A Spinorial proof of the positivity of Quasi-Local mass

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Abstract

We provide a spinor based proof of the positivity of the Wang-Yau [13] quasi-local mass. More precisely we prove that the gravitational mass bounded by a spacelike topological $2-$sphere is non-negative in a generic spacetime verifying dominant energy condition and vanishes only if the surface is is embedded in the Minkowski space. This proof is purely quasi-local in nature and in particular does not rely on the Bartanik's gluing and asymptotic extension construction [22] and subsequent application of the Schoen-Yau [1, 2] positive mass theorem.

Contents

1 Introduction

In the framework of general relativity, pure gravitational notion of local mass/energy density is absent due to the equivalence principle. In simpler terms, one can always choose a geodesic normal coordinate chart to make the spacetime metric trivial (Minkowski) at a point. Instead one may ask a quasi-local definition of energy. In fact, Penrose [3] listed a set of major open problems in mathematical general relativity in 1982 that contained the following problem: *find a quasi-local definition of energy-momentum in the context of pure gravity.* To be more precise, one asks to evaluate the energy content in a spacelike region bounded by a topological $2-$sphere. Some desirable physical properties of this definition of energy should include non-negativity under the assumption of an appropriate energy condition on the spacetime and rigidity i.e., it should vanish for a topological $2-$sphere embedded in the Minkowski space. In addition, it should encode the information about pure gravitational energy (Weyl curvature effect) as well as the stress-energy tensor of any source fields present in the spacetime. Based on a Hamilton-Jacobi analysis, Brown-York [8, 9] and Liu-Yau [10, 11] defined a quasi-local mass by isometrically embedding the $2-$surface into the reference Euclidean $3-$ space and comparing the extrinsic geometry (the formulation relied on the embedding theorem of Pogorelov [31] i.e., the topological $2-$sphere needed to posses everywhere non-negative sectional curvature). However, [12] discovered surfaces in Minkowski space that do have strictly positive Brown-York and Liu-Yau mass. It appeared that these formulations were lacking a prescription of momentum information. This led Wang and Yau [13, 14] to define the most consistent notion of the quasi-local mass associated with a space-like topological $2-$sphere (we should mention that the second author together with Alae and Khuri recently provided a new definition of quasi-local mass bounded by a class of surfaces verifying certain topological conditions in [15]). It involves isometric embedding of the topological $2-$sphere bounding a space-like domain in the physical spacetime satisfying dominant energy condition (energy can not flow into a past light cone of an arbitrary point in spacetime; essentially finite propagation speed) into the Minkowski spacetime instead of Euclidean $3-$space. This formulation relies on a weaker condition on the sectional curvature of the $2-$surface of interest and solvability of Jang's equation [16] with prescribed Dirichlet boundary data. The Wang-Yau quasi-local mass is then defined as the infimum of the Wang-Yau quasi-local energy among all physical observers that is found by solving an optimal isometric embedding equation. This Wang-Yau quasi-local mass possesses several good properties that are desired on a physical ground. This mass is strictly positive for $2-$surfaces bounding a space-like domain in a curved spacetime that satisfies the dominant energy condition and it identically vanishes for any such $2-$surface in Minkowski spacetime. In addition, it coincides with the ADM mass at space-like infinity [17] and Bondi-mass at null infinity [18], and reproduces the time component of the Bel-Robinson tensor (a pure gravitational entity) together with the matter stress-energy at the small sphere limit, that is when the $2-$sphere of interest is evolved by the flow of its null geodesic generators and the vertex of the associated null cone is approached [19]. In addition, explicit conservation laws were also discovered at the asymptotic infinity [20].

One of the important question that arises in the context of quasi-local mass is the proof of its non-negativity and rigidity property. [21] provided a proof of the positivity of the Brown-York mass under the assumption of the non-negativity of the scalar curvature of the bulk Ω with strictly convex boundary $\Sigma := \partial\Omega$. They analyzed a non-linear parabolic equations which arises due to the use of Bartnik's gluing and extension construction [22]. This entails suitable infinite asymptotically flat extension of Ω with non-negative scalar curvature by deforming the exterior of the embedding of the boundary $\Sigma \subset \mathbb{R}^3$ of Ω to have the same mean curvature as Σ in Ω and to glue it to the manifold Ω along its boundary Σ. Through this construction, they extend Σ along the radial direction to the spacelike infinity and applies the Schoen-Yau positive mass theorem [1, 2] and non-increasing property of the Brown-York mass as the radius of the boundary Σ is increased. Later Liu-Yau [10,11] and Wang-Yau [13] used this construction in more general settings to prove the non-negativity of Liu-Yau and Wang-Yau mass. One natural question that arises is the following: can one prove the non-negativity of the quasi-local mass by using data on the compact hypersurface Ω and its boundary Σ in the spacetime without the extension construction of [21] and use of positive mass theorem. In fact, Richard Schoen [23] suggested the following problem *find a quasi-local proof of nonnegativity for the Brown-York and related quasi-local masses.* Very recently, Montiel [24] provided a positive solution to this problem for the Brown-York mass associated with a surface Σ under the assumption that the bulk Ω (such that $\partial\Omega = \Sigma$) has non-negative scalar curvature and the boundary Σ has strictly positive mean curvature. Motivated by Montiel's result, we give a spinorial proof of the non-negativity of the Wang-Yau quasi-local mass assuming the dominant energy condition on the physical spacetime. As a special case, the non-negativity of the Liu-Yau mass follows. This essentially completes the quasi-local proof of the non-negativity of the quasi-local masses modulo the new mass defined by [15]. Below we describe the main theorem and idea of the proof. For this we need to introduce the following basic notations.

Let us assume that the mean curvature vector $\mathbf{H}$ of Σ is space-like. Let $\mathbf{J}$ be the reflection of $\mathbf{H}$ through the future outgoing light cone in the normal bundle of Σ. The data that Wang and Yau use to define the quasi-local energy is the triple $(\sigma, |\mathbf{H}|_{\hat{g}}, \alpha_{\mathbf{H}})$ on Σ, where σ is the induced metric on Σ by the Lorentzian metric $\hat{g}$ on the physical spacetime N, $|\mathbf{H}|_{\hat{g}}$ is the Lorentzian norm of $\mathbf{H}$, and $\alpha_{\mathbf{H}}$ is the connection $1-$form of the normal bundle with respect to the mean curvature vector $\mathbf{H}$ and is defined as follows

$$\alpha_{\mathbf{H}}(X) := \hat{g}(\nabla[\hat{g}]_X \frac{\mathbf{J}}{|\mathbf{H}|}, \frac{\mathbf{H}}{|\mathbf{H}|}). \tag{1.1}$$

Choose a basis pair (e_3, e_4) for the normal bundle of Σ in the spacetime that satisfy $\hat{g}(e_3, e_3) = 1, \hat{g}(e_4, e_4) = -1$, and $\hat{g}(e_3, e_4) = 0$. Now embed the $2-$surface Σ isometrically into the Minkowski space with its usual metric η i.e., the embedding map $X : x^a \mapsto X^\mu(x^a)$ satisfies $\sigma(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}) = \eta(\frac{\partial X}{\partial x^a}, \frac{\partial X}{\partial x^b})$, where $\{x^a\}_{a=1}^2$ are the coordinates on Σ. Now identify a basis pair (e_{30}, e_{40}) in the normal bundle of $X(\Sigma)$ in the Minkowski space that satisfy the exact similar property as (e_3, e_4). In addition the time-like unit vector e_4 is chosen to be future directed i.e., $\hat{g}(e_4, \partial_t) < 0$. Let $\tau := -\langle X, \partial_t \rangle_\eta$, a function on Σ be the time function of the embedding X. The Wang-Yau quasi-local energy is defined as follows

$$8\pi M^{wy} := \inf_{\tau} \left(\underbrace{\int_{\Sigma_t} \left(-\sqrt{1+|\nabla\tau|^2_\sigma}\langle \mathbf{H}_0, e_{30}\rangle - \langle \nabla[\eta]_{\nabla\tau} e_{30}, e_{40}\rangle \right) \mu_\Sigma}_{I:=Contribution\ from\ the\ Minkowski\ space} \right. \tag{1.2}$$
$$\left. - \underbrace{\int_{\Sigma_t} \left(-\sqrt{1+|\nabla\tau|^2_\sigma}\langle \mathbf{H}, e_3\rangle - \langle \nabla[\hat{g}]_{\nabla\tau} e_3, e_4\rangle \right) \mu_\Sigma}_{II:=contribution\ from\ the\ physical\ spacetime} \right).$$

Technically speaking, there is still a boost-redundancy left in the normal bundle of Σ in N. [13] minimizes the physical space contribution II over the all possible boost-angles (II is a convex function of the boost-angle in the normal bundle of Σ). However we will prove the non-negativity of $I - II$ for all *admissible* τ and therefore M^{wy} will be non-negative since subtracting minimum of II over boost angles only increases the value of the mass.

