On a Rigidity Result in Scalar Curvature Geometry

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Abstract

I prove a scalar curvature rigidity theorem for spheres. Specifically, I show that geodesic balls of radii strictly less than $\frac{\pi}{2}$ in the $n+1-$ dimensional unit sphere ($n \geq 2$) are rigid under smooth perturbations that increase scalar curvature while preserving the intrinsic geometry and the mean curvature of the boundary. Furthermore, this rigidity property fails in the case of the hemisphere. The proof uses the notion of a real Killing connection and the resolution of the boundary value problem associated with its Dirac operator. This result provides the sharpest possible refinement of the now-disproven Min-Oo conjecture.

Contents

1. Introduction

The positive mass theorem is one of the most important results in differential geometry. This theorem states that any asymptotically flat Riemannian manifold (M, g) of dimension $n \leq 7$ of non-negative scalar curvature has a non-negative ADM mass. Moreover, the ADM mass for such a manifold is strictly positive unless (M, g) is isometric to the Euclidean space. Schoen-Yau [26, 27] proved this theorem first using minimal surface techniques. Later Witten [32], Taubes-Parker [25] proved it using the spinor method. These spinor methods remove the dimensional restriction but assume that the manifold is spin (which is a non-trivial condition and requires the vanishing of the second Steifel-Whitney class). Recently, Schoen-Yau [28]

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gave a proof of the positive mass theorem in any dimensions using the minimal surface technique where the spin condition is not required.

Theorem (Positive Mass Theorem) *[26, 27, 28] Assume that M is an asymptotically flat manifold with scalar curvature $R \geq 0$. We then have that the ADM mass is non-negative. Furthermore, if the mass is zero, then M is isometric to $\mathbb{R}^n$ with the standard Euclidean metric.*

Several rigidity results can be deduced by applying this theorem. The first observation was made by Miao [22] who pointed out that the positive mass theorem implies the following

Theorem *[22] Let D^n be a topological $n-$ball and g be a smooth metric on D^n that has the following properties*
(a) Scalar curvature of g is non-negative i.e., $R[g] \geq 0$, (b) The induced metric on the boundary ∂D^n is the standard round metric on $\mathbb{S}^{n-1}$,
(c) The mean curvature of ∂D^n with respect to g is at least $n-1$,
then g is isometric to the standard Euclidean metric on D^n.

The sketch of the proof of this theorem can be stated as follows. Let us first remove the standard unit ball $\mathbb{B}^n$ from $\mathbb{R}^n$. The boundary is the unit sphere with the standard round metric. Now glue D^n to $\mathbb{R}^n - \mathbb{B}^n$ along the boundary sphere ∂D^n. Let the metric on this new manifold be denoted by $\widetilde{g}$. The new manifold obtained thus verifies the scalar curvature condition $R[\widetilde{g}] \geq 0$ and the unit sphere along which D^n is glued verifies the men curvature condition as well. Now this is trivially asymptotically flat and in fact exactly Euclidean at infinity and therefore has zero ADM mass. Therefore by positive mass theorem, $\widetilde{g}$ is isometric to the standard Euclidean metric on $\mathbb{R}^n$.
Later, Shi-Tam [29] proved a localized version of positive mass theorem. The following theorem was proved by Shi-Tam

Theorem (Non-negativity of Brown-York Mass) *[29] Let M be a strictly convex domain in $\mathbb{R}^n$ with smooth boundary and a smooth Riemannian metric g verifying*
(a) The scalar curvature $R[g] \geq 0$,
(b) The induced metric on the boundary ∂M by g is the same as the restriction of the standard Euclidean metric on ∂M,
(c) The mean curvature of ∂M with respect to g is strictly positive. Then the Brown-York mass associated with ∂M is non-negative i.e.,

$$\int_{\partial M} (K_0 - K_g)\mu_{\sigma_g} \geq 0, \tag{1.1}$$

where K_0 is the mean curvature of ∂M with respect to the Euclidean metric whereas K_g is with respect to that of g. Moreover, if the equality holds, then g is isometric to the standard Euclidean metric.

Importantly note that the expression $\int_{\partial M}(K_0 - K_g)\mu_{\sigma_g}$ approaches the ADM mass for (M, g) in the limit where ∂M approaches infinity. This entity also called the Brown-York mass, arises in time-symmetric general relativistic settings. Analogous results have also been proven in the asymptotically hyperbolic framework where the manifold under study verifies a negative lower bound of $-n(n-1)$ for the scalar curvature [1, 11, 12]. The other model geometry is the sphere with strictly positive curvature. Motivated by the relevant results on asymptotically flat and hyperbolic settings, Min-Oo [23] conjectured that an analogous should hold true for spheres. More precisely the Min-Oo conjecture states the following

Conjecture 1.1 (Min-Oo conjecture[23]) *Let g be a smooth metric on the upper hemisphere S^n_+ with the following property*
(a) It has scalar curvature $R[g] \geq (n-1)n$,
(b) the induced metric on the boundary sphere ∂S^n_+ is the same as the standard round metric on $\mathbb{S}^{n-1}$,
(c) The boundary ∂S^n_+ is totally geodesic with respect to g. Then g is isometric to the standard round metric on $\mathbb{S}^n_+$.

