factor $a(t)$ obeys a suitable integrability condition and the adiabatic index γ_{ad} lies in the range $(1, \frac{n+1}{n})$ ($(1, 4/3)$ in the physically relevant '3+1' case; $\gamma_{ad} = 1$ and $\gamma_{ad} = 4/3$ correspond to a pressure-less fluid or 'dust' and a 'radiation' fluid, respectively), then the Lyapanuv functional remains uniformly bounded by its initial value and moreover the perturbations decay in the expanding direction. This is in a sense equivalent to the fact that if the perturbations to the matter part are restricted within a smaller cone contained in the 'sound' cone (in the tangent space; to be defined later), that is, they do not propagate at speed higher than $\sqrt{\frac{1}{n}}$ (in the unit of light speed), then the asymptotic stability holds. On the other hand, there exists a seemingly contradictory result in the purely non-relativistic case of a self-gravitating fluid system. For a self gravitating non-relativistic 3-dimensional fluid body (Euler-Poisson system), Chandrasekhar [128] used a virial identity argument to show that the static isolated compact solution of the Euler-Poisson system is stable for $\gamma_{ad} > \frac{4}{3}$. However, if one observes carefully, our result and Chandrasekhar's result are not contradictory since the notions of stability are different in the two contexts. In the context of Chandrasekhar's argument, the stability is defined by the negativity of total energy i.e., the gravitational energy dominates (stable $\iff$ gravitationally bound system). Now of course, a system with strong gravity is stable in the sense of Chandrasekhar (since total energy is dominated by gravitational energy and therefore negative) which is nonsensical in our fully relativistic context (ultra high gravity could focus and blow up forming a singularity, indicating instability in our context).

The stability criteria is satisfied by the universe model in which one includes a positive cosmological constant Λ and in such case, one obtains a uniform decay at the linear level. However, turning off the cosmological constant results in losing the uniform boundedness property simply because one does not have the integrability property for the scale factor. This may be attributed to the fact that a positive cosmological constant induced accelerated expansion wins over the gravitational effect of the fluid source at the level of linear theory. However, we do not claim that turning off the cosmological constant leads to instability but simply we are unable to reach a definite conclusion with the currently available method. In addition, we also note that in the borderline case $\gamma_{ad} = \frac{n+1}{n}$, we can only prove uniform boundedness of the energy functional (no decay). We will investigate the borderline cases of $\gamma_{ad} = 0$ and $\gamma_{ad} = \frac{n+1}{n}$ using special techniques in the future. Considering this stability criterion holds, we hope to extend our analysis in a potential future proof of non-linear stability of (3.1) in the CMCSH gauge using the energy current method developed by Christodoulou [130].

3.0.1 Notations and facts

The '$n+1$' dimensional spacetime manifold is denoted by $\tilde{M}$. We are interested in spacetime manifolds $\tilde{M}$ with the product topology $\mathbb{R}\times M$, where M denotes the n-dimensional spatial manifold diffeomorphic to a Cauchy hypersurface (i.e., every inextendible causal curve intersects M exactly once). We denote the space of the Riemannian metrics on M by $\mathcal{M}$. A subspace $\mathcal{M}_{-1}$ of $\mathcal{M}$ is defined as follows

$$\mathcal{M}_{-1}=\{\gamma\in\mathcal{M}|R(\gamma)=-1\},$$

with $R(\gamma)$ is the scalar curvature associated with $\gamma\in\mathcal{M}$. We explicitly work in the L^2 (with respect to a background metric γ) Sobolev space H^s for $s>\frac{n}{2}+1$. The L^2 inner product on the fibres of the bundle of symmetric covariant 2-tensors on (M,γ) with co-variant derivative $\nabla[\gamma]$ is defined as

$$\langle u|v\rangle_{L^2}:=\int_M u_{ij}v_{kl}\gamma^{ik}\gamma^{jl}\mu_\gamma. \tag{3.2}$$

The standard norms are defined naturally as follows

$$||u||_{L^2}:=\left(\int_M u_{ij}u_{kl}\gamma^{ik}\gamma^{jl}\mu_\gamma\right)^{\frac{1}{2}} \tag{3.3}$$

$$||u||_{L^\infty}:=\sup_M(u_{ij}u_{kl}\gamma^{ik}\gamma^{jl})^{1/2} \tag{3.4}$$

and so on. The inner product on the derivatives is defined

$$\langle\nabla[\gamma]u|\nabla[\gamma]v\rangle_{L^2} \quad = \quad \int_M \nabla[\gamma]_m u_{ij}\nabla[\gamma]_n v_{kl}\gamma^{mn}\gamma^{ik}\gamma^{jl}\mu_\gamma,$$

where μ_γ is the volume form associated with $\gamma\in\mathcal{M}$

$$\mu_\gamma=\sqrt{\det(\gamma_{ij})}dx^1\wedge dx^2\wedge dx^3\wedge ....\wedge dx^n.$$

we will use μ_γ to denote both the volume form as well as $\sqrt{\det(\gamma_{ij})}$ (slight abuse of notation). We define the rough Laplacian Δ_γ on (M,γ) for any $\gamma\in\mathcal{M}$ in the following way so that it has a non-negative spectrum i.e.,

$$\Delta_\gamma\equiv-\gamma^{ij}\nabla_i\nabla_j. \tag{3.5}$$

For $a, b \in R_{>0}$, $a \lesssim b$ is defined to be $a \leq Cb$ for some constant $0 < C < \infty$. This essentially means that a is controlled by b (in this context a and b could be norms of two different entities). For example $||A||_{L^2} \lesssim ||B||_{L^2}$ means $||A||_{L^2} \leq C||B||_{L^2}$, where $0 < C < \infty$. The spaces of symmetric covariant 2-tensors and vector fields on M are denoted by $S^0_2(M)$ and $\mathfrak{X}(M)$, respectively. The adiabatic index in denoted by γ_{ad}.

3.0.2 Overview of the paper

Section 2 provides the necessary detail about the formulation of the gauge fixed Einstein-Euler field equations including a positive cosmological constant Λ, the motivation of the study, and the main result. Note that we do not however discuss the technicalities associated with the main result in this section rather a few physical consequences.

Next in section 3, we introduce the re-scaled system of Einstein-Euler field equations with a positive cosmological constant and state the key properties of the re-scaled background solutions describing a class of fixed points. In addition, we describe the center manifold of the re-scaled dynamics when the Einstein moduli space is non-trivial. Lastly, we describe the kinematics of the perturbations under the assumption of a shadow gauge condition (introduced by [206]) when the center manifold is finite-dimensional in addition to computing a crucial term that appears in one of the elliptic equations obtained through imposing the gauge condition.

In section 4 we linearize the re-scaled field equations about the background solutions described in sections 2 and 3. Let us for the moment denote the field equations by $\mathcal{O}[\mathcal{A}] = 0$ where $\mathcal{A}$ denotes the associated fields with their background values being $\mathcal{A}_{B.G}$. The linearized equations about the background $\mathcal{A}_{B.G}$ are obtained by setting the first variation to zero i.e., $D\mathcal{O}|_{\mathcal{A}_{B.G}} \cdot \delta\mathcal{A} = \frac{d}{dt}\mathcal{O}(A_{B.G} + t\delta\mathcal{A})|_{t=0} = 0$, where the first variations to the fields are denoted by $\delta\mathcal{A}$. The linearized equations are studied with the assumption that the suitable function space norms of the higher-order terms are significantly smaller compared to the linear terms. Note that the relativistic Euler's equations are essentially a hyperbolic system of partial differential equations that are derivable from a Lagrangian (see [131, 132, 133] for a Hamiltonian structure of a relativistic perfect fluid). At the linearized level, this may be shown explicitly by employing Hodge decomposition of the $n-$velocity vector field (projection of the $n + 1-$velocity field onto the spatial hypersurface M). Doing so we reduce the relativistic Euler's equations to a wave equation for the associated scalar potential of the irrotational part (with respect to the associated 'sound' metric) while the rotational part and the harmonic part (a topological contribution) of the velocity field become decoupled and satisfy ordinary differential equations in time. Therefore the linearized coupled Einstein-Euler field equations become an ‘elliptic- hyperbolic’ system in our choice of gauge. Lastly, we close this section by stating a local well-posedness theorem for this coupled system and necessary elliptic estimates.

Section 5 is the most important chapter in terms of the proof of the main theorem of the paper. To address stability issues of a dynamical system, the use of energy functional is indispensable. On this particular occasion, we model the energy functional after the wave equation type energy introduced by [206]. We construct a similar wave equation type energy and its higher-order analog. First, we show the uniform boundedness of this energy functional in the expanding direction, and later utilizing this uniform boundedness we prove the decay property of the desired fields. Doing so we derive the required conditions for the stability results to hold. Finally, we close the section with a sketch of the proof of the geodesic completeness of the perturbed spacetimes at the level of linear perturbation theory.

3.1 Complete Einstein-Euler system and gauge fixing

The ADM formalism splits the spacetime described by an ‘$n+1$’ dimensional Lorentzian manifold $\tilde{M}$ into $\mathbb{R} \times M$. Here each level set $\{t\} \times M$ of the time function t is an orientable n-manifold diffeomorphic to a Cauchy hypersurface (assuming the spacetime admits a Cauchy hypersurface) and equipped with a Riemannian metric. Such a split may be executed by introducing a lapse function N and shift vector field X belonging to suitable function spaces on $\tilde{M}$ and defined such that

$$\partial_t = N\mathbf{n} + X, \tag{3.6}$$

where t and $\mathbf{n}$ are time and a hypersurface orthogonal future directed timelike unit vector i.e., $\hat{g}(\mathbf{n}, \mathbf{n}) = -1$, respectively. The above splitting writes the spacetime metric $\hat{g}$ in local coordinates $\{x^{\alpha}\}_{\alpha=0}^{n} = \{t, x^1, x^2, ..., x^n\}$ as

$$\hat{g} = -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt) \tag{3.7}$$

and the stress-energy tensor as

$$\mathbf{T} = E\mathbf{n} \otimes \mathbf{n} + 2\mathbf{J} \odot \mathbf{n} + \mathbf{S}, \tag{3.8}$$

where $\mathbf{J} \in \mathfrak{X}(M)$, $\mathbf{S} \in S^2_0(M)$, and $A \odot B = \frac{1}{2}(A \otimes B + B \otimes A)$. Here, $\mathfrak{X}(M)$ and $S^2_0(M)$ are the space of vector fields and the space of symmetric covariant 2-tensors, respectively. Here $E := \mathbf{T}(\mathbf{n}, \mathbf{n})$ is the energy density observed by a time-like observer with $n+1$-velocity $\mathbf{n}$, $\mathbf{J}_i = -\mathbf{T}(\partial_i, \mathbf{n})$ is the momentum density, $\mathbf{J}_i = -\mathbf{T}(\mathbf{n}, \partial_i) =$ is the energy flux density, and $\mathbf{S}_{ij} = \mathbf{T}(\partial_i, \partial_j)$ is the momentum flux density (with respect to the chosen constant t hypersurface M). The choice of a spatial slice in the spacetime leads to consideration of the second fundamental form k_{ij} which describes how the slice is curved in the spacetime. The trace of the second fundamental form ($\tau :=_g k$) is the mean extrinsic curvature of the slice, which will play an important role in the analysis. Under such decomposition, the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}R\hat{g}_{\mu\nu} + \Lambda\hat{g}_{\mu\nu} = T_{\mu\nu} \tag{3.9}$$

take the form ($8\pi G = c = 1$)

$$\begin{aligned} \partial_t g_{ij} &= -2Nk_{ij} + L_X g_{ij}, \qquad (3.10) \\ \partial_t k_{ij} &= -\nabla_i\nabla_j N + N\{R_{ij} + \tau k_{ij} - 2k_{ik}k^k_j \\ &\quad -\frac{1}{n-1}(2\Lambda - S + E)g_{ij} - \mathbf{S}_{ij}\} + L_X k_{ij} \end{aligned}$$

along with the constraints (Gauss and Codazzi equations)

$$R(g) - |k|^2 + \tau^2 = 2\Lambda + 2E, \tag{3.11}$$

$$\nabla_j k^j_i - \nabla_i \tau = -\mathbf{J}_i, \tag{3.12}$$

where $S = g^{ij}\mathbf{S}_{ij}$. The vanishing of the covariant divergence of the stress energy tensor i.e., $\nabla_\nu T^{\mu\nu} = 0$ is equivalent to the continuity equation and equations of motion of the matter

$$\frac{\partial E}{\partial t} = L_X E + NE\tau - N\nabla_i\mathbf{J}^i - 2\mathbf{J}^i\nabla_i N + N\mathbf{S}^{ij}k_{ij}, \tag{3.13}$$

$$\frac{\partial \mathbf{J}^i}{\partial t} = L_X\mathbf{J}^i + N\tau\mathbf{J}^i - \nabla_j(N\mathbf{S}^{ij}) + 2Nk^i_j\mathbf{J}^j - E\nabla^i N. \tag{3.14}$$

We want to study the Einstein-Euler system. Let the '$n+1$' velocity field of a perfect fluid be denoted by $\mathbf{u}$ which satisfies the normalization condition $\hat{g}(\mathbf{u}, \mathbf{u}) = -1$. One may for convenience decompose the $n+1$ velocity $\mathbf{u}$ into its component parallel and perpendicular to constant t hypersurface M as follows

$$\mathbf{u} = v - \hat{g}(\mathbf{u}, \mathbf{n})\mathbf{n} \tag{3.15}$$

where v is a vector field parallel to the spatial manifold M i.e., $v \in \mathfrak{X}(M)$. Importantly note that $\mathbf{u}_i = v_i$ but $\mathbf{u}^i \neq v^i$ unless the shift vector field X vanishes. The stress energy tensor for a perfect fluid with $n+1$ velocity field $\mathbf{u}$ reads

$$\mathbf{T} = (P + \rho)\mathbf{u} \otimes \mathbf{u} + P\hat{g}, \tag{3.16}$$

where P and ρ are the pressure and the mass energy density, respectively and $\hat{g}$ is the spacetime metric. After projecting onto the spatial manifold M, several components of the stress energy tensor may be computed as follows

$$E = (P + \rho)(N\mathbf{u}^0)^2 - P, \mathbf{J}^i = (P + \rho)\sqrt{1 + g(v, v)}v^i, \tag{3.17}$$
$$\mathbf{S} = (P + \rho)v \otimes v + Pg.$$

However, notice that the field equations do not close with the information of the stress-energy tensor alone, and therefore one needs an equation of state relating pressure P and mass-energy density ρ. The choice of the equation of state is a non-trivial fact and there is numerous study about the equation of state alone in cosmology literature (e.g., [167, 168, 169]). In the standard model of cosmology, one frequently uses a barotropic equation of state of the type $P = (\gamma_{ad} - 1)\rho$, where γ_{ad} is the adiabatic index. The speed of sound C_s is defined as $C_s^2 := \frac{\partial P}{\partial \rho}$, where the derivative is computed at constant entropy. However, for the barotropic equation of state (for which pressure is only a function of mass-energy density) chosen here, the entropy equation decouples from the field equations and therefore plays no role in our analysis (e.g., see [130]). The speed of sound is then a constant $\sqrt{\gamma_{ad} - 1}$. Due to causality, one must have $0 \leq C_s^2 \leq 1$ yielding $\gamma_{ad} \in [1, 2]$. This is equivalent to the fact that the sound cone is contained within the light cone in the tangent space. Here $\gamma_{ad} = 1$ corresponds to pressure-less fluid or 'dust' and $\gamma_{ad} = 2$ corresponds to a ultra-relativistic stiff fluid. We note an important fact that choosing this equation of state, we ignore the rest energy of the fluid which may be relevant in the late epoch of the universe evolution. In other words, the equation of state $P = (\gamma_{ad} - 1)(\rho - ne)$, n being the baryon number density and e the fluid rest energy per particle, may be more appropriate [134]. Here we will stick with the equation of state $P = (\gamma_{ad} - 1)\rho$ here for simplicity and in such case, the baryon number conservation equation $\nabla_\mu(nu^\mu) = 0$ is a simple consequence of the Euler's equations i.e., $\nabla_\nu T^{\mu\nu} = 0$ [130]. With the barotropic equation of state, the components of the stress energy tensor is computed to be

$$E = (P + \rho)(1 + g(v, v)) - P = \rho + \gamma_{ad}\rho g(v, v) \tag{3.18}$$

$$\mathbf{J}^i = \gamma_{ad}\rho\sqrt{1 + g(v, v)}v^i, \mathbf{S} = \gamma_{ad}\rho v \otimes v + (\gamma_{ad} - 1)\rho g.$$

The complete system of evolution and constraint equations is expressible as follows

$$\begin{aligned}\partial_t\rho + \frac{\gamma\rho N}{[1 + g(v, v)]^{1/2}}\nabla_i v^i = L_X\rho - \frac{\gamma_{ad}\rho g(\partial_t v, v)}{1 + g(v, v)} + \frac{\gamma_{ad}\rho N k(v, v)}{1 + g(v, v)} - \frac{2\gamma_{ad}\rho v_i L_X v^i}{1 + g(v, v)} \\ - \frac{\gamma_{ad} N\rho L_v N}{[1 + g(v, v)]^{1/2}} + N k^i_i\rho - \frac{N L_v\rho}{[1 + g(v, v)]^{1/2}}.\end{aligned}$$

and

$$\begin{aligned}\gamma_{ad}\rho\left[u^0\partial_t v^i - 2Nu^0 k^i_j v^j - u^0(L_X v)^i + v^j\nabla_j v^i + \frac{1 + g(v, v)}{N}\nabla^i N\right] \\ +(\gamma_{ad} - 1)\left[\nabla^i\rho + v^i L_v\rho + u^0 v^i(\partial_t\rho - L_X\rho)\right] = 0\end{aligned} \tag{3.19}$$

$$\partial_t g_{ij} = -2Nk_{ij} + L_X g_{ij}, \tag{3.20}$$

$$\begin{aligned}\partial_t k_{ij} = -\nabla_i\nabla_j N + N\left\{R_{ij} + \tau k_{ij} - 2k_{ik}k^k_j - \frac{2\Lambda}{n-1}g_{ij}\right. \\ \left. -\gamma_{ad}\rho v_i v_j + \frac{\gamma_{ad} - 2}{n-1}\rho g_{ij}\right\} + L_X k_{ij},\end{aligned} \tag{3.21}$$

$$R(g) - |k|^2 + \tau^2 = 2\Lambda + 2\rho\left\{1 + \gamma_{ad} g(v, v)\right\}, \tag{3.22}$$

$$\nabla_i k^{ij} - \nabla^j\tau = -\gamma_{ad}\rho\sqrt{1 + g(v, v)}v^i. \tag{3.23}$$

In order to obtain equations satisfied by the lapse function and the shift vector field, we need to fix a choice of gauge. Setting the mean extrinsic curvature (τ) of the hypersurface M to a constant allows it to be a suitable time function. This choice of temporal gauge yields an elliptic equation for the lapse function N. However, not all spacetimes admit a constant mean extrinsic curvature (CMC) hypersurface. Existence of a CMC slice is far from obvious and is a part of active mathematical research (for detail see [136, 137, 138, 139]). Luckily the background spacetimes (3.1) that we are interested in, do admit a CMC slice which is verified through explicit calculations. Therefore simply setting the mean extrinsic curvature of the perturbed spacetimes to be constant on each spatial slice seems to be a reasonable choice

$$\partial_i \tau \quad = \quad 0. \tag{3.24}$$

Such an assumption can be made without loss of generality for solutions that are close to the background solution [142, 180]. This choice allows τ to play the role of time i.e.,

$$t = monotonic\ function\ of\ \tau. \tag{3.25}$$

To utilize the property $\partial_i \tau = 0$, one may simply compute the entity $\frac{\partial \tau}{\partial t}$ to yield

$$\frac{\partial \tau}{\partial t} = \Delta_\gamma N + \left\{ |k|_g^2 + [\frac{(n\gamma - 2)}{n-1} + \gamma_{ad} g(v,v)]\rho - \frac{2\Lambda}{n-1} \right\} N + L_X \tau. \tag{3.26}$$

Using equation (3.24), we obtain the desired elliptic equation for the lapse function

$$\Delta_g N + \left\{ |k|_g^2 + [\frac{(n\gamma - 2)}{n-1} + \gamma_{ad} g(v,v)]\rho - \frac{2\Lambda}{n-1} \right\} N = \quad \frac{\partial \tau}{\partial t}. \tag{3.27}$$

Once we have fixed the temporal gauge, we need to fix the spatial gauge which would yield an equation for the shift vector field. We follow the work of [151, 206] regarding spatial gauge fixing. Let $\zeta : (M, g) \to (M, \gamma)$ be a harmonic map with the Dirichlet energy $\frac{1}{2} \int_M g^{ij} \frac{\partial \zeta^k}{\partial x^i} \frac{\partial \zeta^l}{\partial x^j} \gamma_{kl} \mu_g$. Since the harmonic maps are the critical points of the Dirichlet energy functional, ζ satisfies the following formal Euler-Lagrange equation

$$g^{ij} \left(\partial_i \partial_j \zeta^k - \Gamma[g]^l_{ij} \partial_l \zeta^k + \Gamma[\gamma]^k_{\alpha\beta} \partial_i \zeta^\alpha \partial_j \zeta^\beta \right) = 0. \tag{3.28}$$

Now, we fix the gauge by imposing the condition that $\zeta = id$, which leads to the following equation

$$-g^{ij} (\Gamma[g]^k_{ij} - \Gamma[\gamma]^k_{ij}) \quad = \quad 0. \tag{3.29}$$

where $\hat{\Gamma}[\gamma]^k_{ij}$ is the connection with respect to some arbitrary background Riemannian metric γ. Choice of this spatial harmonic gauge yields the following elliptic equation for the shift vector field

X after time differentiating equation (3.29)

$$\begin{aligned} \Delta_g X^i - R^i_j X^j &= -2k^{ij}\nabla_j N - 2N\nabla_j k^{ij} + \tau\nabla^i N \\ &\quad +(2Nk^{jk} - 2\nabla^j X^k)(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}) - g^{ij}\partial_t\Gamma[\gamma]^k_{ij}. \end{aligned} \tag{3.30}$$

However, one natural question arises while imposing the spatial harmonic gauge. Does there exist a harmonic map between the manifolds (M, g) and (M, γ)? This is an extremely important and nontrivial issue. There are explicit results in geometry about the existence of harmonic maps between Riemannian manifolds. The one that is relevant to the current study is that of Eells and Sampson [143] who proved if M is a compact manifold with non-positive Riemann curvature, then any continuous map from a compact manifold into M is homotopic to a harmonic map. Andersson and Moncrief [150] applied this gauge to prove a local existence theorem for the vacuum Einstein's equations in arbitrary dimensions. Later they have successfully implemented this gauge to prove a small data nonlinear stability result of the 'Milne' model and its higher-dimensional generalization (so-called 'Lorentz cone spacetimes') [151, 206]. We designate our gauge fixed system as **CMCSH Einstein-Euler-Λ** system.

3.1.1 Motivation of the study and main result

In this section, we describe the motivation for the current study. Firstly, for a universe model to be a viable candidate for a physical cosmological model, it must be stable against perturbations in a suitable sense. Therefore, one should ideally prove the fully non-linear stability assuming certain smallness condition on the data. A natural first step towards such stability would be to establish the stability at the level of linear perturbation theory. In addition to this obvious motivation, based on a few preliminary studies [161, 165, 229] we are led to believe that these expanding (accelerated expansion since $\Lambda > 0$) spacetimes may be stable under small perturbations. From a physical viewpoint, the accelerated expansion tends to kill off the perturbations. Let us describe this point in a slightly more formal way. First, we decompose the second fundamental form k as follows

$$k_{ij} = k^{TT}_{ij} + \frac{tr_g k}{n} g_{ij} + (L_Y g - \frac{2}{n}(\nabla\cdot Y)g)_{ij}, \tag{3.31}$$

where k^{TT} is a transverse-traceless (with respect to the metric g) symmetric $2-$tensor, $tr_g k$ is the trace of the second fundamental form with respect to the metric g, and Y is a vector field tangent to M. Now consider the Hamiltonian constraint (4.19)

$$\frac{n-1}{n}(\tau^2-\frac{2n\Lambda}{n-1})=|k^{TT}|_g^2+|L_Yg-\frac{2}{n}(\nabla\cdot Y)g|_g^2 \\ +2\rho(1+\gamma_{ad}g(v,v))-R(g) \tag{3.32}$$

and construct the following integral

$$\frac{n-1}{n}\int_M(\tau^2-\frac{2n\Lambda}{n-1})\mu_g=\int_M(|k^{TT}|_g^2+|L_Yg-\frac{2}{n}(\nabla\cdot Y)g|_g^2 \\ +2\rho(1+\gamma_{ad}g(v,v))-R(g))\mu_g>0, \tag{3.33}$$

then we readily observe that the integral contains the L^2 norm of $k^{tr}=k_{ij}^{TT}+(L_Yg-\frac{2}{n}(\nabla\cdot Y)g)_{ij}$ in addition to the physical energy density and is strictly positive since the re-scaled volume is positive (note that $\tau^2>\frac{2n\Lambda}{n-1}$ for spacetimes of interest; spatial manifold is of negative Yamabe type; see [229] for detail about Yamabe classification; this inequality can be showed as follows: The left hand side of (3.32) contains non-negative term except $-R(g)$. But since M is assumed to be negative Yamabe, $R(g)<0$ must hold at some point on M. But $\tau^2-\frac{2n\Lambda}{n-1}$ is constant on M due to constant mean extrinsic curvature assumption on each spatial slice. Therefore $\tau^2>\frac{2n\Lambda}{n-1}$). Motivated by such property, [161] studied the monotonic decay of the entity $\int_M\tau^2\mu_g$ for pure vacuum case and [229] studied the same for $\int_M(\tau^2-\frac{2n\Lambda}{n-1})\mu_g$ in the vacuum case with $\Lambda>0$ in the CMC gauge (also see [140, 141]). In each of these cases, this re-scaled volume acted as a Hamiltonian while expressed in terms of suitable "reduced" phase space variables. Subsequently, [165] studied the monotonic decay (in the expanding direction) of the entity $\int_M(\tau^2-\frac{2n\Lambda}{n-1})\mu_g$ in the presence of matter sources satisfying suitable energy conditions in particular perfect fluids, where this entity played the role of a Lyapunov functional.

Naively, monotonic decay of this Lyapunov functional indicates the stability of a class of expanding solutions (on which the Lyapunov functional attains its infimum) against arbitrary large data perturbations. Motivated by this result, we are interested in pursuing the stability property of a perfect fluid-filled spatially compact universe model. Before proceeding to the fully nonlinear study of the stability problem, it is important to prove the linear stability in a rigorous way (not simply the mode stability) and obtain decay estimates if possible. In a sense, the linear stability and decay should these hold provide motivation towards studying the fully non-linear stability and may also become useful in handling the higher-order couplings and defining appropriate energies. However, this does not imply stability since the Lyapunov functional does not control the norm that is required by the local existence theorem (for vacuum and Λ-vacuum the function space required for (g,k) by the local existence is $H^s \times H^{s-1}, s > \frac{n}{2}+1$). It only controls the minimum regularity ($H^1 \times L^2$ of (g,k)) and therefore is not sufficient to state a global existence for arbitrarily large data (one may therefore call this Lyapunov functional as a 'weak' Lyapunov functional). Even though the re-scaled volume functional controlling the minimum norm decays monotonically in the expanding direction and attains its infimum for a particular class of spacetimes (see the relevant sections of [161,165,229] for details), the classical solution may form a finite time singularity via curvature focusing. In other words, a suitable norm of arbitrarily large data (physically interesting solutions) may be obstructed to decay monotonically to its infimum since pure gravity could blow up before the spatial volume of the universe tends to infinity. The addition of perfect fluids adds additional problems since they are known to form shock singularities in the fully non-linear setting. Christodoulou proved that perfect fluids form shock on Minkowski background for arbitrary small data [163]. Of course, Minkowski space does not have the expansion property (accelerated expansion including $\Lambda > 0$) like the space-times (3.1) of the current investigation do. We denote a ball centered at $(x, y, z, ..)$ with radius δ by $B_\delta(x, y, z, ..)$. The Einstein structure of the manifold M (or the deformation space of an arbitrary Einstein metric γ on M) is denoted by $\mathcal{N}$. This will be defined later in a rigorous way. Now we summarize the result in terms of the following 'rough' version which is equivalent to the main theorem stated in the later part of the paper.

A rough version of the main Theorem: *Let* $(g_0, k_0^{tr}, \rho_0, v_0) \in H^s \times H^{s-1} \times H^{s-1} \times H^{s-1},\ s > \frac{n}{2}+2$ *be the initial data for the re-scaled Einstein-Euler-*Λ *system with* $\Lambda > 0$*. Let us also consider that the Cauchy data for the re-scaled background solutions are* $(g_{B.G}, k^{tr}_{B.G}, \rho_{B.G}, v_{B.G})$ *which satisfy* $R[g_{B.G}]_{ij} = -\frac{1}{n}(g_{B.G})_{ij}, k^{tr}_{B.G} = 0, \rho_{B.G} = C_\rho, v^i_{B.G} = 0$*,* C_ρ *is a constant. Now assume that the following smallness condition holds* $||g_0 - g_{B.G}||_{H^s} + ||k_0^{tr} - k^{tr}_{B.G}||_{H^{s-1}} + ||\rho_0 - \rho_{B.G}||_{H^{s-1}} + ||v_0 - v_{B.G}||_{H^{s-1}} < \delta$*,* δ *is chosen sufficiently small. If* $t \mapsto (g(t), k^{tr}(t), \rho(t), v(t))$ *is the maximal development of the Cauchy problem for the linearized re-scaled Einstein-Euler-*Λ *system about the background solutions* $(g_{B.G}, k^{tr}_{B.G}, \rho_{B.G}, v_{B.G})$ *in constant mean extrinsic curvature spatial harmonic gauge (CMCSH) (3.109-3.116) with initial data* $(g_0, k_0^{tr}, \rho_0, v_0)$*, then the following holds in the limit of infinite time*

$$\lim_{t\to\infty} (g(t), k^{tr}(t), \rho(t), v(t)) = (\gamma^\dagger, 0, \rho^{'}, 0), \tag{3.34}$$

where $\gamma^\dagger$ *satisfies* $R[\gamma^\dagger] = -1$ *and* $\rho^{'}$ *depends on the initial re-scaled density* ρ_0 *and* $\partial_i\rho^{'} \neq 0$ *in general.*

Let us now explain the main theorem putting aside the technical details. The statement essentially deals with the *asymptotic* stability of the solutions that are characterized by constant negative scalar curvature (-1 to be precise in this occasion) and sufficiently close to the FLRW solutions In other words this corresponds to a *Lyapunov* stability of the FLRW solutions. In other words, the solutions of linearized Einstein-Euler-Λ equations about the background FLRW solutions (3.1) decay to nearby solutions which have constant negative scalar curvature. While the background solutions the FLRW spacetimes (3.1) are described by the property that the spatial metric $g_{B.G}$ is negative Einstein i.e., $R[g_{B.G}]_{ij} = -\frac{1}{n}(g_{B.G})_{ij}$, the perturbed solutions decay in infinite time to the space which does not share the same property. More specifically, the spatial metric $\gamma^{\dagger}$ of the asymptotic solution satisfies $R[\gamma^{\dagger}] = -1$. Now, the space of negative Einstein metrics is a subset of the space of metrics with constant negative scalar curvature. Simple use of triangle inequality shows that the asymptotic state should be the space of constant negative scalar curvature sufficiently close to and containing the background FLRW solutions. Since the latter is a subspace of the former, one may fine-tune the initial conditions to obtain the asymptotic solution to be exactly of FLRW type (i.e., the solution has a spatial metric that is negative Einstein). But such a procedure proves to be too restrictive and physically undesirable.

If for the moment we focus on the physical 3+1 case, then ideally we want the asymptotic solution to have constant negative sectional curvature (i.e., hyperbolic or negative Einstein by Mostow rigidity) not just constant negative scalar curvature and a spatially uniform background energy density. In other words, if one perturbs an FLRW solution, it does not generically come back to an FLRW solution in infinite time. Now notice that the local spatial homogeneity and isotropy criteria required constancy of the sectional curvature and as such in 3 spatial dimensions, a negative Einstein metric automatically has constant sectional curvature. However, space of metrics with constant negative scalar curvature is a much larger space and does not necessarily exhibit the local spatial homogeneity and isotropy criteria (only a subspace that is described by negative Einstein metrics or hyperbolic metrics do; these correspond to compact hyperbolic manifolds that are locally homogeneous and isotropic by construction). This result is not completely satisfactory but on the other hand, maybe physically expected as these inhomogeneous and anisotropic characteristics may be attributed to the structure formation (see [122, 145] about the role of cosmological perturbation theory in structure formation). This is a remarkable fact that indicates there is a dynamical mechanism at work within the Einstein-Euler-Λ flow that naturally drives the physical universe to an anisotropic and inhomogeneous state leading to cosmological structure formation. We do not of course claim such a strong result only based on our linear stability analysis.

We note that [144] reported this behavior of the FLRW spacetimes as well considering 'dust' as the matter source. They constructed a finite-dimensional dynamical system by spatially averaging out the inhomogeneities and showed that *FLRW cosmologies are unstable in some relevant cases: averaged models are driven away from them through structure formation and accelerated expansion.* However, presently we do not claim that our theorem indicates such a strong result since we need to execute the full nonlinear analysis to incorporate the possibility of shock formation. Nevertheless, the phenomenon of accelerated expansion driving the solutions away from the background is previously noted in [177] which dealt with a particular case of vacuum gravity (background solutions were described by the negative Einstein spaces) including a positive cosmological constant. The solutions decay asymptotically to the nearby ones described by constant negative scalar curvature. On the other hand, in the same problem if the cosmological constant is turned off, then the perturbed solutions do come back to the background solution asymptotically [151, 206]. Nevertheless, the presence of a positive cosmological constant yields universal decay rates that do not depend on the background geometry unlike the zero cosmological constant case, where the decay rate explicitly depended on the spectra of a Lichnerowicz type Laplacian operator defined on the background geometry (see [206] for detail).

3.2 Re-scaled field equations and background solutions

The background solution we are interested in is the constant negative sectional spatial curvature FLRW model in '$3+1$' dimensions. In higher dimensions ($n>3$), however, the spatial metrics are negative Einstein (not necessarily hyperbolic). The solution is expressible in the following warped product form

$$^{n+1}g \quad = \quad -dt\otimes dt + a(t)^2\gamma_{ij}dx^i\otimes dx^j, \tag{3.35}$$

where $R_{ij}(\gamma) = -\frac{1}{n}\gamma_{ij}$ and $t\in(0,\infty)$. Clearly we have $N=1,\ X^i=0,\ g_{ij}=a(t)^2\gamma_{ij}$. These spacetimes are globally foliated by constant mean curvature slices which is evident from the following simple calculations

$$\begin{aligned} k_{ij} &= -\frac{1}{2N}\partial_t g_{ij} = -a\dot{a}\gamma_{ij}, \qquad (3.36)\\ \tau(t) &= k_{ij}g^{ij} = -a\dot{a}\gamma_{ij}\frac{1}{a(t)^2}\gamma^{ij} = -\frac{n\dot{a}}{a}. \end{aligned}$$

Note that in order to study the linearized perturbations, we need to re-scale the field equations. We want to re-scale in such a way as to make the background time independent and then analyze the associated re-scaled dynamical system. For this particular purpose, we will need the background density and $n-$velocity. Let us now explicitly calculate the scale factor $a(t)$, the background density, and the velocity. Note that we are only interested in the asymptotic behaviour of the scale factor at $t \to \infty$ for our stability analysis instead of its precise expression. The following lemma states the asymptotic behaviour of the scale factor.

Lemma 1: *Let γ_{ad} be the adiabatic index. The background energy density, $n-$velocity vector field, and the scale factor satisfy the following*

$$
\begin{aligned}
\rho(t)a(t)^{n\gamma_{ad}} &= C_\rho, \\
v^i &= 0, \\
a \sim e^{\alpha t} \quad for \quad \Lambda &> 0, \\
a \sim t \quad as \quad t \to \infty \quad for \quad \Lambda &= 0,
\end{aligned}
$$

where $\alpha := \sqrt{\frac{2\Lambda}{n(n-1)}}$, $a(0) := a(t=0)$, and $\infty > C_\rho > 0$ is a constant.

Proof: The energy equation yields

$$
\frac{\partial \rho}{\partial t} = -n\gamma_{ad}\frac{\dot{a}}{a}\rho, \tag{3.37}
$$

integration of which leads to

$$
\rho a(t)^{n\gamma_{ad}} = C_\rho, \tag{3.38}
$$

for some finite positive constant C_ρ. The momentum constraint (3.23) equation yields

$$
v^i = 0 \tag{3.39}
$$

since $k^{tr} = 0$. Utilizing the Hamiltonian constraint, we obtain the following ODE for the scale factor

$$
\frac{1}{a}\frac{da}{dt} = \sqrt{\frac{1}{n(n-1)}\left(2\Lambda + \frac{1}{a^2} + \frac{2C_\rho}{a^{n\gamma_{ad}}}\right)}, \tag{3.40}
$$

integration of which yields at $t \to \infty$

$$a \sim e^{\alpha t} \tag{3.41}$$

where $\alpha := \sqrt{\frac{2\Lambda}{n(n-1)}}$. Taking $\Lambda = 0$, integration of (3.40) yields

$$a \sim a(0) + \frac{t}{\sqrt{n(n-1)}} \tag{3.42}$$

as $t \to \infty$, which completes the proof of the lemma.