Theorem 1.1 (Main Theorem) *Let (Σ, σ) be a spacelike topological $2-$sphere that bounds a spacelike topological ball Ω in the physical spacetime N verifying dominant energy condition. Moreover assume Σ is not an apparent horizon and its mean curvature vector is spacelike. Let τ be a function on Σ that verifies the following admissibility criterion*

$$K + (1+|\nabla\tau|^2_\sigma)^{-1}\det(\nabla^2\tau) > 0 \tag{1.3}$$

where K is the Gauss curvature of (Σ, σ) and $\nabla^2\tau$ is the hessian of τ with respect to the Levi-Civita connection $\nabla[\sigma]$ on (Σ, σ). Then the Wang-Yau quasi-local energy M^{wy} is non-negative and it vanishes if the spacetime N is Minkowski along Σ.

The main idea behind the proof is solving the Dirac equation on Ω with Atyiah-Patodi-Singer (APS) boundary condition [25–27] on Σ along with the application of an important spin inequality 3.1. Under the assumption of the dominant energy condition on the spacetime and Σ not being an apparent horizon, we prove the solvability of the Dirac equation with APS boundary condition. This in turn yields the vital spin inequality 3.1. In the next step, we reduce the expression for the Wang-Yau quasi-local energy to an appropriate form so that the application of the spin inequality 3.1 becomes possible. This involves solving the Jang's equation on the physical spacetime. We utilize the result established in the original article by Wang-Yau [13] regarding the solvability of the Jang's equation. The dominant energy condition on the spacetime N yields a curvature condition on the bulk spacelike topological ball Ω. In addition, the assumption that Σ is not an apparent yields the appropriate condition on the mean curvature of Σ. Putting together all these ingredients along with with the solution of Dirac equation with APS boundary condition verifying the spin inequality 3.1 completes the proof. Rigidity of the Wang-Yau quasi-local mass essentially follows from the fact that it becomes zero when Ω admits a parallel spinor and hence Ricci flat. Ricci flatness in 3 dimensions implies flatness and due to being simply connected, a domain in $\mathbb{R}^3$. This together with the dominant energy condition yields vanishing momentum property of Ω which through the structure equation of the embedding $\Omega \hookrightarrow N$ implies N is Minkowski along Ω.

2 Extrinsic Spin Geometry

Let (Ω, g) be a compact connected oriented 3 dimensional C^∞ Riemannian manifold with boundary $\partial\Omega = \Sigma$. Let $\widehat{\nabla}$ be the metric compatible connexion on (Ω, g) and ∇ is the induced connexion on Σ. We denote the spin bundle on Ω by S^Ω while that of Σ is denoted by S^Σ. In our case, Ω is a topological ball, and naturally, Σ is a topological $2-$sphere. On the spin bundle S^Ω there is a natural Hermitian inner product $\langle\cdot,\cdot\rangle$ that is compatible with the connexion $\widehat{\nabla}$ (we denote the Levi-Civita connexion and its spin connexion by the same symbol) and Clifford multiplication $c : Cl(\Omega) \to \mathrm{End}(S^\Omega)$. We denote a spinor a section of the spin bundle by $\psi \in \Gamma(S^\Omega)$. Naturally its restriction to Σ is a section of S^Σ. Let us consider $\{e_i\}_{i=1}^3$ as an orthonormal frame on Σ where $e_3 = \nu$ is the interior pointing normal to the boundary Σ. The following relation holds $\forall X, Y \in \Gamma(T\Omega)$

$$X\langle\psi_1, \psi_2\rangle = \langle\widehat{\nabla}_X\psi_1, \psi_2\rangle + \langle\psi_1, \widehat{\nabla}_X\psi_2\rangle \tag{2.1}$$

$$\widehat{\nabla}_X(c(Y)\psi) = c(\widehat{\nabla}_X Y)\psi + c(Y)\widehat{\nabla}_X Y \tag{2.2}$$

$$\langle c(X)\psi_1, \psi_2\rangle = -\langle\psi_1, c(X)\psi_2\rangle \tag{2.3}$$

$$\langle c(X)\psi_1, c(X)\psi_2\rangle = g(X, X)\langle\psi_1, \psi_2\rangle. \tag{2.4}$$

These are the intrinsic identities that are verified by the geometric entities on Ω. The boundary map $i : \Sigma \hookrightarrow \Omega$, induces a connexion on Σ that verifies

$$\widehat{\nabla}_X Y = \nabla_X Y + \mathcal{H}(X,Y)\nu, \tag{2.5}$$

where $X, Y \in \Gamma(T\Sigma)$ and $\mathcal{H}$ is the second fundamental form of Σ while viewed as an embedded surface in Ω. Now this split extends to the spinor bundles. One can consider the restricted bundle $S^{\Omega}|_{\Sigma}$ which is isomorphic to S^{Σ} since dimension of Σ is even. The spin connexion splits as follows

$$\widehat{\nabla}_X \psi = \nabla_X \psi + \frac{1}{2} c(\mathcal{H}(X,\cdot))c(\nu)\psi \tag{2.6}$$

for $X \in \Gamma(T\Sigma)$. With this split we can write the hypersurface Dirac operator in terms of the bulk Dirac operator and the second fundamental form. The hypersurface Dirac operator is defined as follows

$$\begin{aligned} D\psi &= \sum_{i=1}^{2} c(e_i)c(\nu)\nabla_{e_i}\psi = \sum_{i=1}^{2} c(e_i)c(\nu)\left(\widehat{\nabla}_{e_i}\psi - \frac{1}{2}c(\mathcal{H}(e_i,\cdot))c(\nu)\psi\right) \\ &= \sum_{i=1}^{2} c(e_i)c(\nu)\widehat{\nabla}_{e_i}\psi - \frac{1}{2}\sum_{i=1}^{2} c(e_i)c(\nu)c(\mathcal{H}(e_i,\cdot))c(\nu)\psi \end{aligned} \tag{2.7}$$

Now observe the following ($\mathcal{H}(e_i,\cdot) \in T^*\Sigma$ and therefore $g(\mathcal{H}(e_i,\cdot),\nu) = 0$ and therefore

$$\begin{aligned} D\psi &= \sum_{i=1}^{2} c(e_i)c(\nu)\widehat{\nabla}_{e_i}\psi + \frac{1}{2}\sum_{i=1}^{2} c(e_i)c(\nu)c(\nu)c(\mathcal{H}(e_i,\cdot))\psi \\ &= \sum_{i=1}^{2} c(e_i)c(\nu)\widehat{\nabla}_{e_i}\psi + \frac{1}{2}H\psi = -\sum_{i=1}^{2} c(\nu)c(e_i)\widehat{\nabla}_{e_i}\psi + \frac{1}{2}H\psi \\ &= -c(\nu)\sum_{i=1}^{3} c(e_i)\widehat{\nabla}_{e_i}\psi + c(\nu)c(\nu)\widehat{\nabla}_{\nu}\psi + \frac{1}{2}H\psi \\ &= -c(\nu)\widehat{D}\psi - \widehat{\nabla}_{\nu}\psi + \frac{1}{2}H\psi, \end{aligned} \tag{2.8}$$

where $H = \mathrm{tr}\,[\mathcal{H}]$. Therefore we have the following relation between the hypersurface Dircal operator and the bulk Dirac operator

$$D\psi = -c(\nu)\widehat{D}\psi - \widehat{\nabla}_{\nu}\psi + \frac{1}{2}H\psi. \tag{2.9}$$

Now we derive the following identity

Proposition 2.1 *The following Lichnerowicz type identity is verified by* ψ

$$\int_{\Sigma}\left(\langle D\psi,\psi\rangle - \frac{1}{2}H|\psi|^2\right)\mu_{\Sigma} = \frac{1}{4}\int_{\Omega} R^{\Omega}|\psi|^2\mu_{\Omega} - \int_{\Omega}|\widehat{D}\psi|^2\mu_{\Omega} + \int_{\Omega}|\widehat{\nabla}\psi|^2\mu_{\Omega} \tag{2.10}$$