This conjecture can be thought of as the positive mass theorem for the strictly positive scalar curvature geometry. There have been several results in special circumstances. This is true for the two-dimensional case by a theorem of Toponogov [30]. For $n \geq 3$, Hang and Wang [17] proved that this conjecture holds true for any metric conformal to the standard metric on the sphere. Huang and Hu [20] proved that the Min-Oo conjecture holds true for a class of manifolds that are graphs of hypersurfaces in $\mathbb{R}^{n+1}$. Finally, Brendle-Marques-Neves [9] proved that the conjecture is false generically for $n \geq 3$. In particular, they constructed counterexamples where the scalar curvature in the bulk can be increased keeping the boundary intact (i.e., totally geodesic and isometric to standard $n-1$ sphere). They accomplish this in two steps. First, they perform a perturbation of the standard hemisphere to increase the scalar curvature. This process makes the mean curvature of the boundary strictly positive. In the next step, they perturb to make the boundary totally geodesic. This perturbation is supported close to the boundary and is not strong enough to reduce the scalar curvature below the strict lower bound $(n-1)n$.
Motivated by these results, it is natural to ask whether the rigidity result holds for geodesic balls of radius strictly less than $\frac{\pi}{2}$ instead of the hemisphere. It turns out that indeed such a rigidity result holds true if one imposes certain additional conditions on the spin structures of the geodesic balls and their boundaries. This is accomplished in two steps. First, we consider a general positive scalar curvature connected spin $n+1-$ dimensional Riemannian manifold with a mean convex boundary. Imposing a lower bound on the scalar curvature of the boundary, we prove a new and improved eigenvalue estimate for the boundary Dirac operator (see [16] for a review of the eigenvalue estimates of the Dirac operators). This step involves solving the Dirac equation associated with a real Killing connection using an APS type non-local boundary conditions [2, 3]. We explain the first theorem in this context. Assume (M, g) is a connected oriented spin manifold with boundary Σ. Consider a real Killing spin connection of Bär

$$\widetilde{\nabla}_X\psi = \widehat{\nabla}_X\psi + \alpha\rho(X)\psi, \tag{1.2}$$

for $X \in C^\infty(\Gamma(TM)),\ \psi \in C^\infty(\Gamma(S^M))$. The corresponding Dirac operator $\widetilde{D}$ reads

$$\widetilde{D} = \widehat{D} - (n+1)\alpha. \tag{1.3}$$

From now on we fix $\alpha = \pm\frac{1}{2}$. Define $\widetilde{D}^{\pm}$ to be

$$\widetilde{D}^{\pm} = \widehat{D} \mp \frac{n+1}{2}. \tag{1.4}$$

The central idea of this article is to solve the Dirac equation of this Killing connection. We use H^s to denote the L^2 Sobolev spaces $W^{s,2}$. In particular, $H^1(M)$ norm of a section φ is defined as follows

$$||\varphi||^2_{H^1(M)} := ||\widehat{\nabla}\varphi||^2_{L^2(M)} + ||\varphi||^2_{L^2(M)} \tag{1.5}$$

The canonical volume form of the boundary (Σ, σ) is denoted by μ_σ. To this end let us define the following two spaces. Naturally $H^1(M)$ has trace $H^{\frac{1}{2}}(\Sigma)$ on Σ which is defined in the spectral side. On Σ, one performs a L^2-orthogonal spectral decomposition (detail in section 4)

$$\varphi = \sum_i a_i \Phi_i^+ + \sum_i d_i \widetilde{\Phi}_i^-, \ a_i, \ d_i \in \mathbb{C}. \tag{1.6}$$

Here Φ_i^+ and $\widetilde{\Phi}_i^-$ are the positive and negative eigenspinors of certain boundary (Σ) Dirac operators (see section 4 for the existence and uniqueness of such decomposition). $\widetilde{P}^+_{\geq 0}$ and $\widetilde{P}^-_{\leq 0}$ denote the respective L^2-projection operators to the subspaces spanned by Φ_i^+ and $\widetilde{\Phi}_i^-$, respectively. Also let H^s denote the L^2 Sobolev space of spinors on M or Σ and $\xi := \Phi_1^+$ such that $\widetilde{P}^+_{\geq 0}\xi = \xi$. First, define the following two spaces.

$$\mathcal{D}^+ := \{\varphi \in H^1(\Gamma(S^M)) \cap H^{\frac{1}{2}}(\Gamma(S^\Sigma)) | \widetilde{D}^+\varphi = 0, on\ M, \widetilde{P}^+_{\geq 0}\psi = \xi, on\ \Sigma, \Re(\int_\Sigma \langle \varphi|_\Sigma, \widetilde{\Phi}_1^- \rangle \mu_\sigma) \geq 0\}$$

$$\mathcal{D}^- := \{\varphi \in H^1(\Gamma(S^M)) \cap H^{\frac{1}{2}}(\Gamma(S^\Sigma)) | \widetilde{D}^+\varphi = 0, on\ M, \widetilde{P}^+_{\geq 0}\psi = \xi, on\ \Sigma, \Re(\int_\Sigma \langle \varphi|_\Sigma, \widetilde{\Phi}_1^- \rangle \mu_\sigma) < 0\},$$

where $\Re$ and $\Im$ denote the real and imaginary parts of a $\mathbb{C}-$number, respectively. Here $\widetilde{P}^+_{\geq 0}$ denotes an appropriate spectral projection operator on Σ detail of which is defined in the section 4. The important point to note here is that these projection operators define new non-local boundary conditions that we shall use to solve the Dirac equation. The first theorem that we prove is that $\mathcal{D}^+ \cup \mathcal{D}^-$ is non-empty.