Now let us focus on obtaining a re-scaled system of evolution and constraint equations. Recall that the metric in local coordinate (t, x) may be written as

$$\hat{g} \quad = \quad -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt) \tag{3.43}$$

and the background solutions are written as

$${}^{n+1}g_{background} = -dt \otimes dt + a^2 \gamma_{ij} dx^i \otimes dx^j, \tag{3.44}$$

where $R[\gamma]_{ij} = -\frac{1}{n}\gamma_{ij}$ and $t \in (0, \infty)$. If we assign a the dimension of length (natural) (similar to the re-scaling used by [206]), the dimensions of the rest of the entities follow $t \sim length$, $x^i \sim (length)^0$, $g_{ij} \sim (length)^2$, $N \sim (length)^0$, $X^i \sim (length)^{-1}$, $\tau \sim (length)^{-1}$, $\kappa^{tr}_{ij} \sim length$. Therefore, the following scaling follows naturally

$$\tilde{g}_{ij} = a^2 g_{ij}, \tilde{N} = N, \tilde{X}^i = \frac{1}{a} X^i, \tilde{k^{tr}}_{ij} = a k^{tr}_{ij}. \tag{3.45}$$

Here, we denote the dimension-full entities with a tilde sign and dimensionless entities are left alone for simplicity. The Hamiltonian constraint (4.19) and the momentum constraint (3.23) take the following forms

$$\begin{aligned} R(g) - |k^{tr}|^2 + 1 + 2C_\rho a^{2-n\gamma} &= 2a(t)^2 \tilde{\rho}\left(1 + \gamma_{ad}\tilde{g}(\tilde{v}, \tilde{v})\right), \\ \nabla_j k^{trij} &= a(t)^3 \gamma_{ad} \tilde{\rho} \sqrt{1 + \tilde{g}(\tilde{v}, \tilde{v})}\, \tilde{v}^i. \end{aligned} \tag{3.46}$$

Notice that we are yet to re-scale the matter fields i.e., (ρ, v). Now, of course, in lemma 1, we have already seen that the background energy density satisfies $\rho(t) = \frac{C_\rho}{a^{n\gamma_{ad}}}$ for some constant $C_\rho > 0$

depending on the initial density. Therefore, we naturally scale the energy density by $\frac{1}{a^{n\gamma_{ad}}}$ i.e.,

$$\tilde{\rho} \;=\; \frac{1}{a(t)^{n\gamma_{ad}}}\rho \tag{3.47}$$

so that the background density becomes time independent. Notice that C_ρ is a constant with dimension of $(length)^{n\gamma_{ad}-2}$. now, since, the background $n-$velocity vanishes for the background solution, we do not have a straightforward scaling. However, we have the normalization condition for the $n+1-$velocity field $\mathbf{u} = -^{n+1}g(u,n)n + v$

$$-(\tilde{N}\tilde{\mathbf{u}}^0)^2 + \tilde{g}_{ij}\tilde{v}^i\tilde{v}^j \;=\; -1, \tag{3.48}$$

and therefore scaling of the metric $\tilde{g}_{ij} = a^2 g_{ij}$ yields the following natural scaling of v^i

$$\tilde{v}^i \;=\; \frac{1}{a}v^i, \tag{3.49}$$

since the right hand side is simply a constant. The complete evolution and constraints of the gravity coupled fluid system may be expressed in the following re-scaled form

$$\frac{\partial g_{ij}}{\partial t} \;=\; 2\frac{\dot{a}}{a}(N-1)\,g_{ij} - \frac{2}{a}Nk^{tr}_{ij} + \frac{1}{a}(L_X g)_{ij}, \tag{3.50}$$

$$\begin{aligned}\frac{\partial k^{tr}_{ij}}{\partial t} \;=\;& -\underbrace{\frac{\dot{a}}{a}\left((N-1)(n-2)+n-1\right)}_{\textit{should be <0 to generate decay}}k^{tr}_{ij} - \frac{2}{a}Nk^{tr}_{ik}k^{trk}_{j} + \underbrace{\frac{1}{a}NR_{ij}}_{\textit{gauge fixing is required}} \\ & -\Bigg(\frac{a}{n^2}(\tau^2 - \frac{2n\Lambda}{n-1}) + \frac{n\gamma_{ad}-2}{n(n-1)}a^{1-n\gamma_{ad}}C_\rho - \frac{N(1+2C_\rho a^{2-n\gamma})}{a(n-1)} \\ & + \frac{\gamma_{ad}-2}{n-1}a^{1-n\gamma_{ad}}N\rho g_{ij}\Bigg)g_{ij} - \frac{1}{a}\nabla_i\nabla_j N - a^{1-n\gamma_{ad}}N\gamma\rho v_i v_j + \frac{1}{a}(L_X k^{tr})_{ij},\end{aligned} \tag{3.51}$$

$$\begin{aligned}&\partial_t\rho + \frac{\gamma_{ad}\rho N\nabla_i v^i}{a[1+g(v,v)]^{1/2}} + \underbrace{\frac{(N-1)n\gamma_{ad}\rho\dot{a}}{a}}_{\textit{should contribute to higher order since } N-1\leq 0} \\ &= \frac{1}{a}L_X\rho - \frac{\gamma_{ad}\rho g(\partial_t v, v)}{1+g(v,v)} - \frac{NL_v\rho}{a[1+g(v,v)]^{1/2}} + \frac{\gamma_{ad}\rho N}{1+g(v,v)}\left(\frac{1}{a}k^{tr}(v,v) - \frac{\dot{a}}{a}g(v,v)\right) \\ &\qquad + \frac{\gamma_{ad}\rho v_i L_X v^i}{a[1+g(v,v)]} - \frac{\gamma_{ad}N\rho L_v N}{a[1+g(v,v)]^{1/2}},\end{aligned} \tag{3.52}$$

$$
\gamma_{ad}\rho u^0\partial_t v^i + \underbrace{\gamma_{ad}\rho u^0[2N-1-n(\gamma_{ad}-1)]\frac{\dot{a}}{a}}_{should\ be\ >0\ to\ generate\ decay} v^i - \frac{2\gamma_{ad}\rho N u^0 k^{tri}_j v^j}{a} - \frac{\gamma_{ad}\rho u^0}{a}L_X v^i
$$

$$
+\frac{\gamma_{ad}\rho}{a}v^j\nabla_j v^i + \frac{\gamma_{ad}\rho[1+g(v,v)]}{aN}\nabla^i N + \frac{(\gamma_{ad}-1)}{a}\left(\nabla^i\rho + v^i L_v\rho + a u^0 v^i\partial_t\rho \right.
$$

$$
\left. -u^0 v^i L_X\rho\right) = 0,
$$

$$
R(g) - |k^{tr}|^2 - 2a^{2-n\gamma_{ad}}\rho\{1+\gamma_{ad}g(v,v)\} + 1 + 2C_\rho a^{2-n\gamma} = 0, \tag{3.53}
$$

$$
\nabla_j k^{trij} = -\gamma_{ad}a^{2-n\gamma_{ad}}\rho\sqrt{1+g(v,v)}v^i, \tag{3.54}
$$

$$
\Delta_g N + \left(\underbrace{|k^{tr}|^2 + a(t)^2(\frac{\tau^2}{n} - \frac{2\Lambda}{n-1}) + a^{2-n\gamma_{ad}}[\frac{(n\gamma_{ad}-2)}{n-1} + \gamma_{ad}g(v,v)]\rho}_{should\ be\ >0\ for\ elliptic\ regularity}\right) N
$$

$$
= a^2\frac{\partial\tau}{\partial t}, \tag{3.55}
$$

$$
\Delta_g X^i - R^i_j X^j = (1-\frac{2}{n})\tau\nabla^i N - 2\nabla^j N k^{tri}_j - 2N\nabla_j k^{trij} \tag{3.56}
$$

$$
+(2Nk^{trjk} - 2\nabla^j X^k)\left(\Gamma[g]^i_{jk} - \Gamma[\gamma]^i_{jk}\right) - a g^{ij}\partial_t\Gamma[\gamma]^k_{ij}.
$$

Note that an straightforward maximum principle argument applied to the lapse equation yields $N \leq 1$. In CMCSH gauge, the time function is obtained by setting τ =monotonic function of t alone. Now, we have τ evaluated for the background solution which is a monotonic function of t alone since utilizing the Hamiltonian constraint and the lapse equation at the background solution one obtains

$$
\begin{aligned}
\frac{\partial\tau}{\partial t} &= \frac{1}{n}(\tau^2 - \frac{2n\Lambda}{n-1}) + \frac{n\gamma_{ad}-2}{n-1}a^{-n\gamma_{ad}}C_\rho \\
&= \frac{1}{a(t)^2(n-1)}\left(1 + n\gamma_{ad}C_\rho a^{2-n\gamma_{ad}}\right) > 0.
\end{aligned} \tag{3.57}
$$

We set the time function to be the solution of this equation

$$
\frac{\partial\tau}{\partial t} = \frac{1}{a(t)^2(n-1)}\left(1 + n\gamma_{ad}C_\rho a(t)^{2-n\gamma_{ad}}\right), \tag{3.58}
$$

i.e., $t = t(\tau)$ which settles the business of choosing a time coordinate. Let us analyze the background solutions in a bit more detail.

Lemma 2(a): *The background re-scaled solution of the Einstein-Euler system verifies the following equations*

$$N_{B.G} = 1, X^i_{B.G} = 0, R[g_{B.G}]_{ij} = -\frac{1}{n} g_{B.Gij}, k^{tr}_{B.G} = 0, v^i_{B.G} = 0, \partial_i \rho_{B.G} = 0, \tag{3.59}$$

and the spacetime is foliated by CMC slices.

Proof: The background spacetime

$$^{n+1}g \quad = \quad -dt \otimes dt + a(t)^2 \gamma_{ij} dx^i \otimes dx^j \tag{3.60}$$

yields $\tilde{N}_{B.G} = 1,\ \tilde{X}^i_{B.G} = 0,\ \tilde{g}_{B.Gij} = a^2\gamma_{ij}$ and therefore yields $N_{B.G} = 1,\ X^i_{B.G} = 0,\ g_{B.Gij} = \gamma_{ij}$. A simple calculation yields

$$\tilde{k}_{B.Gij} = -a\dot{a}\gamma_{ij} \tag{3.61}$$

and therefore

$$k^{tr}_{B.G} = 0,\ \tau = -\frac{n\dot{a}}{a}. \tag{3.62}$$

The momentum constraint (3.54) yields

$$v^i_{B.G} = 0 \tag{3.63}$$

which together with the equations of motion (3.53) for the fluid leads to

$$\partial_i \rho_{B.G} = 0. \tag{3.64}$$

From the previous calculation $\tau = -\frac{\dot{a}}{a}$, it is obvious that the spacetime is foliated by constant mean curvature spatial slices.

Lemma 2(b): *The transverse-traceless part of the traceless second fundamental form vanishes for the background solutions i.e.,* $k^{tr}_{B.G} = k^{TT} + L_Y g_{B.G} - \frac{2}{n}(\nabla[g_{B.G}]_m Y^m) g_{B.G}$ *reduces to* $k^{tr} = k^{TT} = 0$ *and moreover* $Y \equiv 0$ *for background solutions.*

Proof: The vanishing of the re-scaled trace-less second fundamental form $k^{tr} = 0$ for the background solutions yields the following through the decomposition of the symmetric trace-less $2-$ tensors

$$k^{TT} + L_Y\gamma - \frac{2}{n}(\nabla[\gamma]_m Y^m)\gamma = 0, \tag{3.65}$$

which may be converted to

$$\begin{aligned} \int_M (k^{TT} + L_Y\gamma - \frac{2}{n}\nabla[\gamma]_m Y^m\gamma)_{ij}(k^{TT} + L_Y\gamma - \frac{2}{n}\nabla[\gamma]_m Y^m\gamma)^{ij}\mu_\gamma &= 0 \\ \int_M |k^{TT}|^2_\gamma \mu_\gamma + \int_M |L_Y\gamma - \frac{2}{n}\nabla[\gamma]_m Y^m\gamma|^2_\gamma \mu_\gamma &= 0 \end{aligned} \tag{3.66}$$

since k^{TT} and $L_Y\gamma - \frac{2}{n}\nabla[\gamma]_m Y^m\gamma$ are L^2 orthogonal with respect to the metric γ. Therefore we readily obtain

$$k^{TT} \equiv 0, \tag{3.67}$$

$$L_Y\gamma - \frac{2}{n}\nabla[\gamma]_m Y^m\gamma \equiv 0, \tag{3.68}$$

where we have used the fact that k^{TT} and $L_Y\gamma - \frac{2}{n}\nabla[\gamma]_m Y^m$ are L^2 orthogonal (with respect to the background metric γ). Now the covariant divergence of the conformal Killing equation (3.68) yields

$$\begin{aligned} \nabla[\gamma]^i\left(\nabla[\gamma]_i Y_j + \nabla[\gamma]_j Y_i - \frac{2}{n}(\nabla[\gamma]_m Y^m)\gamma_{ij}\right) &= 0 \\ \int_M \left(Y_j\nabla[\gamma]^i\nabla[\gamma]_i Y^j + R^j_i Y^i Y_j + (1-\frac{2}{n})Y_j\nabla[\gamma]^j(\nabla[\gamma]_i Y^i)\right)\mu_\gamma &= 0 \\ -\int_M \left(\nabla[\gamma]_i Y_j\nabla[\gamma]^i Y^j + \frac{1}{n}\gamma_{ij}Y^iY^j + (1-\frac{2}{n})|\nabla[\gamma]_i Y^i|^2\right)\mu_\gamma &= 0 \\ \Rightarrow Y \equiv 0 & \end{aligned} \tag{3.69}$$

throughout M since $n \geq 3$. Here we have used $R(\gamma)_{ij} = -\frac{1}{n}\gamma_{ij}$ together with integration by parts and the Stokes' theorem for a closed (compact with $\partial M = \{0\}$) manifold. This is equivalent to the fact that the isometry group of (M, γ) is discrete. The spacetimes (3.1) admit a timelike conformal Killing field $K := K^\mu\frac{\partial}{\partial x^\mu} = a(t)\frac{\partial}{\partial t}$ i.e.,

$$L_{a(t)\frac{\partial}{\partial t}}\ {}^{n+1}g = 2\dot{a}\ {}^{n+1}g. \tag{3.70}$$

The results obtained so far yields the following theorem characterizing a class of *fixed points* of the *re-scaled* Einstein-Euler-Λ system.

Theorem: *Let M be a closed (compact without boundary) connected orientable manifold that* *admits a negative Einstein metric. Then the fixed point solutions of the re-scaled Einstein-Euler-Λ flow (3.50-3.56) in constant mean extrinsic curvature spatial harmonic gauge (CMCSH) on* $(t_-, t_+)\times M$, $0 \leq t_- < t_+ < \infty$ *have the Cauchy data* $(g_{B.G}, k^{tr}_{B.G}, N_{B.G}, X_{B.G}, \rho_{B.G}, v_{B.G})$ *that satisfy the following equations*

$$R_{ij}[g_{B.G}] = -\frac{1}{n}(g_{B.G})_{ij},\ k^{tr}_{B.G} = 0,\ N_{B.G} = 1,\ X_{B.G} = 0,\ \rho_{B.G} = C_\rho,$$
$$v^i_{B.G} = 0. \qquad (3.71)$$

The physical spacetimes (3.35) constructed by this Cauchy data admit $K := a(t)\partial_t$ *as a globally defined time-like conformal Killing vector field. Here* $a(t)$ *is the scale factor.*

3.2.1 Center Manifold of the dynamics

The following mathematics concerning the center manifold dynamics in the setting of Einstein flow was first established by the work of Andersson-Moncrief [151]. We now present some of this work and adapt it to the Einstein-Euler-Lambda setting. A center manifold of a dynamical system is associated with fixed points of the dynamics. Essentially a center manifold of a fixed point corresponds to the nearby solutions (in phase space) that do not exhibit exponential growth or decay. This may be zero-dimensional or a subspace of the phase space and admits a manifold structure (so the name center 'manifold'). For a more precise and mathematically rigorous definition, we refer the reader to [170, 171]. There is a crucial theorem namely the 'center manifold theorem' which plays a significant role in the analysis of dynamical systems. In general, center manifolds do not have the uniqueness property, unlike stable and unstable manifolds which do [172, 173, 174]. In a finite-dimensional setting, this roughly corresponds to the case when the linearization of the flow vector field has purely imaginary spectra. In an infinite-dimensional setting, moduli spaces naturally play the role of center manifold (in an appropriate sense of course). As we shall see, the center manifold in this particular occasion is played by the Einstein moduli space, which consists of non-isolated fixed points of the Einstein-Euler-Λ flow.

Unlike the $n = 3$ dimensional case where the negative Einstein spaces are hyperbolic (and the center manifold of the gravitational dynamics consists of a point), the higher dimensional case is more interesting. The matter degrees of freedom corresponding to the background solution (in its re-scaled version) in CMCSH gauge are essentially described by $v^i = 0, \rho = \rho_{B.G}$. It becomes more interesting when we consider the gravitational degrees of freedom. In the case of $n = 3$, a negative Einstein structure implies hyperbolic structure through Mostow rigidity theorem and therefore the manifold describing the fixed point of the dynamics is zero-dimensional. In other words, the particular solution ($\{\gamma|Ricci[\gamma]=-\frac{1}{3}\gamma, k^{tr}=0, \rho=C_{\rho}, v^{i}=0$) serves as an isolated fixed point of the Einstein-Euler dynamics. However, in the higher dimensional cases ($n>3$), a new possibility arises of having non-hyperbolic negative Einstein spaces. When we linearize about any member of such a non-isolated family of negative Einstein metrics, the linearized equations will always admit a finite-dimensional space of neutral modes. Naturally, this represents (modulo the matter degrees of freedom) the tangent space to the background spacetimes (3.1). These smooth families of background spacetimes determined by the corresponding families of negative Einstein metrics and zero-dimensional matter degrees of freedom ($v^{i}=0, \rho=C_{\rho}$) form the 'center manifold' for the dynamical system defined by the re-scaled Einstein-Euler-Λ equations. A family of background solutions of the Einstein-Euler-Λ system in CMCSH gauge is the spacetimes as described in the previous section. The spatial metric component of these spacetimes is a negative Einstein metric i.e., the spatial metric satisfies

$$R_{ij}(\gamma)=-\frac{1}{n}\gamma_{ij}. \tag{3.72}$$

Let us denote the space of metrics satisfying equation (4.16) by $\mathcal{E}in_{-\frac{1}{n}}$

Let $\gamma^{*}\in\mathcal{E}in_{-\frac{1}{n}}$ and $\mathcal{V}$ be its connected component. Also consider $\mathcal{S}_{\gamma}$ to be the harmonic slice of the identity diffeomorphism i.e., the set of $\gamma\in\mathcal{E}in_{-\frac{1}{n}}$ for which the identity map $id:(M,\gamma)\rightarrow(M,\gamma^{*})$ is harmonic (since any metric $\gamma\in\mathcal{E}in_{-\frac{1}{n}}$ verifies the fixed point criteria $R[\gamma]_{ij}=-\frac{1}{n}\gamma_{ij}$, it should also satisfy the CMCSH gauge condition with respect to a background metric and in this case the background metric is simply chosen to be γ^{*}, another element of $\mathcal{E}in_{-\frac{1}{n}}$). This condition is equivalent to the vanishing of the tension field $-V^{k}$ that is

$$-V^{k}=-\gamma^{ij}(\Gamma[\gamma]^{k}_{ij}-\Gamma[\gamma^{*}]^{k}_{ij})=0. \tag{3.73}$$

For $\gamma\in\mathcal{E}in_{-\frac{1}{n}}$, $\mathcal{S}_{\gamma}$ is a submanifold of $\mathcal{M}$ for γ sufficiently close to γ^{*} (easily proven using standard procedure and therefore we omit the proof, see [206] for detail). The deformation space $\mathcal{N}$ of $\gamma^{*}\in\mathcal{E}in_{-\frac{1}{n}}$ is defined as the intersection of the $\gamma^{*}-$connected component $\mathcal{V}\subset\mathcal{E}in_{-\frac{1}{n}}$ and the harmonic slice $\mathcal{S}_{\gamma}$ i.e.,

$$\mathcal{N}:=\mathcal{V}\cap\mathcal{S}_{\gamma}. \tag{3.74}$$

$\mathcal{N}$ is assumed to be smooth (i.e., equipped with C^{∞} topology). In the case of $n=3$, following the Mostow rigidity theorem, the negative Einstein structure is rigid [121]. This rigid Einstein structure in $n = 3$ corresponds to the hyperbolic structure up to isometry. For higher genus Riemann surfaces Σ_{genus} ($genus > 1$), space of distinct hyperbolic structures is the classical Teichmüller space diffeomorphic to $\mathbf{R}^{6genus-6}$. Tangent space to any point in the Teichmüller space corresponds to the 'neutral modes' of $2+1$ gravity defined by transverse-traceless tensors. However, $2+1$ gravity is fundamentally different from the higher dimensional cases since the former is devoid of gravitational waves degrees of freedom. For $n > 3$, the centre manifold $\mathcal{N}$ is a finite dimensional submanifold of $\mathcal{M}$ (and in particular of $\mathcal{M}_{-1}$). Following the analysis of [206], the formal tangent space $T_\gamma\mathcal{N}$ in local coordinates is expressible as

$$\frac{\partial\gamma}{\partial q^a} = l_a^{TT||} + L_{W_a^{||}}\gamma, \tag{3.75}$$

where $l_a^{TT||} \in \ker(\mathcal{L}) = C^{TT||}(S^2M) \subset C^{TT}(S^2M)$, $W^{||} \in \mathfrak{X}(M)$ satisfies through the time derivative of the CMCSH gauge condition $-\gamma^{ij}(\Gamma[\gamma]^k_{ij} - \Gamma[\gamma^*]^k_{ij}) = 0$

$$-[\nabla[\gamma]^m\nabla[\gamma]_m W^{||i} + R[\gamma]^i_m W^{||m}] + (l^{TT||} + L_{W^{||}}\gamma)^{mn}(\Gamma[\gamma]^k_{mn} - \Gamma[\gamma^*]^k_{mn}) = 0,$$

and, $\{q^a\}_{a=1}^{dim(\mathcal{N})}$ is a local chart on $\mathcal{N}$, $\mathfrak{X}(M)$ is the space of vector fields on M (in a suitable function space setting), $C^{TT}(S^2M)$ is the space of $\gamma-$fransverse-traceless $2-$tensors on M. An important thing to note is that all known examples of closed negative Einstein spaces have integrable deformation spaces and that the deformation spaces are stable i.e., $Spec\{\mathcal{L}_{\gamma,\gamma}\} \geq 0$. The spectrum of the operator $\mathcal{L}_{\gamma,\gamma}$ plays an important role determining the decay rates in the pure vacuum case considered by [151, 206].

3.2.2 Kinematics of the perturbations at the linear level

As discussed in the previous section, the Einstein moduli space is finite-dimensional for $n > 3$ and serves as the centre manifold of the dynamics (the deformation space $\mathcal{N}$ to be precise). On the other hand, for the $n = 3$ case, following Mostow rigidity, negative Einstein structure implies hyperbolicity and therefore the moduli space reduces to a point. We will consider the general case when the moduli space is finite-dimensional since the zero-dimensional moduli space case is trivial. Let $\mathcal{N}$ be the deformation space of γ^*. Now suppose, we perturb the metric about γ^*. The perturbation is defined to be $h := g - \gamma^*$ such that $||h||_{H^s} < \delta$ for a suitable $\delta > 0$. Due to the finite dimensionality of $\mathcal{N}$, g in fact could be an element of $\mathcal{N}$ lying within a $\delta-$ball of γ^*. But if g lies on $\mathcal{N}$, then corresponding solution satisfies the fixed point condition $Ric(g) = -\frac{1}{n}g$. Therefore these 'trivial' perturbations do not see the dynamics. The question is how to treat such perturbations which are tangent to $\mathcal{N}$. This was precisely handled by invoking a shadow gauge condition in the study of a small data fully nonlinear stability problem of the vacuum Einstein' equations by [206] and later used in [177] to handle the case with a positive cosmological constant. The shadow gauge reads

$$(g-\gamma) \perp \mathcal{N} \tag{3.76}$$

for $\gamma \in \mathcal{N}$, where the orthogonality is in the L^2 sense with respect to the metric γ. This precisely takes care of the motion of the orthogonal perturbations by studying the time evolution of $h := g-\gamma$, while the motion of the tangential perturbations is taken care of by evolving $\gamma \in \mathcal{N}$. This does not really affect the characteristics of the background spacetimes as γ does not leave the moduli space i.e., it is still a negative Einstein metric with Einstein constant $-\frac{1}{n}$. However, if we write the perturbations as $h_{ij} = g_{ij} - \gamma_{ij}$, then $\frac{\partial g_{ij}}{\partial t} = \frac{\partial h_{ij}}{\partial t} + \frac{\partial \gamma_{ij}}{\partial t}$ and therefore, $\frac{\partial \gamma_{ij}}{\partial t}$ enters into the evolution equation for the metric. But an important question is how exactly to control this term since it will enter into the time derivative of the energy as well. We will obtain the necessary estimates in a later section when we study the energy inequality. But how does the instability (if it occurs) materialize in $\mathcal{N}$? The instability is precisely described by the fact that the curve $\gamma(t)$ runs off the edge of $\mathcal{N}$ (leaves every compact set of $\mathcal{N}$) similar to the case of $2+1$ gravity where the solution curve runs off the edge of the Teichmüller space (played by $\mathcal{N}$ in the current context) at the limit of the big-bang singularity [146, 198]. See figure (1) which depicts the kinematics of the perturbations (notice that this notion of stability/ instability is general; here we specialize to the linearized problem). This notion of stability/instability is true in a general non-linear setting as well. Linearization is a special case which we will consider here.

One subtle point left to discuss is how to control the time derivative term '$g^{ij}\partial_t\Gamma[\gamma]^k_{ij}$' (or '$\gamma^{ij}\partial_t\Gamma[\gamma]^k_{ij}$' at the linear level) in the equation for the shift vector field (3.56). By a simple calculation, we will show that this contribution vanishes at the linear level. $\frac{\partial \gamma_{ij}}{\partial t}$ may be written as (equation (3.75))

$$\frac{\partial \gamma_{ij}}{\partial t} = l^{TT||}_{ij} + (L_{Z^{||}}\gamma)_{ij}, \tag{3.77}$$

where $Z^{||}$ satisfies through the CMCSH gauge condition (3.29),

$$-[\nabla[\gamma]^m\nabla[\gamma]_m Z^{||i} + R[\gamma]^i_m Z^{||m}] + (l^{TT||} + L_{Z^{||}}\gamma)^{mn}(\Gamma[\gamma]^k_{mn} - \Gamma[\gamma^*]^k_{mn}) = 0.$$

At the linear level, however, the term '$(l^{TT||} + L_{Z^{||}}\gamma)^{mn}(\Gamma[\gamma]^k_{mn} - \Gamma[\gamma^*]^k_{mn})$' vanishes since it is of second order. This yields

$$\nabla[\gamma]^m\nabla[\gamma]_m Z^{||i} + R[\gamma]^i_m Z^{||m} = 0 \tag{3.78}$$

leading to

$$-\int_M (\nabla[\gamma]_i Z^{||}_j \nabla[\gamma]^i Z^{||j} + \frac{1}{n}\gamma_{ij} Z^{||i} Z^{||j})\mu_\gamma = 0 \Rightarrow Z^{||} = 0 \tag{3.79}$$

on M. Therefore, the pure gauge part of the $\mathcal{N}-$tangential velocity $\frac{\partial \gamma_{ij}}{\partial t}$ vanishes yielding

$$\frac{\partial \gamma_{ij}}{\partial t} = l^{TT||}_{ij}. \tag{3.80}$$

A direct calculation yields the following

$$\begin{aligned}
\gamma^{ij}\partial_t\Gamma[\gamma]^k_{ij} &= \gamma^{ij} D\Gamma[\gamma]^k_{ij}\cdot\frac{\partial\gamma}{\partial t} \\
&= \gamma^{ki}\nabla[\gamma]^j(\partial_t\gamma_{ij}) - \frac{1}{2}\nabla[\gamma]^k tr_\gamma(\partial_t\gamma) \\
&= \gamma^{ki}\nabla[\gamma]^j l^{TT||}_{ij} - \frac{1}{2}\nabla[\gamma]^k tr_\gamma(l^{TT||}) \\
&= 0,
\end{aligned} \tag{3.81}$$

since $l^{TT||}$ is transeverse-traceless with respect to γ. We are left with one task of estimating the tangential velocity $l^{TT||}$, which will be done in a later chapter.

3.3 Linearized Einstein-Euler system

In this section we linearize the complete Einstein-Euler system and state a local existence theorem for such system. The background (re-scaled) solution obtained previously reads $(g_{B.G}, k^{tr}_{B.G}, N_{B.G}, X_{B.G}, \rho_{B.G}, v_{B.G}) = (\gamma, 0, 1, 0, C_\rho, 0)$, where $R_{ij}(\gamma) = -\frac{1}{n}$ and C_ρ is a positive constant. Of course, in the previous section, we have recalled that the Einstein structure is not rigid for the $n > 3$ cases. The perturbations to the background solutions may be described as follows $h := g-\gamma$, $\delta k^{tr} = k^{tr} - k^{tr}_{B.G} = k^{tr} - 0 = k^{tr}$, $\delta N =$

$N - N_{B.G} = N - 1, \delta X := X - X_{B.G} = X - 0 = X$, $\delta\rho := \rho - \rho_{B.G} = \rho - C_\rho$, $\delta v := v - v_{B.G} = v - 0 = v$. Now we linearize the Einstein-Euler system about this background solution. The linearized Euler equations read

$$\frac{\partial \delta\rho}{\partial t} = -\frac{C_\rho n \gamma_{ad} \dot{a}}{a}\delta N - \frac{1}{a}\gamma_{ad} C_\rho \nabla[\gamma]_i v^i, \tag{3.82}$$

$$\frac{\partial v^i}{\partial t} = \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})v^i - \frac{1}{a}\nabla[\gamma]^i \delta N - \frac{\gamma_{ad}-1}{a\gamma_{ad}C_\rho}\nabla[\gamma]^i \delta\rho. \tag{3.83}$$

Now, an obvious problem is that the $n-$velocity vector field v^i consists of irrotational, rotational and harmonic parts. Utilizing a Hodge decomposition [179] of the $n-$velocity $1-$form on the spatial manifold M, one may simply write the velocity vector field as follows

$$v = (d\phi)^\sharp + (\delta\psi)^\sharp + (\mathcal{H})^\sharp, \tag{3.84}$$

where $\phi \in \Omega^0(M), \psi \in \Omega^2(M), \mathcal{H} \in \Omega^1(M)$, and $\Delta\mathcal{H} = (d\delta + \delta d)\mathcal{H} = 0$ (in the weak sense). Existence of a metric ensures the required isomorphism between the space of vector fields and the space of 1-forms and such isomorphism is denoted by $\sharp$. At the linear level, the covariant divergence of the evolution equation for the velocity (3.83) yields

$$\Delta_\gamma\left(\frac{\partial\phi}{\partial t} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\phi + \frac{1}{a}\delta N + \frac{\gamma-1}{a\gamma_{ad}C_\rho}\delta\rho\right) = 0. \tag{3.85}$$

Now clearly, the kernel of the rough Laplacian Δ_γ on a closed connected M consists of constant functions only and therefore

$$\frac{\partial\phi}{\partial t} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\phi + \frac{1}{a}\delta N + \frac{\gamma-1}{a\gamma_{ad}C_\rho}\delta\rho = C(t). \tag{3.86}$$

One may absorb $C(t)$ in the scalar field ϕ since ϕ is unique up to functions constant on M_t. Let us make the transformation $\phi \mapsto \phi + F(t)$. The evolution equation for ϕ yields

$$\begin{aligned}\frac{\partial\phi}{\partial t} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\phi + \frac{1}{a}\delta N + \frac{\gamma-1}{a\gamma_{ad}C_\rho}\delta\rho + \frac{dF(t)}{dt} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})F(t)\\ = C(t).\end{aligned}$$

Now setting $\frac{dF}{dt} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})F(t) = C(t)$ (which has a unique solution $F(t) = F(t_0)(\frac{a(t)}{a(t_0)})^{n(\gamma_{ad}-\frac{n-1}{n})} + a(t)^{n(\gamma_{ad}-\frac{n-1}{n})}\int_{t_0}^{t}\frac{C(t^{'})}{a(t^{'})^{n(\gamma_{ad}-\frac{n-1}{n})}}dt^{'}$) , the fluid equations reduce to the following wave equation for the scalar potential ϕ (propagating with the speed of sound $c_s = \sqrt{\gamma_{ad} - 1}$)

$$\frac{\partial \phi}{\partial t} = \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\phi - \frac{1}{a}\delta N - \frac{\gamma_{ad} - 1}{a\gamma_{ad}C_\rho}\delta\rho, \tag{3.87}$$
$$\frac{\partial \delta\rho}{\partial t} = -\frac{C_\rho n \gamma_{ad}\dot{a}}{a}\delta N + \frac{1}{a}\gamma_{ad}C_\rho \Delta_\gamma \phi.$$

Notice that this is precisely the wave equation with respect to the 'sound' metric $S_{\mu\nu} := \hat{g}_{\mu\nu} + (1 - c_s^2)\mathbf{u}_\mu \mathbf{u}_\nu$ (this sound metric was first derived by [164] in the context of the stability of a black hole accretion problem). Therefore, we observe that the irrotational part of the $n-$velocity field is decoupled from the rotational and the harmonic parts. The question remains how do we extract the evolution equations for the remaining parts of the velocity vector field. Let us now focus on the harmonic contribution. For closed (closed negative Einstein in this case to be precise) manifolds, the space of harmonic forms (1-forms in this particular case) is a finite-dimensional vector space. Therefore, the harmonic form contribution of the $n-$velocity 1-form field may be written as

$$\mathcal{H} = \sum_{k=1}^{L} m^k \mathcal{A}_k, \tag{3.88}$$

where $\{\mathcal{A}\}_{k=1}^{l}$ form a basis of the space of harmonic forms. Here L is the first Betti number or the dimension of the first de-Rham cohomology group (or the space of the first singular cohomology group $H^1(M)$ with real coefficients). Now we may utilize the L^2 orthogonality property of the three constituents in the Hodge decomposition of the velocity field with respect to the natural Riemannian metric γ i.e.,

$$\int_M \langle d\phi, \mathcal{H}\rangle_\gamma = \int_M \langle \delta\psi, \mathcal{H}\rangle_\gamma = 0 \tag{3.89}$$

since $\mathcal{H} \in \ker(d\delta + \delta d)$. Therefore, the harmonic contribution decouples to yield the following set of ordinary differential equations for each harmonic part

$$\frac{dm^k}{dt} = \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})m^k. \tag{3.90}$$

Integration of this equation yields

$$m^k(t) = m^k(t_0)\left(\frac{a(t)}{a(t_0)}\right)^{n(\gamma_{ad} - \frac{n+1}{n})}. \tag{3.91}$$

Once we have extracted the time evolution of the harmonic contribution, the pure rotational part may be extracted by the application of a double 'curl' operator $\delta \circ d$. Notice that $(\delta\psi)^{\sharp}$, the pure rotational contribution and $\mathcal{H}^{\sharp}$, the harmonic contribution to the velocity field, both have zero covariant divergence (see the appendix). Let us denote the vector field counterparts $(\delta\psi)^{\sharp}$ of $\delta\psi$ and $\mathcal{H}^{\sharp}$ of $\mathcal{H}$ by ξ and χ, respectively i.e.,

$$v^i = \nabla[\gamma]^i\phi + \xi^i + \chi^i. \tag{3.92}$$

Here ξ^i and χ^i satisfy (in the weak sense of course since they belong to $H^{s-1}, s > \frac{n}{2}+1$)

$$\nabla[\gamma]_i\xi^i = 0, \tag{3.93}$$

$$\nabla[\gamma]_i\chi^i = 0. \tag{3.94}$$

(i.e., $\exists\Theta \in C^{\infty}(M)$ such that $\int_M (\nabla[\gamma]_i\Theta)\xi^i = 0 = \int_M (\nabla[\gamma]_i\Theta)\chi^i$) The action of double curl $\delta \circ d$ on the evolution equation for the velocity (3.83) yields

$$\frac{\partial[(\delta \circ dv^{\flat})^{\sharp}]^i}{\partial t} = \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})[(\delta \circ dv^{\flat})^{\sharp}]^i, \tag{3.95}$$

since the action of curl annihilates the gradient by definition (exterior derivative to be precise). Noting $(\delta A)_i = \nabla[\gamma]^j A_{ij}\ \forall A \in S^0_2(M)$, $d^2\phi = 0 = d\chi^{\flat}$, One may explicitly evaluate $[(\delta \circ dv^{\flat})^{\sharp}]^i$ to yield

$$\begin{aligned}
[(\delta \circ dv^{\flat})^{\sharp}]^i &= \nabla[\gamma]_j(\nabla[\gamma]^i\xi^j - \nabla[\gamma]^j\xi^i), \qquad (3.96)\\
&= -\nabla[\gamma]_j\nabla[\gamma]^j\xi^i + \nabla[\gamma]^i\nabla[\gamma]_j\xi^j + R^i_j\xi^j\\
&= (\delta^i_j\Delta_\gamma + R[\gamma]^i_j)\xi^j,
\end{aligned}$$

where we have used the divergence free property of ξ. Therefore, one obtains the following evolution equation for ξ

$$(\delta^i_j\Delta_\gamma + R[\gamma]^i_j)\left(\frac{d\xi^j}{dt} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\xi^j\right) = 0. \tag{3.97}$$

Now, the kernel of the Hodge Laplacian $(d\delta + \delta d)$ is precisely the space of harmonic forms. Therefore,

$$\frac{\partial\xi^i}{\partial t} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\xi^i = \zeta^i(t,x) \tag{3.98}$$

for $\zeta^\flat \in \ker(d\delta + \delta d)$. On the other hand, $\xi^\flat$ is always unique up to a harmonic form $\kappa^\flat$ since $\delta\xi^\flat = \delta(\xi^\flat + \kappa^\flat)$. Therefore, if we make a transformation $\xi(t,x) \mapsto \xi(t,x) + \kappa(t,x)$, the evolution equation becomes

$$\frac{\partial \xi^i}{\partial t} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\xi^i + \left(\frac{\partial \kappa^i}{\partial t} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\kappa^i\right) = \zeta^i(t,x). \tag{3.99}$$

Now setting $\frac{\partial \kappa^i}{\partial t} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\kappa^i = \zeta^i(t,x)$ (which has a unique solution $\kappa^i(t,x) = \kappa^i(t_0,x)(\frac{a(t)}{a(t_0)})^{n(\gamma_{ad}-\frac{n-1}{n})} + a(t)^{n(\gamma_{ad}-\frac{n-1}{n})}\int_{t_0}^{t}\frac{\zeta^i(t',x)}{a(t')^{n(\gamma_{ad}-\frac{n-1}{n})}}dt'$), the evolution equation for the rotational part ξ reduces to

$$\frac{d\xi^i}{dt} - \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\xi^i = 0 \tag{3.100}$$

from which one obtains the following stability criteria

$$\gamma_{ad} < \frac{n+1}{n}. \tag{3.101}$$

Direct integration of the evolution equation for the curl part (3.100) yields

$$\xi^i(t) = \xi^i(t_0)\left(\frac{a(t)}{a(t_0)}\right)^{n(\gamma_{ad}-\frac{n+1}{n})}. \tag{3.102}$$

Clearly, the curl part satisfies an ordinary differential equation in time and therefore we may control its Sobolev norm of arbitrary order. Provided that the adiabatic index γ_{ad} lies in the suitable range (3.101), the curl part of the $n-$velocity field decays.