Proof. First recall the Lichnerowicz identity on Ω

$$\widehat{D}^2\psi = -\widehat{\nabla}^2\psi + \frac{1}{4}R^\Omega\psi, \tag{2.11}$$

for a spinor $\psi \in \Gamma(S^\Omega)$. Let us consider an orthonormal frame $\{e_i\}_{i=1}^3$ on Ω with e_3 being the unit inside pointing normal ν of the boundary surface $\partial\Omega = \Sigma$. Now we evaluate the following using the compatibility property of the Hermitian inner product $\langle\cdot,\cdot\rangle$ on the spin bundle S^Ω

$$\begin{aligned}\widehat{\nabla}_i\langle c(e_i)\widehat{D}_i\psi,\psi\rangle + \widehat{\nabla}_i\langle\widehat{\nabla}_i\psi,\psi\rangle &= \langle c(e_i)\widehat{\nabla}_i\widehat{D}_i\psi,\psi\rangle + \langle c(e_i)\widehat{D}_i\psi,\widehat{\nabla}_i\psi\rangle + \langle\widehat{\nabla}^2\psi,\psi\rangle + \langle\widehat{\nabla}_i\psi,\widehat{\nabla}_i\psi\rangle\\ &= \langle\widehat{D}^2\psi,\psi\rangle - \langle\widehat{D}_i\psi,c(e_i)\widehat{\nabla}_i\psi\rangle + \langle\widehat{\nabla}^2\psi,\psi\rangle + \langle\widehat{\nabla}_i\psi,\widehat{\nabla}_i\psi\rangle\\ &= \langle\widehat{D}^2\psi,\psi\rangle - \langle\widehat{D}_i\psi,\widehat{D}_i\psi\rangle + \langle\widehat{\nabla}^2\psi,\psi\rangle + \langle\widehat{\nabla}_i\psi,\widehat{\nabla}_i\psi\rangle,\end{aligned}$$

where we have used the property 2.1 and 2.3 along with the fact that $\widehat{\nabla}_i(c(e_i)\widehat{D}_i\psi) = c(e_i)\widehat{\nabla}_i\widehat{D}_i\psi$. Now we integrate over Ω to yield

$$\begin{aligned}&\int_\Omega\left(\widehat{\nabla}_i\langle c(e_i)\widehat{D}_i\psi,\psi\rangle + \widehat{\nabla}_i\langle\widehat{\nabla}_i\psi,\psi\rangle\right)\mu_\Omega \\ &= \int_\Omega\left(\langle\widehat{D}^2\psi,\psi\rangle - \langle\widehat{D}_i\psi,\widehat{D}_i\psi\rangle + \langle\widehat{\nabla}^2\psi,\psi\rangle + \langle\widehat{\nabla}_i\psi,\widehat{\nabla}_i\psi\rangle\right)\mu_\Omega\\ \implies -\int_\Sigma\left(\langle c(\nu)\widehat{D}_\nu\psi,\psi\rangle + \langle\widehat{\nabla}_\nu\psi,\psi\rangle\right)\mu_\Sigma &= \int_\Omega\left(\frac{1}{4}R^\Omega|\psi|^2 - |\widehat{D}\psi|^2 + |\widehat{\nabla}\psi|^2\right)\mu_\Omega\\ \implies \int_\Sigma\left(\langle D\psi,\psi\rangle - \frac{1}{2}H|\psi|^2\right)\mu_\Sigma &= \int_\Omega\left(\frac{1}{4}R^\Omega|\psi|^2 - |\widehat{D}\psi|^2 + |\widehat{\nabla}\psi|^2\right)\mu_\Omega,\end{aligned} \tag{2.12}$$

where we have used the Lichnerowicz idenity 2.11 and the relation 2.9 between the Dirac operators on the boundary Σ and the bulk Ω. This concludes the proof of the spin identity that we will frequently use in this article. □

3 Dirac equation and spin inequality

The Lichnerowicz type identity obtained in proposition 2.1 is one of the main ingredients for the proof of the theorem 1.1. Notice that in the identity 2.10, the right-hand side contains a non-positive definite term involving the action of the Dirac operator on the spinor ψ in the bulk Ω. To get rid of this term, we want to impose that $\widehat{D}\psi = 0$ on Ω with a specified boundary condition on $\Sigma := \partial\Omega$. This issue is non-trivial since in dimensions greater than or equal to 2, the Dirac operator on a compact manifold with a boundary has an infinite-dimensional kernel in general. A simple example would be solving the Dirac equation on a disk in $\mathbb{C}$. This would essentially be finding the holomorphic and anti-holomorphic functions that clearly form infinite dimensional vector space. To fix this problem, one imposes a so-called elliptic boundary condition for the spinor on the boundary of the compact manifold Ω. The elliptic boundary condition for the boundary value problem associated with the bulk Dirac operator $\widehat{D}$ is a pseudo differential operator $P : L^2(S^\Sigma) \to L^2(V)$, where V is a hermitian vector bundle on Σ such that the boundary value problem

$$\widehat{D}\psi = \Psi \text{ on } \Omega \tag{3.1}$$

$$P(\psi|_{\Sigma}) = \alpha \text{ on } \Sigma = \partial\Omega \tag{3.2}$$

has a unique solution given smooth data Ψ and α modulo finite dimensional kernel. Here we choose to work with the Atyiah-Patodi-Singer (APS) boundary condition [25–27] that was first utilized to prove the index theorem for compact manifolds with boundary. Here the choice of the pseudo-differential operator P_λ is simply the L^2 orthogonal projection onto the subspace spanned by eigenvectors of the boundary Dirac operator D whose eigenvalues are not smaller than λ (note that Σ is closed and therefore the boundary operator D is self-adjoint). [25–27] showed that the operator P_λ is a zero order pseudo-differential operator. We invoke the following well-known proposition from [25–28] regarding the ellipticity of the APS boundary condition.

Proposition 3.1 *[25–28] Let (Ω, g) be a compact connected oriented ~~spin~~ Riemannian $3-$ manifold with smooth boundary $\Sigma := \partial\Omega$. Then APS boundary condition is elliptic and moreover for $\lambda = 0$, the spectrum of the Dirac operator is unbounded discrete.*

3.1 Solvability of the Dirac equation

Now we prove a vital proposition that essentially guarantees the solvability of the Dirac equation $\widehat{D}\psi = 0$ on Ω with prescribed boundary condition on Σ.

Proposition 3.2 *Let X be a smooth vector field on Ω and the Scalar curvature R^{Ω} of Ω verifies the following point-wise bound*

$$R^{\Omega} \geqslant 2divX - 2|X|^2, \tag{3.3}$$

and the mean curvature H of $\Sigma = \partial\Omega$ while viewed as an embedded surface in Ω verifies

$$H - \langle X, \nu \rangle > 0, \tag{3.4}$$

where ν is the unit spacelike unit outward directed normal to Σ. Then the homogeneous boundary value problem

$$\widehat{D}\psi = 0 \text{ on } \Omega,\ \psi_{\geqslant\lambda} = 0 \text{ on } \Sigma := \partial\Omega \tag{3.5}$$

has no non-trivial smooth solution.

Proof. First note that $\psi_\lambda = \psi_{\geqslant\lambda} + \psi_{<\lambda}$ due to L^2 orthogonality of the projection. We want to show that $\psi_{<\lambda} = 0$ first. Use the identity 2.10 with ψ_λ to yield

$$\int_\Sigma \left(\langle D\psi_\lambda, \psi_\lambda \rangle - \frac{1}{2} H |\psi_\lambda|^2 \right) \mu_\Sigma = \frac{1}{4} \int_\Omega R^\Omega |\psi_\lambda|^2 \mu_\Omega + \int_\Omega |\widehat{\nabla} \psi_\lambda|^2 \mu_\Omega. \tag{3.6}$$

Now addition of $\frac{1}{2}\int_\Sigma \langle X, \nu \rangle |\psi_\lambda|^2 \mu_\Sigma$ to the both side along with the use of the integration by parts identity

$$\int_\Sigma \langle X, \nu \rangle |\psi_\lambda|^2 = \int_\Omega \left(\mathrm{div} X |\psi|^2 + X(|\psi|^2) \right) \mu_\Omega \tag{3.7}$$

yields

$$\begin{aligned} \int_\Sigma \left(\langle D\psi_\lambda, \psi_\lambda \rangle - \frac{1}{2} (H - \langle X, \nu \rangle) |\psi_\lambda|^2 \right) \mu_\Sigma &= \frac{1}{4} \int_\Omega (R^\Omega + 2 \mathrm{div} X) |\psi_\lambda|^2 \mu_\Omega + \int_\Omega |(|\widehat{\nabla} \psi_\lambda|^2 + \frac{1}{2} X(|\psi_\lambda|^2)) \mu_\Sigma \\ &\geqslant \frac{1}{4} \int_\Omega (R^\Omega + 2 \mathrm{div} X - 2|X|^2) |\psi_\lambda|^2 \mu_\Omega + \frac{1}{2} \int_\Omega |\widehat{\nabla} \psi|^2 \mu_\Omega \geqslant 0 \end{aligned}$$

due to the fact that $|\widehat{\nabla}\psi_\lambda|^2 + \frac{1}{2} X(|\psi_\lambda|^2) = |\widehat{\nabla}\psi_\lambda|^2 + \frac{1}{2}(\langle \widehat{\nabla}_X \psi_\lambda, \psi_\lambda \rangle + \langle \psi_\lambda, \widehat{\nabla}_X \psi_\lambda \rangle) \geqslant \frac{1}{2} |\widehat{\nabla}\psi_\lambda|^2 - \frac{1}{2}|X|^2 |\psi_\lambda|^2$. Therefore we have the following

$$\int_\Sigma \left(\langle D\psi_\lambda, \psi_\lambda \rangle - \frac{1}{2} (H - \langle X, \nu \rangle) |\psi_\lambda|^2 \right) \mu_\Sigma \geqslant 0. \tag{3.8}$$