Theorem 1.1 (Existence of a unique twisted Dirac spinor) *Let $\Phi \in L^2(\Gamma(S^M))$ and $\alpha \in H^{\frac{1}{2}}(\Gamma(S^\Sigma))$. Let the scalar curvature of a $n+1$ dimensional smooth connected oriented Riemannian spin manifold (M, g) verify $R[g] \geq (n+1)n$ and the mean curvature K of its boundary (Σ, σ) with respect to the inward-pointing normal vector be strictly positive i.e., Σ is strictly mean convex in M and its scalar curvature verifies $R[\sigma] \geq (n-1)n$. Then the following boundary value problem*

$$\widetilde{D}^+\psi := (\widehat{D} - \frac{n+1}{2})\psi = \Phi, \tag{1.7}$$
$$\widetilde{P}^+_{\geq 0}\psi = \widetilde{P}^+_{\geq 0}\alpha, \tag{1.8}$$

has a unique solution $\psi \in H^1(\Gamma(S^M)) \cap H^{\frac{1}{2}}(\Gamma(S^\Sigma))$ that verifies

$$||\psi||_{H^1(\Gamma(S^M)) \cap H^{\frac{1}{2}}(\Gamma(S^\Sigma))} \leq C(n) \left(||\Phi||_{L^2(\Gamma(S^M))} + ||\widetilde{P}^+_{\geq 0}\alpha||_{H^{\frac{1}{2}}(\Gamma(S^\Sigma))} \right), \tag{1.9}$$

for a constant $C(n)$ depending on the geometry of (M, g) and (Σ, σ) and the dimension n.

Remark 1 *Note that One can also solve the equation $\widetilde{D}^-\psi = (\widehat{D} + \frac{n+1}{2})\psi = 0$ with a modified boundary condition $\widetilde{P}^-_{\geq 0}\psi = \widetilde{\xi}$ for $\widetilde{\xi} \in \Gamma(S^\Sigma)$ belonging to the eigen subspace of an appropriate Dirac operator (see section 4 for details).*

Since $\mathbb{R} = \mathbb{R}_{\geq 0} \cup \mathbb{R}_{<0}$, theorem 1.1 implies $\mathcal{D}^+ \cup \mathcal{D}^-$ is non-empty. The theorem 1.1 is the cornerstone of this article. We prove this in section 4. Using this theorem, we prove an improved eigenvalue estimate for the Dirac operator of the boundary Σ. But before we state the theorem, we need to define the following geometric entity

$$K_0 := \frac{2n}{\sqrt{\sup_{\psi \in \mathcal{D}^-, ||\widetilde{P}^+_{\geq 0}\psi||_{L^2(\Sigma)=1}} ||\psi||^2_{L^2(\Sigma)} - 1}} \ if\ \mathcal{D}^+\ is\ empty. \tag{1.10}$$

By theorem 1.1, if $\mathcal{D}^+$ is empty then $\mathcal{D}^-$ must be non-empty-then $\infty > ||\psi||_{L^2(\Sigma)} > 1$. We define the following ranges for the mean curvature K of the boundary Σ that we will be concerned with

$$\begin{aligned} \inf_\Sigma K &\geq K_0 \; if \; \mathcal{D}^+ \; is \; empty, && (1.11)\\ \inf_\Sigma K &> 0 \; if \; \mathcal{D}^+ \; is \; non-empty && (1.12) \end{aligned}$$

The vital point to note here is hat the non-emptiness of $\mathcal{D}^+ \cup \mathcal{D}^-$ depends only on the conditions $R[g] \geq n(n+1),\ R[\sigma] \geq (n-1)n,\ K > 0$. In particular K_0 is only relevant in the following theorems 1.2 and 1.3.

Theorem 1.2 (Eigenvalue Estimate) *Let (M, g) be an $n+1,\ n \geq 2$ dimensional connected oriented Riemannian spin manifold with smooth spin boundary Σ. Let the induced metric on Σ by g be σ. Assume the following:*
(a) The scalar curvature of (M, g) $R[g] \geq n(n+1)$,
(b) The scalar curvature of (Σ, σ) $R[\sigma] \geq n(n-1)$,
(c) Σ is strictly mean convex with respect to g with mean curvature K verifies the bound 1.11. Then the first eigenvalue λ_1 of the Dirac operator $\mathbf{D}$ on (Σ, σ) verifies the estimate

$$\lambda_1(\mathbf{D})^2 \geq \frac{1}{4}\inf_\Sigma K^2 + \frac{n^2}{4}. \tag{1.13}$$

This eigenvalue estimate is new (see [16] for an overview of the existing results; Montiel [24] attempted to obtain such an estimate but the inequality in theorem A of Montiel's article is impossible precisely because of the impossibility of prescribing a Dirichlet type boundary condition for Dirac equations on a compact manifold with boundary). In the second step, we use the eigenvalue estimate to prove a rigidity result for the geodesic balls of radius strictly less than $\frac{\pi}{2}$. In particular, we stress the fact that this rigidity fails for the geodesic ball of radius $\frac{\pi}{2}$ i.e., the upper hemisphere. The crucial assumption of the equality of the intrinsic and induced spin structure on the boundary Σ is verified automatically in dimensions greater than one. The lack of such a rigidity theorem for the hemisphere is due to the loss of both the injectivity (uniqueness) and the subjectivity (existence) property of the Dirac operator associated with the real Killing connection. The rigidity theorem is stated as follows