Notice another remarkable fact that the three parts of the velocity field are decoupled at the linear level. In other words, the coupling is necessarily of higher order. Now we turn to the linearized Einstein's equations imposing constant mean extrinsic curvature spatially harmonic (CMCSH) gauge. Note that the following lemma holds

Lemma 3: *Let h_{ij} be the perturbation to the metric γ_{ij}. The perturbation to the Ricci tensor in CMCSH gauge satisfies*

$$\delta R_{ij} + \frac{1}{n}h_{ij} = DR[\gamma]_{ij}\cdot h + \frac{1}{n}h_{ij} = \frac{1}{2}\mathcal{L}_{\gamma,\gamma}h_{ij}, \tag{3.103}$$

where $\mathcal{L}_{\gamma,\gamma}h_{ij}$ is defined to be

$$\mathcal{L}_{\gamma,\gamma}h_{ij} := \Delta_\gamma h_{ij} + (R[\gamma]^k{}_{ij}{}^m + R[\gamma]^k{}_{ji}{}^m)h_{km}. \tag{3.104}$$

Proof: Using the Frechet derivative formula for the connection coefficents $D\Gamma[\gamma]^i_{jk}\cdot h = \frac{1}{2}\gamma^{il}(\nabla[\gamma]_j h_{lk}+\nabla[\gamma]_k h_{jl} - \nabla[\gamma]_l h_{ij})$, one may readily obtain the first variation of the Ricci tensor in the direction of h

$$\begin{aligned}\delta R_{ij} = DR[\gamma]_{ij}\cdot h = \frac{1}{2}\{\Delta_\gamma h_{ij} + (R[\gamma]^k{}_{ij}{}^m + R[\gamma]^k{}_{ji}{}^m)h_{km} \\ +\nabla[\gamma]_j\nabla[\gamma]_m h^m_i + \nabla[\gamma]_i\nabla[\gamma]_m h^m_j + R[\gamma]_{kj}h^k_i + R[\gamma]_{ki}h^k_j - \nabla[\gamma]_i\nabla[\gamma]_j tr_\gamma h\}.\end{aligned} \tag{3.105}$$

The linearized version of the spatial harmonic gauge '$id : (M,\gamma+h)\to(M,\gamma)$ is harmonic' yields

$$\begin{aligned}\gamma^{ij}D\Gamma[\gamma]^k_{ij}\cdot h = 0 \\ 2\nabla[\gamma]_j h^{ij} - \nabla[\gamma]^i tr_\gamma h = 0.\end{aligned} \tag{3.106}$$

Therefore, the potentially problematic term $\nabla[\gamma]_j\nabla[\gamma]_m h^m_i + \nabla[\gamma]_i\nabla[\gamma]_m h^m_j$ obstructing the ellipticity of $h_{ij}\mapsto DR[\gamma]_{ij}\cdot h$ becomes

$$\begin{aligned}\nabla[\gamma]_j\nabla[\gamma]_m h^m_i + \nabla_i\nabla_m h^m_j &= \frac{1}{2}(\nabla[\gamma]_i\nabla[\gamma]_j + \nabla[\gamma]_i\nabla[\gamma]_j)tr_\gamma h \\ &= \nabla[\gamma]_i\nabla[\gamma]_j tr_\gamma h,\end{aligned} \tag{3.107}$$

which gets cancelled with $-\nabla[\gamma]_i\nabla[\gamma]_j tr_\gamma h$ and therefore

$$\begin{aligned}\delta R_{ij} = DR[\gamma]_{ij}\cdot h &= \frac{1}{2}\{\Delta_\gamma h_{ij} + (R[\gamma]^k{}_{ij}{}^m + R[\gamma]^k{}_{ji}{}^m)h_{km} + R[\gamma]_{kj}h^k_i \\ &\quad +R[\gamma]_{ki}h^k_j\} \\ &= \frac{1}{2}\{\Delta_\gamma h_{ij} + (R[\gamma]^k{}_{ij}{}^m + R[\gamma]^k{}_{ji}{}^m)h_{km}\} \\ &\quad -\frac{1}{n}h_{ij},\end{aligned}$$

i.e.,

$$\delta R_{ij} + \frac{1}{n}h_{ij} = DR[\gamma]_{ij}\cdot h + \frac{1}{n}h_{ij} = \mathcal{L}_{\gamma,\gamma}h_{ij}, \tag{3.108}$$

where we have used the fact that the background metric is negative Einstein ($R[\gamma]_{ij} = -\frac{1}{n}\gamma_{ij}$). This concludes the proof of the lemma.

Therefore, we establish that $h\mapsto DR[\gamma]_{ij}\cdot h$ is linear elliptic, which will help us to cast the

Einstein evolution equations as a linear hyperbolic system. The linearized Einstein-Euler evolution and constraint equations read

$$
\begin{aligned}
\frac{\partial h_{ij}}{\partial t} &= -\frac{2}{a}k^{tr}_{ij} + \frac{2\dot{a}}{a}\delta N\gamma_{ij} + \underbrace{l^{TT||}_{ij}}_{I} + \frac{1}{a}(L_X\gamma)_{ij}, && (3.109)\\
\frac{\partial k^{tr}_{ij}}{\partial t} &= -(n-1)\frac{\dot{a}}{a}k^{tr}_{ij} - \frac{1}{a}\nabla[\gamma]_i\nabla[\gamma]_j\delta N + +\frac{\gamma-2}{n-1}a^{1-n\gamma_{ad}}\delta\rho\gamma_{ij} + \frac{1}{2a}\mathcal{L}_{\gamma,\gamma}h_{ij}\\
&\quad +\frac{1}{a}\left\{\frac{1}{n-1}(1+\gamma_{ad}a^{2-n\gamma_{ad}}C_\rho)\gamma_{ij} - \frac{1}{n}\gamma_{ij}\right\}\delta N, && (3.110)\\
\frac{\partial \phi}{\partial t} &= \underbrace{\frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})}_{<0}\phi - \frac{1}{a}\delta N - \frac{\gamma_{ad}-1}{C_\rho\gamma_{ad}a}\delta\rho, && (3.111)\\
\frac{\partial \delta\rho}{\partial t} &= -\frac{C_\rho n\gamma_{ad}\dot{a}}{a}\delta N + \frac{C_\rho\gamma_{ad}}{a}\Delta_\gamma\phi, && (3.112)\\
\frac{d\xi^i}{dt} &= \underbrace{\frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})}_{<0}\xi^i, && (3.113)\\
\frac{d\chi^i}{dt} &= \underbrace{\frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})}_{<0}\chi^i, && (3.114)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_\gamma tr_\gamma h + \nabla[\gamma]^i\nabla[\gamma]^j h_{ij} - R[\gamma]_{ij}h^{ij} &= 2a^{2-n\gamma_{ad}}\delta\rho, && (3.115)\\
\nabla[\gamma]_j k^{trij} &= -\gamma_{ad}a^{2-n\gamma_{ad}}C_\rho v^i. && (3.116)
\end{aligned}
$$

along with the elliptic equations for the lapse function (perturbation) and the shift vector field arising as a consequence of the gauge fixing

$$\Delta_\gamma\delta N + \frac{1}{n-1}\left(1 + C_\rho n\gamma_{ad}a^{2-n\gamma_{ad}}\right)\delta N = -\frac{n\gamma_{ad}-2}{n-1}a^{2-n\gamma_{ad}}\delta\rho, \quad (3.117)$$

$$\Delta_\gamma X^i - R[\gamma]^i_j X^j = -(n-2)\frac{\dot{a}}{a}\nabla[\gamma]^i\delta N - 2\nabla[\gamma]_j k^{trij}. \quad (3.118)$$

Notice that the negativity of the underlined terms in the linearized evolution equations is crucial in our analysis since these will generate uniform decay of the matter fields. Here we have used the fact that the term '$g^{ij}\partial_t\Gamma[\gamma]^k_{ij}$'in the shift equation (3.56) vanishes at the linear level (as shown in section 3.2). Notice that a problematic term $\nabla[\gamma]_j k^{trij}$ arises in the right hand side of the shift equation, which requires a higher regularity of the trace-less second fundamental form k^{tr}. However, we may immediately utilize the momentum constraint (3.23) to get rid of this term and obtain

$$\Delta_\gamma X^i - R[\gamma]^i_j X^j = -(n-2)\frac{\dot{a}}{a}\nabla[\gamma]^i\delta N + 2\gamma_{ad}C_\rho a^{2-n\gamma_{ad}}v^i. \quad (3.119)$$

Here $v = (d\phi)^{\sharp} + \xi + \chi$. Observe that the $\mathcal{N}-$tangential velocity $l^{TT||}$ entered into the linearized evolution equation for the metric perturbations h_{ij} and we need to estimate it (term I in the equation). Choosing a basis $\frac{\partial\gamma}{\partial q^a} = m_a^{TT||} + L_{W_a^{||}}\gamma$ for the formal tangent space of $\mathcal{N}$ at γ, the time derivative of the shadow gauge condition yields at the linear level

$$
\begin{aligned}
\langle \partial_t h, m_a^{TT||} + L_{W_a^{||}}\gamma\rangle_\gamma &= 0 \\
\langle -\frac{2}{a}k^{tr} + \frac{2\dot{a}}{a}\delta N\gamma + l^{TT||} + \frac{1}{a}L_X\gamma, m_a^{TT||} + L_{W_a^{||}}\gamma\rangle_\gamma &= 0 \\
\langle -\frac{2}{a}k^{tr} + l^{TT||}, m_a^{TT||} + L_{W_a^{||}}\gamma\rangle_\gamma &= 0 \\
\langle -\frac{2}{a}(k^{TT} + L_Y\gamma - \frac{2}{n}\nabla[\gamma]_m Y^m\gamma) + l^{TT||}, m_a^{TT||} + L_{W_a^{||}}\gamma\rangle_\gamma &= 0 \\
\langle -\frac{2}{a}k^{TT} + l^{TT||}, m_a^{TT||}\rangle_\gamma = 0 \Rightarrow l^{TT||} &= \frac{2}{a}k^{TT||}.
\end{aligned}
\tag{3.120}
$$

Here we have utilized the L^2 orthogonality between A^{TT} and $L_B\gamma$ for $B \in \mathfrak{X}(M)$ and A^{TT} is a TT-tensor with respect to γ. $k^{TT||}$ denotes the projetion of the transverse-traceless part of k^{tr} onto the kernel of $\mathcal{L}_{\gamma,\gamma}$ (as we have discussed in section 3.1). In the energy inequality, we will observe that the term $l^{TT||}$ is rather innocuous at the linear level since action of $\mathcal{L}_{\gamma,\gamma}$ annihilates it (and only contributes at higher order).

3.3.1 Local Well-posedness

[150] proved a well-posedness theorem for the Cauchy problem for a family of elliptic-hyperbolic systems that included the '$n+1$' dimensional vacuum Einstein equations in CMCSH gauge. [229] sketched how to apply the theorem of [150] to a gauge fixed system of 'Einstein-Λ' field equations. However, a local existence theorem for the fully non-linear Einstein-Euler system in CMCSH gauge remains open till today. [180] utilized a CMC transported spatial coordinate condition (zero shift) to establish a local and subsequently a global existence result for the sufficiently small however fully non-linear irrotational Einstein-Euler system on $(0,\infty) \times \mathbb{T}^3$. In the case of an irrotational fluid, one readily has a nonlinear wave equation (in a 'sound metric') for the scalar potential. One potential complication that arises in the case of a general Einstein-Euler system is that the rotational contribution and the harmonic contribution of the $n-$velocity vector field are coupled non-linearly to the scalar field. Such questions are not however relevant at the linear level and left for future study. At this linear level, the system (3.109)-(3.119) forms a coupled linear elliptic hyperbolic system. A straightforward energy argument together with a contraction mapping on the Banach space $C^0([0,T]; H^s \times H^{s-1} \times H^{s-1} \times H^s \times H^{s-1} \times H^{s-1})$ (i.e., $(h(t), k^{tr}(t), \delta\rho(t), \phi(t), \xi(t), \chi(t)) \in$

$H^s \times H^{s-1} \times H^{s-1} \times H^s \times H^{s-1} \times H^{s-1}$) with $s > \frac{n}{2}+1$ would yield an existence result. However, a subtlety is that in the energy expressions, we will have the shift vector field X and the perturbation to the lapse function δN. Therefore, we need to ensure that they are uniquely determined or in other words that the following elliptic equations have unique solutions

$$\Delta_\gamma \delta N + \frac{1}{n-1}\left(1 + C_\rho n\gamma_{ad} a^{2-n\gamma_{ad}}\right)\delta N = -\frac{n\gamma_{ad}-2}{n-1} a^{2-n\gamma_{ad}}\delta\rho, \tag{3.121}$$

$$\Delta_\gamma X^i - R[\gamma]^i_j X^j = -(n-2)\frac{\dot{a}}{a}\nabla[\gamma]^i \delta N + 2\gamma_{ad} C_\rho a^{2-n\gamma_{ad}} v^i \tag{3.122}$$

in $H^{s+1}\times H^{s+1}$. But, this is equivalent to showing that the operators $\Delta_\gamma + \frac{1}{n-1}\left(1 + C_\rho n\gamma_{ad} a^{2-n\gamma_{ad}}\right) id$ and $\delta^i_j \Delta_\gamma - R[\gamma]^i_j$ have trivial kernels on $H^{s+1}\times H^{s+1}$. However, this follows from the fact that $1 + C_\rho n\gamma_{ad} a^{2-n\gamma_{ad}} > 0$ and $R[\gamma]_{ij} = -\frac{1}{n}\gamma_{ij}$ [150, 206]. Note that once the perturbation to the lapse is uniquely determined in terms of $\delta\rho$, the shift vector field is then uniquely determined in terms of v (ϕ, ξ, χ) and $\delta\rho$. The detailed elliptic estimates are stated in the next section. Uniqueness and the continuation criteria is straightforward for this linear elliptic-hyperbolic system. The only difference between our equations here and the vacuum work studied by [150] is the appearance of matter terms. However these appear with good signs and so the analysis proceeds in the same way. For completeness we state the following theorem concerning the local well-posedness.

Local well-posedness theorem: *Let $s > \frac{n}{2}+1$. The CMCSH Cauchy problem of the linearized Einstein-Euler$-\Lambda$ system with initial data $(h_0, k^{tr}_0, \delta\rho_0, \phi_0, \xi_0, \chi_0) \in H^s \times H^{s-1}\times H^{s-1}\times H^s \times H^{s-1}\times H^{s-1}$ and $\Lambda \geq 0$ is strongly well posed in $C^0([0,T^\dagger]; H^s \times H^{s-1}\times H^{s-1}\times H^s\times H^{s-1}\times H^{s-1})$. In particular, there exists a time $T^\dagger > 0$ depending on $||h_0||_{H^s}, ||k^{tr}_0||_{H^{s-1}}, ||\delta\rho_0||_{H^{s-1}}, ||\phi_0||_{H^s}, ||\xi_0||_{H^{s-1}}, ||\chi_0||_{H^{s-1}}$ such that the solution map $(h_0, k^{tr}_0, \delta\rho_0, \phi_0, \xi_0, \chi_0) \mapsto (h(t), k^{tr}(t), \delta\rho(t), \phi(t), \xi(t), \chi(t), N(t), X(t))$ is continuous as a map*

$$\begin{aligned} &H^s \times H^{s-1}\times H^{s-1}\times H^s\times H^{s-1}\times H^{s-1} \to \\ &H^s \times H^{s-1}\times H^{s-1}\times H^s\times H^{s-1}\times H^{s-1}\times H^{s+1}\times H^{s+1}. \end{aligned} \tag{3.123}$$

Let $T^\dagger$ be the maximal time of existence of the solution to the CMCSH Cauchy problem with data $(h_0, k^{tr}_0, \delta\rho_0, \phi_0, \xi_0, \chi_0)$. Then either $T^\dagger = +\infty$ or

$$\begin{aligned} \lim_{t\to T^\dagger} \sup \max(&||h(t)||_{H^s}, ||k^{tr}(t)||_{H^{s-1}}, ||\delta\rho(t)||_{H^{s-1}}, \\ &||\phi(t)||_{H^s}, ||\xi(t)||_{H^{s-1}}, ||\chi(t)||_{H^{s-1}}) = +\infty. \end{aligned} \tag{3.124}$$

In addition to the local well-posedness, one needs to ensure the conservation of gauges and constraints, that is, the following entities must vanish identically along the solution curve

$$A^i := \nabla[\gamma]^i\tau, \tag{3.125}$$

$$B^k := \gamma^{ij}(\Gamma[\gamma+h]^k_{ij} - \Gamma[\gamma]^k_{ij}) = \gamma^{ij} D\Gamma^k[\gamma]_{ij} \cdot h, \tag{3.126}$$

$$D := \Delta_\gamma tr_\gamma h + \nabla^i\nabla^j h_{ij} - R[\gamma]_{ij}h^{ij} - 2a^{2-n\gamma}\delta\rho, \tag{3.127}$$

$$E^i := \nabla[\gamma]_j k^{trij} + \gamma a^{2-n\gamma} C v^i. \tag{3.128}$$

The vanishing of the entity A^i precisely states the imposition of the constant mean extrinsic curvature gauge i.e., τ is a function of time t alone. Vanishing of D is essentially the conservation of the linearized version of the Hamiltonian constraint. One might naturally expect the conservation of the gauges and constraints due to the Bianchi identities. However, utilizing the evolution equations of the relevant fields, one may also directly obtain a system of coupled PDEs after a lengthy but straightforward calculation. Utilizing an energy argument similar to [150], one may show that if $(A, B, D, E) = 0$ for the initial data $(h_0, k_0^{tr}, \delta\rho_0, v_0)$, then $(A, B, D, E) \equiv 0$ along the solution curve $t \mapsto (h(t), k^{tr}(t), \delta\rho(t), v(t), \delta N(t), X(t))$. At the linear level, one sees trivially that $A^i \equiv 0$. Here, of course, by v we mean the triple (ϕ, ξ, χ).

3.3.2 Elliptic Estimates

In this section, we estimate δN and X^i from their respective elliptic equations. These estimates are necessary to derive the desired energy inequality. At each step, we carefully keep track of the time dependent expansion factor.

Lemma 4: *Let $s > \frac{n}{2} + 1$ and assume $(\delta N, X^i, h_{ij}, k^{tr}_{ij}, \delta\rho, v^i)$ solves the linearized Einstein-Euler system expressed in CMCSH gauge. The following estimate holds for the linearized perturbation to the lapse function δN*

$$||\delta N||_{H^{s+1}} \leq C a^{2-n\gamma_{ad}} ||\delta\rho||_{H^{s-1}}, \tag{3.129}$$

where a is the scale factor for the background solution and $C > 0$ is a suitable constant dependent on the background solution (3.1).

Proof: The elliptic equation for the perturbation to the lapse function reads

$$\Delta_\gamma \delta N + \frac{1}{n-1}\left(1 + C_\rho n\gamma_{ad} a^{2-n\gamma_{ad}}\right)\delta N = -\frac{n\gamma_{ad} - 2}{n-1} a^{2-n\gamma_{ad}} \delta\rho. \tag{3.130}$$

Now, we have shown in the previous section that the operator $\Delta_\gamma + \frac{1}{n-1}\left(1 + C_\rho n\gamma_{ad} a^{2-n\gamma_{ad}}\right) id : H^{s+1} \to H^{s-1}$ is injective and following its self adjointness, also surjective (it has closed range). Therefore, following this isomorphism, we immediately obtain

$$||\delta N||_{H^{s+1}} \leq Ca^{2-n\gamma_{ad}}||\delta\rho||_{H^{s-1}}. \tag{3.131}$$

A short proof may be as follows. Let us denote the operator $\Delta_\gamma + \frac{1}{n-1}\left(1 + C_\rho n\gamma_{ad} a^{2-n\gamma_{ad}}\right) id$ by A. Assume that an estimate of the type $||u||_{H^{s+1}} \lesssim ||Au||_{H^{s-1}}$ does not hold. Then there exists a sequence $\{u_i\}_{i=1}^{\infty}$ with $||u_i||_{H^{s+1}} = 1$ and $||Au_i||_{H^{s-1}} \to 0$ as $i \to \infty$. Now M is compact and therefore H^{s+1} is compactly embedded into H^{s-1} (notice $s > \frac{n}{2}+1$). This yields a sub-sequence $\{u_{i_j}\}$ converging to $u^* \in H^{s+1}$ strongly in H^{s-1}, which by construction satisfies $Au^* = 0$. This contradicts the fact that the operator A is injective. Therefore an estimate of the type $||u||_{H^{s+1}} \lesssim ||Au||_{H^{s-1}}$ holds $\forall s \geq 1$ (which is satisfied here since $s > \frac{n}{2} + 1$). Here, the constant C depends on the background entities as evident from the elliptic equation.

Lemma 5: *Let $s > \frac{n}{2} + 1$ and assume $(\delta N, X^i, h_{ij}, k^{tr}_{ij}, \delta\rho, v^i)$ solves the linearized Einstein-Euler system expressed in CMCSH gauge. The following estimate holds for the linearized perturbation to the shift vector field*

$$||X||_{H^{s+1}} \leq C(\dot{a}a^{1-n\gamma_{ad}}||\delta\rho||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}}), \tag{3.132}$$

where a is the scale factor for the background solution and $C > 0$ is a suitable constant dependent on the background solution (3.1).

Proof: The elliptic equation for the shift vector field reads

$$\Delta_\gamma X^i - R[\gamma]^i_j X^j = -(n-2)\frac{\dot{a}}{a}\nabla[\gamma]^i \delta N + 2\gamma_{ad} C_\rho a^{2-n\gamma_{ad}} v^i. \tag{3.133}$$

Now given that γ is negative Einstein i.e., $R[\gamma]_{ij} = -\frac{1}{n}\gamma_{ij}$, the operator $P := \delta^i_j \Delta_\gamma - R[\gamma]^i_j : H^{s+1} \to H^{s-1}$ has trivial kernel. We may use a similar argument as that of the previous case to obtain the following estimate

$$
\begin{aligned}
||X||_{H^{s+1}} &\leq C\left(||\frac{-(n-2)\dot{a}}{a}\nabla[\gamma]^{i}\delta N+2\gamma_{ad}C_{\rho}a^{2-n\gamma_{ad}}v^{i}||_{H^{s-1}}\right) \\
&\leq C\left(\frac{\dot{a}}{a}||\nabla[\gamma]^{i}\delta N||_{H^{s-1}}+C_{\rho}a^{2-n\gamma_{ad}}||v^{i}||_{H^{s-1}}\right) \\
&\leq C\left(\frac{\dot{a}}{a}||\delta N||_{H^{s}}+C_{\rho}a^{2-n\gamma_{ad}}||v^{i}||_{H^{s-1}}\right) \\
&\leq C\left(\dot{a}a^{1-n\gamma_{ad}}||\delta\rho||_{H^{s-1}}+C_{\rho}a^{2-n\gamma_{ad}}||v||_{H^{s-1}}\right).
\end{aligned} \tag{3.134}
$$

Here we have used the fact that $||A||_{H^{s_1}} \lesssim ||A||_{H^{s_2}}$ for $s_1 < s_2$ and the estimate for the perturbation to the lapse function from the previous lemma (4). This concludes the proof of the lemma.

3.4 Energy functional

In the study of hyperbolic PDE, the use of an energy functional is indispensable. Since the linearized Einstein-Euler system is a linear coupled hyperbolic system (except that the rotational and harmonic parts of the velocity field are decoupled), one may naturally define a wave equation type of energy and its higher-order extension which controls the desired norm of the data $(h, k^{tr}, \delta\rho, \phi, \xi, \chi)$. Note that at the linear level, the rotational (ξ) and the harmonic (χ) parts of the velocity field simply satisfy ordinary differential equations in time and therefore, we may control their Sobolev norm of any order. Let us first define the lowest order energy $\mathcal{E}_1$

$$
\begin{aligned}
\mathcal{E}_1 \quad := \quad & \frac{(\gamma_{ad}-1)}{2\gamma_{ad}^2C_\rho^2}\langle\delta\rho,\delta\rho\rangle_{L^2}+\frac{1}{2}\langle\phi,\Delta_\gamma\phi\rangle_{L^2}+\frac{1}{2}\langle\xi,\xi\rangle_{L^2} \\
&+\frac{1}{2}\langle\mathcal{H},\mathcal{H}\rangle_{L^2}+\frac{1}{2}\langle k^{tr},k^{tr}\rangle_{L^2}+\frac{1}{8}\langle h,\mathcal{L}_{\gamma,\gamma}h\rangle_{L^2}.
\end{aligned} \tag{3.135}
$$

Here Δ_γ, Δ, and $\mathcal{L}_{\gamma,\gamma}$ are defined as follows

$$
\begin{aligned}
\Delta_\gamma := -\gamma^{ij}\nabla[\gamma]_i\nabla[\gamma]_j, \Delta\xi^i := \Delta_\gamma\xi^i+R[\gamma]^i_j\xi^j, \mathcal{L}_{\gamma,\gamma}h_{ij} := \Delta_\gamma h_{ij} \\
+(R[\gamma]^k{}_{ij}{}^m+R[\gamma]^k{}_{ji}{}^m)h_{km}.
\end{aligned} \tag{3.136}
$$

Clearly, Δ and $\mathcal{L}_{\gamma,\gamma}$ are the Hodge Laplacian and a Lichnerowicz type Laplacian, respectively. They both have non-negative spectrum on M. The lowest order energy may explicitly be written as follows

$$\begin{aligned}\mathcal{E}_1 = \frac{(\gamma_{ad}-1)}{2\gamma_{ad}^2 C_\rho^2}\int_M (\delta\rho)^2\mu_\gamma + \frac{1}{2}\int_M \nabla[\gamma]_i\phi\nabla[\gamma]_j\phi\gamma^{ij}\mu_\gamma + \frac{1}{2}\int_M \xi^i\xi^j\gamma_{ij}\mu_\gamma \\ +\frac{1}{2}\int_M \mathcal{H}_i\mathcal{H}_j\gamma^{ij}\mu_\gamma + \frac{1}{2}\int_M k^{tr}_{ij}k^{tr}_{kl}\gamma^{ik}\gamma^{jl}\mu_\gamma \\ +\frac{1}{8}\int_M (\nabla[\gamma]_k h_{ij}\nabla[\gamma]_l h_{mn}\gamma^{kl}\gamma^{im}\gamma^{jn} + (R[\gamma]^k{}_{ij}{}^m + R[\gamma]^k{}_{ji}{}^m)h_{km}h_{ln}\gamma^{il}\gamma^{jn})\mu_\gamma.\end{aligned}$$

Using the structure of the lowest order energy, the higher order energies may be defined as follows

$$\begin{aligned}\mathcal{E}_i \quad := \quad & \frac{(\gamma_{ad}-1)}{2\gamma_{ad}^2 C_\rho^2}\langle\delta\rho, \Delta_\gamma^{i-1}\delta\rho\rangle_{L^2} + \frac{1}{2}\langle\phi, \Delta_\gamma^i\phi\rangle_{L^2} + \frac{1}{2}\langle\xi, \Delta^{i-1}\xi\rangle_{L^2} \\ & +\frac{1}{2}\langle\mathcal{H}, \Delta_\gamma^{i-1}\mathcal{H}\rangle_{L^2} + \frac{1}{2}\langle k^{tr}, \mathcal{L}_{\gamma,\gamma}^{i-1}k^{tr}\rangle_{L^2} + \frac{1}{8}\langle h, \mathcal{L}_{\gamma,\gamma}^i h\rangle_{L^2},\end{aligned}$$

for $1 \le i \le s$. The total energy is naturally defined as

$$\mathcal{E} := \sum_{i=1}^s \mathcal{E}_i. \tag{3.137}$$

The energy is positive semi-definite and it only vanishes precisely when $(h, k^{tr}, \delta\rho, \phi, \xi, \chi) \equiv 0$. The first variation of the energy about 0 in the direction of $\mathcal{X} := (u, v, w, x, y, z)$ vanishes

$$D\mathcal{E}(0)\cdot\mathcal{X} = 0, \tag{3.138}$$

that is, 0 is a critical point of $\mathcal{E}$ in the space $\mathcal{Q} := H^s \times H^{s-1} \times H^{s-1} \times H^s \times H^{s-1} \times H^{s-1}$. The Hessian of $\mathcal{E}$ at 0 reads

$$\begin{aligned}D^2\mathcal{E}(0)\cdot(\mathcal{X},\mathcal{X}) = \frac{(\gamma_{ad}-1)}{\gamma_{ad}^2 C_\rho^2}\sum_{i=1}^s\langle w, \Delta_\gamma^{i-1}w\rangle_{L^2} + \sum_{i=1}^s\langle x, \Delta_\gamma^i x\rangle_{L^2} \\ +\sum_{i=1}^s\langle y, \Delta^{i-1}y\rangle_{L^2} + \sum_{i=1}^s\langle z, \Delta_\gamma^{i-1}z\rangle_{L^2} + \sum_{i=1}^s\langle v, \mathcal{L}_{\gamma,\gamma}^{i-1}v\rangle_{L^2} \\ +\frac{1}{4}\sum_{i=1}^s\langle u, \mathcal{L}_{\gamma,\gamma}^i u\rangle_{L^2} \ge 0.\end{aligned} \tag{3.139}$$

The equality hold iff $u = u^{TT||}, v = x = y = z = 0$ (at the linear level $u^{TT||} = 0$). Therefore, the positive semi-definiteness of the Lyapunov function on $B_\delta(0) - \{0\} \subset \mathcal{Q}$ is heavily dependent on the non-negativity of the spectrum of the associated rough Laplacian Δ_γ, Hodge Laplacian Δ, and the

Lichnerowicz type Laplacian $\mathcal{L}_{\gamma,\gamma}$. Now, since the hessian is positive definite on $B_\delta(0)-\{0\}$, the map $D^2\mathcal{E}(0): B_\delta(0)-\{0\} \to Image(D^2\mathcal{E}(0))$ is an isomorphism yielding

$$\begin{aligned} ||h||^2_{H^s}+||k^{tr}||^2_{H^{s-1}}+||\delta\rho||^2_{H^{s-1}}+||\phi||^2_{H^s}+||\xi||^2_{H^{s-1}}+||\chi||^2_{H^{s-1}} \\ \lesssim D^2\mathcal{E}(0)\cdot(\mathcal{X},\mathcal{X}). \end{aligned} \tag{3.140}$$

Now, using a version of the Morse lemma (Hilbert space version) on the non-degenerate critical point $(0,0)$, we obtain that there exists a $\delta>0$ such that for variations lying within $B_\delta(0)$, the following holds up to a possibly non-linear diffeomorphism $\mathcal{E}=\mathcal{E}(0)+D^2\mathcal{E}(0)\cdot(\mathcal{X},\mathcal{X})=D^2\mathcal{E}(0)\cdot(\mathcal{X},\mathcal{X})$ (notice that $\mathcal{E}(0)=0$). Therefore, we prove that the energy controls the desired norm of the perturbations i.e.,

$$||h||^2_{H^s}+||k^{tr}||^2_{H^{s-1}}+||\delta\rho||^2_{H^{s-1}}+||\phi||^2_{H^s}+||\xi||^2_{H^{s-1}}+||\chi||^2_{H^{s-1}} \lesssim \mathcal{E}. \tag{3.141}$$

3.4.1 **Uniform boundedness of the energy functional**

In this section, we prove the uniform boundedness of the energy functional and it's possible in time decay. In the process, we obtain the necessary and sufficient conditions for such estimates to hold. To obtain the uniform boundedness property of the energy, the necessary ingredients such as elliptic estimates are at hand. We therefore explicitly compute the time derivative for the lowest order energy defined in the previous section. An analogous calculation holds for the time derivative of the higher-order energies as well. We will use the Sobolev embedding on a compact domain $||A||_{L^\infty} \lesssim ||A||_{H^a}$ for $a>\frac{n}{2}$ and the following product estimates

$$||AB||_{H^s} \lesssim (||A||_{L^\infty}||B||_{H^s}+||A||_{H^s}||B||_{L^\infty}), s>0, \tag{3.142}$$

$$||AB||_{H^s} \lesssim ||A||_{H^s}||B||_{H^s}, s>\frac{n}{2}, \tag{3.143}$$

$$\begin{aligned} ||[P,A]B||_{H^a} \lesssim (||\nabla A||_{L^\infty}||B||_{H^{s+a-1}}+||A||_{H^{s+a}}||B||_{L^\infty}), \\ P\in\mathcal{OP}^s, s>0, a\geq 0, \end{aligned} \tag{3.144}$$

where $\mathcal{OP}^s$ denotes the pseudo-differential operators with symbol in the Hormander class $S^s_{1,0}$ (see [175] for details). The first and second inequalities essentially emphasize the algebra property of $H^s\cap L^\infty$ for $s>0$ and of H^s for $s>\frac{n}{2}$, respectively. In addition, we of course use integration by parts, Holder's and Minkowski's inequalities whenever necessary.

Lemma 6: *Let $s>\frac{n}{2}+1$, $\gamma\in\mathcal{E}in_{-\frac{1}{n}}$ be the shadow of $g\in\mathcal{M}$ and assume $(h=g-\gamma, k^{tr}, \delta\rho, \phi, \xi, \chi)$*

satisfy the linearized Einstein-Euler system. Also assume there exists a $\delta > 0$ *such that* $(h, k^{tr}, \delta\rho, \phi, \xi, \chi) \in \mathcal{B}_\delta(0) \subset H^s \times H^{s-1} \times H^{s-1} \times H^s \times H^{s-1} \times H^{s-1}$. *The time derivative of the lowest order energy reads*

$$\begin{aligned}
\frac{\partial \mathcal{E}_1}{\partial t} &= -(n-1)\frac{\dot{a}}{a}\langle k^{tr}, k^{tr}\rangle_{L^2} + \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \phi, \Delta_\gamma \phi\rangle_{L^2} \\
&+ \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \xi, \Delta\xi\rangle_{L^2} + \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \mathcal{H}, \Delta_\gamma \mathcal{H}\rangle_{L^2} \\
&+ \mathcal{B},
\end{aligned} \tag{3.145}$$

where $\mathcal{B}$ *satisfies*

$$\begin{aligned}
|\mathcal{B}| &\lesssim (\gamma_{ad} - 1)\frac{\dot{a}}{a}||\delta\rho||_{H^{s-1}}||\delta N||_{H^{s+1}} + \frac{1}{a}||\phi||_{H^s}||\delta N||_{H^{s+1}} + \frac{1}{a}||\delta N||_{H^{s+1}}||k^{tr}||_{H^{s-1}} \\
&+ \frac{\dot{a}}{a}||\delta N||_{H^{s+1}}||h||_{H^s} + \frac{1}{a}||X||_{H^{s+1}}||h||_{H^s} + \frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E}_1.
\end{aligned}$$

Proof: A direct calculation using the evolution equations (3.109)-(3.116) yields

$$\begin{aligned}
\frac{\partial \mathcal{E}_1}{\partial t} &= \frac{\gamma_{ad} - 1}{\gamma_{ad}^2 C_\rho^2}\langle \partial_t \delta\rho, \delta\rho\rangle_{L^2} + \langle \partial_t \phi, \Delta_\gamma \phi\rangle_{L^2} + \langle \partial_t \xi, \xi\rangle_{L^2} \\
&+ \langle \partial_t \chi, \chi\rangle_{L^2} + \langle \partial_t k^{tr}, k^{tr}\rangle_{L^2} + \frac{1}{4}\langle \partial_t h, \mathcal{L}_{\gamma,\gamma} h\rangle_{L^2} + \mathcal{ER} \\
&= \frac{\gamma_{ad} - 1}{\gamma_{ad}^2 C_\rho^2}\langle -\frac{C_\rho n \gamma_{ad}\dot{a}}{a}\delta N + \frac{1}{a}C_\rho \gamma_{ad}\Delta_\gamma \phi, \delta\rho\rangle_{L^2} \\
&+ \langle \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\phi - \frac{1}{a}\delta N - \frac{\gamma_{ad} - 1}{aC_\rho \gamma_{ad}}\delta\rho, \Delta_\gamma \phi\rangle_{L^2} \\
&+ \langle \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\xi, \xi\rangle_{L^2} + \langle \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\chi, \chi\rangle_{L^2} \\
&+ \langle -(n-1)\frac{\dot{a}}{a}k^{tr} - \frac{1}{a}\nabla[\gamma]\otimes\nabla[\gamma]\delta N + \frac{1}{a}\left\{\frac{1}{n-1}(1 + \gamma_{ad}a^{2-n\gamma_{ad}}C_\rho)\gamma - \frac{1}{n}\gamma\right\}\delta N \\
&+ \frac{\gamma_{ad} - 2}{n-1}a^{1-n\gamma_{ad}}\delta\rho\gamma + \frac{1}{2a}\mathcal{L}_{\gamma,\gamma}h_{ij}, k^{tr}\rangle_{L^2} + \frac{1}{4} < -\frac{2}{a}k^{tr} + \frac{2\dot{a}}{a}\delta N\gamma + l^{TT||} \\
&+ \frac{1}{a}L_X \gamma, \mathcal{L}_{\gamma,\gamma}h\rangle_{L^2} + \mathcal{ER} \\
&= \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \phi, \Delta_\gamma \phi\rangle_{L^2} + \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \xi, \xi\rangle_{L^2} \\
&+ \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \chi, \chi\rangle_{L^2} - (n-1)\frac{\dot{a}}{a}\langle k^{tr}, k^{tr}\rangle_{L^2} \\
&- \frac{(\gamma_{ad} - 1)n}{\gamma_{ad}C_\rho}\frac{\dot{a}}{a}\langle \delta N, \delta\rho\rangle_{L^2} - \frac{1}{a}\langle \delta N, \Delta_\gamma \phi\rangle_{L^2} - \frac{1}{a}\langle \nabla[\gamma]\otimes\nabla[\gamma]\delta N, k^{tr}\rangle_{L^2} \\
&+ \frac{1}{2}\frac{\dot{a}}{a}\langle \delta N\gamma, \mathcal{L}_{\gamma,\gamma}h\rangle_{L^2} + \frac{1}{4a}\langle L_X \gamma, \mathcal{L}_{\gamma,\gamma}h\rangle_{L^2} + \mathcal{ER}.
\end{aligned} \tag{3.146}$$

Here $(\nabla[\gamma]\otimes\nabla[\gamma]\delta N)_{ij} := \nabla[\gamma]_i \nabla[\gamma]_j \delta N$. Now the rationale behind choosing the coefficients of different terms in the energy expression (such as $\frac{\gamma_{ad}-1}{\gamma_{ad}^2 C_\rho^2}$ term with $\langle \delta\rho, \delta\rho\rangle_{L^2}$) may be seen as follows.

Note that in the time evolution of the lowest order energy, we have terms like $\langle \delta\rho, \Delta_{\gamma}\phi\rangle_{L^2}$ and $\langle k^{tr}, \mathcal{L}_{\gamma,\gamma}h\rangle_{L^2}$. These terms are dangerous because, when we reach the term with highest Sobolev regularity, the higher order versions of these mixed terms $\langle \delta\rho, \Delta^s_{\gamma}\phi\rangle_{L^2}$, $\langle k^{tr}, \mathcal{L}^s_{\gamma,\gamma}h\rangle_{L^2}$ require regularity more than the maximum available regularity. Therefore, we want these terms to get cancelled with their negative counterparts. Note that with this particular choice of the positive coefficients of different terms in the energy expression, these dangerous terms are cancelled point-wise. Therefore, we are able to control the time derivative of the energy functional in terms of the maximum available Sobolev norms of the perturbations.