But on the boundary Σ, $\psi_{\geqslant\lambda} = 0$. Set $\lambda \leqslant 0$ to yield

$$- \left(\int_\Sigma \sum_i \lambda_i^2 |\psi_i|^2 + \frac{1}{2} (H - \langle X, \nu \rangle) |\psi_{<0}|^2 \right) \mu_\Sigma \geqslant 0 \tag{3.9}$$

which yields $\psi_{<0} = 0$ since $H - \langle X, \nu \rangle > 0$. Therefore $\psi_\lambda = 0$ on Σ. Now since $\widehat{D}\psi_\lambda = 0$ on Ω, using Lichnerowicz formula $\widehat{D}^2 = -\widehat{\nabla}^2 + \frac{1}{4} R^\Omega$ we obtain

$$\int_\Omega |\widehat{\nabla}\psi_\lambda|^2 \mu_\Omega + \frac{1}{4} R^\Omega |\psi_\lambda|^2 \mu_\Omega = 0 \tag{3.10}$$

due to $\psi_\lambda = 0$ on Σ. Now we add $\frac{1}{2}\int_\Sigma \langle X, \nu \rangle |\psi_\lambda|^2$. Since $\psi_\lambda = 0$ on Σ, it does not alter anything. This yields through integration by parts

$$\int_\Omega \left(|\widehat{\nabla}\psi_\lambda|^2 \mu_\Omega + \frac{1}{4} R^\Omega |\psi_\lambda|^2 + \frac{1}{2} \mathrm{div} X |\psi|^2 + \frac{1}{2} X(|\psi_\lambda|^2) \right) = 0 \tag{3.11}$$

or

$$\frac{1}{4} \int_\Omega (R^\Omega + 2 \mathrm{div} X - 2|X|^2) |\psi_\lambda|^2 \mu_\Omega + \frac{1}{2} \int_\Omega |\widehat{\nabla}\psi_\lambda|^2 \mu_\Sigma \leqslant 0 \tag{3.12}$$

yielding $\psi_\lambda = 0$ on Ω since $R^\Omega + 2\mathrm{div} X - 2|X|^2 \geqslant 0$. This follows from the fact that $|\psi_\lambda|^2$ is either identically zero or constant on Ω with vanishing boundary value. Therefore it must have zero length throughout Ω due to continuity. Note that $C^{\infty}(\Omega)$ is dense in Solbolev spaces. Therefore, if we choose the function spaces appropriately (e.g., $\psi_{\lambda} \in H^{s}(\Omega) \cap H^{s-\frac{1}{2}}(\Sigma), s \geqslant 2$) then working with smooth solutions and employing density argument works. Also, due to an argument of [29], the Hausdorff dimension of the zero sets of the solution of the Dirac equation on Ω is at most $1-$ or the solution is identically 0. But $\psi_{\lambda} = 0$ on $\Sigma := \partial\Omega$. So ψ_{λ} must be identically 0 on Ω. This concludes the proof of the proposition. $\square$

Following the previous proposition and the triviality of the co-kernel (i.e., solution of the problem $D\psi = 0$ on Ω and $\psi_{>\lambda} = 0$ on Σ) under the mean curvature condition $H - \langle X, \nu \rangle > 0$, we conclude that there is a unique solution to the Dirac equation on Ω with the prescribed APS boundary condition if the spinors are chosen to lie in appropriate function spaces. First, we note the following properties of the Sobolev spaces on Ω and Σ. A well known fact is that all functions belonging to $H^{s}(\Omega),\ s \in \mathbb{N}$ have traces on the smooth boundary $\Sigma = \partial\Omega$ that belong to $H^{s-\frac{1}{2}}(\Sigma)$. With a bit of technicality this passes to the case of spinors due to the compatibility of the spin connexion with the hermitian inner product $\langle \cdot, \cdot \rangle$ (in fact this generalizes to sections of smooth vector bundles where fiber metrics are compatible with the bundle connexions). More precisely the trace operator $\mathrm{Tr} : H^{s}(\Omega) \to H^{s-\frac{1}{2}}(\Sigma)$ is bounded and surjective. The following proposition completes the discussion of the solvability of the Dirac equation on Ω with the APS boundary condition and the regularity of the solutions.

Proposition 3.3 *Let Ω be a connected oriented $3-$Riemannian manifold with connected smooth boundary Σ and X be a smooth vector field on Ω. Assume that the scalar curvature R^{Ω} of Ω verifies $R^{\Omega} \geqslant 2divX - 2|X|^{2}$ and the mean curvature H of Σ while viewed as an embedded surface in Ω verifies $H - \langle X, \nu \rangle > 0$. Then the inhomogeneous Dirac equation*

$$\widehat{D}\psi = \Psi \ \text{on}\ \Omega \tag{3.13}$$

$$\psi_{\geqslant\lambda} = \alpha_{\geqslant\lambda} \ \text{on}\ \Sigma \tag{3.14}$$

with $\Psi \in H^{s-1}(S^{\Omega})$ and $\alpha_{\geqslant\lambda} \in H^{s-\frac{1}{2}}(S^{\Sigma}), s \geqslant 2$ has a unique solution $\psi_{\lambda} \in H^{s}(S^{\Omega}) \cap H^{s-\frac{1}{2}}(S^{\Sigma})$ for every $\lambda \leqslant 0$. We have an estimate of the type

$$||\psi_{\lambda}||_{H^{s}(\Omega)} \lesssim ||\Psi||_{H^{s-1}(\Omega)} + ||\alpha_{\geqslant\lambda}||_{H^{s-\frac{1}{2}}(\Sigma)},\ s \geqslant 2, \tag{3.15}$$

where $A \lesssim B$ denotes $A \leqslant CB$, for a constant C dependent on background geometry of Ω and Σ.

The previous proposition works with $s \geqslant 1$. Here we choose $s \geqslant 2$ to ensure continuity which will be important while obtaining the spin inequality 3.1. With the help of this proposition, we will obtain a crucial inequality in the next section that will be vital towards proving the positivity of the Wang-Yau quasi-local mass.

3.2 An important spin inequality

Now consider a vector field X that verifies $R^{\Omega} \geqslant 2\mathrm{div}X - 2|X|^2$. We prove the following theorem.

Theorem 3.1 *Let Ω be a spacelike topological ball in the spacetime N. Let X be a vector field on Ω that verifies the following point-wise inequality*

$$R^{\Omega} \geqslant 2|X|^2 - 2divX, \tag{3.16}$$

where R^{Ω} is the scalar curvature of Ω. Let the boundary of Ω be Σ with mean curvature H while viewed as an embedded surface in Ω verifying

$$H - \langle X, \nu \rangle > 0. \tag{3.17}$$

Then for every spinor $\alpha \in H^s(S^{\Sigma}),\ s \geqslant \frac{3}{2}$ we have the following inequality

$$\int_{\Sigma} \langle D\alpha, \alpha \rangle \mu_{\Sigma} \geqslant \frac{1}{2} \int_{\Sigma} (H - \langle X, \nu \rangle)|\alpha|^2 \mu_{\Sigma}. \tag{3.18}$$

The equality holds only if $X = 0$, and α is the boundary condition of a parallel spinor ψ on Ω solving Dirac equation. In such case

$$D\alpha = \frac{1}{2} H\alpha. \tag{3.19}$$

Proof. The proof relies on the proposition 2.1 and solving the Dirac equation on Ω with Atyiah-Singer-Patodi boundary condition. Now suppose, we have $\widehat{D}\psi = 0$, then

$$\int_{\Sigma} \left(\langle D\psi, \psi \rangle - \frac{1}{2} H|\psi|^2 \right) \mu_{\Sigma} = \frac{1}{4} \int_{\Omega} R^{\Omega} |\psi|^2 \mu_{\Omega} + \int_{\Omega} |\widehat{\nabla}\psi|^2 \mu_{\Omega} \tag{3.20}$$