Theorem 1.3 (Rigidity of Geodesic Balls) *Let M be a $n+1,\ n \geq 2$ dimensional cap with boundary Σ and g be a smooth Riemannian metric on M that induces metric σ on Σ. Let the following conditions are verified by (M, g) and (Σ, σ)*
(a) Scalar curvature of (M, g) $R[g] \geq (n+1)n$,
(b) The induced metric σ on Σ by g agrees with the metric σ_0 of the boundary of a geodesic ball of standard unit sphere of radius less than or equal to $l := \frac{\pi}{2} - \epsilon$ for any $\epsilon > 0$ if $\mathcal{D}^+$ is non-empty and $\epsilon \geq \sin^{-1}\sqrt{\frac{K_0^2}{n^2+K_0^2}}$ if $\mathcal{D}^+$ is empty.
(c) The mean curvature of Σ with respect to g and the standard round metric on unit sphere $\mathbb{S}^{n+1}$ coincide, constant, and strictly positive,
Then g is isometric to the standard round metric on the unit sphere $\mathbb{S}^{n+1}$. In particular, M is isometric to a geodesic ball of radius l in a standard unit sphere.

Definition 1 *Let (M, g_0) be a geodesic ball of $\mathbb{S}^{n+1}$ of radius strictly less than $\frac{\pi}{2}$. We call a smooth deformation $\Psi : (M, g_0) \to (M, g)$ of (M, g_0) an admissible deformation if $R[g] \geq n(n+1), R[\sigma] \geq n(n-1)$ and the mean curvature K of the boundary Σ of M wit respect to g is strictly positive. σ is the metric induced on the boundary Σ of M by g.*

Remark 2 *The rigidity of spheres is conditional as per theorem 1.3. If we consider any geodesic ball* (M, g_0) *of radius strictly less than* $\frac{\pi}{2}$*, then there are two cases :*
(a) If the class of smooth admissible deformations $\{\Psi : (M, g_0) \to (M, g)\}$ *producing* (M, g) *admit non-empty* $\mathcal{D}^+$ *on* (M, g)*, then such deformations are necessarily isometry,*
(b) If the class of smooth admissible deformations $\{\Psi : (M, g_0) \to (M, g)\}$ *producing* (M, g) *admit empty* $\mathcal{D}^+$ *on* (M, g)*, then* $\mathcal{D}^-$ *is non-empty and such deformations are isometry if the mean curvature of the boundary* Σ *verifies the bound 1.11. Therefore not every geodesic ball of radius strictly less than* $\frac{\pi}{2}$ *verifies the rigidity property but there exists an upper bound on the radius. This is consistent with the result of [9] and [10, 13].*
Moreover, all such smooth admissible deformations produce non-empty $\mathcal{D}^+ \cup \mathcal{D}^-$ *due to theorem 1.1.*

This can be interpreted as the sharpest possible weaker version of the Min-Oo conjecture in a generic setting. The failure of rigidity of the hemisphere implies a lack of positive mass theorem in the strictly positive scalar curvature geometry. This has implications for the existence of black holes in de-Sitter spacetime in relativity. Compare this with the result of Brendle-Marques-Neves [10] and subsequent work by Cox-Miao-Tam [13]. The connection of the present result with that of [10] and [13] remains to be understood. Moreover, using the result stated in the theorem 1.3 together with the rigidity result stated in [10] (and the improved one in [13]), one can obtain a definite bound on the norm of the twisted Harmonic spinors on geodesic balls of $\mathbb{S}^{n+1}$ (i.e., the spinors that solve the Dirac equation associated with the real Killing connection).
Lastly, we define a notion of mass for geometries with positive scalar curvature. Unlike the positive mass theorem with reference space flat or hyperbolic space [1, 26], the rigidity property of this new mass spans over Einstein spaces admitting a real Killing connection. Alternatively, in light of the theorems 1.2 and 1.3, one may perceive the difference between the first eigenvalues of the Dirac operator of the boundary of (M, g) and that of the geodesic balls (of appropriate radius depending on the emptiness of $\mathcal{D}^+$) of $\mathbb{S}^{n+1}$. But that definition turns out to be too restrictive in the sense that spherical geometry is not rigid uniformly and rigidity depends on the diameter of the boundary contrary to flat or hyperbolic space. To define the mass, we introduce the following non-self-adjoint boundary Dirac operators

$$\widetilde{\mathbf{D}}^+ := \mathbf{D} + \frac{n}{2}\rho(\nu), \ \widetilde{\mathbf{D}}^- := \mathbf{D} - \frac{n}{2}\rho(\nu), \tag{1.14}$$

where $\mathbf{D}$ is the boundary Dirac operator on Σ induced by (M, g), ρ is the Clifford multiplication operator, and ν is the inward pointing unit normal vector to Σ (the details are in section 3).