There are additional terms that appear when we apply ∂_t on the volume form μ_γ since $\partial_t\gamma = l^{TT||} = \frac{1}{a}k^{TT||}$. However, since $\partial_t\mu_\gamma = \frac{1}{2}\mu_\gamma\gamma(\partial_t\gamma)$, this term does not contribute as a result of $\gamma-$trace-less property of $k^{TT||}$. In addition there are terms that arise due to appearance of the metric γ in the definition of the inner product. We denote these additional terms by $\mathcal{ER}$. Making use of the $\gamma-$traceless property of k^{tr} and $k^{TT||}$ i.e., $\gamma^{ij}k^{tr}_{ij} = 0 = \gamma^{ij}k^{TT||}_{ij}$ and $\mathcal{L}_{\gamma,\gamma}k^{TT||} = 0$, the error term $\mathcal{ER}$ may be written as follows (using Sobolev embedding $||k^{TT||}||_{L^\infty} \lesssim ||k^{TT||}||_{H^{s-1}}$ for $s > \frac{n}{2} + 1$)

$$|\mathcal{ER}| \lesssim \frac{1}{a}||k^{TT||}||_{H^{s-1}}\mathcal{E}_1. \tag{3.147}$$

Now using the momentum constraint (3.116), $k^{tr}_{ij} = k^{TT}_{ij} + (L_Y\gamma - \frac{2}{n}\nabla_m Y^m\gamma)_{ij}$, $(L_Y\gamma)_{ij} := \gamma_{ik}\nabla[\gamma]_j Y^k + \gamma_{ki}\nabla[\gamma]_i Y^k$, and $2\nabla_m Y^m =_\gamma (L_Y\gamma)$, we obtain

$$|\mathcal{ER}| \lesssim \frac{1}{a}\left(||k^{tr}||_{H^{s-1}} + ||Y||_{H^s}\right)\mathcal{E}_1 \tag{3.148}$$

where Y satisfies the following elliptic equation

$$\nabla[\gamma]^i\nabla[\gamma]_i Y^j + R[\gamma]^j_i Y^i + (1-\frac{2}{n})\nabla[\gamma]^j(\nabla[\gamma]_i Y^i) = a^{2-n\gamma_{ad}}C_\rho v^i. \tag{3.149}$$

Elliptic regularity yields the following estimate for Y

$$||Y||_{H^{s+1}} \lesssim C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}}. \tag{3.150}$$

Therefore, $\mathcal{ER}$ satisfies $|\mathcal{ER}| \lesssim \frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E}_1$. Putting everything together, we obtain

$$\begin{aligned}\frac{\partial \mathcal{E}_1}{\partial t} &= -(n-1)\frac{\dot a}{a} < k^{tr}, k^{tr} >_{L^2} + \frac{n\dot a}{a}(\gamma_{ad} - \frac{n+1}{n}) < \phi, \Delta_\gamma \phi >_{L^2} \\ &\quad + \frac{n\dot a}{a}(\gamma_{ad} - \frac{n+1}{n}) < \xi, \xi >_{L^2} + \frac{n\dot a}{a}(\gamma_{ad} - \frac{n+1}{n}) < \mathcal{H}, \mathcal{H} >_{L^2} \\ &\quad + \mathcal{B}.\end{aligned} \tag{3.151}$$

Trivial power counting of a and $\dot a$ yields the desired estimate of $\mathcal{B}$

$$\begin{aligned}|\mathcal{B}| &\lesssim (\gamma_{ad} - 1)\frac{\dot a}{a}||\delta\rho||_{H^{s-1}}||\delta N||_{H^{s+1}} + \frac{1}{a}||\phi||_{H^s}||\delta N||_{H^{s+1}} + \frac{1}{a}||\delta N||_{H^{s+1}}||k^{tr}||_{H^{s-1}} \\ &\quad + \frac{\dot a}{a}||\delta N||_{H^{s+1}}||h||_{H^s} + \frac{1}{a}||X||_{H^{s+1}}||h||_{H^s} + \frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E}_1.\end{aligned}$$

Similarly, the time derivatives of the higher order energies may be computed. The following lemma states the time derivatives of the higher order energies.

Lemma 7: *Let $s > \frac{n}{2}+1$, $\gamma \in \mathcal{E}in_{-\frac{1}{n}}$ be the shadow of $g \in \mathcal{M}$ and assume $(h = g-\gamma, k^{tr}, \delta\rho, \phi, \xi, \chi)$ satisfy the linearized Einstein-Euler system. Also assume there exists a $\delta > 0$ such that $(h, k^{tr}, \delta\rho, \phi, \xi, \chi) \in \mathcal{B}_\delta(0) \subset H^s \times H^{s-1} \times H^{s-1} \times H^s \times H^{s-1} \times H^{s-1}$. The time derivative of the higher order energy reads*

$$\begin{aligned}\frac{\partial \mathcal{E}_i}{\partial t} &= -(n-1)\frac{\dot a}{a}\langle k^{tr}, \mathcal{L}^{i-1}_{\gamma,\gamma} k^{tr}\rangle_{L^2} + \frac{n\dot a}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \phi, \Delta^i_\gamma \phi\rangle_{L^2} \\ &\quad + \frac{n\dot a}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \xi, \Delta^{i-1}\xi\rangle_{L^2} + \frac{n\dot a}{a}(\gamma_{ad} - \frac{n+1}{n})\langle \mathcal{H}, \Delta^{i-1}_\gamma \mathcal{H}\rangle_{L^2} \\ &\quad + \mathcal{B}_i,\end{aligned}$$

where $\mathcal{B}$ satisfies

$$\begin{aligned}|\mathcal{B}_i| &\lesssim (\gamma_{ad} - 1)\frac{\dot a}{a}||\delta\rho||_{H^{s-1}}||\delta N||_{H^{s+1}} + \frac{1}{a}||\phi||_{H^s}||\delta N||_{H^{s+1}} + \frac{1}{a}||\delta N||_{H^{s+1}}||k^{tr}||_{H^{s-1}} \\ &\quad + \frac{\dot a}{a}||\delta N||_{H^{s+1}}||h||_{H^s} + \frac{1}{a}||X||_{H^{s+1}}||h||_{H^s} + \frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E}_i\end{aligned}$$

for $1 \leq i \leq s$.

Proof: With a similar type of calculation using the evolution equations of the Einstein-Euler system together with the inequalities (3.142-3.144) whenever necessary, we obtain the time derivatives of the higher order energies. The estimate of the residue terms $\mathcal{B}_i$ may be obtained by carefully keeping track of the explicitly time dependent terms a and $\dot a$.

Utilizing the previous two lemmas, we may now obtain the time derivative of the total energy. The total energy satisfies the following differential equation

$$
\begin{aligned}
&\frac{\partial \mathcal{E}}{\partial t} \qquad (3.152)\\
&= -(n-1)\frac{\dot{a}}{a}\sum_{i=1}^{s}\langle k^{tr}, \mathcal{L}_{\gamma,\gamma}^{i-1}k^{tr}\rangle_{L^2} + \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\sum_{i=1}^{s}\langle \phi, \Delta_{\gamma}^{i}\phi\rangle_{L^2} \\
&+\frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\sum_{i=1}^{s}\langle \xi, \Delta^{i-1}\xi\rangle_{L^2} + \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\sum_{i=1}^{s}\langle \mathcal{H}, \Delta_{\gamma}^{i-1}\mathcal{H}\rangle_{L^2} \\
&+\mathcal{G},
\end{aligned}
$$

where $\mathcal{G}$ satisfies

$$
\begin{aligned}
|\mathcal{G}| &\lesssim (\gamma_{ad}-1)\frac{\dot{a}}{a}||\delta\rho||_{H^{s-1}}||\delta N||_{H^{s+1}} + \frac{1}{a}||\phi||_{H^s}||\delta N||_{H^{s+1}} + \frac{1}{a}||\delta N||_{H^{s+1}}||k^{tr}||_{H^{s-1}} \\
&+\frac{\dot{a}}{a}||\delta N||_{H^{s+1}}||h||_{H^s} + \frac{1}{a}||X||_{H^{s+1}}||h||_{H^s} + \frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E}.
\end{aligned}
$$

Now we make use of the elliptic estimates (lemma 4-lemma 5). An estimate of the shift vector field reads

$$
||X||_{H^{s+1}} \leq C(\dot{a}a^{1-n\gamma_{ad}}||\delta\rho||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}}). \qquad (3.153)
$$

Now the $n-$velocity field v may be estimated in terms of its irrotational, rotational, and harmonic parts as follows

$$
\begin{aligned}
||v||_{H^{s-1}} &\leq ||(d\phi)^{\sharp}||_{H^{s-1}} + ||\xi||_{H^{s-1}} + ||\mathcal{H}||_{H^{s-1}} \qquad (3.154)\\
&\leq ||\phi||_{H^s} + ||\xi||_{H^{s-1}} + ||\mathcal{H}||_{H^{s-1}}.
\end{aligned}
$$

Therefore, the shift vector field may be estimated as follows

$$
||X||_{H^{s+1}} \lesssim \dot{a}a^{1-n\gamma_{ad}}||\delta\rho||_{H^{s-1}} + a^{2-n\gamma_{ad}}(||\phi||_{H^s} + ||\xi||_{H^{s-1}} + ||\mathcal{H}||_{H^{s-1}}) \qquad (3.155)
$$

which together with the estimate of the lapse perturbation (lemma 4)

$$
||\delta N||_{H^{s+1}} \lesssim a^{2-n\gamma_{ad}}||\delta\rho||_{H^{s-1}}, \qquad (3.156)
$$

yields

$$
\begin{aligned}
\mathcal{G} &\lesssim \dot{a}a^{1-n\gamma_{ad}}||\delta\rho||^2_{H^{s-1}} + a^{1-n\gamma_{ad}}(||\phi||^2_{H^s} + ||\delta\rho||^2_{H^{s-1}}) + a^{1-n\gamma_{ad}}(||\delta\rho||^2_{H^{s-1}} \\
&\quad + ||k^{tr}||^2_{H^{s-1}}) + \dot{a}a^{1-n\gamma_{ad}}(||\delta\rho||^2_{H^{s-1}} + ||h||^2_{H^s}) + a^{1-n\gamma_{ad}}(||\phi||^2_{H^s} + ||\xi||^2_{H^{s-1}} \\
&\quad + ||\chi||^2_{H^{s-1}} + ||h||^2_{H^s}) + \frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E} \\
&\lesssim (\dot{a}a^{1-n\gamma_{ad}} + a^{1-n\gamma_{ad}})||\delta\rho||^2_{H^{s-1}} + a^{1-n\gamma_{ad}}||\phi||^2_{H^s} + (\dot{a}a^{1-n\gamma_{ad}} + a^{1-n\gamma_{ad}})||h||^2_{H^s} \\
&\quad + a^{1-n\gamma_{ad}}||k^{tr}||^2_{H^{s-1}} + \frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E} \\
&\lesssim (\dot{a}a^{1-n\gamma_{ad}} + a^{1-n\gamma_{ad}})\mathcal{E} + \frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E}.
\end{aligned}
\tag{3.157}
$$

In the view of small data we may absorb $||k^{tr}||_{H^{s-1}}$ and $||v||_{H^{s-1}}$ in $\frac{1}{a}(||k^{tr}||_{H^{s-1}} + C_\rho a^{2-n\gamma_{ad}}||v||_{H^{s-1}})\mathcal{E}$ with a suitable constant. Therefore, the energy inequality becomes

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} \leq\; & -(n-1)\frac{\dot{a}}{a}\sum_{i=1}^{s}\langle k^{tr}, \mathcal{L}^{i-1}_{\gamma,\gamma}k^{tr}\rangle_{L^2} + \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\sum_{i=1}^{s}\langle\phi, \Delta^i_\gamma\phi\rangle_{L^2} \\
& + \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\sum_{i=1}^{s}\langle\xi, \Delta^{i-1}\xi\rangle_{L^2} + \frac{n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\sum_{i=1}^{s}\langle\mathcal{H}, \Delta^{i-1}_\gamma\mathcal{H}\rangle_{L^2} \\
& + C(\dot{a}a^{1-n\gamma_{ad}} + a^{1-n\gamma_{ad}} + a^{-1})\mathcal{E},
\end{aligned}
$$

which upon utilizing the first stability criterion

$$
\gamma_{ad} \leq \frac{n+1}{n} \tag{3.158}
$$

becomes

$$
\frac{d\mathcal{E}}{dt} \leq C(\dot{a}a^{1-n\gamma_{ad}} + a^{1-n\gamma_{ad}} + a^{-1})\mathcal{E}. \tag{3.159}
$$

Integration of the energy inequality yields

$$
\mathcal{E}(t) \lesssim \mathcal{E}(t_0)e^{C\left(-\frac{1}{n\gamma_{ad}-2}a^{2-n\gamma_{ad}} + \int_{t_0}^t a^{1-n\gamma_{ad}}dt' + \int_{t_0}^t a^{-1}dt'\right)}. \tag{3.160}
$$

Noting that $n\gamma_{ad} - 2 > 0$ $(n \geq 3)$, a second stability criterion in the expanding universe is obtained as

$$
\int_{t_0}^{\infty} a^{1-n\gamma_{ad}}dt < \infty, \;\; \int_{t_0}^{\infty} a^{-1}dt < \infty. \tag{3.161}
$$

Analysis up to now holds for $\Lambda \geq 0$. However, in order to conclude that the FLRW models considered are stable, we need to consider two separate cases: 1. $\Lambda > 0$, 2 $\Lambda = 0$.

3.4.2 $\Lambda > 0$ case

We may immediately check whether the stability is satisfied by the spacetimes under consideration including a positive cosmological constant Λ. From lemma (1), we know that the scale factor exhibits the following asymptotic behaviour for $\Lambda > 0$

$$a \sim e^{\sqrt{\frac{2\Lambda}{n(n-1)}}t} \tag{3.162}$$

as $t \to \infty$. Let $\sqrt{\frac{2\Lambda}{n(n-1)}} = \alpha > 0$. The integrability conditions $\int_{t_0}^{\infty} a^{1-n\gamma_{ad}}dt < \infty$ and $\int_{t_0}^{\infty} a^{-1}dt < \infty$ are trivially satisfied since, $1 - n\gamma_{ad} < 0$ for $\gamma_{ad} \in (1, \frac{n+1}{n}), n \geq 3$ and $\int_{t_0}^{\infty} e^{-(n\gamma_{ad}-1)\alpha t}dt < \infty, \int_{t_0}^{\infty} e^{-\alpha t}dt < \infty$. We may in fact establish the decay of the re-scaled perturbations (in a suitable norm) once we have established the uniform boundedness condition, that is,

$$\begin{aligned} &||h(t)||_{H^s} + ||k^{tr}(t)||_{H^{s-1}} + ||\delta\rho(t)||_{H^{s-1}} + ||v(t)||_{H^{s-1}} \\ \lesssim\ &||h(t_0)||_{H^s} + ||k^{tr}(t_0)||_{H^{s-1}} + ||\delta\rho(t_0)||_{H^{s-1}} + ||v(t_0)||_{H^{s-1}} < \delta \end{aligned} \tag{3.163}$$

for $t > t_0$ and sufficiently small δ. Following this uniform boundedness of the appropriate norm of the perturbations, we may immediately apply the local existence theorem to yield a global existence result. However, we may achieve better than the uniform boundedness in a rapidly expanding background since expansion is expected to 'kill' off the perturbations. We may in fact obtain decay of the perturbations in an appropriate norm. Since, the rotational and harmonic parts of the $n-$velocity field exhibit decay, we will focus on the scalar field ϕ, density perturbation $\delta\rho$, and the geometric entities h and k^{tr}. First we obtain the estimates for the lapse and the shift. Utilizing the elliptic estimates (lemma 4 and 5) and boundedness of $||\delta\rho||_{H^{s-1}}$,$||v||_{H^{s-1}}$ (3.163) one may obtain the following estimates in a straightforward way

$$||\delta N||_{H^{s+1}} \lesssim e^{-\alpha(n\gamma_{ad}-2)t}, \tag{3.164}$$

$$||X||_{H^{s+1}} \lesssim e^{-\alpha(n\gamma_{ad}-2)t}. \tag{3.165}$$

Now let us evaluate the time derivative of the following entities

$$\mathcal{E}_k := \frac{1}{2}\sum_{i=1}^{s-1}\langle k^{tr}, \mathcal{L}^{i-1}_{\gamma,\gamma}k^{tr}\rangle_{L^2} \tag{3.166}$$

and

$$\mathcal{E}_\phi := \frac{1}{2}\sum_{i=1}^{s-1}\langle \phi, \Delta^i_\gamma \phi\rangle_{L^2}. \tag{3.167}$$

An explicit calculation yields

$$\begin{aligned}\frac{\partial \mathcal{E}_k}{\partial t} &\leq -2\frac{n\dot{a}}{a}\mathcal{E}_k + C(\frac{1}{a}||k^{tr}||_{H^{s-2}}||h||_{H^s} + \frac{1}{a}||\delta N||_{H^s}||k^{tr}||_{H^{s-2}} \\ &\quad + a^{1-n\gamma_{ad}}(t)||\delta\rho||_{H^{s-2}}||k^{tr}||_{H^{s-2}}) + \frac{C}{a}, \\ \frac{\partial \mathcal{E}_\phi}{\partial t} &\leq \frac{2n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\mathcal{E}_\phi + C(\frac{1}{a}||\delta N||_{H^{s-1}}||\phi||_{H^{s-1}} + \frac{1}{a}||\delta\rho||_{H^{s-1}}||\phi||_{H^{s-1}}) \\ &\quad + \frac{C}{a}\end{aligned}$$

Now notice the following extremely important fact. Since the dangerous term $\langle k^{tr}, \mathcal{L}^i_{\gamma,\gamma}h\rangle_{L^2}$ does not get cancelled in $\frac{\partial \mathcal{E}_k}{\partial t}$, we lose one order of regularity i.e., we may only obtain a decay of $||k^{tr}||_{H^{s-2}}$ instead of $||k^{tr}||_{H^{s-1}}$ (similarly for ϕ, we may only control $||\phi||_{H^{s-1}}$). However, since we can consider $s > \frac{n}{2} + 2$, we will still obtain the desired point-wise decay. Utilizing the boundedness of the fields as $t \to \infty$, we may write the previous inequalities as follows

$$\frac{\partial \mathcal{E}_k}{\partial t} \leq -2\frac{n\dot{a}}{a}\mathcal{E}_k + \frac{C}{a}, \tag{3.168}$$

$$\frac{\partial \mathcal{E}_\phi}{\partial t} \leq \frac{2n\dot{a}}{a}(\gamma_{ad} - \frac{n+1}{n})\mathcal{E}_\phi + \frac{C}{a}. \tag{3.169}$$

Noticing $a \sim e^{\alpha t}$ and $\frac{\dot{a}}{a} \sim \alpha$ as $t \to \infty$, we have

$$\frac{\partial \mathcal{E}_k}{\partial t} \leq -2n\alpha\mathcal{E}_k + Ce^{-\alpha t}, \tag{3.170}$$

$$\frac{\partial \mathcal{E}_\phi}{\partial t} \leq 2n\alpha(\gamma_{ad} - \frac{n+1}{n})\mathcal{E}_\phi + Ce^{-\alpha t} \tag{3.171}$$

yielding

$$\mathcal{E}_k \lesssim e^{-\alpha t}, \tag{3.172}$$

$$\mathcal{E}_\phi \lesssim e^{-\zeta t}, \tag{3.173}$$

where $0 < \zeta := \min\big(\alpha, 2n\alpha(\frac{n+1}{n} - \gamma_{ad})\big)$. Note here that we need $\gamma_{ad} < \frac{n+1}{n}$ for a decay. This is extremely important. These decay estimates yield

$$||k^{tr}||_{H^{s-2}} \lesssim e^{-\frac{\alpha}{2}t}, ||\phi||_{H^{s-1}} \lesssim e^{-\frac{\zeta}{2}t}. \tag{3.174}$$

One may actually obtain a better decay estimate for $||k^{tr}||_{H^{s-2}}$ by an iterative argument. Going back to the evolution equation of $\mathcal{E}_k$

$$\frac{\partial \mathcal{E}_k}{\partial t} \leq -2\frac{n\dot{a}(t)}{a}\mathcal{E}_k + C(\frac{1}{a}||k^{tr}||_{H^{s-2}}||h||_{H^s} + \frac{1}{a}||\delta N||_{H^s}||k^{tr}||_{H^{s-2}}$$
$$+a^{1-n\gamma_{ad}}(t)||\delta N||_{H^s}||k^{tr}||_{H^{s-2}} + a^{1-n\gamma_{ad}}(t)||\delta\rho||_{H^{s-2}}||k^{tr}||_{H^{s-2}})$$

and substituting the estimate $||k^{tr}||_{H^{s-2}} \lesssim e^{-\alpha t/2}$ yields

$$\frac{\partial \mathcal{E}_k}{\partial t} \leq -2n\alpha\mathcal{E}_k + Ce^{-\alpha(1+\frac{1}{2})t} \tag{3.175}$$

as $t \to \infty$. Integrating the previous inequality, one immediately obtains

$$\mathcal{E}_k \lesssim e^{-\alpha(1+\frac{1}{2})t} \tag{3.176}$$

and therefore,

$$||k^{tr}||_{H^{s-2}} \lesssim e^{-\frac{\alpha}{2}(1+\frac{1}{2})t}. \tag{3.177}$$

Notice that we have gained an extra factor $\alpha/4$ in the rate of decay. Continuing in a similar way, we obtain the final decay rate of $\mathcal{E}_k$ to be the following sum

$$\alpha\left\{1+\frac{1}{2}(1+\frac{1}{2}(1+\frac{1}{2}(1+.........\right\} = \alpha\sum_{k=0}^{\infty}\frac{1}{2^k} = 2\alpha \tag{3.178}$$

and therefore we obtain

$$||k^{tr}||_{H^{s-2}} \lesssim e^{-\alpha t} \ or \ ||k^{tr}||_{L^{\infty}} \lesssim e^{-\alpha t}. \tag{3.179}$$

Now using the equation of motion for the metric g_{ij} (3.50), the estimates for lapse and shift (3.164,3.165), and the uniform boundedness result (3.163), one readily obtains

$$||\frac{\partial g}{\partial t}||_{H^{s-1}} \lesssim e^{-\alpha t} \tag{3.180}$$

which upon integration yields

$$||g(t) - g^{\dagger}||_{H^{s-1}} \lesssim e^{-\alpha t} \; as \; t \to \infty, \tag{3.181}$$

where $g^{\dagger} \in H^{s-1}$. We need to identify the limit metric $g^{\dagger}$. Since M is compact, H^{I} is compactly imbedded in H^{I-1}, $\forall I \geq 1$. From uniform boundedness (3.163), the limit metric $g^{\dagger}$ must therefore be in H^{s}. Now we need to estimate the density function. Using the evolution equation for $\rho = C_{\rho} + \delta\rho$, we obtain

$$||\rho(t) - \rho^{'}||_{H^{s-2}} \lesssim e^{-\beta t} \;\; as \;\; t \to \infty \tag{3.182}$$

for $\rho^{'} \in H^{s-2}$ and $\beta = \min(\alpha(n\gamma_{ad} - 2), \alpha + \zeta/2)$. From uniform boundedness (3.163) and compact imbedding, $\rho^{'} \in H^{s-1}$. Now utilizing the Hamiltonian constraint (3.53), Sobolev embedding $(H^{I}(M) \hookrightarrow L^{\infty}(M),\; I > \frac{n}{2})$, and the decay estimate of the relevant fields, one readily obtains

$$\lim_{t \to \infty} R(g) = -1 \tag{3.183}$$

i.e.,

$$||g - g^{\dagger}||_{H^{s-1}} \lesssim e^{-\alpha t} \tag{3.184}$$

as $t \to \infty$. Here $g^{\dagger} \in \mathcal{M}^{\epsilon}_{-1}$, where $\mathcal{M}^{\epsilon}_{-1}$ is a sufficiently small neighbourhood of the space of Einstein metrics $\mathcal{N}$ in the space of constant negative scalar curvature -1. Application of Sobolev embedding on compact domains (since we are now considering $s > \frac{n}{2} + 2$), the final pointwise decay estimates of the fields are as follows

$$||g(t) - g^{\dagger}||_{L^{\infty}} \lesssim e^{-\alpha t}, ||k^{tr}(t)||_{L^{\infty}} \lesssim e^{-\alpha t}, ||\rho(t) - \rho^{'}||_{L^{\infty}} \lesssim e^{-\beta t}, \tag{3.185}$$
$$||N(t) - 1||_{L^{\infty}} \lesssim e^{-\alpha(n\gamma_{ad}-2)t}, ||X(t)||_{L^{\infty}} \lesssim e^{-\alpha(n\gamma_{ad}-2)t}, ||l^{TT||}(t)||_{L^{\infty}} \lesssim e^{-2\alpha t}.$$

The individual constituents of h and k^{tr} may be estimated by using constraints. Using the momentum constraint (3.116) and $k^{tr}_{ij} = k^{TT}_{ij} + (L_Y g - \frac{2}{n}\nabla_m Y^m g)_{ij}$, one obtains

$$\nabla[\gamma]^i\nabla[\gamma]_i Y^j + R[\gamma]^j_i Y^i + (1-\frac{2}{n})\nabla[\gamma]^j(\nabla[\gamma]_i Y^i) = a^{2-n\gamma_{ad}} C_\rho v^i \tag{3.186}$$

Now of course the kernel of the elliptic operator on the left is trivial since the background metric is negative Einstein. Therefore, noting $||v||_{H^{s-1}} < C$, the following elliptic estimate holds

$$||Y||_{H^{s+1}} \lesssim e^{-\alpha(n\gamma_{ad}-2)t} \tag{3.187}$$

as $t \to \infty$ and therefore

$$||k^{TT}||_{H^{s-2}} \lesssim e^{-\alpha t}. \tag{3.188}$$

Notice that, by virtue of the transverse-traceless property of k^{TT}, we do not lose regularity of Y. Similarly, the Hamiltonian constraint (4.19) together with the spatial harmonic gauge condition may be utilized to obtain decay of the individual parts of $h_{ij} = h^{TT}_{ij} + f g_{ij} + (L_W g)_{ij}$ [184]. The first variation of the Hamiltonian constraint (4.19) with respect to (g, k^{tr}, ρ, v) at $(g=\gamma,\ k^{tr}=0, \rho = C_\rho, v=0)$ yields

$$\begin{aligned} DR[\gamma]\cdot h &= 2a^{2-n\gamma_{ad}}\delta\rho, \\ \Delta_\gamma tr_g h + \nabla^i\nabla^j h_{ij} - R[\gamma]_{ij}h^{ij} &= 2a^{2-n\gamma_{ad}}\delta\rho, \end{aligned} \tag{3.189}$$

where $R[\gamma]_{ij} = \frac{R(\gamma)}{n}\gamma_{ij} = -\frac{1}{n}\gamma_{ij}$. Upon substituting the decomposition $h_{ij} = h^{TT}_{ij} + f\gamma_{ij} + (L_W\gamma)_{ij}$ into the variation of the Hamiltonian constraint (4.19) and noticing that $\Delta_\gamma tr_\gamma(L_W g) + \nabla^i\nabla^j(L_W\gamma)_{ij} - R[\gamma]_{ij}(L_W\gamma)^{ij} \equiv 0$ yields

$$\begin{aligned} n\Delta_\gamma f + \gamma^{ij}\nabla_i\nabla_j f - R[\gamma]f &= 2a^{2-n\gamma_{ad}}\delta\rho \\ (n-1)\Delta_\gamma f - R[\gamma]f &= 2a^{2-n\gamma_{ad}}\delta\rho. \end{aligned} \tag{3.190}$$

Following the injectivity of $2\Delta_\gamma - R[\gamma]$ for $R[\gamma] < 0$ (which is the case here), we immediately obtain the following estimate

$$||f||_{H^{s+1}} \lesssim a^{2-n\gamma_{ad}}||\delta\rho||_{H^{s-1}}. \tag{3.191}$$

The remaining is the estimation of the vector field W. The vector field W is estimated in terms of f through the following elliptic equation, which follows from the spatial harmonic gauge, that is, '$id : (M, \gamma + h) \to (M, \gamma)$ is harmonic'. The linearized version of the spatial harmonic gauge yields

$$\gamma^{ij} D\Gamma^i_{jk}[\gamma] \cdot h = 0 \tag{3.192}$$
$$2\nabla[\gamma]_j h^{ij} - \nabla^i tr_\gamma h = 0$$
$$\Delta_\gamma W^i - R[\gamma]^i_j W^j = -\frac{n-2}{2}\nabla^i f.$$

Once again, $\delta^i_j \Delta_\gamma - R[\gamma]^i_j$ is injective since γ is negative Einstein. The estimate of W reads

$$||W||_{H^{s+2}} \lesssim ||f||_{H^{s+1}}. \tag{3.193}$$

Since, $||\delta\rho||_{H^{s-1}} \lesssim C$ and $a \sim e^{-\alpha t}$, one immediately obtains the following decay

$$||f||_{H^{s+1}} \lesssim e^{-(n\gamma_{ad}-2)\alpha t}, \tag{3.194}$$
$$||W||_{H^{s+2}} \lesssim e^{-(n\gamma_{ad}-2)\alpha t} \tag{3.195}$$

as $t \to \infty$. Using Sobolev embedding on compact domains, we may of course deduce the point-wise decay

$$||k^{TT}(t)||_{L^\infty} \lesssim e^{-\alpha t}, ||Y||_{L^\infty} \lesssim e^{-\alpha(n\gamma_{ad}-2)t}, ||f(t)||_{L^\infty} \lesssim e^{-(n\gamma_{ad}-2)\alpha t}, \tag{3.196}$$
$$||W(t)||_{L^\infty} \lesssim e^{-(n\gamma_{ad}-2)\alpha t}, ||l^{TT||}(t)||_{L^\infty} \lesssim e^{-2\alpha t}$$

as $t \to \infty$. Notice that the decay property of the matter fields are only possible if the adiabatic index lies in the interval $(1, \frac{n+1}{n})$. In the border line case of $\gamma_{ad} = \frac{n+1}{n}$, one can only prove a uniform boundedness at the linear level. The uniform boundedness and decay property lead to the following main theorem.

Main Theorem: *Let $(g_0, k^{tr}_0, \rho_0, v_0) \in B_\delta(\gamma, 0, C_\rho, 0) \subset H^s \times H^{s-1} \times H^{s-1} \times H^{s-1}$, where $s > \frac{n}{2} + 2$ and $\delta > 0$ is sufficiently small. Also consider that $\mathcal{N}$ is the integrable deformation space of γ and $\Lambda > 0$ is the cosmological constant. Assume that the adiabatic index γ_{ad} lie in the interval $(1, \frac{n+1}{n})$. Let $t \mapsto (g(t), k^{tr}(t), \rho(t), v(t))$ be the maximal development of the Cauchy problem for the linearized Einstein-Euler-Λ system about (3.71) in constant mean extrinsic curvature spatial harmonic gauge (CMCSH) (3.109-3.116) with initial data $(g_0, k^{tr}_0, \rho_0, v_0)$. Then there exists* *a* $\gamma^{\dagger} \in \mathcal{M}^{\epsilon}_{-1} \cap \mathcal{S}_{\gamma}$ *and* $\gamma^{*} \in \mathcal{N}$ *such that* $(g, \gamma, k^{tr}, \rho, v)$ *flows toward* $(\gamma^{\dagger}, \gamma^{*}, 0, \rho^{'}, 0)$ *in the limit of infinite time, that is,*

$$\lim_{t \to \infty} (g(t,x), \gamma(t,x), k^{tr}(t,x), \rho(t,x), v(t,x)) = (\gamma^{\dagger}(x), \gamma^{*}(x), 0, \rho^{'}(x), 0)$$

and moreover either $\gamma^{\dagger} = \gamma^{*}$ *or* γ^{*} *is the shadow of* $\gamma^{\dagger}$*. Here* $\mathcal{M}^{\epsilon}_{-1}$ *denotes the space of metrics of constant negative* (-1) *scalar curvature sufficiently close to and containing the deformation space* $\mathcal{N}$*.* $\mathcal{S}_{\gamma}$ *denotes a harmonic slice through the metric* γ*.* $\rho^{'}$ *is dependent on the initial density. The convergence is understood in the strong sense i.e., with respect to the available Sobolev norms.*

3.4.3 $\Lambda = 0$ **Case**

In our linearized analysis, the scale factor a associated with the background solution plays an important role. In the time coordinate t, we needed an integrability condition on $a(t)$ (3.161) in addition to the suitable range of the adiabatic index γ_{ad} (3.101). This integrability condition was satisfied by the scale factor associated with our background solutions since it exhibits asymptotically exponential behavior (3.42)

$$a \sim e^{\alpha t} \; as \; t \to \infty, \tag{3.197}$$

where $\alpha = \sqrt{\frac{2\Lambda}{n(n-1)}}$. Now an important question arises. Do we need a positive cosmological constant? In other words: do we need accelerated expansion to stabilize the self-gravitating relativistic fluid on a background that is already expanding? If we turn off the cosmological constant, then we have

$$a \sim t \; as \; t \to \infty, \tag{3.198}$$

and therefore a satisfies the first integrability condition of (3.161) (i.e., $\int_{t_0}^{\infty} a^{1-n\gamma_{ad}} dt < \infty$) but $\int_{t_0}^{\infty} a^{-1} dt < \infty$ is no longer satisfied. Therefore, we do not even obtain boundedness. However, we cannot claim that turning off the cosmological constant leads to instability but are simply unable to establish stability with the current method. Note that recently, using a special technique, [183] showed asymptotic stability of Milne universe in the presence of dust, where the cosmological constant is turned off (the scale factor behaves $\sim t$). However, we want to point out that the FLRW models considered here are different from the Milne model since the latter has a vanishing background energy density.

3.4.4 **Geodesic Completeness**

In order to establish the future completeness of the perturbed spacetime, we need to show that the solutions of the geodesic equation must exist for an infinite interval of the associated affine parameter. Here we need to be more precise about what we mean by the perturbed spacetimes. If the first variation of the metric is denoted by $\delta\hat{g}$ which satisfies the linearized equations along with a certain smallness condition (a re-scaled version of the physical perturbations to be precise). We show the geodesic completeness for the perturbed spacetimes $\hat{g} = \hat{g}_{FLRW} + \delta\hat{g}$. We provide a rough sketch of the proof in this context following [151]. Let's designate a timelike geodesic in the homotopy class of a family of timelike curves by $\mathcal{C}$. The tangent vector $\beta = \frac{d\mathcal{C}}{d\lambda} = \beta^{\mu}\partial_{\mu}$ to $\mathcal{C}$ for the affine parameter λ satisfies $\hat{g}(\beta, \beta) = -1$, where $\hat{g}$ is the spacetime metric. As $\mathcal{C}$ is causal, we may parametrize it as $(t, \mathcal{C}^i)$, $i = 1, 2, 3$. We must show that $\lim_{t\to\infty} \lambda = +\infty$, that is,

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{d\lambda}{dt'} dt' = +\infty. \tag{3.199}$$

Noting that $\beta^0 = \frac{dt}{d\lambda}$, we must show

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{\beta^0} dt' = +\infty. \tag{3.200}$$

Showing that $|\tilde{N}\beta^0|$ is bounded and therefore $\lim_{t\to\infty} \int_{t_0}^{t} \tilde{N} dt' = +\infty$ is enough to ensure the geodesic completeness. We first show that $|\tilde{N}\beta^0|$ is bounded. Proceeding the same way as that of [177] together with the estimates obtained in section 5.2, we obtain

$$\tilde{N}^2(\beta^0)^2 \leq C \tag{3.201}$$

as $t \to \infty$ for some $C < \infty$. Therefore, we only need to show that the following holds

$$\lim_{t\to\infty} \int_{t_0}^{t} \tilde{N} dt = \infty \tag{3.202}$$

in order to finish the proof of timelike geodesic completeness. Notice that $\tilde{N} = N = 1 + \delta N$. Now $|\delta N(t)| \leq C^{'} e^{-\alpha(n\gamma_{ad}-2)t}, 0 < C^{'} < \infty$ as $t \to \infty$ yielding

$$\lim_{t\to\infty}\left(\int_{t_0}^{t} dt - C^{'}\alpha(n\gamma - 2)(e^{-\alpha(n\gamma_{ad}-2)t_0} - e^{-\alpha(n\gamma_{ad}-2)t})\right) \leq \lim_{t\to\infty}\int_{t_0}^{t}\tilde{N}dt$$
$$\leq \lim_{t\to\infty}\left(\int_{t_0}^{t} dt + C^{'}\alpha(n\gamma_{ad} - 2)(e^{-\alpha(n\gamma_{ad}-2)t_0} - e^{-\alpha(n\gamma_{ad}-2)t})\right)$$
$$=> \lim_{t\to\infty}\int_{t_0}^{t}\tilde{N}dt = \infty.$$

This completes the proof of the geodesic completeness of the spacetimes of interest under linearized perturbations. The following theorem summarizes the results obtained so far.

Theorem: *Let $\mathcal{N}$ be the integrable deformation space of γ^*. Then $\exists\delta > 0$ such that for any $(g(t_0), k^{tr}(t_0), \delta\rho(t_0), \phi(t_0), \xi(t_0), \chi(t_0) \in B_\delta(\gamma^*, 0, C_\rho, 0, 0, 0) \subset H^s \times H^{s-1} \times H^{s-1} \times H^s \times H^{s-1} \times H^{s-1} \times H^{s-1}$,$s > \frac{n}{2} + 1$, the Cauchy problem for the re-scaled linearized Einstein-Euler system in constant mean extrinsic curvature (CMC) and spatial harmonic (SH) gauge is globally well posed to the future and the space-time is future complete.*

3.5 Concluding Remarks

We have proved a global existence result for the linearized Einstein-Euler-Λ system about a spatially compact negative spatial curvature FLRW solution (and its higher-dimensional generalization). In addition to the uniform boundedness of the evolving data in terms of its initial value, we proved a decay (in suitable norm) of the perturbations as well. This is an indication of the fact that the accelerated expansion-induced linear decay term may dominate the nonlinearities at the level of small data in a potential future proof of the fully non-linear stability. However, a serious difficulty - namely the possibility of shock formation - arises at the non-linear level in the perfect fluid matter model even without the presence of gravity. Coupling to gravity could make the scenario even more troublesome since pure gravity could itself blow up in finite time through curvature concentration. On the other hand, [166, 176] utilized the accelerated expansion (i.e., included a positive Λ) of the background solution on the spacetime of topological type $\mathbb{T}^3 \times \mathbb{R}$ to avoid the shock formation at the small data limit. Our study is significantly different from that of Rodnianski and Speck [166] or Speck [176] in a sense that we are concerned with negatively curved background (for physically interesting three spatial dimensions, we consider compact hyperbolic manifolds) unlike their study where compact flat torus $\mathbb{T}^3$ is used. As mentioned earlier, our study is motivated by the recent observations that do tend to indicate a negatively curved spatial universe. Secondly, they study the fully nonlinear stability (of small data perturbations) while we restrict ourselves with linear theory only. In terms of choice of gauge, [166] and [176] used a space-time harmonic gauge or wave gauge first by Choquet-Bruhat to prove a local existence theorem for vacuum Einstein's equations. We, on the other hand, used a constant mean extrinsic curvature spatial harmonic gauge (CMCSH) introduced by [150]. An advantage of choosing the CMCSH gauge is that one readily has nice elliptic estimates for the associated lapse function and the shift vector field. Despite these distinctions, we will observe a similar decay property in the current study with the presence of a positive cosmological constant. We also obtain a similar stability criterion that is similar to [166, 176]. A fully nonlinear analysis of the spacetimes of our interest is currently under intense investigation.