We want to solve the equation $\widehat{D}\psi = 0$ on Ω with Atyiah-Singer boundary condition on Σ. Let us denote by $\psi_{\geqslant\lambda}$ the spinor with eigenvalues $\geqslant \lambda$. We solve the following boundary value problem

$$\widehat{D}\psi = 0 \text{ on } \Omega \tag{3.21}$$

$$\psi_{\geqslant\lambda} = \alpha_{\geqslant\lambda} \text{ on } \Sigma. \tag{3.22}$$

In the end, we want to take the limit $\lambda \to -\infty$. First recall the Sobolev trace inequality

$$||\psi_{\geqslant\lambda}||_{H^s(\Sigma)} \lesssim ||\psi_{\geqslant\lambda}||_{H^{s+\frac{1}{2}}(\Omega)},\ s > \frac{1}{2} \tag{3.23}$$

assuming smooth Σ. For the point-wise convergence to hold, we need to work in the following function space $\psi \in H^s(\Omega),\ s \geqslant 2$ and from trace inequality $\alpha_{\geqslant\lambda} \in H^{s-\frac{1}{2}}(\Sigma)$ and the proposition 3.3

we have

$$||\psi_\lambda||_{H^s(\Omega)} \lesssim ||\alpha_{\geqslant\lambda}||_{H^{s-\frac{1}{2}}(\Sigma)} \lesssim ||\alpha||_{H^{s-\frac{1}{2}}(\Sigma)} \ \forall\lambda \leqslant 0. \tag{3.24}$$

Due to $\alpha_{\geqslant\lambda}$ is $H^{s-\frac{1}{2}}(\Sigma),\ s \geqslant 2$, we have $H^{s-\frac{1}{2}}(\Sigma) \hookrightarrow C^0(\Sigma)$ and therefore point-wise convergence as well i.e.,

$$\psi|_\Sigma := \lim_{\lambda\to-\infty} \psi_{\geqslant\lambda}|_\Sigma = \lim_{\lambda\to-\infty} \alpha_{\geqslant\lambda} = \alpha. \tag{3.25}$$

Therefore, we have a unique solution to the boundary value problem following proposition 3.3

$$\widehat{D}\psi = 0 \text{ on } \Omega \tag{3.26}$$

$$\psi|_{\partial\Omega=\Sigma} = \alpha \tag{3.27}$$

Now we do the following manipulations to the identity 3.20. First we add $\frac{1}{2}\int_\Sigma \langle X,\nu\rangle\mu_\Sigma|\psi|^2$ to the both sides of 3.20. This yields

$$\int_\Sigma \left(\langle D\psi,\psi\rangle - \frac{1}{2}H|\psi|^2\right)\mu_\Sigma + \frac{1}{2}\int_\Sigma \langle X,\nu\rangle|\psi|^2\mu_\Sigma = \frac{1}{4}\int_\Omega R^\Omega|\psi|^2\mu_\Omega + \int_\Omega |\widehat{\nabla}\psi|^2\mu_\Omega + \frac{1}{2}\int_\Sigma \langle X,\nu\rangle|\psi|^2\mu_\Sigma.$$

Now recall the following integration by parts identity

$$\int_\Sigma \langle X,\nu\rangle|\psi|^2 = \int_\Omega \left(\text{div}X|\psi|^2 + X(|\psi|^2)\right)\mu_\Omega \tag{3.28}$$

to yield

$$\int_\Sigma \left(\langle D\psi,\psi\rangle - \frac{1}{2}(H - \langle X,\nu\rangle)|\psi|^2\right)\mu_\Sigma = \frac{1}{4}\int_\Omega (R^\Omega + 2\text{div}X)|\psi|^2\mu_\Omega + \int_\Omega |(|\widehat{\nabla}\psi|^2 + \frac{1}{2}X(|\psi|^2))\mu_\Sigma.$$

Now note the following calculations

$$|\widehat{\nabla}\psi|^2 + \frac{1}{2}X(|\psi|^2) = |\widehat{\nabla}\psi|^2 + \frac{1}{2}(\langle\widehat{\nabla}_X\psi,\psi\rangle + \langle\psi,\widehat{\nabla}_X\psi\rangle) \geqslant \frac{1}{2}|\widehat{\nabla}\psi|^2 - \frac{1}{2}|X|^2|\psi|^2 \tag{3.29}$$

and therefore we have

$$\begin{aligned}\int_\Sigma \left(\langle D\psi,\psi\rangle - \frac{1}{2}(H - \langle X,\nu\rangle)|\psi|^2\right)\mu_\Sigma &\geqslant \frac{1}{4}\int_\Omega (R^\Omega + 2\text{div}X - 2|X|^2)|\psi|^2\mu_\Omega + \frac{1}{2}\int_\Omega |\widehat{\nabla}\psi|^2\mu_\Omega \\ &\geqslant \frac{1}{4}\int_\Omega (R^\Omega + 2\text{div}X - 2|X|^2)|\psi|^2\mu_\Omega \geqslant 0\end{aligned}$$

by the assumption $R^\Omega \geqslant 2|X|^2 - 2\text{div}X$. Now the equality holds only when

$$\widehat{\nabla}\psi = 0, \tag{3.30}$$

$$R^\Omega + 2\text{div}X - 2|X|^2 = 0. \tag{3.31}$$

Now $\widehat{\nabla}\psi = 0$ implies ψ is a parallel Dirac spinor on Ω. Therefore Ω must be Ricci flat and therefore flat. Moreover, the identity 2.9 $D\psi = -c(\nu)\widehat{D}\psi - \widehat{\nabla}_{\nu}\psi + \frac{1}{2}H\psi$ yields

$$D\psi = \frac{1}{2}H\psi \text{ on } \Sigma. \tag{3.32}$$

Therefore

$$\int_{\Sigma}\langle X, \nu\rangle|\psi|^2\mu_{\Sigma} = 0. \tag{3.33}$$

which after integration by parts and parallelity condition $\widehat{\nabla}\psi = 0$ implies

$$\int_{\Omega}\mathrm{div}X|\psi|^2\mu_{\Omega} = 0 \tag{3.34}$$

Moreover, flatness implies $R^{\Omega} = 0$ yielding

$$\mathrm{div}X - |X|^2 = 0 \tag{3.35}$$

and therefore

$$\int_{\Omega}|X|^2|\psi|^2\mu_{\Omega} = 0 \implies X = 0. \tag{3.36}$$

This concludes the proof of the theorem. □

Remark 1 *In the case of equality in the previous theorem, Ω admits parallel spinor and hence it is Ricci flat and hence flat in dimension three. Due to being simply connected, Ω is a domain in $\mathbb{R}^3$.*

4 Wang-Yau Quasi-local mass and its positivity

We first define the Wang-Yau quasi-local mass. The details can be found in the original article of Wang & Yau [13]. Here we state the necessary ideas that lead to the construction of the Wang-Yau quasi-local mass. The aim is to define the gravitational energy contained in a spacelike region bounded by a topological $2-$sphere in a $3+1$ dimensional physical spacetime $(N, \widehat{g})$ verifying dominant energy condition. We assume that the spacetime M is globally hyperbolic, that is, it can be expressed as the product $M \times \mathbb{R}$ where M is a Cauchy hypersurface. We write the spacetime metric $\widehat{g}$ in the following form

$$\widehat{g} := -\beta^2 dt \otimes dt + g_{ij}(dx^i + \gamma^i dt) \otimes (dx^j + \gamma^j dt), \tag{4.1}$$

in a local chart $\{t, x^i\}_{i=1}^3$. Here β and $\gamma := \gamma^i \frac{\partial}{\partial x^i}$ are the lapse function and the shift vector field, respectively. Here we will be interested in Einstein's constraint equations primarily. Note that the

Gauss and Codazzi equations for the spacetime N foliated by M yield the constraint equations on each $t=$ constant slice

$$R(g)-P_{ij}P^{ij}+(\mathrm{tr}_gP)^2=2\mu \tag{4.2}$$

$$\widehat{\nabla}^j k_{ij}-\nabla_i\mathrm{tr}_gP=J_i, \tag{4.3}$$

where μ and $J:=J_idx^i$ are the local energy and momentum density of the matter (or radiation) sources present in the spacetime. The dominant energy condition states $\mu\geqslant\sqrt{g(J,J)}$. P_{ij} is the extrinsic curvature or the second fundamental form of M in $N=M\times\mathbb{R}$. Once we are equipped with this information about the spacetime N (which is assumed to be Einsteinian spacetime i.e., the spacetime metric verifies Einstein's constraint and evolution equations), we can define the energy contained within a membrane $\Sigma\subset M$ i.e., a spacelike topological 2−sphere. The Wang-Yau quasi-local mass precisely quantifies this energy through a Hamilton-Jacobi analysis (see [13, 14] for the detail on how such an energy is obtained).