Theorem 1.4 (A spinor mass for positive geometry) *Let* (M, g) *be an* $n + 1$, $n \geq 2$ *dimensional connected oriented Riemannian spin manifold with smooth spin boundary* Σ*. Let the induced metric on* Σ *by* g *be* σ*. Assume the following:*
(a) The scalar curvature of (M, g) $R[g] \geq n(n + 1)$*,*
(b) The scalar curvature of (Σ, σ) $R[\sigma] \geq n(n - 1)$*,*
(c) Σ *is strictly mean convex with respect to* g*. Then the mass* m *of* (M, g) *is defined as*

$$m := \inf_{\varphi \in \mathcal{D}^+ \cup \mathcal{D}^-} \int_\Sigma \langle \widetilde{\mathbf{D}}^\pm \varphi - \frac{1}{2} K\varphi, \varphi \rangle \mu_\sigma \tag{1.15}$$

verifies

$$m \geq 0 \tag{1.16}$$

and $m = 0$ *if and only if* (M, g) *is an Einstein manifold that admits a real Killing spinor i.e., belongs to the classification of Bär [5].*

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2. Preliminaries

We denote by (M, g) an $n+1,\ n \geq 2$ dimensional smooth connected Riemannian spin manifold. Let (Σ, σ) be the smooth boundary of M and σ the induced Riemannian metric on it by g. Let the spin structure of M be denoted by Spin_M. Let $\{e_I\}_{I=1}^{n+1}$ be an orthonormal basis for the $SO(n+1)-$ frame bundle of (M, g). The last basis vector e_{n+1} is set to be the interior pointing normal vector to the boundary (Σ, σ) and is globally defined on (M, g) along (Σ, σ) and denote it by ν. This allows us to induce a spin structure on (Σ, σ) from Spin_M.
Let $\mathbb{C}l_n$ be the n dimensional complex Clifford algebra and its even part is denoted by $\mathbb{C}l_n^0$. Using ν we can induce an isomorphism between $\mathbb{C}l_n$ and $\mathbb{C}l_{n+1}^0$ as follows

$$\mathcal{I} : \mathbb{C}l_n \to \mathbb{C}l_{n+1}^0 \tag{2.1}$$

$$e_I \mapsto e_I \cdot \nu,\ I = 1, ..., n. \tag{2.2}$$

Now the principal $SO(n)$ bundle of oriented orthonormal frames on Σ can be identified as a sub-bundle of $SO(n+1)$ bundle on M along Σ that leaves ν invariant. Let $\mathcal{J} : SO(n)_\Sigma \hookrightarrow SO(n+1)_M|_\Sigma$ denote such an identification map. We can pull back the fiber bundle the spin bundle $\mathrm{Spin}_M|_\Sigma$ on $SO(n+1)_M|_\Sigma$ as a spin structure on Σ. Let us denote this induced spin bundle on Σ by Spin_Σ. Note the commutative diagram below.

$$\begin{array}{ccc} \mathrm{Spin}_\Sigma & \xrightarrow{\mathcal{J}^*} & \mathrm{Spin}_M|_\Sigma \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ SO(n)_\Sigma & \xrightarrow{\mathcal{J}} & SO(n+1)_M|_\Sigma \end{array}$$

Let S^M denote the spinor bundle of (M, g). As a bundle S^M may be identified with the trivial bundle $\mathrm{Spin}_M \times \Delta_{n+1} \to \Delta_{n+1}$, Δ_{n+1} is the space of spinors, of complex dimension $2^{\lfloor \frac{n+1}{2} \rfloor}$. Each tangent vector $X \in T_x M$ induces a skew-adjoint map $\rho(X) \in \mathrm{End}(S_x^M)$. For the orthonormal basis $\{e_I\}_{I=1}^{n+1}$, Clifford relation holds

$$\rho(e_I)\rho(e_J) + \rho(e_J)\rho(e_I) = -2\delta_{IJ}\mathrm{id} \tag{2.3}$$

for $i, j = 1, ..., n+1$. For $n+1$ odd, the representation ρ is chosen such that the complex volume form acts as identity on Δ_{n+1}. Locally a section ψ of the spinor bundle S^M over an open set $M \subset M$ may be represented by the double

$$\psi = [s, \beta], \tag{2.4}$$

where $s : M \to \mathrm{Spin}_M$ and $\beta : M \to \Delta_{n+1}$. For an element $A \in \mathrm{Spin}(n+1)_M$, the gauge transformation is written as

$$s \mapsto sA,\ \beta \mapsto \rho(A^{-1})\beta. \tag{2.5}$$

One may restrict the spinor bundle S^M on Σ by using the $\mathcal{I}$ map i.e., define the restricted bundle as $S^M|_\Sigma := \mathrm{Spin}_\Sigma \times_{\rho\circ\mathcal{I}} \Delta$. The Clifford multiplication on Σ is defined as follows. For $X \in \Gamma(T\Sigma)$, the Clifford multiplication $\rho^\Sigma(X) = \rho(X) \circ \rho(\nu)$. The intrinsic spinor bundle of Σ $S^\Sigma = \mathrm{Spin}_\Sigma \times_{\rho^\Sigma} \Delta_n$ can be identified with $S^M|_\Sigma$ if $n+1$ is odd. For $n+1$ even, one may obtain the identification $S^M|_\Sigma \simeq S^\Sigma \oplus S^\Sigma$. This implicitly uses the assumption that the intrinsic spin structure on Σ and that induced from (M, g) are the same. For details about the theory of induced spin structures, readers are referred to [18, 7].
Now we discuss the extrinsic Riemannian geometry of (Σ, σ) as a hypersurface in (M, g). Let $\widehat{\nabla}$ be the metric compatible connection on (M, g) and ∇ is the induced connection on Σ. For two smooth vector fields $A, B \in T\Sigma$, $\widehat{\nabla}$ and ∇ are related by