Even though there are numerous studies of cosmological perturbation theory [122, 123, 124, 125, 126, 127], they are primarily 'formal' mode stability results and in a sense (described in the introduction) do not address the 'true' linear stability problem. Our study, on the other hand, casts the Einstein-Euler-Λ system into a coupled initial value PDE problem (a 'Cauchy' problem to be precise). This facilitates the study of the 'true' linear stability of suitable background solutions via energy methods. In addition, even the 'formal' mode stability analysis is absent in the case of the spatially compact negatively curved FLRW model. This model is particularly interesting since there are infinitely many topologically distinct manifolds mathematically constructible by taking quotients of $\mathbb{H}^3$ by discrete and torsion-free subgroups of $SO^+(1,3)$ acting properly discontinuously. The existence of non-trivial harmonic forms precisely encodes such topological information, which as we have seen, inevitably appear in the field equations. In addition, we study the more general '$n+1$' model where the spatial part is described by a compact negative Einstein manifold (which, while restricted to the $n = 3$ case becomes compact hyperbolic through the application of the Mostow rigidity theorem). Our study recovers the mode stability results in a straightforward way (briefly described in the appendix). One interesting aspect of our study is that it hints towards the following fact: the FLRW spacetimes with a compact spatial manifold of constant negative sectional do not exhibit an asymptotic stability property with $\Lambda > 0$. Simply put, any perturbations to FLRW solutions may not return to solutions of the same type rather converge to nearby solutions characterized by a compact spatial manifold with constant negative scalar curvature. Simultaneously, the matter density perturbations may not approach zero as well. Since the local homogeneity and isotropy criteria require constancy of *sectional* curvature, not just the *scalar* curvature and a homogeneous matter distribution, a natural conclusion would be that the Einsteinian evolution asymptotically drives the homogeneous and isotropic universe (local) to an inhomogeneous and anisotropic state. This may indicate a possibility of 'structure formation' [145] which tends to break the homogeneity and isotropy of the physical universe. We do not however claim that such a mechanism is yet so compelling based on only a linear stability analysis.

Lastly, it is worth mentioning that our results indicate the linear stability if the adiabatic index γ_{ad} lies in the range $(1, \frac{n+1}{n})$. Such a result, while is a necessary condition for stability, is only valid in the linear regime. It would certainly be interesting to check if this criterion persists in the fully nonlinear level as well. As mentioned previously, in the presence of accelerated expansion, Rodnianski and Speck [166] and Speck [176] obtained the same condition for nonlinear stability of small data on $\mathbb{R} \times \mathbb{T}^3$. In addition to [166] and [176], avoidance of shock formation by exponential expansion was also studied by [182] for radiation fluid ($\mathbb{S}^3$ spatial topology), by [181] for $\gamma_{ad} \in (4/3, 3/2)$ ($\mathbb{T}^3$ spatial topology), [185] for dusts ($\mathbb{T}^3$ spatial topology). Recently by [183] removed the positive cosmological constant and proved the asymptotic stability of the Milne universe including dust. We will extend our study to the fully non-linear case in the future.

Chapter 4

The Thurston boundary of Teichmüller space is the space of big bang singularities of 2+1 gravity

We study the asymptotic behaviour of the solution curves of the dynamics of spacetimes of the topological type $\Sigma_g \times \mathbb{R}$, $g > 1$, where Σ_g is a closed Riemann surface of genus g, in the regime of $2+1$ classical general relativity. The configuration space of the gauge fixed dynamics is identified with the Teichmüller space ($\mathcal{T}\Sigma_g \approx \mathbb{R}^{6g-6}$) of Σ_g. Utilizing the properties of the Dirichlet energy of certain harmonic maps, estimates derived from the associated elliptic equations in conjunction with a few standard results of surface theory, we show that every non-trivial solution curve runs off the edge of the Teichmüller space at the limit of the big bang singularity and approaches the space of projective measured laminations/foliations ($\mathcal{PML}$ $\mathcal{PMF}$), the Thurston boundary of the Teichmüller space. This result which identifies the complete solution space of the Einstein equations on flat spacetimes of the type $\Sigma_g \times \mathbb{R}$, also yields yet another way to compactify the Teichmüller space.

4.1 Introduction

2+1 gravity formulated for spacetimes of the type $\Sigma_g \times \mathbb{R}$, where Σ_g is the closed (compact without boundary) Riemann surface of genus $g > 1$, is of considerable interest in mathematical relativity despite the fact that it does not allow for gravitational wave degrees of freedom and as such is devoid of straightforward physical significance. However, it becomes extremely important while studying '$3+1$' gravity on spacetimes of a certain topological type. [194] studied the Einstein's equations for $U(1)$ symmetric vacuum spacetimes with spatial topology being a circle bundle over $\mathbb{S}^2$. Later [191, 192, 193] studied the vacuum Einstein equations for $U(1)$ symmetric spacetimes with spatial topology being circle bundles over higher genus Riemann surfaces ($g > 1$), where $3+1$ gravity is reduced to $2+1$ gravity coupled to a wave map which has the hyperbolic plane as its target space. In addition to these classical analyses, considerable attention has been paid quantum mechanically [189, 190, 195, 196], where $2+1$ gravity is essentially treated as a toy model for $3+1$ quantum gravity.

Despite such physical motivations to study $2+1$ gravity as a tool for studying physically interesting $3+1$ gravity, $2+1$ gravity is itself a mathematically rich topic with several open issues even at the purely classical level. A considerable amount of work has been done on purely classical $2+1$ gravity. Moncrief [186] reduced the Einstein equations in $2+1$ dimensions to a Hamiltonian system over Teichmüller space, where the phase space of the dynamics was identified with the co-tangent bundle of Teichmüller space ($\approx \mathbb{R}^{12g-12}$). Later [197] proved the global existence of the Einstein equations on spacetimes of the topological type $\Sigma_g \times \mathbb{R}, g > 1$ by controlling the Dirichlet energy (a proper function on Teichmüller space) of an associated harmonic map. Moncrief's extensive analysis of $2+1$ gravity (using constant mean curvature spatial harmonic gauge) in [198] led to several classical results of Teichmüller theory, which were obtained by means of purely relativistic/Riemannian geometric analysis. This included, e.g., the homeomorphism between the Teichmüller space and the space of holomorphic quadratic differentials (transverse-traceless tensors in the context of relativity) etc. In the same paper, the term '*Relativistic Teichmüller theory*' was coined. Through studying a Hamilton Jacobi equation whose complete solution determines all the solution curves of the reduced Einstein equations and a Monge-Ampere type equation which allows for a more explicit characterization of these solution curves, he defined a family of ray structures on the Teichmüller space of Σ_g. Studying the behaviour of the associated Dirichlet energy, Moncrief [198] conjectured that *each of these non-trivial solution curves runs off the edge of Teichmüller space at the limit of the big-bang singularity and attaches to the Thurston boundary of the Teichmüller space, that is, the space of projective measured laminations or foliations ($\mathcal{PML}$, $\mathcal{PMF}$).* This, in principle, if it holds true, then classifies the big bang singularities of '$2+1$' gravity as the points on the Thurston boundary and serves as another means to compactify Teichmüller space.

[199] studied the spacetimes of simplicial type (a dense subset in the space of all such spacetimes) in cosmological time gauge and obtained a similar result that the past singularity corresponds to the isometric action of the fundamental group of Σ_g on a certain real tree, in other words, that a point on the Thurston boundary is associated to the initial singularity. Later, based on the work of [199], [200] used barrier arguments to control the constant mean curvature slices relative to the cosmic time ones near the big bang singularities and thereby to show that Thurston boundary points are attained in the limit, by the former as well as the latter. Despite the fact that these results conform to the conjecture of Moncrief to a large extent, they lack direct arguments and also differ in the choice of gauge. Whether this result is gauge invariant is currently unknown. Therefore, it is

worth proving the conjecture by a direct analysis of the Einstein evolution and constraint equations in CMCSH gauge.

In addition to the general relativistic perspective, M. Wolf [215] established the homeomorphism between the space of holomorphic quadratic differential and the Teichmüller space of Σ_g by utilizing the complex analytic properties such as the Beltrami differential (stretching) of the associated harmonic map. One may naively expect that Wolf's result might be directly applicable to the relativistic case since the transverse-traceless tensor of GR may be associated to a holomorphic quadratic differential. However, in Wolf's case, the domain is kept fixed while the dynamics occurs on the target surface and therefore the available machinery from complex analysis became useful. But, in the relativistic case, the domain (conformal structure) is varying while the target is fixed (an interior point of the Teichmüller space). Therefore, the traditional machinery becomes useless and we are left with tools which are only seemingly accessible through GR.

In this paper, we aim to study the '$2+1$' gravity on vacuum spacetimes of topological type $\Sigma_g \times \mathbb{R}$ in constant mean extrinsic curvature spatial harmonic gauge (CMCSH). Utilizing the direct estimates from the Einstein evolution and constraint equations in conjunction with a few established results from [198] and the theory of Riemann surfaces, we show via a direct argument that indeed Moncrief's conjecture holds true, that is, at the limit of the big-bang singularity, the conformal geometry degenerates and every corresponding non-trivial solution curve attaches to the Thurston boundary. The structure of the paper is as follows. We begin with introducing necessary background for the theory of Riemann surfaces such as harmonic maps, holomorphic quadratic differentials, the associated measured foliations and their transverse measures etc. Then we study the reduced Einstein equations through a conformal technique and obtain the estimates necessary from the associated elliptic PDEs. Finally, we state the relativistic interpretation of the concepts introduced from surface theory and show using the estimates obtained that the conjecture holds true, that is, at the limit of big-bang singularity, every non-trivial solution curve runs off the edge of the Teichmüller space and attaches to the space of projective measured foliations/laminations and exhausts these spaces. We conclude by discussing the potential validity of the conjecture with the inclusion of a cosmological constant and suitable matter sources.

4.2 Notations and facts

We denote the '$2+1$' spacetime by $\tilde{M}$ with its topology being $\Sigma_g \times \mathbb{R}$. Here, Σ_g is a closed (compact without boundary) Riemann surface with genus $g > 1$. The space of Riemannian metrics on Σ_g is denoted by $\mathcal{M}$ and its closed submanifold $\mathcal{M}_{-1}$ is defined as follows

$$\mathcal{M}_{-1} = \{\gamma \in \mathcal{M} | R(\gamma) = -1\}, \tag{4.1}$$

where $R(\gamma)$ is the scalar curvature of the metric γ. The space of symmetric 2-tensor fields is denoted by $S_2^0(\Sigma_g)$. The L^2 inner product with respect to the metric $\gamma \in \mathcal{M}$ between any two elements A and B of $S_2^0(\Sigma_g)$ is defined as

$$< A, B >_{L^2} := \int_{\Sigma_g} A_{ij} B_{kl} \gamma^{ik} \gamma^{jl} \mu_\gamma, \tag{4.2}$$

where $\mu_\gamma = \sqrt{\det(\gamma_{ij})} dx^1 \wedge dx^2$ is the volume form on Σ_g. Abusing notation we will use μ_γ for both $\sqrt{\det(\gamma_{ij})}$ and the volume form. Unless otherwise stated, we will consider an element of $\mathcal{M}$ in isothermal coordinates that is $\mathcal{M} \ni \gamma := e^{\eta(z)} |dz|^2, \eta : \Sigma_g \to \mathbb{R}$. The Laplacian Δ_γ is defined so as to have non-negative spectrum on Σ_g, that is, $\Delta_\gamma := -\gamma^{ij} \nabla_i \nabla_j$. For, $a, b > 0$, $a \leq Cb$ ($\geq$ resp.) for some $\infty > C > 0$ is denoted by $a \lesssim b$ ($a \gtrsim b$ resp.). $C_1 b \leq a \leq C_2 b$, $0 < C_1 < C_2 < \infty$ is denoted by $a \approx b$ (this essentially denotes that one is controlled by the other or two entities are bounded by one another). By a nontrivial element of $\pi_1(\Sigma_g)$, we will always mean a non-trivial closed curve since, there is a one to one correspondence between the homotopy classes of essential (not homotopic to a point or neighbourhood of a puncture) closed curve and the conjugacy classes of non-trivial elements in $\pi_1(\Sigma_g)$. The group of diffeomorphisms (of Σ_g) and its identity component are denoted by $\mathcal{D}$ and $\mathcal{D}_0$, respectively.

4.3 Background on Teichmüller space

Teichmüller space is studied from an algebraic topologic perspective [201, 202], a complex analytic perspective [201, 203], and a Riemannian geometric perspective [204]. Here, we will focus mainly on the latter as the Teichmüller space while viewed from a Riemannian geometric perspective naturally appears as the configuration space of vacuum Einstein gravity (with or without a positive cosmological constant) on $\Sigma_g \times R$. Nevertheless, we will state the algebraic topologic definition of Teichmüller space and show how this is connected to Einstein gravity. The Teichmüller space of Σ_g is defined as the space of homomorphisms (more accurately the discrete and faithful representations) of the fundamental group of Σ_g into the isometry group of its universal cover that is the hyperbolic plane modulo the action of the isometry group by conjugation. If the Poincaŕe disk model of the hyperbolic plane is assumed, then the Teichmüller space is defined to be

$$\begin{aligned} \mathcal{T}\Sigma_g &:= \rho(\pi_1(\Sigma_g), PSL_2R)/PSL_2Rconj \\ &\subset Hom(\pi_1(\Sigma_g), PSL_2R)/PSL_2Rconj, \end{aligned} \tag{4.3}$$

where ρ denotes the space of discrete and faithful representations (sometimes representations abusing notation). Here note that $Hom(\pi_1(\Sigma_g), PSL_2R)/PSL_2Rconj$ is also called character variety and $\mathcal{T}\Sigma_g$ is a connected component of this character variety. Dimension of $\mathcal{T}\Sigma_g$ may be calculated as follows. The space of homomorphisms $Hom(\pi_1(\Sigma_g), PSL_2R)$ is moded out by the PSL_2R conjugation so as to remove the base point of the homotopy (at the level of loops). This definition precisely identifies the ways to equip Σ_g with distinct conformal structures (or hyperbolic structures). The fundamental group $\pi_1(\Sigma_g)$ is to be viewed as a discrete and faithful subgroup of PSL_2R and as such is finitely generated ($2g$ generators). The dimension of PSL_2R is 3 and action by conjugation by an element of PSL_2R produces equivalence classes (with respect to gauge transformation in physics terminology). In addition, the generators $(A_i, B_i)_{i=1}^g$ satisfy the commutation relation $\prod_{i=1}^g A_i B_i A_i^{-1} B_i^{-1} = id$ implying the representation $\rho \in Hom(\pi_1(\Sigma_g), PSL_2R)/PS_2R\ conj$ would satisfy $\prod_{i=1}^g \rho(A_i)\rho(B_i)\rho(A_i)^{-1}\rho(B_i)^{-1} = id$ as well. Therefore we lose $3+3=6$ degrees of freedom out of $2g \times 3 = 6g$ and the dimension of the Teichmuller space turns out to be $6g-6$. Let us now show how this is related to vacuum Einstein dynamics. The vacuum Einstein equations in $2+1$ dimension reads

$$R_{\mu\nu} = 0, \tag{4.4}$$

where (μ, ν) correspond to the spacetime indices. Now, in $2+1$ dimension, vanishing of the Ricci tensor ($R_{\mu\nu}$) implies vanishing of the full Riemann tensor (or the sectional curvature) and therefore, the solutions of the Einstein equations are necessarily the flat spacetimes and consequently are locally isometric to the Minkowski spacetime. Now we are interested in flat spacetimes foliated by Σ_g. In order to obtain the solution space, we therefore need to identify the space of homomorphisms (once again space of discrete and faithful representations to be precise) of $\pi_1(\Sigma_g \times R)$ into the isometry group of the flat spacetimes, which in this case is the full Poincare group $ISO(2,1)$. Now $\pi_1(\Sigma_g \times R) \approx \pi_1(\Sigma_g)$ and therefore the solution space is described as

$$Ein_S = \rho(\pi_1(\Sigma_g), ISO(2,1))/ISO(2,1)conj, \tag{4.5}$$

where Ein_S is the space of solutions of the equation (4.4). In the similar way, we may compute the dimension of Ein_S. Note that now the isometry group $ISO(2,1)$ has dimension 6 and therefore following the exact same procedure, we obtain the dimension of Ein_S to be $12g - 12$. Therefore, the full solution space is twice the dimension of the Teichmüller space. One immediate guess would be that the co-tangent bundle $T^*\mathcal{T}\Sigma_g$ of the Teichmüller space acts as the full solution space, which is precisely the case as shown in [186,198]. $T^*\mathcal{T}\Sigma_g$ is indeed the phase space of the reduced dynamics. We will get back to this point in detail later. Let us first develop the concepts of geodesic currents, measured laminations and foliations, which will be required to prove the conjecture.

Let us now introduce a few elementary concepts from the theory of Riemann surfaces. From elementary hyperbolic geometry, we know that there exists a unique geodesic between any two distinct points lying on the boundary of the Poincaŕe disc (in this model of the hyperbolic 2-space). Therefore, we define the set of all un-oriented geodesics on $\tilde{\Sigma}_g$ (lift of Σ_g to its universal cover) as the $\mathbb{Z}_2$ graded double boundary of $\tilde{\Sigma}$ i.e., $G(\tilde{\Sigma}_g)$={The set of all un-oriented geodesics on $\tilde{\Sigma}$} $\approx (\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty - \Delta)/\mathbb{Z}_2$, where Δ represents the diagonal. A geodesic current is a radon measure on $G(\tilde{\Sigma})$ which is invariant under the $\pi_1(\Sigma_g)$ action (see [209, 210] for more details and see [207] for details about radon measures). The property of a radon measure which would be of particular interest to us is that it is locally finite. In a sense, a geodesic currents is essentially an assignment of a radon measure to the open sets of $G(\tilde{\Sigma})$, which remain invariant under the action of the fundamental group $\pi_1(\Sigma_g)$. This $\pi_1(\Sigma_g)$ invariance property of the geodesic currents allows one to define it on the space of geodesics on Σ_g i.e., $G(\Sigma_g) = G(\tilde{\Sigma}_g)/\pi_1(\Sigma_g)$ (note that the action of $\pi_1(\Sigma_g)$ extends continuously to $\partial\tilde{\Sigma}_g$). Now, for a closed hyperbolic surface of genus greater than 1, $\pi_1(\Sigma_g)$ while viewed as a proper discrete subgroup of the isometry group of the hyperbolic plane that is PSL_2R, consists of hyperbolic (also called loxodromic) elements only (see [202, 208] for a detailed classification of the types of isometries of $\mathbb{H}^2$). Each element of $\pi_1(\Sigma_g)$ has an axis geodesic along which it acts by translation and in general it has two fixed points: one attracting, one repelling. Therefore each element of $\pi_1(\Sigma_g)$, a homotopy class of nontrivial loops (rectifiable), has a unique geodesic representative. Whenever we will consider the length of a non-trivial closed curve on Σ_g we will always mean the length of the geodesic in its homotopy class. A geodesic lamination is a closed subset of Σ_g which is the union of disjoint geodesics. A measured lamination is defined as a geodesic lamination equipped with a transverse measure (invariant under translations along the leaves of the lamination). Clearly, the space of measured laminations is a subset of the space of geodesic currents. A geodesic foliation may be thought of as the union of the geodesics which are also integral curves of a vector field. Zeros of the vector field correspond to the singularities of the foliation. One may similarly assign a transverse measure to the foliation promoting it to a measured foliation. There is a natural homeomorphism between the space of measured laminations and measured foliations (via a straightening map; see Fig [**??**]). This homeomorphism persists at the level of corresponding projective spaces. This projective space (projective measured laminations or foliations) is the Thurston boundary of the Teichmüller space. At this point, it suffices to know this fact and therefore, we do not dwell on this matter further rather provide a small detail in the appendix. Interested readers are referred to the same.

4.3.1 Homeomorphism between $\mathcal{ML}$, $\mathcal{MF}$, and $\mathcal{QD}$

Let us first define a holomorphic quadratic differential on a Riemann surface Σ_g. A holomorphic quadratic differential is a holomorphic section of the symmetric square of the holomorphic cotangent bundle of Σ_g. It may be defined locally as follows. Let $\{z_a : U_a \to \mathbb{C}\}$ be an atlas for Σ_g. A holomorphic quadratic differential Φ on Σ_g is locally expressible on the chart z_a as $\Phi_a(z_a)dz_a^2$ with the following properties: [1] $\Phi_a : z_a(U_a) \to \mathbb{C}$ is holomorphic, i.e., $\frac{\partial \Phi_a}{\partial \bar{z}_a} = 0$, and [2] $\Phi_a(z_a)(\frac{dz_a}{dz_b})^2 = \Phi_b(z_b)$ for two different overlapping charts $z_a : U_a \to \mathbb{C}$ and $z_b : U_b \to \mathbb{C}$. The second condition precisely states the invariance of Φdz^2 under coordinate transformations. Let us denote the space of holomorphic quadratic differentials on Σ_g by $\mathcal{QD}$. By the famous theorem of Hubbard and Masur [216], there is a homeomorphism between the space of holomorphic quadratic differential $\mathcal{QD}$ and the space of measured foliations $\mathcal{MF}$ on Σ_g. One may simply associate a vertical or horizontal foliation with $\Phi \in \mathcal{QD}$ (up to isotopy and Whitehead moves; see [215] for details about Whitehead moves). For details, the reader is referred to [217]. For now we will only need this homeomorphism property. Given a holomorphic quadratic differential $\Phi(z)dz^2$ in some chart, the transverse measures of a non-trivial element $\mathcal{A}$ of $\pi_1(\Sigma_g)$ (except at the zeros of Φ, which correspond to the singularities of the foliation) with respect to the vertical foliation and the horizontal foliation associated with Φ are defined as follows

$$\mu_{vert}(\mathcal{A}) := \int_{\mathcal{A}} |\mathcal{R}\left(\sqrt{\Phi(z)}dz\right)|, \tag{4.6}$$

$$\mu_{hor}(\mathcal{A}) := \int_{\mathcal{A}} |\mathcal{I}\left(\sqrt{\Phi(z)}dz\right)|, \tag{4.7}$$

where $\mathcal{R}$ and $\mathcal{I}$ denote the real and imaginary parts, respectively. We will use these definitions later while considering the Einstein flow on Σ_g exclusively. Given a measured foliation, one may obtain a measured lamination via a suitable straightening map [214, 218] (or collapsing a lamination yields a foliation). Therefore, there is a homeomorphism between $\mathcal{MF}$ and $\mathcal{ML}$. Figure (**??**) depicts the mechanism of yielding a lamination from a foliation. For our purposes, we will only use the homeomorphism between $\mathcal{QD}$ and $\mathcal{MF}$. All of these spaces remain homeomorphic to each other at the level of projective spaces.

4.3.2 Harmonic Maps

Let us now introduce another essential ingredient of our analysis: the harmonic maps. These will be crucial later in studying the Einsteinian dynamics. Let us consider a map $\mathcal{E} : (M, g) \to (N, \rho)$ (where M and N are considered to be two closed Riemann surfaces) and define the Dirichlet energy

$$E[\mathcal{E}; g, \rho] \quad = \quad \frac{1}{2}\int_M \rho_{\alpha\beta}\frac{\partial\mathcal{E}^\alpha}{\partial x^i}\frac{\partial\mathcal{E}^\beta}{\partial x^j}g^{ij}\mu_g. \tag{4.8}$$

From the expression of the Dirichlet energy, it is obvious that it only depends on the conformal structure of the domain, that is, a conformal transformation $g_{ij} \mapsto e^{2\delta}g_{ij}, \delta : M \to \mathbb{R}$ leaves E invariant. Harmonic maps are defined to be the critical points of this Dirichlet energy functional in the space of $\mathcal{E}$. The critical points of E may be computed as follows. On the bundle $T^*M \otimes \mathcal{E}_*^{-1}TN$ (while restricted to the image), one has the following connection

$$\nabla_i A^\alpha_j := \partial_i A^\alpha_j +^N \Gamma^\alpha_{\beta\gamma}A^\beta_j\frac{\partial\xi^\gamma}{\partial x^i} -^M \Gamma^k_{ij}A^\beta_k, \tag{4.9}$$

for $A \in \{\text{sections}(T^*M \otimes \mathcal{E}_*^{-1}TN)\}$. Using this definition of the connection, a few lines of calculation yields the harmonicity condition

$$g^{ij}\partial_i\partial_j\xi^\alpha - g^{ij}\ {}^M\Gamma^k_{ij}\partial_k\xi^\alpha +^N \Gamma^\alpha_{\beta\gamma}\partial_i\xi^\beta\partial_j\xi^\gamma g^{ij} \quad = \quad 0. \tag{4.10}$$

From [219, 220], we know that there is a harmonic map homotopic to the identity i.e., $\mathcal{E} \in \mathcal{D}_0$ and in fact such a map is an orientation preserving diffeomorphism. If we take $\mathcal{E}$ to be the identity map, then the harmonicity condition reduces to the following

$$-g^{ij}\left(\Gamma[g]^\alpha_{ij} - \Gamma[\rho]^\alpha_{ij}\right) = 0. \tag{4.11}$$

This condition will be of extreme importance when we fix the spatial gauge of the Einstein equations and also in the later part of the analysis. The Dirichlet energy of this identity map is computed to be

$$E[id; g, \rho] \quad = \quad \frac{1}{2}\int_{\Sigma_g} \rho_{ij} g^{ij} \mu_g. \tag{4.12}$$

Note that the conformal and diffeomorphism invariance of $E[id; g, \rho]$ allow it to be a function on the Teichmüller space of Σ_g and in particular a proper function (that is the inverse images of the compact sets are compact) [204, 215, 220, 221].

4.4 Einstein flow on $\Sigma_g \times \mathbb{R}$

We will use the ADM formalism of general relativity in order to obtain a Cauchy problem for '$2+1$' gravity. The ADM formalism of '2+1' gravity splits the spacetime described by a '2+1' dimensional Lorentzian manifold $\tilde{M}$ into $\mathbb{R} \times \Sigma_g$ with each level set $\{t\} \times \Sigma_g$ of the time function t being an orientable 2-manifold diffeomorphic to a Cauchy hypersurface (assuming the spacetime admits a Cauchy hypersurface) and equipped with a Riemannian metric. Such a split may be executed by introducing a lapse function N and shift vector field X belonging to suitable function spaces and defined such that

$$\partial_t \quad = \quad N\hat{n} + X \tag{4.13}$$

with t and $\hat{n}$ being time and a hypersurface orthogonal future directed timelike unit vector i.e., $\tilde{g}(\hat{n}, \hat{n}) = -1$, respectively. The above splitting writes the spacetime metric $\tilde{g}$ in local coordinates $\{x^\alpha\}_{\alpha=0}^2 = \{t, x^1, x^2\}$ as

$$\tilde{g} \quad = \quad -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt) \tag{4.14}$$

where $g_{ij} dx^i \otimes dx^j$ is the induced Riemannian metric on Σ_g. In order to describe the embedding of the Cauchy hypersurface Σ_g into the spacetime $\tilde{M}$, one needs the information about how the hypersurface is curved in the ambient spacetime. Thus, one needs the second fundamental form k defined as

$$K_{ij} \quad = -\tfrac{1}{2N}(\partial_t g_{ij} - (L_X g)_{ij}), \tag{4.15}$$

the trace of which ($tr_g K = \tau = g^{ij}K_{ij}$, $g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} := g^{-1}$) is the mean extrinsic curvature of Σ_g in $\tilde{M}$ and L denotes the Lie derivative operator. The vacuum Einstein equations

$$R_{\mu\nu}(\tilde{g}) - \frac{1}{2}R(\tilde{g})\tilde{g}_{\mu\nu} = 0 \tag{4.16}$$

may now be expressed as the evolution and constraint (Gauss and Codazzi equations) equations of g and k

$$\begin{aligned}
\partial_t g_{ij} &= -2NK_{ij} + (L_X g)_{ij}, && (4.17)\\
\partial_t K_{ij} &= -\nabla_i \nabla_j N + N(R_{ij} + \tau K_{ij} - 2k_i^k k_{jk}) + (L_X k)_{ij}, && (4.18)\\
0 &= R(g) - |K|^2 + (tr_g K)^2, &&\\
0 &= \nabla^i K_{ij} - \nabla_j tr_g K. && (4.19)
\end{aligned}$$

Note that there is no canonical way way to split the spacetimes, that is, the choice of a spacelike hypersurface is not unique. In order to choose a slice and study its evolution under the Einstein flow, we must fix the gauge. In our case, the most convenient choice is the constant mean extrinsic curvature spatial harmonic gauge used by [222]. In this gauge, $\tau = tr_g K$ is constant throughout the hypersurface ($\partial_i \tau = 0$) and therefore is chosen to play the role of time

$$t = monotonic\ function\ of\ \tau. \tag{4.20}$$

Spatial harmonic gauge is precisely the vanishing of the tension vector field $-g^{ij}\left(\Gamma[g]^k_{ij} - \Gamma[\hat{g}]^k_{ij}\right)$, where $\hat{g}$ is an arbitrary background metric or in other words, the harmonicity of the identity map defined in the previous section. This choice of gauge yields the following two elliptic equations for the lapse function and the shift vector field, respectively

$$\begin{aligned}
\Delta_g N + N(|K^{TT}|_g^2 + \frac{\tau^2}{2}) &= \partial_t \tau, && (4.21)\\
\Delta_g X^i - R^i_j X^j = (\nabla^i N)\tau - 2\nabla^j N K^i_j &+ (2NK^{jk} - 2\nabla^j X^k) && (4.22)\\
&(\Gamma[g]^i_{jk} - \hat{\Gamma}[\hat{g}]^i_{jk}). &&
\end{aligned}$$

This Cauchy problem (with initial data (g_0, k_0)) with constant mean extrinsic curvature and spatially harmonic gauge is referred to as ?CMCSH Cauchy? problem.

4.4.1 **Well-posedness:**

[222] proved a local well posedness theorem for the Cauchy problem for a family of elliptic- hyperbolic systems that included the '$n+1$' dimensional vacuum Einstein equations in CMCSH gauge, $n \geq 2$. They also proved the conservation of gauges and constraints. In addition to the local well-posedness, [197] proved a global existence theorem for the expanding solutions in the same gauge through controlling the Dirichlet energy of an associated harmonic map for any $\tau \in (-\infty, 0)$. Therefore, the well-posedness of the Cauchy problem is established and we do not wish to repeat the same here. Interested readers are referred to these articles.

4.4.2 **Reduced Dynamics**

Given a scalar function $\varphi : \Sigma_g \to \mathbb{R}$, we define a set of conformal variables (γ, k^{TT}) (k^{TT} is transverse-traceless with respect to the metric γ) in terms of the physical variables (g, k^{TT}) by setting

$$(g_{ij}, K^{TTij}) \quad = \quad (e^{2\varphi}\gamma_{ij}, e^{-4\varphi}k^{TTij}), \tag{4.23}$$

where $R(\gamma) = -1$ (the Uniformization theorem guarantees that such γ exists if genus$(\Sigma_g) > 1$)) and the second fundamental form is written as follows

$$K = K^{TT} + \frac{\tau}{2}g, \tag{4.24}$$

by using the momentum constraint with K^{TT} being transverse-traceless with respect to g. Here k^{TT} is transeverse-traceless with respect to γ, that is,

$$\nabla[\gamma]_j k^{TTij} \quad = \quad 0, \tag{4.25}$$

$$\gamma_{ij}k^{TTij} \quad = \quad 0, \tag{4.26}$$

if and only if K^{TT} is transverse-traceless with respect to g. Naturally

$$k^{TT}_{ij} = K^{TT}_{ij}, \tag{4.27}$$

$$k^{TTij} = \gamma^{ik}\gamma^{jl}k^{TT}_{kl}. \tag{4.28}$$

φ can be found by solving the Hamiltonian constraint which now takes the form of the following semilinear elliptic PDE namely the Lichnerowicz equation

$$-2\Delta_{\gamma}\varphi + 1 + e^{-2\varphi}|k^{TT}|^2_{\gamma} - \frac{e^{2\varphi}\tau^2}{2} \quad = \quad 0. \tag{4.29}$$

Using the sub and super solution technique [223, 224], it is established that there is a unique solution $\varphi[\gamma, k^{TT}, \tau]$ of the Lichnerowicz equation. Indeed, this equation will be crucial to our analysis towards proving the main theorem. The phase space of the reduced dynamics now may be defined as $\{(\gamma_{ij}, k^{TTij})|\gamma \in \mathcal{M}_{-1}, tr_{\gamma}k^{TT} = 0 = \nabla[\gamma]_j k^{TTij}\}$. In reality, the true dynamics assumes a metric lying in the orbit space $\mathcal{M}_{-1}/D_0$, D_0 being the group of diffeomorphisms (of Σ_g) isotopic to identity. This is a consequence of the fact that if $\gamma_{ij} \in \mathcal{M}_{-1}, k^{TTij}, \varphi, N$, and X^i solve the Einstein equations, so do $((\phi^{-1})^*\gamma)_{ij}, (\phi_* k^{TT})^{ij}, (\phi^{-1})^*\varphi = \varphi \circ \phi^{-1}, (\phi^{-1})^* N = N \circ \phi^{-1}$, and $(\phi_* X)^i$, where $\phi \in D_0$ and *, and $_*$ denote the pullback and push-forward operations (time independent) on the cotangent and tangent bundles of M, respectively. Let us now consider a time dependent $\phi_t \in \mathcal{D}_0$ and go back to the un-scaled dynamical equation (4.17) (note that the un-scaled fields (g, K, N, X)) solve the Einstein's dynamical and constraint equations (4.17-4.19) iff $(\gamma, k, \varphi, N, X)$ solve the reduced equations)

$$\begin{aligned}
\partial_t((\phi_t^{-1})^*g)_{ij} &= -2(\phi_t^{-1})^*)(NK_{ij}) + (L_{\phi_{t*}X}(\phi_t^{-1})^*g)_{ij}, \qquad (4.30)\\
(\phi_t^{-1})^*\partial_t g_{ij} + (\partial_t(\phi_t^{-1})^*)g_{ij} &= -2(\phi_t^{-1})^*(NK_{ij}) + \frac{\partial}{\partial s}((\phi_t^{-1}\varphi_s^X\phi_t)^*(\phi_t^{-1})^*g)|_{s=0},\\
(\phi_t^{-1})^*\partial_t g_{ij} + (\partial_s(\phi_{t+s}^{-1})^*)g_{ij}|_{s=0} &= -2(\phi_t^{-1})^*(NK_{ij}) + (\phi_t^{-1})^*(L_X g)_{ij},\\
(\phi_t^{-1})^*\partial_t g_{ij} + (\phi_t^{-1})^*(L_Y g)_{ij} &= -2(\phi_t^{-1})^*(NK_{ij}) + (\phi_t^{-1})^*(L_X g)_{ij},\\
(\phi_t^{-1})^*\left\{\partial_t g_{ij} = -2NK_{ij} + (L_{X-Y}g)_{ij}\right\}.
\end{aligned}$$

Here Y is the vector field associated with the flow ϕ_t and φ_s^X is the flow of the shift vector field X. A similar calculation for the evolution equation for the second fundamental form shows that if we make a trasformation $X \mapsto X + Y$, the Einstein evolution and constraint (due to their natural spatial covariance nature) equations are satisfied by the transformed fields. Action of ϕ_t on the un-scaled fields naturally extends to the conformally scaled fields. Therefore, the true reduced dynamics occurs on the quotient space $\mathcal{M}_{-1}/\mathcal{D}_0$. Now, $\mathcal{M}_{-1}/D_0$ is precisely the Teichmüller space of Σ_g and following [204], the transverse-traceless tensor k^{TT} models the tangent space at γ. Therefore, we obtain the Teichmüller space ($6g-6$ dimensional) $\mathcal{T}\Sigma_g$ as the configuration space, while the cotangent bundle ($12g-12$ dimensional) of $\mathcal{T}\Sigma_g$ serves as the phase space of the reduced dynamics.

This is precisely what was stated previously in section 2 while relating the full solution space of the vacuum Einstein equations and the Teichmüller space through its algebraic topologic definition.

Now we will obtain a series of estimates which will be useful for the later analysis. The following lemma provides a point-wise estimate for the conformal factor $e^{2\varphi}$.

Lemma 1: *Let $\varphi : \Sigma_g \to \mathbb{R}$ solves the Lichnerowicz equation (4.29). Then $e^{2\varphi} := e^{2\varphi(\tau, k^{TT}, \gamma)}$ verifies the following point-wise estimate*

$$\frac{2}{\tau^2} \leq e^{2\varphi} \leq \frac{1 + \sqrt{1 + 2\tau^2 \sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)}}{\tau^2} \qquad \forall\ \tau \in (-\infty, 0). \tag{4.31}$$

Proof: A standard maximum principle argument while applied to the Lichnerowicz equation (4.29), yields the following

$$\tau^2 e^{4\varphi} - 2e^{2\varphi} - 2\sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau) \leq 0. \tag{4.32}$$

Noting that the discriminant of the quadratic form $\tau^2 e^{4\varphi} - 2e^{2\varphi} - 2\sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)$, $4 + 8\tau^2 \sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)$, is strictly positive, the inequality is satisfied only for a specific range of $e^{2\varphi}$ i.e.,

$$\left(e^{2\varphi} - \frac{1 + \sqrt{1 + 2\tau^2 \sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)}}{\tau^2}\right) \tag{4.33}$$
$$\left(e^{2\varphi} - \frac{1 - \sqrt{1 + 2\tau^2 \sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)}}{\tau^2}\right) \leq 0.$$

But, $e^{2\varphi} > 0$ and therefore, we must have

$$e^{2\varphi} \leq \frac{1 + \sqrt{1 + 2\tau^2 \sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)}}{\tau^2}. \tag{4.34}$$

Similarly, at a minimum, the following holds

$$\tau^2 e^{4\varphi} - 2e^{2\varphi} \geq 0, \tag{4.35}$$

that is,

$$e^{2\varphi} \geq \frac{2}{\tau^2}, \tag{4.36}$$

where the equality holds if and only if

$$k^{TT} \equiv 0. \tag{4.37}$$

In summary, we have the following estimate of the conformal factor from the Lichnerowicz equation

$$\frac{2}{\tau^2} \leq e^{2\varphi} \leq \frac{1+\sqrt{1+2\tau^2 \sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)}}{\tau^2}, \tag{4.38}$$

which will be useful later. This concludes the proof of the lemma. $\square$

Now we will obtain an estimate for $|K^{TT}|^2_g = e^{-4\varphi}|k^{TT}|^2_\gamma$. In order to do so, we first obtain an elliptic equation for $|K^{TT}|^2_g$. The following lemma provides the necessary elliptic equation $|K^{TT}|^2_g$.