4.1 Definition of Wang-Yau Quasi-local Energy

First, we recall the following definition of a generalized mean curvature as given in [13].

Definition 1 (Wang-Yau) *[13] Suppose $i:\Sigma\hookrightarrow N$ is an embedded space-like two-surface with induced metric σ. Given a smooth function τ on Σ and a space-like normal e_3, the generalized mean curvature $h(\Sigma,i,\tau,e_3)$ associated with these data is defined to be*

$$h(\Sigma,i,\tau,e_3):=-\sqrt{1+|\nabla\tau|^2}\langle\mathbf{H},e_3\rangle-\alpha_{e_3}(\nabla\tau), \tag{4.4}$$

where $\mathbf{H}$ is the mean curvature vector of Σ in N and and α_{e_3} is the connexion form (see Definition 1.1) of the normal bundle of Σ in N determined by e_3 and the future-directed time-like unit normal e_4 orthogonal to e_3.

Now we consider the isometric embedding of (Σ,σ) into the Minkowski space $(\mathbb{R}^{1,3},\eta)$ with prescribed mean curvature in a fixed time direction. Let us denote the standard Lorentzian inner product on $\mathbb{R}^{1,3}$ by $\langle\cdot,\cdot\rangle$ and let T_0 be a constant timeline unit vector in $\mathbb{R}^{1,3}$. Wang & Yau [13] proved the following existence and uniqueness theorem for such an isometric embedding

Theorem 4.1 (Wang-Yau) *[13] Let χ be a function on (Σ,σ) with $\int_\Sigma\chi\mu_\Sigma=0$. Let τ be a potential function of χ i.e., $\Delta_\sigma\tau=\chi$. Suppose the Gaussian curvature K of (Σ,σ) verifies the inequality $K+(1+|\nabla\tau|^2_\sigma)^{-1}\det(\nabla^2\tau)>0$, then there exists a space-like isometric embedding $i_0:\Sigma\to\mathbb{R}^3$ such that the mean curvature $\boldsymbol{H}_0$ of the embedded surface $(i_0(\Sigma),\sigma)$ verifies $\langle\boldsymbol{H}_0,T_0\rangle=-\chi$.*

The main idea behind the proof of this theorem is to consider the projection of $\widehat{i}_0:\Sigma\to\mathbb{R}^3$ onto the orthogonal complement of T_0 i.e., $i_0:=i-\tau T_0$. Let us denote the projected surface by $\widehat{\Sigma}$. The induced metric on $\widehat{\Sigma}$ is given by $\widehat{\sigma} := \sigma + d\tau \otimes d\tau$ and the Gauss curvature $\widehat{K}$ as $(1+|\nabla\tau|^2_\sigma)^{-1}(K + (1+|\nabla\tau|^2_\sigma)^{-1}\det(\nabla^2\tau))$ which is positive by the assumption of the theorem 4.1. Therefore, the existence of $\widehat{i}_0$ follows by the classic theorem of Nirenberg and Pogolerov [31]. Now existence of i_0 can be established as follows. One starts with a metric σ and a function χ and solve for $\Delta_\sigma\tau = \lambda$ (unique solution exists since Σ is compact, the kernel of Δ_σ is trivial modulo constants). Now one constructs the new metric $\widehat{\sigma} = \sigma + d\tau \otimes d\tau$ and finds that its Gauss curvature $\widehat{K}$ is again $(1+|\nabla\tau|^2_\sigma)^{-1}(K + (1+|\nabla\tau|^2_\sigma)^{-1}\det(\nabla^2\tau))$ which is strictly positive by the hypothesis. Therefore the embedding into $\mathbb{R}^3$ is established. The desired embedding i_0 is found as $i_0 = \widehat{i}_0 + \tau T_0$.

Now the generalized mean curvature of the surface $(i_0(\Sigma), \sigma)$ in the Minkowski space is defined in a similar fashion

$$h(\Sigma, i_0, \tau, (e_3)_0) := -\sqrt{1+|\nabla\tau|^2}\langle \mathbf{H}_0, (e_3)_0\rangle - \widehat{\alpha}_{(e_3)_0}(\nabla\tau) \tag{4.5}$$

where $\widehat{\alpha}$ is connection of the normal bundle of $i_0(\Sigma)$ induced by the flat connection $\nabla^{\mathbb{R}^{1,3}}$ of the Minkowski space, $(e_3)_0$ is the spacelike unit normal vector to $(i_0(\Sigma), \sigma)$, and $\mathbf{H}_0$ is the mean curvature of $i_0(\Sigma)$ as an embedded spacelike surface in the Minkowski space. The Wang-Yau quasi-local energy is defined to be the following

$$8\pi M^{wy} := \inf_\tau \int_\Sigma \left(h(\Sigma, i_0, \tau, (e_3)_0) - h(\Sigma, i, \tau, e_3)\right)\mu_\Sigma. \tag{4.6}$$

Strictly speaking, in the definition of $h(\Sigma, i, \tau, e_3)$, there is still redundancy left due to the boost transformation of the normal bundle of Σ in the physical spacetime N. In fact, [13] minimizes $h(\Sigma, i, \tau, e_3)$ over the boost-angle ($h(\Sigma, i, \tau, e_3)$ is a convex function of the boost angle). Here, we aim to prove the positivity of the entity $h(\Sigma, i_0, \tau, (e_3)_0) - h(\Sigma, i, \tau, e_3)$ for any τ verifying the condition of theorem 4.1. This would suffice to show the positivity of M^{wy} since subtracting the minimum of $h(\Sigma, i, \tau, e_3)$ only increases the mass. So from now on if there are no confusions, we will denote $\int_\Sigma \left(h(\Sigma, i_0, \tau, (e_3)_0) - h(\Sigma, i, \tau, e_3)\right)\mu_\Sigma$ by $8\pi M^{wy}$. The main idea is to convert the integrals $\int_\Sigma h(\Sigma, i_0, \tau, (e_3)_0)\mu_\Sigma$ and $\int_\Sigma h(\Sigma, i, \tau, e_3)\mu_\Sigma$ to integrals of mean curvatures of surfaces in $\mathbb{R}^3$ and in the Cauchy slice of the physical spacetime. Then we can use the theorem 3.1 to prove the positivity.

The most important observation one makes is the following two reductions obtained by Wang-Yau [13]. They proved that the Minkowski entity $\int_\Sigma h(\Sigma, i_0, \tau, (e_3)_0)\mu_\Sigma$ can be re-written as the integral of the mean curvature of the projection $\widehat{\Sigma}$ of Σ onto the compliment of T_0 i.e., onto $\mathbb{R}^3$. First consider an embedding $i_0 : \Sigma \hookrightarrow \mathbb{R}^{1,3}$ is an embedding and $\tau := \langle i_0, T_0\rangle$ is the restriction of the time function associated with the time vector field T_0. The outward unit normal vector $\widehat{\nu}$ of $\widehat{\Sigma}$ in $\mathbb{R}^3$ and T_0 form a basis of the normal bundle of $\widehat{\Sigma}$ in $\mathbb{R}^{1,3}$. One can parallel transport $\widehat{\nu}$ along the integral curves of T_0 to make it a vector field in $\mathbb{R}^{1,3}$ and denote this vector field as $\widehat{e}_3$. Let the mean curvature of $\widehat{\Sigma}$ in $\mathbb{R}^3$ be $\widehat{k}$. Then the following proposition holds due to [13].

Proposition 4.1 *The following identity is verified by $h(\Sigma, i_0, \tau, (e_3)_0)$*

$$\int_{\Sigma} h(\Sigma, i_0, \tau, (e_3)_0)\mu_{\Sigma} = \int_{\widehat{\Sigma}} \widehat{k}. \tag{4.7}$$

Here the induced metric on $\widehat{\Sigma}$ is $\sigma + d\tau \otimes d\tau$ and σ is the metric on Σ.