$$\widehat{\nabla}_A B = \nabla_A B + \mathcal{H}(A, B)\nu, \tag{2.6}$$

where ν is interior directed unit normal vector to the boundary Σ and $\mathcal{H}$ is the second fundamental form of Σ in M.
On the spinor bundle S^M there is a natural Hermitian inner product $\langle\cdot,\cdot\rangle$ that is compatible with the connection $\widehat{\nabla}$ (we denote the Levi-Civita connection and its spin connection by the same symbol for convenience) and Clifford multiplication $\rho : Cl(M, \mathbb{C}) \to \mathrm{End}(S^M)$. We denote a spinor as a section of the spin bundle by $\psi \in \Gamma(S^M)$. Naturally, its restriction to Σ is a section of S^Σ by previous discussion. The following relations holds $\forall X, Y \in \Gamma(TM)$

$$\begin{aligned} X\langle\psi_1, \psi_2\rangle &= \langle\widehat{\nabla}_X\psi_1, \psi_2\rangle + \langle\psi_1, \widehat{\nabla}_X\psi_2\rangle & (2.7)\\ \widehat{\nabla}_X(\rho(Y)\psi) &= \rho(\widehat{\nabla}_X Y)\psi + \rho(Y)\widehat{\nabla}_X Y & (2.8)\\ \langle\rho(X)\psi_1, \psi_2\rangle &= -\langle\psi_1, \rho(X)\psi_2\rangle & (2.9)\\ \langle\rho(X)\psi_1, \rho(X)\psi_2\rangle &= g(X, X)\langle\psi_1, \psi_2\rangle. & (2.10)\end{aligned}$$

These are the intrinsic identities that are verified by the geometric entities on (M, g). Intrinsically, we can define the Dirac operator on M as the composition map

$$\Gamma(S^M) \xrightarrow{\widehat{\nabla}} \Gamma(T^*M \otimes S^M) \xrightarrow{\rho} \Gamma(S^M). \tag{2.11}$$

In local coordinates, the above decomposition gives the Dirac operator $\widehat{D}$

$$\widehat{D} := \sum_{I=1}^{n+1} \rho(e_I)\widehat{\nabla}_{e_I}. \tag{2.12}$$

The boundary map $i : \Sigma \hookrightarrow M$, induces a connection on Σ that verifies

$$\widehat{\nabla}_X Y = \nabla_X Y + \mathcal{H}(X, Y)\nu, \tag{2.13}$$

where $X, Y \in \Gamma(T\Sigma)$ and $\mathcal{H}$ is the second fundamental form of Σ while viewed as an embedded hypersurface in M. Now this split extends to the spinor bundles. The spin connection splits as follows

$$\widehat{\nabla}_X\psi = \nabla_X\psi + \frac{1}{2}\rho(\mathcal{H}(X, \cdot))\rho(\nu)\psi \tag{2.14}$$

for $X \in \Gamma(T\Sigma)$. With this split, we can write the hypersurface Dirac operator in terms of the bulk Dirac operator and the second fundamental form. We have the following proposition relating the Dirac operator of M and that of its boundary Σ

Proposition 2.1 *Let Σ be the smooth boundary of a smooth connected oriented $n+1$ dimensional Riemannian spin manifold (M,g) and H be the mean curvature of Σ in M. Under the identification $S^M|_\Sigma \simeq S^\Sigma$ or $S^M|_\Sigma \simeq S^\Sigma \oplus S^\Sigma$, The boundary Dirac operator $\mathbf{D}$ on Σ induced by the Riemannian structure of (Σ,σ) and the bulk Dirac operator $\widehat{D}$ on M verify*

$$\mathbf{D}\psi = -\rho(\nu)\widehat{D}\psi - \widehat{\nabla}_\nu\psi + \frac{1}{2}H\psi \tag{2.15}$$

for a C^∞ section ψ of the bundle S^M. Here H is the mean curvature (trace of the second fundamental form $\mathcal{H}$) of Σ in M defined with respect to ν.

Proof. The proof is a result of straightforward calculations. Let $\psi \in C^\infty(S^M)$. We denote the restriction of ψ to the boundary Σ by ψ as well for notational convenience. Consider the orthonormal frame $\{e_I\}_{I=1}^{n+1}$ on M along Σ such that $\{e_I\}_{I=1}^{n}$ are tangent to Σ while $e_{n+1} = \nu$ is the interior pointing normal to Σ. The hypersurface Dirac operator $\mathbf{D}$ is defined as follows

$$\begin{aligned}
\mathbf{D}\psi &= \sum_{I=1}^{n}\rho(e_I)\rho(\nu)\nabla_{e_I}\psi \qquad (2.16)\\
&= \sum_{I=1}^{n}\rho(e_I)\rho(\nu)\left(\widehat{\nabla}_{e_I}\psi - \frac{1}{2}\rho(\mathcal{H}(e_I,\cdot))\rho(\nu)\psi\right)\\
&= \sum_{I=1}^{n}\rho(e_I)\rho(\nu)\widehat{\nabla}_{e_I}\psi - \frac{1}{2}\sum_{I=1}^{n}\rho(e_I)c(\nu)\rho(\mathcal{H}(e_I,\cdot))\rho(\nu)\psi
\end{aligned}$$