Lemma 2: *Let $K := K^{TT} + \frac{1}{2}{}_gKg$ solves the momentum constraint (4.19) in CMC gauge $\partial_i\tau := \partial_{ig}K = 0$, then $|K^{TT}|^2_g$ satisfies the following quasi-linear elliptic equation on a constant time hypersurface*

$$-\Delta_g(|K^{TT}|^2_g) - 2|K^{TT}|^2_g(|K^{TT}|^2_g - \frac{1}{2}\tau^2) \quad = \quad 2\nabla[g]_k(K^{TT}_{ij})\nabla[g]^k(K^{TTij}). \tag{4.39}$$

Proof: Note that in 2 dimensions, the momentum constraint

$$\nabla[g]_j K^j_i - \nabla_i tr_g K = 0 \tag{4.40}$$

implies that K is a Codazzi tensor [197, 198] i.e.,

$$\nabla[g]_j K^i_k - \nabla[g]_k K^i_j = 0. \tag{4.41}$$

After substituting the decomposition $K = K^{TT} + \frac{{}_gK}{2}g$ in the Codazzi equation, Covariant divergence yields

$$\nabla[g]^j\nabla[g]_j K^{TTi}_k - \nabla[g]^j\nabla[g]_k K^{TTi}_j = 0, \tag{4.42}$$

$$\nabla[g]^j\nabla[g]_j K^{TTi}_k - \nabla[g]_k\nabla[g]_j K^{TTij} - R[g]^i{}_{mjk}K^{TTmj} - R[g]^j{}_{mjk}K^{TTim} = 0,$$

which upon utilizing $\nabla[g]_j K^{TTij} = 0$ and $R[g]^i{}_{mjk} = \frac{R(g)}{2}(\delta^i_j g_{mk} - \delta^i_k g_{mj})$ reduces to

$$\nabla[g]^j\nabla[g]_j K^{TTi}_k = R(g)K^{TTi}_k. \tag{4.43}$$

$\Delta_g(|K^{TT}|_g^2)$ may be evaluated as follows

$$\begin{aligned}\Delta_g(|K^{TT}|_g^2) = -\nabla[g]^j\nabla[g]_j|K^{TT}|_g^2 = -\nabla[g]^j\nabla[g]_j(K_k^{TTi}K_i^{TTk}) \\ = -2(\nabla[g]^j\nabla[g]_jK_k^{TTi})K_i^{TTk} - 2\nabla[g]^jK_k^{TTi}\nabla[g]_jK_i^{TTk}, \\ = -2R(g)|K^{TT}|_g^2 - 2\nabla[g]^jK_k^{TTi}\nabla[g]_jK_i^{TTk}, \\ = -2(|K^{TT}|_g^2 - \frac{\tau^2}{2})|K^{TT}|_g^2 - 2\nabla[g]^jK_k^{TTi}\nabla[g]_jK_i^{TTk},\end{aligned} \tag{4.44}$$

i.e.,

$$-\Delta_g(|K^{TT}|_g^2) - 2|K^{TT}|_g^2(|K^{TT}|_g^2 - \frac{1}{2}\tau^2) \quad = \quad 2\nabla[g]_k(K_{ij}^{TT})\nabla[g]^k(K^{TTij}). \tag{4.45}$$

Here, we have used the Hamiltonian constraint (4.19) $|K^{TT}|_g^2 = R(g) + \frac{\tau^2}{2}$. This concludes the proof of the lemma. $\square$

Remarkably, the quasi-linear term appearing in the right hand side of the elliptic equation (4.38) has a favourable sign that is conducive to an application of a standard maximum principle.

Lemma 3: $|K^{TT}|_g^2$ *satisfies the estimate*

$$|K^{TT}|_g^2 \leq \frac{\tau^2}{2} \tag{4.46}$$

for all $\tau \in (-\infty, 0)$.

Proof: The quasi-linear term satisfies $\nabla[g]_k(K_{ij}^{TT})\nabla[g]^k(K^{TTij}) \geq 0$. Application of a standard maximum principle argument yields

$$|K^{TT}|_g^2 \leq \frac{\tau^2}{2}. \qquad \square \tag{4.47}$$

Lastly, we will obtain an estimate for the lapse function after choosing the following time coordinate

$$t := -\frac{1}{\tau}. \tag{4.48}$$

The allowed time range in this coordinate is $(0, \infty)$. The lapse equation (4.21) now reads

$$\Delta_g N + N(|K^{TT}|_g^2 + \frac{\tau^2}{2}) = \tau^2. \tag{4.49}$$

Once again, a standard maximum principle argument applied to the lapse equation together with the estimate (4.47) yields the following estimate of N

$$1 \leq N \leq 2. \tag{4.50}$$

Now we will describe Moncrief's ray structure [198] of the Teichmüller space, which will be of crucial in obtaining the main result. The ray structure defined by Moncrief is the following equation

$$\begin{aligned} \rho_{ij} &= |K|_g^2 g_{ij} + 2\tau(K_{ij} - \frac{1}{2}\tau g_{ij}) \\ &= (|K^{TT}|_g^2 + \frac{\tau^2}{2}) g_{ij} + 2\tau K_{ij}^{TT} \\ &= (e^{-4\varphi}|k^{TT}|_\gamma^2 + \frac{\tau^2}{2}) e^{2\varphi}\gamma_{ij} + 2\tau k_{ij}^{TT} \end{aligned} \tag{4.51}$$

together with an associated Hamilton-Jacobi equation. Here ρ is a fixed metric satisfying $R(\rho) = -1$ (and therefore lies inside the Teichmüller space) and g_{ij} is solved in terms of ρ_{ij}. This computes the end point of a ray in terms of the data along the ray. For the detailed derivation of this expression, one may consult the relevant section of [198]. This is designated in [198] as the '**Gauss**' map equation. For our purpose, the derivation of this map is tangential and hence, we do not wish to repeat the same here. The vital question is whether such (g_{ij}, K^{TTij}, N, X) actually solves the Einstein equations for all τ given an initial $(g_{ij0}, K_0^{TTij}, N_0, X_0)$ satisfying the constraint equations. This is equivalent to solving for conformal variables $(\gamma_{ij}, k^{TTij}, \varphi)$ and associated lapse function N and shift vector field X. This is exactly shown in [198] through studying the associated Hamilton Jacobi equation for the reduced dynamics. When this lagrangian formulation is cast into a more natural Hamiltonian one, one clearly sees that the original Einstein-Hilbert action may be written as follows

$$S = \int_{I\subset\mathbb{R}}\int_{\Sigma_g} \left(\mu_g(-K^{ij} + \tau g^{ij})\frac{\partial g_{ij}}{\partial t} - N\mathcal{H} - X^i\mathcal{P}_i\right) d^2x dt, \tag{4.52}$$

where $\mathcal{H} := \mu_g K_{ij}^{TT} K^{TTij} - \frac{1}{2}\tau^2\mu_g - \mu_g R(g)$, and $\mathcal{P}_i := 2\nabla[g]_j(\mu_g K_i^j - \tau\mu_g\delta_i^j)$. Note that vanishing of $\mathcal{H}$ and $\mathcal{P}_i$ is precisely equivalent to (g_{ij}, K^{ij}) satisfying the Hamiltonian and momentum constraints. When both of these constraints are satisfied we obtain the reduced action

$$S_{reduced} = \int_{I\subset\mathbb{R}}\int_{\Sigma_g} \mu_g(-K^{ij} + \tau g^{ij})\frac{\partial g_{ij}}{\partial t} d^2x dt, \tag{4.53}$$

which through the conformal transformation (4.23) becomes

$$S_{reduced} = \int_{I\subset\mathbb{R}}\left(\int_{\Sigma_g}(-\mu_\gamma k^{TTij}\frac{\partial\gamma_{ij}}{\partial t} - \frac{\partial\tau}{\partial t}\mu_g)d^2x\right)dt, \tag{4.54}$$

where the boundary in time terms are ignored, because, they do not contribute to the equations of motions at the classical level. The Hamiltonian of this reduced dynamics can be read off as follows from the expression of the previous action

$$H_{reduced} = \int_{\Sigma_g}\frac{\partial\tau}{\partial t}\mu_g. \tag{4.55}$$

Substituting the time coordinate from equation (4.48) into the expression of the reduced Hamiltonian together with the Hamiltonian constraint yields

$$H_{reduced} = 2\int_{\Sigma_g}|K^{TT}|_g^2 - 8\pi\chi, \tag{4.56}$$

where $\chi = 2(1-g) < 0$ is the Euler characteristics of Σ_g. This reduced Hamiltonian can be related to the Dirichlet energy of the Gauss map. The Dirichlet energy (conformally invariant on the domain) associated to the Gauss map (4.51) is given as

$$\begin{aligned} E[id;g,\rho] &= \frac{1}{2}\int_{\Sigma}\mu_g g^{ij}\rho_{ij} = \frac{1}{2}\int_{\Sigma}\mu_\gamma\gamma^{ij}\rho_{ij} = E[id;\gamma,\rho] \\ &= 2\int_{\Sigma}|K^{TT}|_g^2\mu_g - 4\pi\chi. \end{aligned} \tag{4.57}$$

Therefore, we have the following relation between the Dirichlet energy of the Gauss map and the reduced Hamiltonian of the dynamics

$$H_{reduced} = E[id;\gamma,\rho] - 4\pi\chi. \tag{4.58}$$

Let us consider that the Teichmüller space $\mathcal{T}\Sigma_g$ is parametrized by $\{q_\alpha\}_{alpha=1}^{6g-6}$, which may be of the Fenchel-Neilsen type (see [202] for details about Fenchel-Neilsen parametrization). Now we observe the following

$$\frac{\partial E[id;\gamma(q),\rho]}{\partial q_\alpha} = \frac{1}{4}\int_{\Sigma_g}\mu_\gamma\left(\gamma^{mn}\gamma^{ij}\rho_{ij} - 2\gamma^{im}\gamma^{jn}\rho_{ij}\right)\frac{\partial\gamma_{mn}}{\partial q_\alpha}, \tag{4.59}$$

which after substituting $\rho_{ij} = (|K^{TT}|_g^2 + \frac{\tau^2}{2})g_{ij} + 2\tau K_{ij}^{TT} = (e^{-4\varphi}|k^{TT}|_\gamma^2 + \frac{\tau^2}{2})e^{2\varphi}\gamma_{ij} + 2\tau k_{ij}^{TT}$ yields

$$\frac{\partial E[id;\gamma(q),\rho]}{\partial q_\alpha} = -\tau \int_{\Sigma_g} \mu_\gamma k^{TTij} \frac{\partial \gamma^{mn}}{\partial q_\alpha}. \tag{4.60}$$

Now let us go back to equation (4.54) and substitute $\gamma = \gamma(q)$. We immediately obtain

$$\begin{aligned} S_{reduced} &= \int_{I\subset\mathbb{R}} \left((\int_{\Sigma_g} -\mu_\gamma k^{TTij} \frac{\partial \gamma_{ij}}{\partial q_\alpha}) \dot{q}_\alpha - H_{reduced}(\gamma(q), p, \rho) \right) dt \\ &= \int_{I\subset\mathbb{R}} (p^\alpha \dot{q}_\alpha - H_{reduced}(\gamma(q), p, \rho))\, dt, \end{aligned} \tag{4.61}$$

which upon utilizing equations (4.58) and (4.60) leads to

$$\frac{\partial H_{reduced}(\gamma(q), p, \rho)}{\partial q_\alpha} = \tau p^\alpha. \tag{4.62}$$

Here $\{(q_\alpha, p^\alpha)\}_{\alpha=1}^{6g-6}$ parametrizes the phase space i.e., the co-tangent bundle of $\mathcal{T}\Sigma_g$. Now using the time defined in (4.48), we may construct a principle functional after substituting $T = -\frac{1}{\tau}$

$$\mathcal{S}(q, \gamma(q), \rho) = -T(E[id;\gamma(q),\rho] - 4\pi\chi) \tag{4.63}$$

which then clearly satisfies

$$p^\alpha = \frac{\partial \mathcal{S}}{\partial q_\alpha}, \tag{4.64}$$

$$-\frac{\partial \mathcal{S}}{\partial T} = E[id;\gamma(q),\rho] - 4\pi\chi = H_{reduced}(q, p, \gamma(q)), \tag{4.65}$$

that is, $\mathcal{S}$ satisfies the Hamilton-Jacobi equation

$$-\frac{\partial \mathcal{S}}{\partial T} = H_{reduced}(q, p, \gamma(q)) \tag{4.66}$$

for all $T \in (0, \infty)$. In other words $\mathcal{S}$ is dynamically complete. For detailed analysis (arguments underlying dynamical completeness of $\mathcal{S}$), the reader is referred to the relevant sections of [198]. Here we only require the fact that through the solution of this Hamilton-Jacobi equation, the Gauss map equation defined in (4.51) solves the Einstein equation for all $T \in (0, \infty)$ or equivalently for all $\tau \in (-\infty, 0)$ and defines a ray-structure based at ρ of the Teichmüller space parametrized by the transverse-traceless conformally invariant 2-tensor k_{ij}^{TT}. The conformal metric $\gamma_{ij} = e^{-2\varphi} g_{ij} \in \mathcal{T}\Sigma_g$ indeed approaches to ρ_{ij} in the limit $\tau \to 0^-$. Therefore, if we run the Einstein flow in the reverse direction, then the expression of γ_{ij} in terms of ρ_{ij} and k^{TT}_{ij} obtained from the Gauss map equation defines a ray-structure of the Teichmüller space parametrized by k^{TT}_{ij} i.e., for a fixed ρ, two different k^{TT}_{ij} correspond to two different rays. An implicit solution [198] of the Gauss map equation (4.51) gives

$$\gamma^{ij} = e^{2\varphi} g^{ij} = e^{2\varphi}\left(\frac{2\tau^3}{\mu_\rho}\frac{\rho^{ik}\mu_\gamma\gamma^{jl}k^{TT}_{kl}}{1+\sqrt{1+\frac{2\tau^2\mu_\gamma^2|k^{TT}|^2_\gamma}{\mu_\rho^2}}} + \tau^2\frac{\sqrt{1+\frac{2\tau^2\mu_\gamma^2|k^{TT}|^2_\gamma}{\mu_\rho^2}}}{1+\sqrt{1+\frac{2\tau^2\mu_\gamma^2|k^{TT}|^2_\gamma}{\mu_\rho^2}}}\rho^{ij}\right).$$

Using this equation (which is effectively the same as the Gauss map equation), [198] constructed a fully non-linear elliptic equation of Monge-Ampere type and showed that a unique solution of such equation exists. Recently [226] showed using a direct analytic technique that such a unique solution exists for all $\tau \in (-\infty, 0)$. Essentially, these analyses are in a sense complementary to the Hamilton-Jacobi theory and provide a more explicit description of the ray structure of the Teichmüller space. Analyzing the associated Monge-Ampere equation, [198] explicitly showed that every non-trivial solution curve of the reduced dynamics in the configuration space ($\mathcal{T}\Sigma_g$) approaches a point (ρ) lying in the interior of the Teichmüller space, that is,

$$\lim_{\tau\to 0^-}\gamma^{ij} \quad = \quad \rho^{ij}. \tag{4.67}$$

Note that the choice of ρ is arbitrary as long as it does not leave the compact sets of $\mathcal{T}\Sigma_g$, and therefore, one may vary ρ over $\mathcal{T}\Sigma_g$ to obtain the full ray-structure of the Teichmüller space. We do not provide the complete calculations regarding the $\tau \to 0^-$ behavior of the solution curve as it is derived and described in detail by Moncrief in [198]. Readers are referred to the relevant sections of the same. We only need the information that the Gauss map equation together with the Hamilton-Jacobi equation indeed describes ray structures of the Teichmüller space and every such ray solves the reduced Einstein equation. Forward time asymptotics of each such ray corresponds to an interior point which also realizes the infimum of the Dirichlet energy (and the reduced Hamiltonian). Each member of a family of rays which asymptotically approach the point $\rho \in \mathcal{T}\Sigma_g$ corresponds to a unique choice of k^{TT} and none of the two rays of a same family intersect each other (except at ρ, where they approach as $\tau \to 0^-$).

The forward in time limit of the solution curves is well studied in [198]. Therefore, without repeating the same here, we will proceed to study the other limit, that is, the $\tau \to -\infty$ limit which corresponds to the big bang singularity. In this limit the solution curve leaves every compact set of the Teichmüller space, a conclusion which may be obtained through studying the time evolution of the Dirichlet energy (a proper function on $\mathcal{T}\Sigma_g$) of the Gauss map. The time is chosen to be $t = -\frac{1}{\tau}$ (4.48). From equation (4.57), the time derivative of the $|K^{TT}|_g^2$ reads

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma |K^{TT}|_g^2\mu_g &= \frac{d}{dt}\int_\Sigma (\frac{\tau^2}{2} + R(g))\mu_g, && (4.68)\\
&= \tau^2\frac{d}{d\tau}\int_\Sigma (\frac{\tau^2}{2} + R(g))\mu_g, && (4.69)\\
&= \tau^3\int_\Sigma \mu_g + \frac{\tau^2}{2}\int_\Sigma \mu_g(-2N\tau), && (4.70)\\
&= \tau\int_\Sigma N|K^{TT}|_g^2\mu_g,\\
&= -\frac{1}{t}\int_\Sigma N|K^{TT}|_g^2\mu_g,
\end{aligned}$$

where, we have used the lapse equation $\Delta_g N + N(|K^{TT}|_g^2 + \frac{\tau^2}{2}) = \tau^2$, the Hamiltonian constraint $|K^{TT}|_g^2 = \frac{\tau^2}{2} + R(g)$, and the evolution equation $\frac{\partial g_{ij}}{\partial t} = -2NK_{ij} + (L_X g)_{ij}$. Utilizing the estimate of the lapse function (4.50), we immediately obtain

$$-\frac{2}{t}\int_\Sigma |K^{TT}|_g^2\mu_g \leq \frac{d}{dt}\int_\Sigma |K^{TT}|_g^2\mu_g \leq -\frac{1}{t}\int_\Sigma |K^{TT}|_g^2\mu_g, \tag{4.71}$$

integration of which yields at the $t \to 0$ limit

$$\frac{const.}{t} \leq \int_\Sigma |K^{TT}|_g^2\mu_g \leq \frac{const.}{t^2}. \tag{4.72}$$

Using the expression of the Dirichlet energy $E[id;\gamma,\rho]$ from equation (4.57), the following estimate is obtained in the limit $\tau \to -\infty$ i.e., $t \to 0$

$$\frac{2C_2}{t} - 4\pi\chi \leq E_\gamma \leq \frac{2C_3}{t^2} - 4\pi\chi, \tag{4.73}$$

which clearly implies that the Dirichlet energy blows up at the limit of the big bang singularity i.e., in the limit $t \to 0$ or equivalently $\tau \to -\infty$ unless $\int_{\Sigma_g} |K^{TT}|_g^2 \equiv 0$, $0 < C_2, C_3 < \infty$. An immediate interpretation of such limiting behavior would be that the corresponding Einstein solution curve leaves every compact set in the Teichmüller space (configuration space). This is precisely a consequence of the fact that the Dirichlet energy is a proper function on the Teichmüller space (see [204] for the detailed proof of the properness of the Dirichlet energy). Therefore, every non-trivial solution curve leaves the Teichmüller space at the limit of the big-bang. However, we do not know where they converge in the space of projective currents. Note that the space of projective currents is compact and therefore every sequence has a convergent subsequence (since for a metric space, compactness and sequential compactness are equivalent). However, in our context, convergence is a bit more subtle since we are necessarily dealing with curves. We have to extract a sequence $\{l_{\tau_i}\}_{i=1}^{\infty}$ and show that it converges in the space of projective currents in the limit $i \to \infty$ ($\lim_{i\to\infty} \tau_i = -\infty$) and that the limit does not depend on the choice of the sequence. We want to identify this limit set in the space of projective currents. In fact we would like to show in the following sections that every non-trivial solution curve indeed attaches to the Thruston boundary of the Teichmüller space. In addition to the backward in time asymptotic behavior of the Dirichlet energy, we also observe the monotonic decay of the same in the time forward direction

$$\begin{aligned}\frac{d}{dt}E[id;\gamma,\rho] &= 2\frac{d}{dt}\int_{\Sigma}|K^{TT}|_g^2\mu_g, \\ &= 2\tau\int_{\Sigma}N|K^{TT}|_g^2\mu_g < 0.\end{aligned} \tag{4.74}$$

$\frac{d}{dt}E[id;\gamma,\rho] \equiv 0$ if and only if $K^{TT} \equiv 0$ (or $k^{TT} \equiv 0$) and $\frac{d}{dt}E[id;\gamma,\rho] \to 0$ at the limit $\tau \to 0$. The solutions corresponding to $k^{TT} \equiv 0$ are nothing but the fixed points of the reduced Einstein evolution equations (i.e., the trivial solutions). Substituting $k^{TT} = 0$ into the Lichnerowicz equation (4.29) yields

$$-2\Delta_{\gamma}\varphi + 1 - \frac{e^{2\varphi}\tau^2}{2} = 0, \tag{4.75}$$

which has a unique solution

$$e^{2\varphi} = \frac{2}{\tau^2}. \tag{4.76}$$

The reduced evoluion equation reads

$$\frac{\partial\gamma_{ij}}{\partial t} = e^{-2\varphi}\left(-(\partial_t e^{2\varphi}\gamma_{ij} - 2Nk_{ij}^{TT} - e^{2\varphi}N\tau\gamma_{ij} + (L_X e^{2\varphi}\gamma)_{ij}\right), \tag{4.77}$$

which, upon substituting $e^{2\varphi} = \frac{2}{\tau^2}$, $k^{TT} = 0$ and utilizing the lapse equation, shift equation, and Hamiltonian constraint yields

$$\frac{\partial\gamma_{ij}}{\partial t} = 0. \tag{4.78}$$

A few lines of simple calculation yields $\partial_t k_{ij}^{TT} = 0$ as well. These fixed points characterized by $(\gamma_{ij}, k_{ij}^{TT} = 0, N = 2, X^i = 0)$, $R(\gamma) = -1$, are indeed fixed points for arbitrary large data (even though the Dirichlet energy controls the $(H^1 \times L^2)$ norm of the data (γ, k^{TT}), finite dimensionality of the phase space implies that control on this norm is sufficient). This is precisely a consequence of the monotonic decay of the Dirichlet energy and $d_t E[id; \gamma, \rho] \equiv 0$ precisely at these fixed points (the Dirichlet energy acts as a Lyapunov function). Every solution curve approaches one of this fixed points in forward infinite time t. This point even though it is described in detail in [197, 198], is extremely important and will be of use in obtaining the main result. Summarizing this section together with the results of [198], we have the following theorem

Theorem 1: *Let Σ_g be a closed (compact without boundary) Riemann surface of genus $g > 1$. The data $(\gamma, k^{TT}, \tau, N, X)$ defined through the Gauss map equation (4.51) and elliptic equations (4.21-4.22) solve the Einstein Einstein dynamical equations iff they also solve the constraints and the associated Hamilton-Jacobi equation (4.66) is satisfied. Such a solution asymptotically approaches the fixed point solution $(R(\gamma) = -1, k^{TT} = 0, N = 2, X^i = 0)$ of the dynamical equations in the limit $\tau \to 0$ and every such non-trivial solution curve runs off the edge of the configuration space (Teichmüller space) in the limit of the big-bang singularity ($\tau \to -\infty$).*

Now we enter into the final phase where we utilize available results stated in the previous sections and obtain the main result.

4.5 Asymptotic behavior of the solution curve at big-bang and Thurston boundary

In the previous section, we have established that every non-trivial solution curve runs off the edge of the Teichmüller space. However, we do not apriori know whether they actually attach to the Thurston boundary. However, when realizing the Teichmüller space as a subset of the space of projective currents (which is compact), if we extract a sequence from the solution curve, this must converge somewhere at the limit $\tau \to -\infty$ (after passing to a subsequence and the limit should not depend on the choice of the sequence). Here, we will show that this limit set will be characterized by $\int_{\Sigma_g} \sqrt{|k^{TT}|_\gamma^2} \mu_\gamma = C$, $C < \infty$ is an uniform constant. Let us designate this boundary as the 'Einstein boundary' of the Teichmüller space and denote it by Ein_g. Our goal in this section is to show that this boundary is indeed equivalent to the Thurston boundary that is $\mathcal{T}\bar{\Sigma}_g^{Th} = \mathcal{T}\Sigma_g \cup \partial\mathcal{T}\Sigma_g^{Th} \approx \mathcal{T}\Sigma_g \cup Ein_g$. Note that Michael Wolf [215] obtained a compactification of Teichmüller space through the use of holomorphic quadratic differentials and he proved that his compactification is indeed equivalent to the Thurston compactification. In our case, we are automatically equipped with a holomorphic quadratic differential k^{TT} (the transverse-traceless tensor). However, importantly, Wolf's analysis is quite different from ours (and complementary in nature) in a sense that the Einsteinian dynamics occurs in the domain of the associated harmonic map while Wolf's dynamics materializes in the target space.

Now we will show the boundedness of $|k^{TT}|^2_\gamma$ in the limit $\tau \to -\infty$.

Lemma 4: *Let (k^{TT}, γ) solve the reduced Einstein equations after imposing the constraints and gauge conditions. Then the following estimates hold for $|k^{TT}|^2_\gamma$ at the limit $\tau \to -\infty$*

$$0 < \lim_{t \to 0} \int_\Sigma \sqrt{|k^{TT}|^2_\gamma}\mu_\gamma \le C < \infty. \tag{4.79}$$

Proof: Note that the following entity is conformally invariant

$$P = \int_{\Sigma_g} \sqrt{|K^{TT}|^2_g}\mu_g = \int_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}\mu_\gamma. \tag{4.80}$$

Applying the Cauchy-Swartz inequality, Hamiltonian constraint $|K^{TT}|^2_g = \frac{\tau^2}{2} + R(g)$, and time defined in (4.48), we immediately obtain

$$\begin{aligned}
\left(\int_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}\mu_\gamma\right)^2 &= \left(\int_{\Sigma_g} \sqrt{|K^{TT}|^2_g}\mu_g\right)^2 \\
\left(\int_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}\mu_\gamma\right)^2 &\le \left(\int_{\Sigma_g} |K^{TT}|^2_g\mu_g\right)\left(\int_{\Sigma_g} \mu_g\right) \\
&= \left(\int_{\Sigma_g} (\frac{\tau^2}{2} + R(g))\mu_g\right)\left(\int_{\Sigma_g} \mu_g\right) \\
&= \frac{\tau^2}{2}\left(\int_{\Sigma_g} \mu_g\right)^2 + 4\pi\chi \int_{\Sigma_g} \mu_g \le \frac{1}{2t^2}V(g)^2,
\end{aligned}$$

where we have the used Gauss-Bonet theorem $\int_{\Sigma_g} R(g)\mu_g = 4\pi\chi$, where $\chi = 2(1-g) < 0$ is the Euler characteristic. On the other hand, we know that the volume $V(g)$ of (Σ_g, g) approaches zero at the big-bang. However, we will study the evolution of $V(g)$ and obtain a more precise estimate in terms of $|\tau|$. Time differentiating $V(g) = \int_{\Sigma_g} \mu_g$ (here 'g' in Σ_g denotes genus while g in μ_g denotes the volume form associated to metric g) yields

$$\frac{dV(g)}{dt} = \frac{1}{2}\int_{\Sigma_g} g^{ij}\partial_t g_{ij}\mu_g, \tag{4.81}$$

which together with the evolution equation $\partial_t g_{ij} = -2N(K^{TT}_{ij} + \frac{\tau}{2} g_{ij}) + (L_X g)_{ij}$ yields

$$\frac{dV(g)}{dt} = \int_{\Sigma_g} (-N\tau + \nabla[g]_i X^i)\mu_g = -\tau \int_{\Sigma_g} N\mu_g, \tag{4.82}$$

where the total covariant divergence term is dropped following Stokes' theorem. Utilizing the estimate of the lapse function $1 \leq N \leq 2$ (4.50) and $t = -\frac{1}{\tau}$ (4.48), we immediately achieve the following bound for the time derivative of the volume $V(g)$

$$\frac{1}{t}V(g) \leq \frac{dV(g)}{dt} \leq \frac{2}{t}V(g), \tag{4.83}$$

integration of which yields the following at the limit $\tau \to -\infty$ or $t \to 0$

$$constant_1 \cdot\ t^2 \leq V(g(t)) \leq\ constant_2 \cdot\ t \tag{4.84}$$

Therefore, by using the inequality $0 < \left(\int_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}\mu_g\right)^2 \leq \frac{1}{2t^2}(V(g))^2$, we obtain

$$0 < \lim_{t\to 0} \int_{\Sigma} \sqrt{|k^{TT}|^2_\gamma}\mu_\gamma \leq C < \infty, \tag{4.85}$$

for a uniform constant C (uniform over the conformal structure). Since, $k^{TT} \equiv 0$ implies convergence to a point lying in the interior of $\mathcal{T}\Sigma_g$, the left inequality in (4.85) is strict (by the blow up of Dirichlet energy). This concludes the proof of the lemma. $\square$

More importantly, $\sup_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}(\tau)$ appears explicitly in a later part where we analyze the Gauss-map equation. In that particular analysis, we require a point-wise control of $\sqrt{|k^{TT}|^2_\gamma}$. Notice that this is the L^∞ norm of the holomorphic quadratic differential $\phi_\tau := (k^{TT}_{11} - ik^{TT}_{12})dz^2$ with respect to the metric γ. The obvious problem is that ϕ_τ may have singularities on measure zero sets. Remarkably, an integrable holomorphic quadratic differential enjoys the property of possessing at most simple poles at punctures of Σ_g. Now, even though Σ_g in question does not have punctures, it forms $\delta-$thin regions in the limit of $\tau \to -\infty$. This is precisely the consequence of the blow up of the Dirichlet energy. If we parameterize the Teichmüller space in the space of projective currents by the lengths of $9g-9$ nontrivial elements of $\pi_1(\Sigma_g)$ ($9g-9$ theorem), then the properness of the Dirichlet energy yields geodesics with large hyperbolic length. As a consequence of the Collar lemma [204], a geodesic transverse to such a long geodesic shrinks (again relative to hyperbolic length) leading to development of a $\delta-$thin region. We will shortly show via the thick-thin decomposition of Σ_g that an integrable holomorphic quadratic differential is bounded (in the sense of L^∞ norm with respect to the metric γ) even if Σ_g develops "bad" parts. Let us first define the norms we are interested in. The L^1 norm and the L^∞ norm (with respect to $\gamma := e^{2\eta}(dx \otimes dx + dy \otimes dy)$) or Ber's supremum norm of ϕ_τ are defined as follows

$$
\begin{aligned}
||\phi_\tau||_{L^1(\Sigma_g)} &:= \tfrac{1}{\sqrt{2}} \int_{\Sigma_g} |\phi_\tau| = \int_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}\mu_\gamma, && (4.86)\\
||\phi_\tau||_{L^\infty(\Sigma_g)} &:= \sup_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma} && (4.87)\\
&:= \sup_{\Sigma_g} \sqrt{\gamma^{ik}\gamma^{jl}k^{TT}_{ij}k^{TT}_{kl}}\\
&= \sup_{\Sigma_g} \sqrt{e^{-4\eta}\delta^{ik}\delta^{jl}k^{TT}_{ij}k^{TT}_{kl}}\\
&= \sqrt{2}\sup_{\Sigma_g} e^{-2\eta}\sqrt{(k^{TT}_{11})^2 + (k^{TT}_{12})^2}.
\end{aligned}
$$

Here, we have used the symmetry and traceless property of k^{TT} i.e., $k^{TT}_{12} = k^{TT}_{21}$, and $k^{TT}_{11} + k^{TT}_{22} = 0$. These norms are the natural ones defined for sections of vector bundles defined on Σ_g. Both norms make the space of holomorphic quadratic differentials on Σ_g to be a Banach space. Since, the dimension (real) of this space is $6g-6$ (therefore, finite), the L^1 norm is equivalent to Ber's supremum norm. However, let us explicitly establish the equivalence between L^1 and L^∞ in the case when Σ_g contains "bad" parts by invoking the thick-thin decomposition of Σ_g.

Let Σ_g be a hyperbolic Riemann surface. We will think of $\pi_1(\Sigma_g)$ as the set of the non-trivial loops up to homotopy. For $\delta > 0$, the thin and thick parts of Σ_g are defined as follows

$$
\begin{aligned}
\Sigma_{g(0,\delta]} &:= \{x \in \Sigma_g : \exists <\alpha> \in \pi_1(\Sigma_g) |\ l_\gamma(\alpha) \leq \delta\} && (4.88)\\
\Sigma_{g[\delta,\infty)} &:= \{x \in \Sigma_g : \exists <\alpha> \in \pi_1(\Sigma_g) |\ l_\gamma(\alpha) \geq \delta\}.
\end{aligned}
$$

Here, $l_\gamma(\alpha)$ indicates the length of the geodesic in the homotopy class $<\alpha>$ with respect to the hyperbolic metric γ. The thin part may consist of cusps and Margulis tubes. Since, we are dealing with the compact case, the thin part contains a Margulis tube ($\mathbb{S}^1 \times I$, $I \subset \mathbb{R}$) only. The obvious problem arises in the $\delta-$thin region since, in this region, the length of a geodesic decreases without bound as we approach the big-bang. Here we fix a $\delta > 0$ and focus on the behaviour of the L^∞ norm (w.r.t γ_τ) of the holomorphic quadratic differential (ϕ_τ) in the $\delta-$thin region since, in the $\delta-$thick region, the L^∞ norm is always controlled by the L^1 norm. We now state two lemmas which conclude the business of controlling the L^∞ norm (with respect to γ) in terms of the L^1 norm of ϕ_τ. Note that an integrable holomorphic quadratic differential on a closed (no punctures, no boundary components) Riemann surface does not have poles and therefore has zero principle part. For such an integrable holomorphic quadratic differential on Σ_g (since it is compact without boundary), the following lemma holds. Proof of this lemma uses results from elementary complex analysis such as the maximum principle for holomorphic functions.

Lemma 5a: [225] *For $\delta > 0$ and any closed Riemann surface Σ_g, there exists a constant $C < \infty$ depending only on the genus g of Σ_g and independent of δ such that for every hyperbolic metric γ on Σ_g the following holds for the holomorphic quadratic differential in the $\delta-$thin region*

$$||\phi||_{L^\infty(M_{(0,\delta]})} \leq Ce^{-\pi/\delta}/\delta^2||\phi||_{L^1(\Sigma_g)}. \tag{4.89}$$

Of course the boundedness follows from the boundedness of $e^{-\pi/\delta}/\delta^2$. In the $\delta-$thick region, L^∞ (with respect to the metric γ) control in terms of L^1 (with respect to the metric γ) is trivial and follows from the following lemma.

Lemma 5b: [225] *For any $\delta > 0$ and any closed Riemann surface Σ_g, there exists a constant $C_\delta < \infty$ depending only on δ and the genus g of Σ_g such that for every hyperbolic metric γ on Σ_g the following holds for the holomorphic quadratic differential in the $\delta-$thick region*

$$||\phi||_{L^\infty(M_{[\delta,\infty)})} \leq C_\delta||\phi||_{L^1(\Sigma_g)}. \tag{4.90}$$

Now consider the case when Σ_g completely degenerates and forms punctures (From the Deligne-Mumford compactness theorem, punctured surfaces are achieved as a limit of a Riemann surface degenerating via collapsing non-trivial simply closed geodesics). Now, the analysis becomes a little more subtle since integrable holomorphic quadratic differentials may have a simple pole at a puncture (at worst). A neighbourhood of this puncture corresponds to a cusp and is equivalent to a punctured open disc (punctured at 0) equipped with the metric $e^{2\eta}(dx^2 + dy^2) = \frac{1}{|z|^2(\log(|z|))^2}|dz|^2$. Now, roughly it is clear that the simple pole of ϕ_τ cancels in the norm $||\phi||_{L^\infty} := e^{-2\eta}\sqrt{k_{11}^2 + k_{12}^2}$. But for completeness we state the following lemma from [225].

Lemma 5c: *Let (Σ_g, γ) be a hyperbolic Riemann surface with finite area. Then for any holomorphic quadratic differential ϕ of Σ_g the following are equivalent*

1. $||\phi||_{L^1(\Sigma_g)} \lesssim C, C < \infty$

2.$||\phi||_{L^\infty(\Sigma_g)} \lesssim C, C < \infty$

3. At each of the punctures of Σ_g the differential ϕ has at worst a simple pole.

Now let us consider a sequence $\{(\gamma_{\tau_j}, \phi_{\tau_j}, \tau_j\}_{i=1}^{\infty}$ lying on the solution curve of the Einstein flow (on the phase space) with $\lim_{j\to\infty} \tau_j = -\infty$. If each member of the sequence satisfies $||\phi_{\tau_j}||_{L^1(\Sigma_g)} \leq C$ with the limit satisfying $\lim_{i\to\infty} ||\phi_{\tau_j}||_{L^1(\Sigma_g)} \leq C$ (this is precisely what we have derived in (4.85)), then from the lemmas 1a,b,c, we conclude that the L^∞ norm of the limit is also bounded, that is, the following is satisfied

$$\lim_{j\to\infty} \sup_{\Sigma_g} \sqrt{|k^{TT}|^2_{\gamma_{\tau_j}}} \leq C, C < \infty. \tag{4.91}$$

Since $||\phi_{\tau_j}||_{L^1(\Sigma_g)}$ can only increase as $\tau_j \to -\infty$, we may set C as the uniform upper bound for the L^1 norm of each element of the sequence. We may therefore conclude the following boundedness of the L^∞ norm of the limit of the sequence $\{\phi_{\tau_j}\}$ (w.r.t $\{\gamma_{\tau_j}\}$) i.e.,

$$\lim_{\tau_j\to-\infty} ||\phi_{\tau_j}||_{L^\infty} \leq \sqrt{2}C_\infty \tag{4.92}$$

or

$$\lim_{\tau_j\to-\infty} \sup_{\Sigma_g} \sqrt{|k^{TT}|^2_{\gamma_{\tau_j}}} \leq C_\infty, \tag{4.93}$$

for some uniform $C_\infty < \infty$. Now we go back to the following point-wise inequality (4.31)

$$\frac{2}{\tau^2} \leq e^{2\varphi} \leq \frac{1 + \sqrt{1 + 2\tau^2 \sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)}}{\tau^2}. \tag{4.94}$$

Using the fact that $\sup_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma} \leq C_\infty < \infty$, we may conclude that the following estimate of $\sup_{\Sigma_g} e^{2\varphi}$ holds in the limit $\tau \to -\infty$

$$\frac{2}{\tau^2} \leq e^{2\varphi} \leq \frac{C_\varphi}{|\tau|}, \tag{4.95}$$

for a suitable constant $0 < C_\varphi < \infty$.