Now one hopes to find a similar expression for the physical spacetime entity $\int_{\Sigma} h(\Sigma, i, \tau, e_3)\mu_{\Sigma}$. But in the physical spacetime, one needs to account for the momentum information (i.e., the second fundamental form of the hypersurface Ω in N). [13] derives an almost similar identity by projecting Σ onto a constant time hypersurface with extra terms accounting for the momentum information. As we shall see, the appearance of this extra momentum contribution would necessitate invoking the dominant energy condition. Consider the initial data set (Ω, g_{ij}, P_{ij}), where P_{ij} is the second fundamental form of Ω in the physical spacetime N and plays the role of momentum for the metric g_{ij}. Now consider the product $\Omega \times \mathbb{R}$ and extend P_{ij} by parallel translation along the Ω-normal field e_4. We denote this entity by P_{ij} as well. Now we seek a hypersurface $\widetilde{\Omega}$ in $\Omega \times \mathbb{R}$ defined as the graph of a function f over Ω, such that the mean curvature of $\widetilde{\Omega}$ in $\Omega \times \mathbb{R}$ is the same as the trace of the restriction of P to $\widetilde{\Omega}$. This is accomplished in [13] by solving the Jang's equation verified by f (Jang's equation enjoys a rich history: Jang's equation was proposed by Jang [16] in an attempt to solve the positive energy conjecture. Yau and Schoen came up with different geometric interpretations, studied the equation in full, and applied them to their proof of the positive mass theorem [1, 2]). First, pick an orthonormal basis $\{\widetilde{e}_\alpha\}_{\alpha=1}^{4}$ for the tangent space of $\Omega \times \mathbb{R}$ along $\tilde{\Omega}$ such that $\{\widetilde{e}_i\}_{i=1}^{3}$ is tangent to $\widetilde{\Omega}$ and $\widetilde{e}_4$ is the downward unit normal to $\widetilde{\Omega}$. If the induced metric on Ω is g_{ij}, then the induced metric on $\widetilde{\Omega}$ is $\widetilde{g}_{ij} = g_{ij} + \partial_i f \partial_j f$. Jang's equations states

$$\sum_{i=1}^{3} \langle \widetilde{\nabla}_{\widetilde{e}_i} \widetilde{e}_4, \widetilde{e}_i \rangle = \sum_{i=1}^{3} P(\widetilde{e}_i, \widetilde{e}_i) \tag{4.8}$$

or more explicitly

$$\sum_{i,j=1}^{3} \left(g^{ij} - \frac{f^i f^j}{1 + |\widehat{\nabla} f|^2} \right) \left(\frac{\widehat{\nabla}_i \partial_j f}{(1 + |\widehat{\nabla} f|^2)^{\frac{1}{2}}} - P_{ij} \right) = 0 \tag{4.9}$$

where $\widetilde{\nabla}$ is the usual Levi-Civita connexion of the product spacetime $\Omega \times \mathbb{R}$. Now let τ be a smooth function on Σ. We consider a solution f to Jangs equation 4.8 in $\Omega \times \mathbb{R}$ that verifies the Dirichlet boundary condition $f = \tau$ on Σ. Let $\widetilde{\Sigma}$ be the graph of τ over Σ and $\tilde{\Omega}$ be the graph of f over Ω such that $\partial \widetilde{\Omega} = \widetilde{\Sigma}$. From now on we denote $\widetilde{\Sigma}$ by $\widehat{\Sigma}$ since they are isometric. More precisely the metric on $\widetilde{\Sigma}$ is $\sigma + d\tau \otimes d\tau$. Jang's equation is a quasi-linear elliptic PDE for f and typically its solution blows up in the presence of an apparent horizon. Here we assume that such an apparent horizon is absent or $\widetilde{k} - \mathrm{tr}_{\widehat{\Sigma}} P > 0$. Under such an assumption, a solution to Jang's equation exists and the foregoing construction makes sense. Under such circumstances, one can relate the generalized total mean curvature of Σ in $\mathbb{R}^{1,3}$ to that of $\widehat{\Sigma}$ in $\widetilde{\Omega}$ together with the momentum information. In

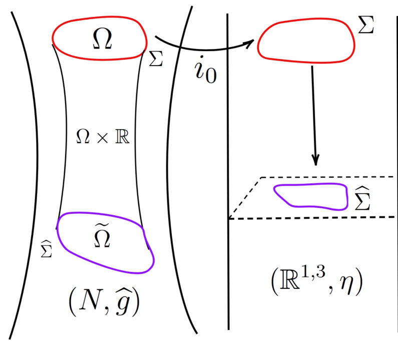

Figure 1: Schematics of the isometric embedding and reduction procedure in Wang-Yau construction. Isometric surfaces are denoted by the same symbol (e.g., $\widehat{\Sigma}$). Jang's equation is solved on Ω with boundary data $f = \tau$ on $\Sigma := \partial\Omega$. $\widetilde{\Omega}$ is the graph of f over Ω and $\widehat{\Sigma} := \partial\widetilde{\Omega}$ is the graph of τ over Σ. i_0 denotes the isometric embedding of the surface (Σ, σ) in $(N, \widehat{g})$ into $(\mathbb{R}^{1,3}, \eta)$. Image of this embedded surface is projected onto T_0 (constant unit timeline vector field in $(\mathbb{R}^{1,3}, \eta)$) orthogonal hyperplane and the projection is denoted by $\widehat{\Sigma}$ as well since it is isometric to $\partial\widetilde{\Omega}$. Wang-Yau energy definition is then reduced to a form that depends on the data on $\widehat{\Sigma}$ and $\partial\widetilde{\Omega}$ (also denoted by $\widehat{\Sigma}$) and spacetime momentum information.

particular, the following proposition holds due to [13]

Proposition 4.2 *Let τ be a smooth function on $\Sigma = \partial\Omega$. We consider a solution f of Jang's equation in $\Omega \times \mathbb{R}$ that satisfies the Dirichlet boundary condition $f = \tau$ on Σ. Denote the graph of τ over Σ by $\widehat{\Sigma}$. Then there exists a unit space-like normal field $e_3^{'}$ to Σ in N such that the following holds*

$$\int_{\widehat{\Sigma}} \left(\widetilde{k} - \langle \widetilde{\nabla}_{\widetilde{e}_4} \widetilde{e}_4, \widetilde{e}_3 \rangle + P(\widetilde{e}_4, \widetilde{e}_3) \right) \mu_{\widehat{\Sigma}} = \int_{\Sigma} \left(-\sqrt{1 + |\nabla\tau|^2} \langle \mathbf{H}, e_3^{'} \rangle - \alpha_{e_3^{'}}(\nabla\tau) \right) \mu_{\Sigma} \tag{4.10}$$

where $\widetilde{k}$ is the mean curvature of $\widehat{\Sigma}$ in $\widetilde{\Omega}$ and $\widetilde{e}_4$ and $\widetilde{e}_3$ are the basis of the normal bundle of $\widehat{\Sigma}$ in N.

4.2 Proof of the main theorem

Following the two propositions 4.1 and 4.2, the expression of the quasi-local energy reduces to the following

$$8\pi M^{wy} := \int_{\widehat{\Sigma}} \widehat{k} \mu_{\widehat{\Sigma}} - \int_{\widehat{\Sigma}} \left(\widetilde{k} - \langle \widetilde{\nabla}_{\widetilde{e}_4} \widetilde{e}_4, \widetilde{e}_3 \rangle + P(\widetilde{e}_4, \widetilde{e}_3) \right) \mu_{\widehat{\Sigma}}, \tag{4.11}$$

where induced metric on $\widehat{\Sigma}$ is simply $\sigma + d\tau \otimes d\tau$. This form of the mass M^{wy} is conducive to the application of the theorem 3.1. Let us first consider the surface $\widehat{\Sigma}$ in $\mathbb{R}^3$. From theorem 3.1, we know that for this surface embedded in $\mathbb{R}^3$ the equality is satisfied and the following holds

$$D\alpha = \frac{1}{2}\widehat{k}\alpha \tag{4.12}$$

for a spinor α in $S^{\widehat{\Sigma}}$. Now denote the vector dual to the 1−form $-P(\widetilde{e}_4, \cdot) + \langle \widetilde{\nabla}_{\widetilde{e}_4}\widetilde{e}_4, \cdot\rangle$ by X. Therefore the integral $\int_{\widehat{\Sigma}} \left(\widetilde{k} - \langle \widetilde{\nabla}_{\widetilde{e}_4}\widetilde{e}_4, \widetilde{e}_3\rangle + P(\widetilde{e}_4, \widetilde{e}_3)\right) \mu_{\widehat{\Sigma}}$ is nothing but $\int_{\widehat{\Sigma}} \left(\widetilde{k} - \langle X, \widetilde{e}_3\rangle\right) \mu_{\widehat{\Sigma}}$, where $\widetilde{e}_3$ is the unit normal to $\widehat{\Sigma}$ parallel to a constant time hypersurface $\widetilde{\Omega}$. Now according to the theorem 3.1, the following entity

$$M_\alpha := \int_{\widehat{\Sigma}} \langle D\alpha, \alpha\rangle \mu_{\widehat{\Sigma}} - \frac{1}{2}\int_{\widehat{\Sigma}} \left(\widetilde{k} - \langle X, \widetilde{e}_3\rangle\right) |\alpha|^2 \mu_{\widehat{\Sigma}} \tag{4.13}$$