Now observe the following ($\mathcal{H}(e_I,\cdot) \in T^*\Sigma, I = 1,...,n$ and therefore $g(\mathcal{H}(e_I,\cdot),\nu) = 0$ and therefore

$$\begin{aligned}
\mathbf{D}\psi &= \sum_{I=1}^{n}c(e_I)c(\nu)\widehat{\nabla}_{e_I}\psi + \frac{1}{2}\sum_{I=1}^{n}\rho(e_I)\rho(\nu)\rho(\nu)\rho(\mathcal{H}(e_I,\cdot))\psi \qquad (2.17)\\
&= \sum_{I=1}^{n}\rho(e_I)\rho(\nu)\widehat{\nabla}_{e_I}\psi + \frac{1}{2}H\psi\\
&= -\sum_{I=1}^{n}\rho(\nu)\rho(e_I)\widehat{\nabla}_{e_I}\psi + \frac{1}{2}H\psi\\
&= -\rho(\nu)\sum_{I=1}^{n+1}\rho(e_I)\widehat{\nabla}_{e_I}\psi + \rho(\nu)\rho(\nu)\widehat{\nabla}_\nu\psi + \frac{1}{2}H\psi\\
&= -\rho(\nu)\widehat{D}\psi - \widehat{\nabla}_\nu\psi + \frac{1}{2}H\psi,
\end{aligned}$$

where we have used the Clifford algebra relation 2.3 and the mean curvature $H = \mathrm{tr}\mathcal{H}$ and is defined with respect to the inward-pointing normal to Σ. □

Remark 3 *Note the following relation between the induced Dirac operator $\mathbf{D}$ on Σ and the intrinsic Dirac operator D on Σ. If Σ is even dimensional, then the induced spinor bundle $S^M|_\Sigma \simeq S^\Sigma$ and correspondingly $\mathbf{D} = D$, spec$(\mathbf{D})$ = spec(D). If the dimension of Σ is odd, then $S^M|_\Sigma \simeq S^\Sigma \oplus S^\Sigma$ and correspondingly $\mathbf{D} = D \oplus -D$, spec$(\mathbf{D})$ = spec$(D) \cup -$spec(D).*

In addition to the relationship between the extrinsic and intrinsic Dirac operators, we have the usual Schrödinger-Lichnerowicz formula for the Dirac operator $\widehat{D}$ on M

$$\widehat{D}^2 = -\widehat{\nabla}^2 + \frac{1}{4}R[g], \tag{2.18}$$

where $R[g]$ is the scalar curvature of (M,g). As an obvious consequence, one observes that a closed spin manifold with strictly positive scalar curvature can not have non-trivial harmonic spinors. We explicitly work with L^2 Sobolev spaces on M defined with respect to the Riemannian metric g. We denote by $W^{s,2}(S^M)$ the Sobolev space of spinors on M whose first s weak derivatives are in $L^2(M)$. $W^{s,2}$ completes C_c^∞ i.e., space of compactly supported sections of S^M with respect to the norm

$$||\psi||_{W^{s,2}} := ||\psi||_{L^2(M)} + \sum_{I=1}^{s} ||\widehat{\nabla}^I \psi||_{L^2(M)}, \tag{2.19}$$

where

$$||\psi||^2_{L^2(M)} := \int_M \langle \psi, \psi \rangle \mu_g, \tag{2.20}$$

$$||\widehat{\nabla}^I \psi||^2_{L^2(M)} := \int_M g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} \dots . g^{i_I j_I} \langle \widehat{\nabla}_{i_1} \widehat{\nabla}_{i_2} \widehat{\nabla}_{i_3} \dots . \widehat{\nabla}_{i_I} \psi, \widehat{\nabla}_{j_1} \widehat{\nabla}_{j_2} \widehat{\nabla}_{j_3} \dots . \widehat{\nabla}_{j_I} \psi \rangle \mu_g.$$

For a manifold with a boundary such as M, we also need to be concerned about the boundary regularity of the spinor. First, recall that all functions belonging to $W^{s,2}$, $s \in \mathbb{N}$ have traces on smooth boundary Σ. This passes to the spinors since one may think of $|\psi|^2 := \langle \psi, \psi \rangle$ as a function on (M,g) and the spin connection $\widehat{\nabla}$ is compatible with the Hermitian inner product $\langle \cdot, \cdot \rangle$. More precisely there is a trace-map

$$\text{Tr} : W^{s,2}(M) \to W^{s-\frac{1}{2},2}(\Sigma), \;\; s \in \mathbb{N} \tag{2.21}$$

that is bounded and surjective. We will mostly be concerned with spinors that are in $W^{s,2}(M) \cap W^{s-\frac{1}{2},2}(\Sigma)$, $s \geq 1$. Oftentimes, we will denote $W^{s,2}$ by H^s for convenience. The background geometry is always assumed to be smooth.