In summary, we have obtained the two following crucial estimate which will be utilized later

$$\sup_{\Sigma_g} |k^{TT}|^2_\gamma \leq C^2_\infty, \tag{4.96}$$

$$\frac{2}{\tau^2} \leq e^{2\varphi} \leq \frac{C_\varphi}{|\tau|}, \tag{4.97}$$

as $\tau \to -\infty$. We have now obtained the necessary estimates from Einstein's equations in CMCSH gauge. Utilizing these estimates we want to establish a relation between the hyperbolic length of a nontrivial element of $\pi_1(\Sigma_g)$ and its transverse measure against the measured foliation associated with the holomorphic quadratic differential. As mentioned previously, we have a natural holomorphic quadratic differential associated to the Einstein flow due to the fact that corresponding to each transverse-traceless tensor k^{TT}, we may associate a holomorphic quadratic differential. Here, we define the following quadratic differential

$$\begin{aligned}
\phi_\tau &= (k^{TT}_{11} - ik^{TT}_{12})dz^2 \quad (4.98)\\
&= \phi_\tau(z)dz^2, \quad (4.99)\\
&= k^{TT}_{11}(dx^2 - dy^2) + 2k^{TT}_{12}dxdy + \\
&\quad -i(k^{TT}_{12}(dx^2 - dy^2) - 2k^{TT}_{11}dxdy),\\
&= k + i\xi,
\end{aligned}$$

where $i = \sqrt{-1}$. Note that the transverse-traceless tensor k^{TT} may be recovered as follows

$$k^{TT} = \mathcal{R}\left(\phi_\tau(z)dz^2\right). \tag{4.100}$$

The transverse-traceless property of k^{TT}_{ij} precisely implies $\frac{\partial\phi_\tau}{\partial\bar{z}} = 0$ i.e., ϕ_τ is holomorphic. This establishes the well known homeomorphism between the space of holomorphic quadratic differentials and the space of transverse-traceless tensors. In addition, we have a natural homeomorphism between the space of transverse traceless tensors on (Σ_g, γ) and the Teichmüller space from the Einstein flow (for detailed analysis see [198]). Once we have a quadratic differential we immediately obtain horizontal and vertical measured foliations associated with this holomorphic quadratic differential. The transverse measures of the vertical measured foliation and horizontal measured foliation are (as follows from (4.6) and (4.7))

$$\mu_{vert}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{\frac{k + \sqrt{k^2 + \xi^2}}{2}}, \tag{4.101}$$

$$\mu_{hor}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{\frac{\sqrt{k^2 + \xi^2} - k}{2}}, \tag{4.102}$$

respectively. Let us consider that the tangent vector field to the curve $\mathcal{C}$ be $u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$ and denote this by $(u^1, u^2)^T$. The term k^2 may be written as the bi-linear form $k^{TT}_{ij}u^iu^j(d\lambda)^2$, where λ is the parameter along $\mathcal{C}$. Similarly, the term ξ may be written as $k^{TT}_{im}J^m_j u^iu^j = k^{TT}_{im}u^iv^m$, where

$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $v^m = J^m_j u^j$ that is $v = (-u^2, u^1)^T$. More importantly, we see that the following holds in isothermal coordinates ($\gamma = \gamma(z)|dz|^2, \gamma(z) = e^{\delta(z)}, \delta(z) : \Sigma_g \to \mathbb{R}$)

$$\gamma(u, v) = \gamma(z)(-u^1u^2 + u^2u^1) = 0, \tag{4.103}$$

that is, u and v are orthogonal to each other with respect to the metric γ. This is precisely a consequence of the existence of an isothermal chart around any point on Σ_g and since $\gamma(u, v)$ is a scalar, vanishing in one coordinate chart implies vanishing in every coordinate chart (as mentioned in the begining, we use isothermal coordinates throughout). The transverse measure to the vertical foliation may be written as follows

$$\mu_{vert}(\mathcal{C}) \quad = \quad \oint_{\mathcal{C}} \sqrt{\left|\frac{k^{TT}_{ij}u^iu^j + \sqrt{(k^{TT}_{ij}u^iu^j)^2 + (k^{TT}_{ij}u^iv^j)^2}}{2}\right|}d\lambda. \tag{4.104}$$

Let us now compute the $\gamma-$length of a geodesic in the homotopy class $[\mathcal{C}]$ and relate it to its transverse measure associated to the measured foliation of the holomorphic quadratic differential ϕ_τ. Through the unique solution of the Monge-Ampere equation, the Gauss map equation defines a ray structure of the Einstein equations. Therefore analyzing the asymptotic behaviour of the Monge-Ampere equation is in principle the same as analysing the Gauss map equation while satisfying the Einstein's equations through the associated Hamilton-Jacobi equation. In addition, analysis of the Gauss map equation seems more tractable (and relevant) because we have a handful of estimates from the elliptic equations associated with the Einstein dynamics. Using the Gauss map equation, we obtain

$$\begin{aligned} \rho_{ij}u^iu^j &= |K|^2_g g_{ij}u^iu^j + 2\tau K_{ij}u^iu^j \\ &\quad -\tau^2 g_{ij}u^iu^j, \\ &= (|K^{TT}|^2_g + \frac{\tau^2}{2})g_{ij}u^iu^j + 2\tau K^{TT}_{ij}u^iu^j \\ &= (e^{-4\varphi}|k^{TT}|^2_\gamma + \frac{\tau^2}{2})e^{2\varphi}\gamma_{ij}u^iu^j \\ &\quad +2\tau k^{TT}_{ij}u^iu^j. \end{aligned} \tag{4.105}$$

We do know the fact that $\rho \in \mathcal{T}\Sigma_g$ is fixed and the $\rho-$length of $\mathcal{C}$ is bounded (due to the properness of the Dirichlet energy which remains finite in the interior of the Teichmüller space).

Following the Gauss map equation, we have the following

$$\begin{aligned} |\rho_{ij}u^iu^j| &= |\left\{|K^{TT}|_g^2+\frac{\tau^2}{2}\right\}g_{ij}u^iu^j+2\tau k_{ij}^{TT}u^iu^j|, \\ &\geq |\left\{|K^{TT}|_g^2+\frac{\tau^2}{2}\right\}g_{ij}u^iu^j|-2|\tau k_{ij}^{TT}u^iu^j|, \\ &= ||k^{TT}|_\gamma^2e^{-2\varphi}\gamma_{ij}u^iu^j+\frac{\tau^2}{2}e^{2\varphi}\gamma_{ij}u^iu^j|-2|\tau k_{ij}^{TT}u^iu^j|, \\ &\geq 2|\sqrt{\frac{|k^{TT}|_\gamma^2\tau^2}{2}(\gamma_{ij}u^iu^j)^2}|-2|\tau k_{ij}^{TT}u^iu^j|, \end{aligned} \tag{4.106}$$

that is,

$$\frac{1}{\sqrt{2}}\sqrt{|k^{TT}|_\gamma^2}\gamma_{ij}u^iu^j\leq|k_{ij}^{TT}u^iu^j|+\frac{1}{2|\tau|}\rho_{ij}u^iu^j. \tag{4.107}$$

Here, we have used $a^2+b^2\geq 2ab$ for $a,b\in\mathbb{R}$. Now point-wise norm of $\sqrt{|k^{TT}|_\gamma^2}$ satisfies

$$0\leq|k^{TT}|_\gamma^2\leq C_\infty^2. \tag{4.108}$$

Notice that the infimum of $|k^{TT}|_\gamma^2$ may be zero since the holomorphic quadratic differential $\phi_\tau:=(k_{11}^{TT}-ik_{12}^{TT})dz^2$ has finite number of zeros. Let us consider that the infimum of $\sqrt{|k^{TT}|_\gamma^2}$ be C_f which is strictly positive provided that we stay away from the zeros (finite number) of the quadratic differential ϕ_τ (which correspond to the singularities of the associated measured foliation). Let us consider that the quadratic differential has zeros at $(z_1,z_2,....,z_n)$, $n<\infty$. Consider ϵ disks $D_\epsilon(z_i)$ around each of the zeros. As these zeros correspond to the singularities of the associated measured foliation, we will consider the trasnverse measure on $\Sigma_g^{'}=\Sigma_g-\{\cup_{i=1}^nD_\epsilon(z_i)\}$ (a detailed rationale is sketched in Wolf's paper and therefore we do not repeat the same here). On $\Sigma_g^{'}$, the previous inequality becomes

$$\frac{|C_f|}{\sqrt{2}}\gamma_{ij}u^iu^j\leq|k_{ij}^{TT}u^iu^j|+\frac{1}{2|\tau|}\rho_{ij}u^iu^j, \tag{4.109}$$

$$\lim_{\tau\to-\infty}\frac{|C_f|}{\sqrt{2}}\gamma_{ij}u^iu^j\leq\lim_{\tau\to-\infty}|k_{ij}^{TT}u^iu^j|+\lim_{\tau\to-\infty}\frac{1}{2|\tau|}\rho_{ij}u^iu^j. \tag{4.110}$$

Now notice the fact that $\rho-$length of $\mathcal{C}$ is finite and independent of τ since ρ lies in the interior of the Teichmüller space and therefore

$$\lim_{\tau\to-\infty}\frac{1}{2|\tau|}\rho_{ij}u^iu^j=0. \tag{4.111}$$

We obtain the following inequality

$$\lim_{\tau\to-\infty}\frac{|C_f|}{\sqrt{2}}\gamma_{ij}u^iu^j \leq \lim_{\tau\to-\infty}|k^{TT}_{ij}u^iu^j|. \tag{4.112}$$

Let us analyze the Gauss-map equation in a different way

$$|\rho_{ij}u^iu^j - 2\tau k^{TT}_{ij}u^iu^j| \quad = \quad |\left\{|K^{TT}|^2_g + \frac{\tau^2}{2}\right\}g_{ij}u^iu^j. \tag{4.113}$$

Now utilizng the estimate of $|K^{TT}|^2_g$ from (4.31), we obtain

$$|2\tau k^{TT}_{ij}u^iu^j| - |\rho_{ij}u^iu^j| \quad \leq \quad \tau^2 e^{2\varphi}\gamma_{ij}u^iu^j, \tag{4.114}$$

which utilizing the estimate (4.31) yields

$$|2\tau k^{TT}_{ij}u^iu^j| - |\rho_{ij}u^iu^j| \quad \leq \quad \left(1+\sqrt{1+2\tau^2\sup_{x\in\Sigma'_g}|k^{TT}|^2_\gamma(\tau)}\right)\gamma_{ij}u^iu^j. \tag{4.115}$$

Substituting the estimate (4.96) into the previous inequality leads to

$$|k^{TT}_{ij}u^iu^j| \leq \left(\frac{1}{2\tau}+\frac{\sqrt{1+2C^2_\infty\tau^2}}{2\tau}\right)\gamma_{ij}u^iu^j + \frac{1}{2|\tau|}\rho_{ij}u^iu^j,$$

that is,

$$|k^{TT}_{ij}u^iu^j| \leq \left(\frac{1}{2\tau}+\frac{\sqrt{1+2C^2_\infty\tau^2}}{2\tau}\right)\gamma_{ij}u^iu^j + \frac{1}{2|\tau|}\rho_{ij}u^iu^j \tag{4.116}$$

and therefore, in the limit $\tau \to -\infty$

$$\begin{aligned}\lim_{\tau\to-\infty}\frac{|k^{TT}_{ij}u^iu^j|}{\gamma_{ij}u^iu^j} \quad &\leq \quad \lim_{\tau\to-\infty}\left(\frac{1}{2\tau}+\frac{\sqrt{1+2C^2_\infty\tau^2}}{2\tau}\right) \\ &= \quad \frac{|C_\infty|}{\sqrt{2}}.\end{aligned} \tag{4.117}$$

In a sense, we have as $\tau \to -\infty$

$$\frac{|C_f|}{\sqrt{2}}\gamma_{ij}u^iu^j \leq |k^{TT}_{ij}u^iu^j| \leq \frac{|C_\infty|}{\sqrt{2}}\gamma_{ij}u^iu^j, \tag{4.118}$$

with $0 < C^2_f < C^2_\infty < \infty$. This is an important expression obtained at the limit of the big-bang ($\tau \to -\infty$). On the other hand, the expression for the transverse measure of the vertical foliation reads (4.104)

$$\mu_{vert}(\mathcal{C}) \quad = \quad \oint_{\mathcal{C}} \sqrt{\left|\frac{k_{ij}^{TT}u^iu^j + \sqrt{(k_{ij}^{TT}u^iu^j)^2 + (k_{ij}^{TT}u^iv^j)^2}}{2}\right|} d\lambda. \tag{4.119}$$

We still need to obtain an estimate for the term $k^{TT}u^iv^j$. In addition to the transverse measure to the vertical foliation, we also have the following transverse measure to the horizontal foliation of the holomorphic quadratic differential ϕ_τ

$$\mu_{hor}(\mathcal{C}) \quad = \quad \oint_{\mathcal{C}} \sqrt{\left|\frac{\sqrt{(k_{ij}^{TT}u^iu^j)^2 + (k_{ij}^{TT}u^iv^j)^2} - k_{ij}^{TT}u^iu^j}{2}\right|} d\lambda. \tag{4.120}$$

In the analysis of Wolf [215], it is shown that this transverse measure associated to the horizontal foliation collapses asymptotically. In Wolf's [215] construction, the domain is fixed while the target is varied, that is the dynamics occurs in the target space. In our case, the dynamics takes place in the domain. Therefore, we can not utilize the available machinery such as the Beltrami differential $\nu := \frac{|W_{\bar{z}}|}{|W_z|}$ ($W : \Sigma_g(\gamma) \to \Sigma_g(\rho)$ and harmonic) or the associated Bochner equation controlling the behaviour of ν to show that μ_{hor} vanishes and therefore, $k_{ij}^{TT}u^iv^j$ approaches zero asymptotically. Once again the Gauss map equation (4.51) together with the Lichnerowicz equation (relativistic version of the Bochner equation) come to the rescue and notably they are of purely relativistic origin. The Gauss-map equation reads

$$\rho_{ij} \quad = \quad (e^{-4\varphi}|k^{TT}|_\gamma^2 + \frac{\tau^2}{2})e^{2\varphi}\gamma_{ij} + 2\tau k_{ij}^{TT}, \tag{4.121}$$

which upon contracting with ζ and η yields

$$\rho_{ij}\zeta^i\eta^j \quad = \quad (e^{-4\varphi}|k^{TT}|_\gamma^2 + \frac{\tau^2}{2})e^{2\varphi}\gamma_{ij}\zeta^i\eta^j + 2\tau k_{ij}^{TT}\zeta^i\eta^j. \tag{4.122}$$

Performing an exactly similar analysis as before, we may arrive without much difficulty to the following relation in the limit of the big-bang ($\tau \to -\infty$)

$$C^a|\gamma_{ij}\zeta^i\eta^j| \leq |k_{ij}^{TT}\zeta^i\eta^j| \leq C^b|\gamma_{ij}\zeta^i\eta^j|, \tag{4.123}$$

for $0 < C^a < C^b < \infty$. Now using (4.104), we immedeately obtain the following at the limit when

τ approaches $-\infty$

$$\oint_{\mathcal{C}} \sqrt{|\frac{k^{TT}_{ij}u^iu^j + \sqrt{(k^{TT}_{ij}u^iu^j)^2 + C^{a2}(\gamma_{ij}u^iv^j)^2}}{2}|}d\lambda \qquad (4.124)$$
$$\leq \mu_{vert}(\mathcal{C}) \leq$$
$$\oint_{\mathcal{C}} \sqrt{|\frac{k^{TT}_{ij}u^iu^j + \sqrt{(k^{TT}_{ij}u^iu^j)^2 + C^{b2}(\gamma_{ij}u^iv^j)^2}}{2}|}d\lambda.$$

But, from the orthogonality of u and v i.e., $\gamma_{ij}u^iv^j = 0$ (4.103) and using the bound $C^a|\gamma_{ij}u^iv^j| \leq |k^{TT}_{ij}u^iv^j| \leq C^b|\gamma_{ij}u^iv^j|$, we immediately observe that $|k^{TT}_{ij}u^iv^j| = 0$ which leads to the following expression for the transverse measure of curve $\mathcal{C}$ with respect to the vertical foliation defined by the holomorphic quadratic differential ϕ_τ

$$\mu_{vert}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{|k^{TT}_{ij}u^iu^j|}d\lambda. \qquad (4.125)$$

The asymptotic vanishing of the term $\xi = k^{TT}_{ij}u^iv^j$ precisely implies that the transverse measure of the associated horizontal folitaion vanishes i.e.,

$$\mu_{hor}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{|\frac{\sqrt{(k^{TT}_{ij}u^iu^j)^2 + (k^{TT}_{ij}u^iv^j)^2} - k^{TT}_{ij}u^iu^j}{2}|}d\lambda \qquad (4.126)$$
$$= 0.$$

Thus, the high Dirichlet energy limit (while viewed as a proper function on the Teichmüller space of the domain) precisely indicates that the transverse measure to the horizontal foliation associated to the quadratic differential ϕ_τ defined in terms of k^{TT} (or equivalently k^{TT}) vanishes. Note that the metric γ, the quadratic differential ϕ_τ, and the dynamics of the associated measured foliations are related to each other via the Einstein flow. In a sense, the Einstein flow drives the solution curve in such a way that the measured foliation behaves in this way at the limit of the big-bang singularity.

Therefore, we obtain the following crucial relation in the big-bang limit ($\tau \to -\infty$)

$$\mu_{vert}(\mathcal{C}) = C\oint_{\mathcal{C}} \sqrt{\gamma_{ij}u^iu^j}d\lambda = Cl_\gamma(\mathcal{C}), \qquad (4.127)$$

for a suitable constant $\frac{|C_f|^{1/2}}{2^{1/4}} \leq C \leq \frac{|C_\infty|^{1/2}}{2^{1/4}}$. An important point to notice is that the constants C_f and C_∞ are uniform in a sense that they do not depend on the chosen homotopy class $[\mathcal{C}]$. Now this does not imply that C is independent of the homotopy class chosen. In fact we need to show that C does not depend on the homotopy class of loops at the limit $\tau \to -\infty$. This follows since we will show that $|k^{TT}|^2_\gamma$ behaves as a constant modulo factors involving inverse power of the mean extrinsic curvature τ as τ approaches $-\infty$ (i.e., big bang). We claim the following.

Lemma 6: *The following is a solution of the Lichnerowicz equation (4.29) as $\tau \to -\infty$ i.e.,*

$$e^{2\varphi} = \frac{\sqrt{2}a}{|\tau|} + \frac{2}{\tau^2} + O(\frac{1}{|\tau|^3}), \tag{4.128}$$

where $a^2 := \lim_{\tau\to-\infty} \sup_\Sigma |k^{TT}|^2_\gamma = C_f$ (as shown previously by the equivalence of norm property) if

$$|k^{TT}|^2_\gamma = a^2 + O(\frac{1}{|\tau|^2}) \text{ a.e on } \Sigma_g \text{ as } \tau \to -\infty. \tag{4.129}$$

The corresponding lapse function N and the shift vector field X satisfy

$$N = 1 + \frac{\sqrt{2}}{|\tau|a} + O(\frac{1}{\tau^2}) \text{ a.e. on } \Sigma_g \text{ as } \tau \to -\infty. \tag{4.130}$$

and

$$\gamma(X, X) = O(\frac{1}{|\tau|}) \text{ a.e on } \Sigma_g \text{ as } \tau \to -\infty. \tag{4.131}$$

Proof: Substitute this form of $e^{2\varphi}$ in the Lichnerowicz equation (4.29) to yield

$$-2\Delta_\gamma\varphi + e^{-2\varphi}(|k^{TT}|^2_\gamma - a^2) = O(\frac{1}{|\tau|}) \tag{4.132}$$

integration of which yields at $\tau \to -\infty$

$$\int_\Sigma e^{-2\varphi}(|k^{TT}|^2_\gamma - a^2)\mu_\gamma = O(\frac{1}{|\tau|}). \tag{4.133}$$

However, by definition, $\sup_{\Sigma_g} |k^{TT}|^2_\gamma = a^2 \geq |k^{TT}|^2_\gamma$ everywhere on Σ_g and therefore

$$|k^{TT}|^2_\gamma = a^2 + O(\frac{1}{|\tau|^2}) \text{ a.e on } \Sigma \text{ as } \tau \to -\infty. \tag{4.134}$$

Substituting $|k^{TT}|^2_\gamma = a^2 + O(\frac{1}{|\tau|^2})$ *a.e on* Σ *as* $\tau \to -\infty$ into the lapse equation yields

$$N = 1 + \frac{\sqrt{2}}{|\tau| a} + O(\frac{1}{\tau^2}) \ as\ \tau \to -\infty \tag{4.135}$$

which yields through the shift equation and an integration by parts argument (4.22)

$$\int_{\Sigma_g} (\nabla[\gamma]_i X^j \nabla[\gamma]_k X^l \gamma^{ik}\gamma_{jl} + \gamma_{ij} X^i X^j)\mu_\gamma = O(\frac{1}{|\tau|}) \tag{4.136}$$

yielding

$$\gamma(X, X) = O(\frac{1}{|\tau|}) \ a.e\ on\ \Sigma_g\ as\ \tau \to -\infty. \qquad \square \tag{4.137}$$

Now we have to show that $|k^{TT}|^2_\gamma$ obtained in (4.134) satisfies the evolution equation at the limit $\tau \to -\infty$.

Lemma 7: *$g-$ norm of the transverse-traceless tensor k^{TT} i.e., $|k^{TT}|^2_g$ satisfies the following evolution equation*

$$\partial_t |k^{TT}|^2_g = L_X |k^{TT}|^2_g + 2N\tau |k^{TT}|^2_g + 2\nabla[g]^i \nabla[g]^j N k^{TT}_{ij}. \tag{4.138}$$

Moreover $|k^{TT}|^2_\gamma = e^{4\varphi}|k^{TT}|^2_g = a^2 + O(\frac{1}{\tau^2})$ satisfies this evolution equation almost everywhere on Σ_g up to $O(\frac{1}{|\tau|})$ as $\tau \to -\infty$.

Proof: Explicit calculation using $|k^{TT}|^2_g = g^{ij}g^{kl}k^{TT}_{ik}k^{TT}_{jl}$ and the Einstein evolution equations (4.17-4.18) yields the desired evolution equation for $|k^{TT}|^2_g$. Now utilizing the conformal transformation $g_{ij} = e^{2\varphi}\gamma_{ij}$ and noting k^{TT}_{ij} is conformally invariant, the evolution equation for $|k^{TT}|^2_g$ may be transformed into an evolution equation for $|k^{TT}|^2_\gamma$

$$\begin{aligned} \partial_t |k^{TT}|^2_\gamma + e^{4\varphi}|k^{TT}|^2_\gamma \partial_t e^{-4\varphi} &= 2N\tau |k^{TT}|^2_\gamma + L_X |k^{TT}|^2_\gamma - e^{4\varphi} L_X e^{-4\varphi} \\ +\gamma^{ik}\gamma^{jl}\nabla[\gamma]_i \nabla_j N k^{TT}_{kl} &- \frac{1}{2} e^{-2\varphi}(\gamma^{ik}\gamma^{ml}\partial_i e^{2\varphi} + \gamma^{mk}\gamma^{jl}\partial_j e^{2\varphi})\nabla_m N k^{TT}_{kl}, \end{aligned} \tag{4.139}$$

where we have utilized the fact $\nabla[g]_i \nabla_j N = \nabla[\gamma]_i \nabla_j N - \frac{1}{2} g^{mn}(\nabla[\gamma]_i g_{nj} + \nabla[\gamma]_j g_{in} - \nabla[\gamma]_n g_{il})$. Using $|k^{TT}|^2_\gamma = a^2 + O(\frac{1}{\tau^2})$, $N = 1 + \frac{\sqrt{2}}{|\tau| a} + O(\frac{1}{\tau^2})$, $e^{2\varphi} = \frac{\sqrt{2}a}{|\tau|} + \frac{2}{\tau^2} + O(\frac{1}{|\tau|^3})$, $e^{-2\varphi} = \frac{|\tau|}{\sqrt{2}a} - \frac{1}{a^2} + O(\frac{1}{|\tau|})$, and $\gamma(X,X)=O(\frac{1}{|\tau|})$, we may compute each term of the evolution equation (4.139) as follows

$$
\begin{aligned}
& e^{4\varphi}|k^{TT}|^2_\gamma\partial_t e^{-4\varphi} \qquad (4.140)\\
&= \left((\frac{2a^2}{\tau^2}+\frac{4}{\tau^4}+\frac{4\sqrt{2}a}{\tau^3})+O(\frac{1}{\tau^4})\right)(a^2+O(\frac{1}{\tau^2}))(\frac{\tau^3}{a^2}-\frac{\sqrt{2}\tau^2}{a^3}+O(|\tau|))\\
&= 2a^2\tau+2\sqrt{2}a+O(\frac{1}{|\tau|}),\\
& 2N\tau|k^{TT}|^2_\gamma = 2\tau a^2+2\sqrt{2}a+O(\frac{1}{|\tau|}) \qquad (4.141)
\end{aligned}
$$

$$e^{4\varphi}L_Xe^{-4\varphi}=O(\frac{1}{\tau^{5/2}}), \tag{4.142}$$

$$\gamma^{ik}\gamma^{jl}\nabla[\gamma]_i\nabla_jNk^{TT}_{kl}=O(\frac{1}{\tau^2}), \tag{4.143}$$

$$\frac{1}{2}e^{-2\varphi}(\gamma^{ik}\gamma^{ml}\partial_ie^{2\varphi}+\gamma^{mk}\gamma^{jl}\partial_je^{2\varphi})\nabla_mNk^{TT}_{kl}=O(\frac{1}{\tau^4}). \tag{4.144}$$

After substituting into the evolution equation (4.139), we observe that the dangerous terms ($O(|\tau|)$ and $O(1)$ terms) are precisely cancelled with their respective negative counterparts. Therefore we observe that the evolution equation for $|k^{TT}|^2_\gamma$ is solved by $|k^{TT}|^2_\gamma=a^2+O(\frac{1}{\tau^2})$ almost everywhere on Σ_g as $\tau\to-\infty$. $\square$

This property is extremely important and indicates an asymptotically velocity dominated behaviour i.e., the evolution equations are effective ordinary differential equations in time as one approaches the big-bang since the spatial parts are weighted by the inverse power of the mean curvature. Velocity term dominated behaviour (VTD) has also been previously noted in the context of big-bang singularity [230]. Since $|k^{TT}|^2_\gamma$ asymptotically approaches a constant a^2, we obtain at the limit $\tau\to-\infty$, $C_\infty=C_f=a$ and therefore the constant C in equation (4.127) is independent of the homotopy class of curve i.e., $C=\frac{a}{2^{1/4}}$ =constant on Σ_g as $\tau\to-\infty$.

In the proof of the compactification, we will need to choose a sequence ($\{\gamma_{\tau_j},k^{TT}_{\tau_j},\tau_j\}$) from the Einstein solution curve (here $f_\tau\equiv f(\tau)$ and both of these notations are used interchangeably). However, the limit should not depend on the sequence chosen as long as $\lim_{j\to\infty}\tau_j=-\infty$. In other words, the ratio of the transverse measure of a curve $\lambda\mapsto\mathcal{C}(\lambda)$ with respect to the vertical foliation corresponding to k^{TT} and its length with respect to the hyperbolic metric γ does not depend on the chosen sequence on the solution curve as one approaches the big-bang singularity. The following lemma establishes such a property.

Lemma 8: *Let $\{\gamma_{\tau_j},k^{TT}_{\tau_j},\tau_j\}$ be any sequence on the solution curve $\tau\mapsto(\gamma_\tau,k^{TT}_\tau,\tau)$ such that*

$\lim_{j\to\infty} = -\infty$. *Let $\mathcal{C}$ be a curve in any homotopy class. Then the following holds*

$$\lim_{j\to\infty} \mu_{vert,\tau_j}(\mathcal{C}) = \frac{C^{1/2}}{2^{1/4}} \lim_{j\to\infty} l_{\gamma_{\tau_j}}(\mathcal{C}), \tag{4.145}$$

where the constant $C = \lim_{\tau\to\infty} |k^{TT}|^2_\gamma$ and therefore is universal.

Proof: Let us consider a slightly general case where $\tau \in (-\infty, 0)$ instead of $\tau \to -\infty$. Let us define the following entities

$$\mathcal{I}n(\tau) := \inf_{\Sigma'_g} \sqrt{|k^{TT}|^2_\gamma(\tau)} \tag{4.146}$$

$$\mathcal{S}(\tau) := \sup_{\Sigma'_g} \sqrt{|k^{TT}|^2_\gamma(\tau)}. \tag{4.147}$$

Both $\mathcal{I}n(\tau)$ and $\mathcal{S}(\tau)$ are continuous functions of τ by existence-uniqueness-continuity (or well-posedness) of the Einstein's equations [197]. Clearly the following holds by continuity and the result of lemma 7

$$\lim_{\tau\to-\infty} \mathcal{I}n(\tau) = C_f = a \tag{4.148}$$

$$\lim_{\tau\to-\infty} \mathcal{S}(\tau) = C_\infty = a. \tag{4.149}$$

Performing a similar analysis on the Gauss-map equation as previously, we obtain

$$\frac{1}{\sqrt{2}}\mathcal{I}n(\tau)\gamma_\tau(u,u) \leq |k^{TT}_\tau(u,u)| + \frac{1}{|\tau|}\rho(u,u) \tag{4.150}$$

$$|k^{TT}_\tau(u,u)| \leq \left(\frac{1}{2|\tau|} + \frac{\sqrt{1+2\mathcal{S}(\tau)^2\tau^2}}{2|\tau|}\right)\gamma_\tau(u,u) + \frac{1}{|\tau|}\rho(u,u) \tag{4.151}$$

$$|k^{TT}_\tau(u,v)| \leq \frac{1}{|\tau|}|\rho(u,v)|, \tag{4.152}$$

where the metric ρ is fixed i.e., independent of τ and lies in the interior of the Teichmüller space. Let us now define

$$\mathcal{A}(\tau) := \frac{1}{2|\tau|} + \frac{\sqrt{1+2\mathcal{S}(\tau)^2\tau^2}}{2|\tau|}. \tag{4.153}$$

Thus, the second inequality of the three inequalities stated above reads

$$|k^{TT}_\tau(u,u)| \leq \mathcal{A}(\tau)\gamma_\tau(u,u) + \frac{1}{|\tau|}\rho(u,u) \tag{4.154}$$

Now, we go back to the formula for the transverse measure to the vertical foliation and obtain the following inequality for $\tau \in (-\infty, 0)$

$$\oint_{\mathcal{C}} \sqrt{k_{ij}^{TT}(\tau)u^i u^j}\, d\lambda \leq \mu_{vert,\tau}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{\left|\frac{|k_{ij}^{TT}(\tau)u^i u^j| + \sqrt{(k_{ij}^{TT}(\tau)u^i u^j)^2 + (k_{ij}^{TT}(\tau)u^i v^j)^2}}{2}\right|}\, d\lambda \tag{4.155}$$

since $(k_{ij}^{TT}(\tau)u^i v^j)^2 \geq 0$. Now utilizing (4.152), we obtain

$$\oint_{\mathcal{C}} \sqrt{|k_{ij}^{TT}(\tau)u^i u^j|}\, d\lambda \leq \mu_{vert,\tau}(\mathcal{C}) \leq \oint_{\mathcal{C}} \sqrt{\left|\frac{|k_{ij}^{TT}(\tau)u^i u^j| + \sqrt{(k_{ij}^{TT}(\tau)u^i u^j)^2 + (\frac{1}{|\tau|}\rho_{ij}u^i u^j)^2}}{2}\right|}\, d\lambda. \tag{4.156}$$

Now consider any sequence $\{\gamma_{\tau_j}, k_{\tau_j}^{TT}, \tau_j\}$ on the solution curve $\tau \mapsto (\gamma_\tau, k_\tau^{TT}, \tau)$ such that $\lim_{j\to\infty} \tau_j = -\infty$. We have the following limits as $j \to \infty$

$$\lim_{j\to\infty} A(\tau_j) = \lim_{j\to\infty} \left(\frac{1}{2|\tau_j|} + \frac{\sqrt{1 + 2\mathcal{S}(\tau_j)^2 \tau_j^2}}{2|\tau_j|} \right) = \frac{1}{\sqrt{2}} \lim_{j\to\infty} \mathcal{S}(\tau_j) = \frac{C_\infty}{\sqrt{2}}$$

$$\lim_{j\to\infty} \frac{1}{|\tau_j|}\rho(u,u) = 0, \tag{4.157}$$

$$\frac{C_f}{\sqrt{2}} \lim_{j\to\infty} \gamma_{\tau j}(u,u) \leq \lim_{j\to\infty} |k_{\tau_j}^{TT}(u,u)| \leq \frac{C_\infty}{\sqrt{2}} \lim_{j\to\infty} \gamma_{\tau_j}(u,u) \tag{4.158}$$

and therefore

$$\lim_{j\to\infty} \mu_{vert,\tau_j} = \lim_{j\to\infty} \oint_{\mathcal{C}} \sqrt{|k_{ik}^{TT}(\tau_j)u^i u^k|}\, d\lambda \tag{4.159}$$

(The last equality may be obtained from (4.156) more formally as $\oint_{\mathcal{C}} \sqrt{\left|\frac{|k_{ij}^{TT}(\tau_k)u^i u^j| + \sqrt{(k_{ij}^{TT}(\tau_k)u^i u^j)^2 + (\frac{1}{|\tau_k|}\rho_{ij}u^i u^j)^2}}{2}\right|}\, d\lambda = \oint_{\mathcal{C}} \sqrt{|k_{ij}^{TT}(\tau_k)u^i u^j|} \sqrt{\left|\frac{1+\sqrt{1+\frac{|\rho_{ij}u^i u^j|^2}{|\tau_k||k_{ij}^{TT}(\tau_k)u^i u^j|^2}}}{2}\right|}\, d\lambda$ $\leq \sup_{\mathcal{C}} \sqrt{\left|\frac{1+\sqrt{1+\frac{|\rho_{ij}u^i u^j|^2}{|\tau_k||k_{ij}^{TT}(\tau_k)u^i u^j|^2}}}{2}\right|} \oint_{\mathcal{C}} \sqrt{|k_{ij}^{TT}(\tau_k)u^i u^j|}$ and noting that $\sup_{\mathcal{C}} \sqrt{\left|\frac{1+\sqrt{1+\frac{|\rho_{ij}u^i u^j|^2}{|\tau_k||k_{ij}^{TT}(\tau_k)u^i u^j|^2}}}{2}\right|}$ approaches 1 as $\tau_k \to -\infty$). Finally, we have the following result as $j \to \infty$

$$\frac{|C_f|^{1/2}}{2^{1/4}} \lim_{j\to\infty} l_{\gamma_{\tau_j}}(\mathcal{C}) \leq \lim_{j\to\infty} \mu_{vert,\tau_j} \leq \frac{|C_\infty|^{1/2}}{2^{1/4}} \lim_{j\to\infty} l_{\gamma_{\tau_j}}(\mathcal{C}). \tag{4.160}$$

However $C_f = C_\infty = C$ yielding

$$\lim_{j\to\infty} \mu_{vert,\tau_j} = \frac{C^{1/2}}{2^{1/4}} \lim_{j\to\infty} l_{\gamma_{\tau_j}}(\mathcal{C}). \tag{4.161}$$

and clearly both of these entities have the same limit at the level of projective space as they are proportional to each other by finite constant (as will be discussed in detail in the next section). This completes the analysis that the limiting behaviour does not depend on the chosen sequence as long as $\lim_{j\to\infty} \tau_j = -\infty$. □

Summarizing this section, we state the following theorem.

Theorem 2: *Let Σ_g be a closed (compact without boundary) Riemann surface of genus $g > 1$ and the data $(\gamma, k^{TT}, \tau, e^{\varphi}, N, X)$ defined by the solution of the Gauss map equation (4.51), Lichnerowicz equation (4.29), and the elliptic equations (4.21-4.22) solve the reduced Einstein equations via the associated Hamilton-Jacobi equation. The ratio of the transverse measure of any non-trivial element $\mathcal{C}$ of $\pi_1(\Sigma_g)$ with respect to the vertical measured foliation of the natural holomorphic quadratic differential $\phi_\tau := (k^{TT}_{11} - ik^{TT}_{12})dz^2$, and its hyperbolic length that is the length with respect to the metric γ approaches to a finite constant independent of any homotopy class in the limit of the big-bang singularity i.e., $\tau \to -\infty$ along every sequence on the solution curve. The transverse measure associated with the horizontal foliation collapses to zero in the same limit.*

4.6 Compactification

In this section we claim that the Thurston compactification of the Teichmüller space is equivalent to our relativistic compactification. Let us denote the Einstein compactification of $\mathcal{T}\Sigma_g$ by $\bar{\mathcal{T}\Sigma_g}^{Ein}$. In this section, we claim that the following theorem holds

Theorem 3: $\bar{\mathcal{T}\Sigma}_g^{Th} \approx \bar{\mathcal{T}\Sigma}_g^{Ein}$.