is non-negative. Now substitute $D\alpha = \frac{1}{2}\widehat{k}\alpha$ and set $|\alpha|^2 = 1$ yields

$$0 \leqslant M_1 = \frac{1}{2}\int_{\widehat{\Sigma}} \widehat{k} - \frac{1}{2}\int_{\widehat{\Sigma}} \left(\widetilde{k} - \langle X, \widetilde{e}_3\rangle\right) \mu_{\widehat{\Sigma}} = 4\pi M^{wy} \tag{4.14}$$

and therefore $M^{wy} \geqslant 0$. This concludes the proof of the theorem. Now the only thing that remains to be justified is the validity of the curvature condition $R^{\tilde{\Omega}} \geqslant 2|X|^2 - 2\mathrm{div}X$. Now recall the geometry of $\tilde{\Omega}$ in the product $\Omega \times \mathbb{R}$. On $\tilde{\Omega}$ the induced metric is $\widetilde{g}_{ij} := g_{ij} + \partial_i f \partial_j f$ and let us denote its second fundamental form by h_{ij} (note that $h_{\mu\nu} = \frac{1}{2}(\widetilde{\nabla}_\mu \widetilde{n}_\mu + \widetilde{\nabla}_\nu \widetilde{n}_\mu)$,$\mu$, $\nu = 0, 1, 2, 3, 4$, where $\widetilde{n} = \widetilde{e}_4$ is the unit downward normal of $\tilde{\Omega}$ in $\Omega \times \mathbb{R}$). Similarly, let P_{ij} be the second fundamental form of Ω in $\Omega \times \mathbb{R}$. On $\tilde{\Omega}$, one has the following inequality verified by the scalar curvature $R^{\tilde{\Omega}}$ if the Hamiltonian and the momentum constraints are satisfied on $\tilde{\Omega}$

$$2(\mu - \sqrt{\widetilde{g}(J, J)}) \leqslant R^{\tilde{\Omega}} - \sum_{i,j}(h_{ij} - P_{ij})^2 - 2\sum_i (h_{i4} - P_{i4})^2 + 2\sum_i \widetilde{\nabla}_i (h_{i4} - P_{i4}) \tag{4.15}$$

But since the spacetime verifies the dominant energy condition, we have $\mu \geqslant \sqrt{\widetilde{g}(J, J)}$, one has

$$\begin{aligned} R^{\tilde{\Omega}} &\geqslant \sum_{i,j}(h_{ij} - P_{ij})^2 + 2\sum_i (h_{i4} - P_{i4})^2 - 2\sum_i \widetilde{\nabla}_i (h_{i4} - P_{i4}) \\ &\geqslant 2\sum_i (h_{i4} - P_{i4})^2 - 2\sum_i \widetilde{\nabla}_i (h_{i4} - P_{i4}). \end{aligned} \tag{4.16}$$

But note that $h_{i4} - P_{i4}$ is nothing but the $\widetilde{e}_i$th components of the 1−form $-P(\widetilde{e}_4, \cdot) + \langle \widetilde{\nabla}_{\widetilde{e}_4}\widetilde{e}_4, \cdot\rangle$. Therefore $\sum_i (h_{i4} - P_{i4})^2 = |X|^2$ and $\sum_i \widetilde{\nabla}_i (h_{i4} - P_{i4}) = \mathrm{div}X$. Therefore, under the assumption of the dominant energy condition, we have

$$R^{\tilde{\Omega}} \geqslant 2|X|^2 - 2\mathrm{div}X. \tag{4.17}$$

In addition, the assumption of no apparent horizon gives existence of bonded solution of Jang's equation and $\widetilde{k} - \langle X, \widehat{e}_3 \rangle > 0$. This concludes the proof the proof of the main theorem. One vital property to note is the non-negativity holds for any admissible τ, that is, a τ that verifies the condition stated in the theorem 4.1. In addition to the non-negativity of the Wang-Yau quasi-local, one obtains a rigidity property as well.

Corollary 4.1 (Rigidity) *Under the assumption of the theorem 4.1 and if the set of admissible τ is non-empty, then the Wang-Yau quasi-local energy bounded by the spacelike topological $2-$sphere Σ is non-negative. Moreover, if Σ is a spacelike topological $2-$sphere embedded in $\mathbb{R}^{1,3}$, then Wang-Yau quasi-local energy M^{wy} is zero. Also, if M^{wy} is zero then the physical spacetime N is Minkowski along Σ.*

Proof. The proof of the non-negativity is trivial if the admissible set of τ is non-empty and essentially a consequence of the proof of the main theorem stated earlier in this section. If the topological $2-$sphere Σ is embedded in $\mathbb{R}^{1,3}$, then one can comply take the embedding i_0 to be i and therefore the vanishing of M^{wy} follows trivially. Now if M^{wy} vanishes, then we have $\widetilde{\Omega}$ to be Ricci flat and therefore flat and more specifically $R^{\tilde{\Omega}} = 0$. Now if the equality holds in 4.14, then according to the theorem 3.1, we have $X = 0$. Therefore $h_{i4} = P_{i4}$ which yields together with $R^{\tilde{\Omega}} = 0$

$$0 \leqslant 2(\mu - |J|) \leqslant -\sum_{i,j}(h_{ij} - P_{ij})^2 \leqslant 0 \tag{4.18}$$

and therefore $\mu = |J|$ and $h_{ij} = P_{ij}$. Therefore $\widetilde{\Omega}$ coincides with Ω and $f = 0$ is a solution to Jang's equation yielding $\mathrm{tr}_g P = 0$ and $R^{\Omega} = 0$. Now write the following for $P_{ij} = P^{tr}_{ij} + \frac{1}{3}\mathrm{tr}_g P g_{ij}$ together with $R^{\Omega} = 0$ and substitute in the Hamiltonian constraint to yield

$$-|P^{tr}|^2_g = 2\mu \tag{4.19}$$

but $\mu \geqslant 0$ (since $\mu \geqslant \sqrt{g(J,J)}$) yielding $P_{ij} = 0 = \mu$ and therefore $J = 0$ as well. Therefore, N is a vacuum spacetime i.e., $\mathbf{Ric}[\widehat{g}] = 0$ Now recall the structure equations for the for Ω embedded in N

$$R_{ij} - P_{im}P^m_j + P_{ij}\mathrm{tr}_g P = E_{ij} + \mathbf{Ric}_{ij} \tag{4.20}$$

$$\nabla_i P_{jm} - \nabla_j P_{km} = \epsilon_{ij}{}^{l} B_{lm}, \tag{4.21}$$

where E_{ij} and B_{ij} are the electric and magnetic fields that completely determine the Riemann curvature (Weyl curvature since N is vacuum). It follows trivially now that $E_{ij} = B_{ij} = 0$ since $R_{ij} = 0$ due to Ricci flatness of Ω and $\mathbb{R}\beth_{ij} = 0$ in vacuum, and therefore $\mathbf{Riem}[\widehat{g}] = 0$ implying N is Minkowski along Ω due to being simply connected. This completes the proof. $\square$

We have the obvious corollary

Corollary 4.2 *Liu-Yau mass is non-negative.*

Proof. Set τ = constant. Then the Wang-Yau quasi-local energy reduces to Liu-Yau energy. The embedding i_0 is in $\mathbb{R}^3$. Then one needs a stronger condition of strict positivity of the Gauss curvature of Σ i.e., $K > 0$. Now the result follows from the main theorem. □

5 Conclusions

We provided a complete quasi-local proof of the non-negativity of the Wang-Yau quasi-local energy and its rigidity property by using spinors that solve Dirac equation on the bulk with Atyiah-Patodi-Singer (APS) boundary condition. Analogous result for the Brown-York mass is recently provided by [24]. In the asymptotically flat setting, the Wang-Yau quasi-local mass approaches ADM mass at spacelike infinity and its positivity proof using spinor (after Schoen-Yau [1,2] proof was announced) in that circumstance was provided by Witten [4] which was subsequently made rigorous by Parker-Taubes [30]. We would like to understand if the spinors can be used to define quasi-local mass contained within compact spacelike hypersurface that has disconnected boundary, for example, a topologically spherical annulus. Another interesting problem would be to obtain a spinorial quasi-local proof of the positivity and rigidity of the new quasi-local mass defined by Alaee-Khuri-Yau [15] where the boundary surface Σ is allowed to have non-trivial topology. These problems are under investigation.

6 Acknowledgement

This work was supported by the Center of Mathematical Sciences and Applications (CMSA), Department of Mathematics at Harvard University.

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