Before proving this theorem, we need a few additional concepts and two lemmas. Let us consider the functional space $\Omega = \mathbb{R}^{G(\Sigma_g)}_{>0}$, where the space of geodesics on Σ_g is denoted as $G(\Sigma_g)$, which may be obtained by the $\pi_1(\Sigma_g)$ action on the space of geodesics on $\mathbb{H}^2$, that is $\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty - \Delta$, Δ being the diagonal. Essentially Ω consists of functionals which take elements of $G(\Sigma_g)$ and associate a positive number to each (in this case, length of the geodesic representative of each homotopy class to be precise). It can essentially be viewed as the space of geodesic currents given that the association of a measure (Radon measure to be precise) $G(\Sigma_g)$ is $\pi_1(\Sigma_g)$ invariant. We may construct the following map

$$l : \mathcal{T}\Sigma_g \to \Omega \tag{4.162}$$
$$\gamma \mapsto l_\gamma : G(\Sigma_g) \to R_{>0}.$$

We may projectivize the space Ω as follows

$$\mathcal{P}\Omega = \Omega/(\beta \sim t\beta, t > 0, \beta \in \Omega) \tag{4.163}$$

and subsequently obtain the following map

$$\pi \circ l : \mathcal{T}\Sigma_g \to \mathcal{P}\Omega. \tag{4.164}$$

Clearly, the map l can be identified with the Liouville currents (see appendix) (or the '$9g-9$' map). The injectivity of $\pi \circ l$ follows from the injectivity of the map L of section (2). Similarly, we may construct the following map from the space of measured laminations to Ω

$$\nu : \mathcal{MF} \to \Omega \tag{4.165}$$
$$\mathcal{F} \mapsto (i(\mathcal{F}, \gamma) = \oint_{[\gamma] \in G(\Sigma_g)} \mu_{\mathcal{F}}).$$

Here, $\mu_{\mathcal{F}}$ corresponds to the transverse measure associated with $\mathcal{F} \in \mathcal{MF}$ and $[\gamma]$ is a homotopy class. Clearly the space of measured geodesic foliations is a subset of the space of all geodesics and therefore, we have the following

$$\mathcal{PMF} := (\mathcal{MF} - \{0\})/(\mathcal{F} \sim t\mathcal{F}, t > 0, \mathcal{F} \in \mathcal{MF}) \tag{4.166}$$
$$= \pi \circ \nu(\mathcal{MF}) \subset \mathcal{P}\Omega.$$

Note that $\pi \circ \nu$ is injective. Let the space of holomorphic quadratic differentials with respect to the conformal structure (M, γ) and (M, ρ) be defined as $\mathcal{HQD}(\gamma)$ and $\mathcal{HQD}(\rho)$, respectively. Now we define another important map namely the Hubbard-Masur homeomorphism

$$\mathcal{F} : \mathcal{HQD}(\gamma) \to \mathcal{MF} \tag{4.167}$$
$$\phi \mapsto \mathcal{F}(\phi). \tag{4.168}$$

Recall that $\mathcal{PMF}$ and $\mathcal{T}\Sigma_g$ are disjoint in the space of projective currents i.e., $\mathcal{P}\Omega$. The Thurston compactification, essentially, is given as $\bar{\mathcal{T}\Sigma_g}^{Th} = \mathcal{T}\Sigma_g \cup \mathcal{PMF}$. Now we state the following crucial lemma.

Lemma 9: *Let the sequence $\{\gamma_{\tau_j}\}$ leave all the compact sets in $\mathcal{T}\Sigma_g$ at the limit of big-bang i.e., $j \to \infty$ ($\tau_j \to -\infty$). Then $\pi \circ l(\gamma_{\tau_j})$ converges if and only if $\pi \circ \nu(\mathcal{F}_{\tau_j})$ converges and subsequently both have the same limit in $\mathcal{P}\Omega$. Here $\mathcal{F}_{\tau_j}$ is the vertical measured foliation associated to the holomorphic quadratic differential $\phi_{\tau_j} = (k^{TT}_{11\tau_j} - ik^{TT}_{12\tau_j})dz^2$ through the Hubbard-Masur homeomorphism $\mathcal{F}$.*

Proof: $\{\gamma_{\tau_j}\}$ diverges in $\mathcal{T}\Sigma_g$ (identified with its image in $\mathcal{P}\Omega$ under the map $\pi \circ l$) and therefore $\lim_{j\to\infty} l_{\gamma_{\tau_j}}(\mathcal{C}) = \infty$ for some $\mathcal{C} \in G(\Sigma_g)$. Now, It must converge to $\mathcal{P}\Omega$ due to the fact that the later is compact (passing to the level of a subsequence). Therefore, $\exists$ $\{\lambda_{\tau_j}\}$ with $\lim_{j\to\infty} \lambda_{\tau_j} = 0$ such that $\lim_{j\to\infty} \lambda_{\tau_j} l_{\gamma_{\tau_j}}(\mathcal{C}) = \mathcal{L} < \infty$. Now, utilizing the theorem 2 or the following equality (4.145) derived in lemma 8, we have in the limit $j \to \infty$

$$l_\gamma(\mathcal{C}) = \frac{2^{1/4}}{C^{1/2}} i(\mathcal{F}, \mathcal{C}) = i(\frac{2^{1/4}}{C^{1/2}}\mathcal{F}, \mathcal{C}), \tag{4.169}$$

and since the constant C does not depend on the homotopy class of curves $\mathcal{C}$, we may immediately obtain $\lim_{j\to\infty} \lambda_{\tau_j} i(\frac{2^{1/4}}{C^{1/2}}\mathcal{F}_{\tau_j}, \alpha) = \mathcal{L}$ i.e., equal to the limit of $\lambda_{\tau_j} l_{\gamma_{\tau_j}}$. Moreover, $\lambda_{\tau_j} \frac{2^{1/4}}{C^{1/2}}\mathcal{F}_{\tau_j}$ converges in $\mathcal{P}\Omega$ or $\frac{2^{1/4}}{C^{1/2}}\mathcal{F}_{\tau_j}$ converges in $\mathcal{PMF}$ (the image of $\mathcal{PMF}$ in $\mathcal{P}\Omega$ under $\pi \circ \nu$ is identified with $\mathcal{PMF}$ and multiplication of a measured foliation by an overall constant yields the same foliation only measure gets scaled. But it makes no difference at the level of projective space by definition). The reverse may be obtained in a similar way. This lemma essentially tells us that a sequence on solution curve (of Einstein's reduced equations) diverging (leaving every compact set) in the configuration space ($\mathcal{T}\Sigma_g$), converges in the space of projective measured foliations as $\tau_j \to -\infty$ (big-bang limit). The space of projective measured foliations is well known to be the Thurston boundary of the Teichmüller space. This establishes the fact that any solution curve approaches the Thurston boundary at the big-bang singularity.

Proof of Theorem 3: We now want to establish the homeomorphism between the Thurston compactified Teichmüller space $\bar{\mathcal{T}\Sigma_g}^{Th}$ and the Einstein compactified Teichmüller space $\bar{\mathcal{T}\Sigma_g}^{Ein}$. Let us first define the space of holomorphic quadratic differential in our setting. Recall that the holomorphic quadratic differential $\phi_\tau := (k^{TT}_{11} - ik^{TT}_{12})dz^2$ is defined with respect to the metric γ_τ. Using ϕ_τ, we may obtain a holomorphic quadratic differential $\Phi(\phi_\tau)$ with respect to the fixed metric ρ as follows. First, decompose k^{TT} into $\rho-$transverse-traceless part and $\rho-$Lie derivative part in the following $\rho - L^2$ orthogonal sense

$$k_{ij}^{TT} = \zeta_{ij}^{TT} + (L_Y \rho)_{ij}, \tag{4.170}$$

where $\nabla[\rho]^j \zeta_{ij}^{TT} = 0 = \rho^{ij}\zeta_{ij}^{TT}$ and $Y \in \mathfrak{X}(\Sigma_g)$. Define the homolorphic quadratic differential with respect to ρ as $\Phi(\phi_\tau) := (\zeta_{11} - i\zeta_{12})dz^2$. This identification is not obvious in the current context since k^{TT} is transverse-traceless with respect to γ where as ζ^{TT} is transverse-traceless with respect to ρ. Therefore, one may simply add $Y^{'}$ to Y to yield another $\gamma-$transverse-traceless tensor $k^{TT'}$ while keeping ζ^{TT} fixed. Therefore, two different $\gamma-$transverse-traceless tensor can have same $\rho-$transverse-traceless part. However, in the current context, this decomposition is not arbitrary. Through imposition of the spatial harmonic gauge condition '$id : (\Sigma, \gamma) \to (\Sigma, \rho)$ is harmonic', Y and ζ^{TT} are related to each other through a truly non-linear elliptic PDE (Monge-Ampere type equation). Moncrief [198] proved the existence of a unique solution of this Monge-ampere type equation (see sections 6 through 8 of [198]). Given a ζ^{TT}, one may retrieve k^{TT} through solving Moncrief's Monge-Ampere equation. k^{TT} always have a unique ζ^{TT} due to the fact that (Σ_g, ρ) does not have Killing/conformal Killing vector fields. Through the well-posedness of Moncrief's Monge-ampere equation, k^{TT} depends continuously on ζ^{TT} or ζ^{TT} depends continuously on k^{TT}. In other words, we may also write $\phi_\tau = \phi_\tau(\Phi) := (k_{11}^{TT}(\zeta^{TT}) - ik_{12}^{TT}(\zeta^{TT}))dz^2$. In summary

$$\Phi(\phi_\tau) := (\zeta_{11} - i\zeta_{12})dz^2, \phi_\tau = \phi_\tau(\Phi) := (k_{11}^{TT}(\zeta^{TT}) - ik_{12}^{TT}(\zeta^{TT}))dz^2. \tag{4.171}$$

This identification yields a curve in the space of holomorphic quadratic differentials with respect to ρ corresponding to a solution curve $\tau \mapsto (\gamma_\tau, k_\tau^{TT}, \tau)$ and also allows us to talk about the boundary of the space of holomorphic quadratic differentials without any ambiguity .

Now let $\mathcal{QD}(\rho)$ be the space of homlomorphic quadratic differentials with respect to ρ. Let $\{\gamma_{\tau_j}, k_{\tau j}^{TT}, \tau_j\}$ be a sequence. Through the correspondence (4.171), this identifies a sequence $\{\zeta_{\tau_j}^{TT}\}$ in $\mathcal{QD}(\rho)$. In the limit $\tau_j \to -\infty$, we have established in theorem 1 that the Dirichlet energy corresponding to the associated harmonic map blows up and subsequently every sequence $\{\gamma_{\tau_j}\}$ leaves all the compact sets of $\mathcal{T}\Sigma_g$. On the other hand, we also have the associated "velocity" variable, the transverse-traceless tensor $\{k_{\tau_j}^{TT}\}$. It was shown in section 5 that its L^1 norm remains bounded as $\tau_j \to -\infty$ and moroever

$$\lim_{\tau_j \to -\infty} \int_{\Sigma_g} \sqrt{|k_{\tau_j}^{TT}|^2_{\gamma_{\tau_j}}} \mu_{\gamma_{\tau_j}} = C. \tag{4.172}$$

Note that the constant C is universal and does not depend on the chosen solution ray. Following the boundedness of the L^1 norm of $k^{TT}_{\tau_j}$ at the limit τ_j, we immediately obtain that the L^1 norm of the associated quadratic differential ϕ_{τ_j} remains bounded as well (norms of ϕ were defined in section 5 and considerable detail was presented there) i.e.,

$$\lim_{\tau_j \to -\infty} ||\phi_{\tau_j}|| = C \tag{4.173}$$

which essentially means that the space of $\gamma_\tau-$holomorphic quadratic differential is $\mathbb{S}^{6g-7}$ at the limit $\tau \to -\infty$ and through the homeomorphic correspondence (4.171), this can be identified with a boundary of a ball $\bar{B}$ in $\mathcal{QD}(\rho)$ radius of which may be set to 1 after a suitable re-normalization i.e., $||\Phi|| = 1$. Let us denote this unit ball in $\mathcal{QD}(\rho)$ by $\bar{B}(1)$. In order to prove theorem 3, it is sufficient to establish the homeomorphism between the ball $\bar{B}(1)$ and the Thurston compactified Teichmüller space $\mathcal{T}\bar{\Sigma}_g{}^{Th}$. The Einstein boundary of the Teichmüller space is therefore $\mathcal{PQD}(\rho)$ and the Einstein compactification of the Teichmüller space is $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$ (naturally through the Einstein flow and the correspondence (4.171)). The procedure to demonstrate the homeomorphism between $\mathcal{T}\bar{\Sigma}_g{}^{Th}$ and $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$ is routine and similar to the one described in Wolf's work. Nevertheless, we sketch the proof for the sake of completeness. More specifically, let us explicitly define the following spaces

$$\mathcal{HQD}(\rho) := \{\Phi \in QD(\rho)|\ ||\Phi|| < 1\} \tag{4.174}$$

$$\mathcal{PQD}(\rho) := \{\Phi \in QD(\rho)|\ ||\Phi|| = 1\} \tag{4.175}$$

$$\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho) := \mathcal{HQD}(\rho) \cup \mathcal{PQD}(\rho). \tag{4.176}$$

Following the homeomorphism between $\mathcal{T}\Sigma_g^{Ein}$ and $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$, our problem reduces to establishing the homeomorphism between $\mathcal{T}\Sigma_g^{Th}$ and $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$. Let us define the following map after using polar coordinates (r, θ) for the space $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$

$$\begin{aligned} \varphi : \mathcal{T}\bar{\Sigma}_g^{Th} \subset \mathcal{P}\Omega \quad &\to \quad \mathcal{HQD}(\rho) \cup \mathcal{PQD}(\rho) \approx \mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho) \approx \mathcal{T}\Sigma_g^{Ein} \subset \mathcal{P}\Omega \\ x \quad &\mapsto \quad \left(||\Phi(x)||, \frac{\Phi(x)}{||\Phi(x)||}\right), \forall \pi \circ l(x) \in \mathcal{T}\Sigma_g \subset \mathcal{P}\Omega \\ &\mapsto \quad \left(1, \lim_{n\to\infty} \frac{\Phi(x_n)}{||\Phi(x_n)||}\right),\ \pi \circ l(x_n) \to \partial\mathcal{T}\Sigma_g^{Th} \subset \mathcal{P}\Omega. \end{aligned} \tag{4.177}$$

Here $\mathcal{T}\Sigma_g$ and $\partial\mathcal{T}\Sigma_g$ are realized as the image of $\pi \circ l$ in $\mathcal{P}\Omega$. In addition, through the Hubbard-Masur homeomorphism $\mathcal{F}$ and $\pi \circ \nu$, $\mathcal{HQD}(\rho)$ and $\mathcal{PQD}(\rho)$ may also be realized as their images in $\mathcal{P}\Omega$. First we want to show that this map is well defined. Consider two solutions $x_n, x_n^{'} \in \mathcal{T}\Sigma_g$ that approach the boundary

$$\lim_{n\to\infty} \pi \circ l(x_n) = \lim_{n\to\infty} \pi \circ l(x_n^{'}) = y \in \partial\mathcal{T}\Sigma_g. \tag{4.178}$$

But, then following lemma 9 and the correspondence (4.171), we immediately have

$$\lim_{n\to\infty} \pi \circ \nu \circ \mathcal{F} \circ \phi_{\tau_n} \circ \Phi(x_n) = \lim_{n\to\infty} \pi \circ \nu \circ \mathcal{F} \circ \phi_{\tau_n} \circ \Phi(x_n^{'}) \tag{4.179}$$

and following the Hubbard-Masur homeomorphism

$$\lim_{n\to\infty} \frac{\Phi(x_n)}{||\Phi(x_n)||} = \lim_{n\to\infty} \frac{\Phi(x_n^{'})}{||\Phi(x_n^{'})||}. \tag{4.180}$$

Now we establish the injectivity, surjectivity and continuity of Ψ as well as continuity of Ψ^{-1}. Here, by a sequence, we will mean a sequence chosen from a solution curve.

Continuity of Ψ: Since, $\lim_{n\to\infty} ||\Phi(x_n)|| = 1$, the continuity in the first component follows. Continuity in the second component is obvious.

Injectivity of Ψ: Injectivity of Ψ on $\mathcal{T}\Sigma_g$ is clear (corresponding to each metric, Φ assigns exactly one holomorphic quadratic differential or transverse-traceless tensor). Now, suppose $x_n, x_n^{'} \in \mathcal{T}\Sigma_g$ such that $\pi\circ l(x_n) \to \partial\mathcal{T}\Sigma_g, \pi\circ l(x_n^{'}) \to \partial\mathcal{T}\Sigma_g$ and $\Psi(x) = \Psi(x^{'})$. We want to show that $\lim_{n\to\infty} \pi \circ l(x_n) = \lim_{n\to\infty} \pi \circ l(x_n^{'}) \in \partial\mathcal{T}\Sigma_g$. Following $\Psi(x) = \Psi(x^{'})$, we have

$$\lim_{n\to\infty} \frac{\Phi(x_n)}{||\Phi(x_n)||} = \lim_{n\to\infty} \frac{\Phi(x_n^{'})}{||\Phi(x_n^{'})||} \tag{4.181}$$

which following the Hubbard-Masur homeomorphism implies

$$\lim_{n\to\infty} \pi \circ \nu \circ \mathcal{F} \circ \phi_{\tau_n} \circ \Phi(x_n) = \lim_{n\to\infty} \pi \circ \nu \circ \mathcal{F} \circ \phi_{\tau_n} \circ \Phi(x_n^{'}). \tag{4.182}$$

But, then lemma 9 (together with the correspondence (4.171)) implies

$$\lim_{n\to\infty} \pi \circ l(x_n) = \lim_{n\to\infty} \pi \circ l(x_n^{'}) \in \partial\mathcal{T}\Sigma_g \tag{4.183}$$

concluding injectivity of Ψ.

Surjectivity of Ψ: Obviously, Ψ is onto from $\mathcal{T}\Sigma_g$ to $\mathcal{HQD}$. Let $(1,\alpha) \in \mathcal{PQD}$ and $a_n\alpha \to \alpha$. Now, following lemma 9, since, $\pi \circ \nu \circ \mathcal{F} \circ \phi_{\tau_n}(a_n\alpha) = constant$, $\pi \circ l(\phi^{-1}(a_n\alpha))$ converges to $y \in \partial\mathcal{T}\Sigma_g$.

Therefore, using continuity of Ψ

$$\Psi(y) = \left(1, \lim_{n\to\infty} \frac{\Phi(\Phi^{-1}(a_n\alpha))}{||\Phi(\Phi^{-1}(a_n\alpha))||}\right) = (1, \alpha), \tag{4.184}$$

which concludes surjectivity.

Continuity of Ψ^{-1}: For, the continuity of Ψ^{-1}, we only need to verify the continuity on $\mathcal{PQD}$. Let us consider that $(a_n, \alpha_n) \to (1, \alpha)$. Following the Hubbard-Masur homeomorphism, $\pi \circ \nu \circ \mathcal{F} \circ \phi_{\tau_n}(a_n\alpha_n)$ converges. On the other hand, lemma 9 implies

$$\lim_{n\to\infty} \pi \circ \nu \circ \mathcal{F} \circ \phi_{\tau_n}(a_n\alpha_n) = \lim_{n\to\infty} \pi \circ l(\Phi^{-1}(a_n\alpha_n)) \in \partial\mathcal{T}\Sigma_g. \tag{4.185}$$

But, from the definition of Ψ, we obtain

$$\Psi(\pi \circ l(\Phi^{-1}(a_n\alpha_n))) = \left(||\Phi(\Phi^{-1}(a_n\alpha_n))||, \frac{\Phi(\Phi^{-1}(a_n\alpha_n))}{||\Phi(\Phi^{-1}(a_n\alpha_n))||}\right) = (a_n, \alpha_n),$$

since $\lim_{n\to\infty} ||\alpha_n|| = 1$. On the other hand, we also have

$$\Psi(\lim_{n\to\infty} \pi \circ l(\Phi^{-1}(a_n\alpha_n))) = \left(1, \lim_{n\to\infty} \frac{\Phi(\Phi^{-1}(a_n\alpha_n))}{||\Phi(\Phi^{-1}(a_n\alpha_n))||}\right) = (1, \alpha). \tag{4.186}$$

Therefore, we finally have

$$\Psi^{-1}(1, \alpha) = \lim_{n\to\infty} \pi \circ l(\Phi^{-1}(a_n\alpha_n)) = \lim_{n\to\infty} \Psi^{-1}(a_n, \alpha_n) \tag{4.187}$$

which concludes the continuity of Ψ^{-1}. which completes the proof of

$$\bar{\mathcal{T}\Sigma}_g^{Th} \approx \bar{\mathcal{T}\Sigma}_g^{Ein}. \qquad \square \tag{4.188}$$

An important observation would be that we are essentially showing the homoemorphism between $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}$ and $\mathcal{T}\bar{\Sigma}_g{}^{Th}$. Simultaneously, Einsteinian dynamics provides a natural homeomorphism between $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}$ and $\mathcal{T}\bar{\Sigma}_g{}^{Ein}$. Therefore, one might naively expect that Wolf's analysis would be directly applicable to obtain the desired result. However, as mentioned previously, Wolf's dynamics occurs in the target space while ours does in the domain. In addition, to establish the homeomorphism between $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}$ and $\mathcal{T}\bar{\Sigma}_g{}^{Th}$, lemma 6 and the correspondence (4.171) play the most important role. However, lemma 6 is obtained by completely relativistic means i.e., by utilizing the Gauss map and Hamilton-Jacobi equation in addition to the estimates derived from the elliptic equations associated to the Gauge and constraints of the Einstein's equations. Similarly the correspondence (4.171) is made solely through solving Moncrief's Monge-Ampere equation which once again is of purely relativistic origin. Let us explain the mechanism in a little less technical way. Notice the diagram above. Moncrief has shown in [198] that no two solution rays originating at an interior point ρ (defined by the Gauss-map, satisfying constraint and gauges, and uniquely satisfying the reduced Einstein equations through the Hamilton-Jacobi equation) intersect each other (except at the limit $\tau \to 0$, where they may asymptotically approach each other). This gives a homoemorphism between the Teichmüller space $\mathcal{T}\Sigma_g$ and the space of transverse-traceless tensors. However, each such transverse-traceless tensor k^{TT} has a holomorphic quadratic differential ϕ_τ associated to it. Morover, each such holomorphic quadratic differential represents a measured foliation (with zeros of the quadratic differential being the singularities of the foliation), which follows from the classical result of Hubbard and Masur [216]. Now let us consider that $\{\gamma_{\tau_j}\}$ leaves every compact set in the Teichmüller space and converges to the $\partial\mathcal{T}\Sigma_g^{Ein}$. Associated with the sequence $\{\gamma_{\tau_j}\}$, there is a sequence of quadratic differentials $\{\phi_{\tau_j} dz^2\}$ (defined in 4.98) from relativistic dynamics and such a unique sequence satisfies

$$\lim_{\tau_j \to -\infty} ||\phi_{\tau_j}||_{L^1} = C, \tag{4.189}$$

for some suitable constant $0 < C < \infty$. Through the correspondence (4.171), this is precisely equivalent to saying that as the sequence $\{\gamma_{\tau_j}\}$ converges in $\partial\mathcal{T}\Sigma_g^{Ein}$, $\{\Phi(\phi_{\tau_j})\}$ approaches the $6g-7$ dimensional sphere $\mathcal{PQD}(\rho)$ in the space of holomorphic quadratic differential $\mathcal{HQD}(\rho)$ and is defined by

$$||\Phi||_{L^1} = 1 \tag{4.190}$$

after suitable re-scaling. Now associated to the sequence $\{\phi_{\tau_j}\}$, there exists a unique sequence of measured foliations $\{\mathcal{F}_{\tau_j}\}$ corresponidng to $\Phi(\phi_{\tau_j})$. Now, the lemma 6 enters into the picture. lemma 6 precisely states that the limits of the sequence $\{\gamma_{\tau_j}\}$ and the sequence $\{\mathcal{F}_{\tau_j}\}$ are the same in the space of projective currents ($\mathbb{P}$Curr) and lie in the space of projective measured foliations ($\mathcal{PMF}$). Therefore, the space $\partial\mathcal{T}\Sigma_g^{Ein}$ is precisely the space of projective measured foliations $\mathcal{PMF}$. But, $\mathcal{PMF}$ is nothing but the Thurston boundary of the Teichmüller space in $\mathbb{P}$Curr. Therefore

$$\partial\mathcal{T}\Sigma_g^{Ein} \approx \partial\mathcal{T}\Sigma_g^{Th}. \tag{4.191}$$

In addition note that each of $\partial\mathcal{T}\Sigma_g^{Ein}$ and $\mathcal{PMF}$ are homeomorphic to $\mathcal{PQD}$. Is a sense the maps $\mathcal{TT}$, $\mathcal{F}$, $\mathcal{F}\circ\mathcal{TT}$ are all homeomorphisms. In a sense, we thus obtained a proof of Moncrief's conjecture that *each of the non-trivial solution curves of the reduced Einsteinian dynamics runs off the edge of the Teichmüller space at the limit of big-bang singularity and attaches to the Thurston boundary of the Teichmüller space, that is, the space of projective measured laminations or foliations ($\mathcal{PML}$, $\mathcal{PMF}$).* As a bonus, we also have in this relativistic setting that the space $\mathcal{PQD}\subset\mathcal{QD}$ is homoemorphic to $\partial\mathcal{T}\Sigma_g^{Ein}$ and therefore $\mathcal{PMF}$. In a sense, we also recover Wolf's result. Now we will describe the possible two mechanisms of approaching the boundary of the Teichmüller space in the next section.

4.7 Approaching $\partial\mathcal{T}\Sigma_g$

Let us consider the Fenchel Neilsen coordinates of the Teichmüller space. Figure (**??**) shows the pants decomposition of the Teichmüller space and the associated Fenchel-Neilen co-ordinates (see [217] for the details of the Fenchel-Neilsen parametrization and pants decomposition). Such parametrization is given by the lengths of $3g-3$ nontrivial (nontrivial in $\pi_1(\Sigma_g)$) geodesics $\{l_i\}_{i=1}^{3g-3}$ along with $3g-3$ associated twist parameters $\{\theta_{i=1}^{3g-3}\}$ (twist is performed about the same geodesic). The two possible mechanisms of attaining the boundary of the Teichmüller space are descried below.

4.7.1 Pinching of Σ

Let $\gamma(l_i^n)$ denotes a sequence of hyperbolic metrics and let $\theta_i=0\ \forall i=1,2,3,....,3g-3$. Letting any one of the l_i tend to infinity i.e., $\lim_{n\rightarrow\infty}l_i^n=\infty$ implies approaching the boundary $\partial\mathcal{T}\Sigma$. Using the collar lemma (see [204] for the detailed proof of the collar lemma), we immediately obtain that there is a non-trivial geodesic transverse to l_i^n with length $\approx\lim_{n\rightarrow\infty}e^{-l_i^n}$. This is the pinching mechanism described in figure (**??**). Note that the nontrivial (in $\pi_1(\Sigma_g)$) geodesic γ_2 collapses while the hyperbolic length l_1 of γ_1 approaches infinity. Now, the Dirichlet energy of the harmonic map $id:(\Sigma_g,\gamma(l_i^n))\rightarrow(\Sigma_g,\rho)$ i.e., between fixed domain (with metric ρ) and the varying target (with metric $\gamma(l_i)$) defined is a continuous proper function on the Teichmüller space. Therefore, the sequence of Dirichlet energies associated with the diverging sequence of metrics (or degenerating to be precise) $\gamma(l_i^n)$ can not stay in a compact set; that is the sequence blows up. Therefore we have the following correspondence

$$\lim_{n\rightarrow\infty}l_i^n\rightarrow\infty=>\lim_{n\rightarrow\infty}E_{\gamma(l_i^n)}\rightarrow\infty. \tag{4.192}$$

Notice that multiple non-trivial geodesics γ_i (and the corresponding transverse ones) may show the pinching behavior at once and each such limit corresponds to distinct points on $\partial\mathcal{T}\Sigma_g$.

4.7.2 Wringing of Σ_g by its neck

In order to explain the approach to $\partial\mathcal{T}\Sigma_g$ through wringing of Σ_g, we need to introduce the symplectic geometry of the Teichmuller space [227,228]. Using the parametrization $(l_i, \theta_i)_{i=1}^{3g-3}$ of Teichmuller space, define the symplectic form

$$\omega \quad = \quad \sum_{i=1}^{3g-3} dl_i \wedge d\theta_i, \tag{4.193}$$

which is preserved under the flow of the vector field $v = -\frac{\partial}{\partial\theta_i}$ and satisfies

$$\omega(-\frac{\partial}{\partial\theta_i}, \cdot) \quad = \quad dl_i. \tag{4.194}$$

The conserved Hamiltonian is nothing but the length l_i. Here, θ_i is the twist parameter about the ith geodesic. Therefore, flow of the vector field $-\frac{\partial}{\partial\theta_i}$ preserves the length l_i of the geodesic about which Σ_g is twisted. After n such twists, the length of the geodesic transverse to the ith geodesic increases by nl_i. The wringing of Σ_g about the ith geodesic corresponds to the limit $n \to \infty$. Let the length of the transverse geodesic before the twist be L^T. After performing n twists, the length becomes $\sim L^T + nl_i$ and therefore, the wringing corresponds to the fact that $\lim_{n\to\infty} \frac{l_i}{L_T + nl_i} = 0$. This is the other mechanism to approach the boundary of the Teichmüller space. Note that every point on the boundary $\partial\mathcal{T}\Sigma_g$ can be obtained through a combination of these two basic operations and in every situation, the Dirichlet energy approaches infinity.

4.8 Conclusion

Despite the fact that '2+1' gravity is devoid of a straightforward physical significance due to the lack of gravitational wave degrees of freedom, it is of extreme importance while studying '3+1' gravity on spacetimes of certain topological type ($\mathbb{S}^2 \times \mathbb{S}^1 \times \mathbb{R}, \mathbb{T}^2 \times \mathbb{S}^1 \times \mathbb{R}$, and $\Sigma_g \times \mathbb{S}^1 \times \mathbb{R}$, non-trivial $\mathbb{S}^1$ bundles over $\Sigma_g \times \mathbb{R}$). As mentioned in the introduction, several studies have been done of this topic where the '$3+1$' gravity has been realized as the '$2+1$' gravity coupled to a wave map,and where the Teichmüller space of Σ_g plays a crucial role. In the $2+1$ case, the configuration space is the Teichmüller space and we've shown here that the space of big-bang singularities is realized as the Thurston boundary of Teichmüller space. At the big-bang, the conformal geometry degenerates via pinching and wringing of (Σ_g, γ). This result essentially characterizes the complete solution space as well as verifies that the reduced Einstein flow can naturally be used to compactify Teichmüller space. While such a result is obtained by studying puerly vacuum gravity, a natural question arises whether inclusion of a positive cosmological constant might yield the same result. [197, 229, 231] studied vacuum GR with a positive cosmological constant in $2+1$ case, where the future in time behavior seems to persist. Therefore, it would be interesting to include a positive cosmological constant and check whether the Thurston boundary is approached in the big-bang limit. In addition, if one includes matter source and focuses on the evolution of the gravitational degrees of freedom (due to the presence of matter sources, the configuration space is now infinite dimensional), can the big-bang limit be realized as the Thurston boundary? Could the Teichmüller degrees of freedom of '3+1' gravity on $U(1)$ symmetric $\mathbb{S}^1$ bundles over $\Sigma_g \times \mathbb{R}$ realize the Thurston boundary in the same limit? What is the implication of such limiting behavior at the classical level in quantizing '2+1' gravity or '3+1' gravity on these special topologies? Can this characterization of the space of singularities be extended to higher dimensional gravity? Can the Einstein flow be used further to study classical Teichmüller theory?

Appendix

Space of projective laminations as the Thurston boundary of the Teichmüller space

In this section, we provide a rough sketch of the proof of Thurston compactification of the Teichmüller space by the space of projective measured laminations ($\mathcal{PML}$). The details may be found in [211, 212]. Here we show that a sequence diverging in Teichmüller space converges in the space of projective measured laminations ($\mathcal{PML}$), which is a compact subset of the space of geodesic currents. A $\pi_1(\Sigma_g)-$invariant measure on $G(\tilde{\Sigma}_g)$ may be defined as

$$\hat{L} = \frac{d\alpha d\beta}{|e^{i\alpha} - e^{i\beta}|^2}, \tag{4.195}$$

where $(e^{i\alpha}, e^{i\beta}) \in (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta$ and Δ represents diagonal. This measure is called the Liouville measure.The Liouville measure corresponding to X is denoted by $\hat{L}_X$, which satisfies the following for any $\gamma \in G(\Sigma_g)$

$$i(\gamma, \hat{L}_X) = l_X(\gamma), \tag{4.196}$$

where i denotes the bilinear function 'intersection number' and $l_X(\gamma)$ denotes the length of γ with respect to the hyperbolic metric on X. The intersection property may be interpreted as follows. Let us consider a closed non-trivial geodesic $\gamma \in G(\Sigma_g)$. Lift γ to the universal cover and consider its intersection with the set of geodesics transverse to its lift $\tilde{\gamma}$ that is $i(\gamma, \hat{L})$ is defined as $\int_E \hat{L}(E \cap \tilde{\gamma})$, where $E \subset G(\tilde{\Sigma}_g)$ is the set of geodesics transverse to $\tilde{\gamma}$. A few lines of calculations show that this integral is indeed the length of γ with respect to the hyperbolic metric (scalar curvature =-1). Note that Liouville measure may be used to define a geodesic currents on $G(\Sigma_g)$ due to its $\pi_1(\Sigma_g)-$invariance property. Now, let (X, f) be a hyperbolic surface (and thus $\in \mathcal{T}\Sigma$) such that $f : \Sigma_g \to X = \mathbb{H}^2/\pi_1(\Sigma_g)$ is a homeomorphism. Liouville measure provides a well defined map from the Teichmüller space $\mathcal{T}\Sigma$ to the space of currents. Here we just provide a brief description of the Thurston compactification of the Teichmüller space, necessary for the currents purpose. For details, the readers are referred to the excellent book [211], where the proof of the stated theorems may be found.

Lemma 0 [211,212] *The map $(X, f) \to \hat{L}_X$ is a proper embedding of $\mathcal{T}\Sigma_g$ into the space of currents $Curr(\Sigma_g)$ given by the intersection number i that is, for all closed curves α in Σ_g,*

$$i(\alpha, \hat{L}_X) = l_X(\alpha) \tag{4.197}$$

defines a proper embedding of $\mathcal{T}\Sigma_g$ into $\mathrm{Curr}(\Sigma_g)$.

Proof: See [211, 212].

We are now ready to establish the Thurston compactification. Let us first state a lemma.

Lemma 1 [211,212] For any hyperbolic surface Σ_g with the marking (X, f), we have the following result

$$i(\hat{L}_X, \hat{L}_X) = \pi^2|\chi(\Sigma_g)|, \tag{4.198}$$

where $\chi(\Sigma_g) = 2(1-g))$ is the Euler characteristics of Σ_g. Remarkably, this is a topological invariant. Let's denote the map $(X, f) \to \hat{L}_X$ by $\hat{L}$. We have the following lemma

Lemma 2:

$$\hat{L} : \mathcal{T}\Sigma_g \to \mathrm{I\!PCurr(\Sigma_g)} = (\mathrm{Curr(\Sigma_g)} - 0)/(\mu \sim \mathrm{t}\mu, \mu \in \mathrm{Curr(\Sigma_g)}, \mathrm{t} \in \mathrm{I\!R}_{>0})$$

is injective.

Proof: Let $[f : \Sigma_g \to X]$ and $[h : \Sigma_g \to Y]$ be two elements of $\mathcal{T}\Sigma_g$. Then

$$[\hat{L}_X] = [\hat{L}_Y] => \hat{L}_X = t\hat{L}_Y. \tag{4.199}$$

Now we use the previous lemma and obtain

$$\begin{aligned} \pi^2|\chi(\Sigma_g)| &= i(\hat{L}_X, \hat{L}_X) = i(t\hat{L}_Y, t\hat{L}_Y) \\ &= t^2 i(\hat{L}_Y, \hat{L}_Y) = t^2\pi^2|\chi(\Sigma_g)|, \end{aligned} \tag{4.200}$$

i.e.,

$$t = 1, \tag{4.201}$$

as $t \in \mathbb{R}_{>0}$ and therefore $L_X = L_Y$.

As we have defined earlier, a lamination $\mathcal{L}$ on Σ_g is a closed subset which is the union of disjoint simple geodesics and the geodesics in $\mathcal{L}$ are called the leaves of the lamination. An important property of these leaves is that they do not intersect each other that is if $\lambda, \alpha \in \mathcal{L}$, then the following is satisfied

$$i(\lambda, \alpha) = 0. \tag{4.202}$$

If we associate a transverse measure to the leaves of $\mathcal{L}$, then we obtain a measured lamination denoted by $\mathcal{ML}$. We may of course construct the projective measured laminations $\mathcal{PML}$ through the following identification

$$\mathcal{PML} = (\mathcal{ML} - \{0\})/(\lambda \sim t\lambda, \lambda \in \mathcal{ML}, t > 0). \tag{4.203}$$

Clearly the leaves of a measured lamination define a subset in the space of all geodesics and therefore, the projective measured lamination $\mathcal{PML}$ may be identified as a subset of the space of geodesic currents. It is in fact a compact subset, which may be proven utilizing an elementary result from topology namely Tychonof's theorem [213]. Another important observation is to note that the image of the Teichmüller space under the map $\hat{L}$ i.e., $\hat{L}(\mathcal{T}\Sigma_g)$ and $\mathcal{PML}$ are disjoint. This follows from the definition of the geodesic lamination that is $i(\lambda, \lambda) = 0\ \forall \lambda \in \mathcal{PML}$, while

$i(\hat{L}_X, \hat{L}_X) = \pi^2|\chi(\Sigma_g)| \neq 0,\ \forall X \in \mathcal{T}\Sigma_g$. Now we finish the Thurston compactification

Lemma 3: The closure of $\mathcal{T}\Sigma_g \subset \mathbb{P}\mathrm{Curr}(\Sigma)$ is precisely $\mathcal{T}\Sigma_g \cup \mathbb{P}\mathcal{ML}$.

Proof: Let say $[f_n : \Sigma_g \to X_n]$ is a sequence that diverges in $\mathcal{T}\Sigma_g$. Then obviously, $\{[\hat{L}_{X_n}]\} \subset \mathcal{P}\mathrm{Curr}(\Sigma_g)$ converges to some element of $\mathcal{P}\mathrm{Curr}(\Sigma_g)$ due to the fact that $\mathcal{P}\mathrm{Curr}(\Sigma_g)$ is a compact subset of $\mathrm{Curr}(\Sigma_g)$ (passing to a subsequence). Then $\exists\ t_n$ such that $Lim_{n\to\infty} t_n \hat{L}_{X_n} = \mu \in$PCurr$(\Sigma_g)$. Now from the divergence criteria, there exists a simply closed curve $\alpha \in \Sigma_g$, such that

$$Lim_{n\to\infty} l_{X_n}(\alpha) = \infty. \tag{4.204}$$

But, $\infty > i(\alpha, \mu) = i(\alpha, t_n \hat{L}_{X_n}) = t_n l_{X_n}(\alpha)$ and thus we must have

$$Lim_{n\to\infty} t_n = 0. \tag{4.205}$$

Now we see the following

$$\begin{aligned} i(\mu, \mu) &= i(lim_{n\to\infty} t_n \hat{L}_{X_n}, lim_{n\to\infty} t_n \hat{L}_{X_n}), && (4.206)\\ &= lim_{n\to\infty} t_n^2 i(\hat{L}_{X_n}, \hat{L}_{X_n}), && (4.207)\\ &= lim_{n\to\infty} t_n^2 \pi^2 |\chi(\Sigma)|, && (4.208)\\ &= 0, && (4.209)\end{aligned}$$

and therefore, $\mu \in \mathcal{PML}$.

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