3. Spinors on positive scalar curvature geometry

A spinor field $\psi \in \Gamma(S^M)$ is called a Killing spinor with Killing constant α if for all vectors X tangent to M, the following equation is verified

$$\widehat{\nabla}_X \psi = \alpha \rho(X) \psi. \tag{3.1}$$

If (M,g) carries a Killing spinor, the first integrability condition of 3.1 forces (M,g) to be an Einstein manifold with Ricci curvature $\text{Ric}[g] = 4n\alpha^2 g$. We have three cases: α can be real and non-zero, then M is compact and ψ is called a *real* Killing spinor, α can be purely imaginary, then M is non-compact and ψ is called imaginary Killing spinor, and α can be 0 in which case α is called a parallel spinor. The manifolds with real Killing spinors have been classified by Bär [5]. A prototypical example is the unit sphere with the standard round metric. Hitchin [19] showed that manifolds with parallel spinor fields can be characterized by their holonomy groups. Baum [8] classified the manifolds with imaginary Killing spinors while Rademacher later extended the classification to include generalized imaginary Killing spinors where α is allowed to be purely imaginary functions.
In this article, we are concerned with real Killing spinors with Killing constant $\alpha = \pm\frac{1}{2}$. On (M,g), let us define a modified connection called Killing connection á la Bär [5]

$$\widetilde{\nabla}_X \psi = \widehat{\nabla}_X \psi + \alpha \rho(X) \psi, \tag{3.2}$$

for $X \in C^\infty(\Gamma(TM))$, $\psi \in C^\infty(\Gamma(S^M))$. The corresponding Dirac operator $\widetilde{D}$ reads

$$\widetilde{D} = \widehat{D} - (n+1)\alpha. \tag{3.3}$$

From now on we fix $\alpha = \pm\frac{1}{2}$. Define $\widetilde{D}^\pm$ to be

$$\widetilde{D}^\pm = \widehat{D} \mp \frac{n+1}{2}. \tag{3.4}$$

We have the following proposition

Proposition 3.1 *Let (M, g) be a $n+1$, $n \geq 2$ dimensional smooth connected oriented Riemannian spin manifold and $\psi \in C^\infty(\Gamma(S^M))$. Let $R[g]$ be the scalar curvature of (M, g) and K be the mean curvature of the boundary (Σ, σ) of (M, g) with respect to the interior pointing unit normal vector ν. Then the following Witten-type identity is verified by ψ*

$$\begin{aligned}\int_\Sigma \left(\langle \widetilde{\mathbf{D}}^\pm\psi, \psi\rangle - \frac{1}{2}K|\psi|^2\right)\mu_\sigma &= \frac{1}{4}\int_M \left(R[g] - n(n+1)\right)|\psi|^2\mu_M - \frac{n}{n+1}\int_M \langle \widetilde{D}^\pm\psi, \widetilde{D}^\mp\psi\rangle \\ &\quad + \int_M |\mathcal{Q}\psi|^2,\end{aligned}$$

where $\mathcal{Q}_X$ is the Penrose operator defined as

$$\mathcal{Q}_X := \widehat{\nabla}_X + \frac{1}{n+1}\rho(X)\widehat{D} \tag{3.5}$$

for $X \in C^\infty(\Gamma(TM))$ and the modified hypersurface Dirac operator $\widetilde{\mathbf{D}}^\pm$ is defined as

$$\widetilde{\mathbf{D}}^\pm := \mathbf{D} \pm \frac{n}{2}\rho(\nu). \tag{3.6}$$

Proof. We prove every identity assuming the data are in C^∞. First, we prove the following well-known identity of the written type. This is standard, nevertheless, we present it for completeness

$$\int_\Sigma \left(\langle \mathbf{D}\psi, \psi\rangle - \frac{1}{2}H|\psi|^2\right)\mu_\Sigma = \int_M \left(\frac{1}{4}R[g]|\psi|^2 - |\widehat{D}\psi|^2 + |\widehat{\nabla}\psi|^2\right)\mu_M. \tag{3.7}$$

First, recall the Schrödinger-Lichnerowicz identity on M

$$\widehat{D}^2\psi = -\widehat{\nabla}^2\psi + \frac{1}{4}R[g]\psi, \tag{3.8}$$

for a spinor $\psi \in C^\infty(\Gamma(S^M))$. Let us consider an orthonormal frame $\{e_I\}_{I=1}^{n+1}$ on M with e_{n+1} being the unit inside pointing normal ν of the boundary surface $\partial M = \Sigma$. Now we evaluate the following using the compatibility property of the Hermitian inner product $\langle\cdot,\cdot\rangle$ on the spin bundle S^M

$$\begin{aligned}&\sum_{I=1}^{n+1}\widehat{\nabla}_I\langle\rho(e_I)\widehat{D}\psi, \psi\rangle + \sum_{I=1}^{n+1}\widehat{\nabla}_I\langle\widehat{\nabla}_I\psi, \psi\rangle \qquad (3.9)\\ &= \sum_{I=1}^{n+1}\langle\rho(e_I)\widehat{\nabla}_I\widehat{D}\psi, \psi\rangle + \sum_{I=1}^{n+1}\langle\rho(e_I)\widehat{D}\psi, \widehat{\nabla}_I\psi\rangle + \langle\widehat{\nabla}^2\psi, \psi\rangle + \sum_{I=1}^{n+1}\langle\widehat{\nabla}_I\psi, \widehat{\nabla}_I\psi\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \widehat{D}^2\psi,\psi\rangle - \sum_{I=1}^{n+1}\langle \widehat{D}\psi, \rho(e_I)\widehat{\nabla}_I\psi\rangle + \langle \widehat{\nabla}^2\psi,\psi\rangle + \sum_{I=1}^{n+1}\langle \widehat{\nabla}_I\psi, \widehat{\nabla}_I\psi\rangle \\
&= \langle \widehat{D}^2\psi,\psi\rangle - \langle \widehat{D}\psi, \widehat{D}\psi\rangle + \langle \widehat{\nabla}^2\psi,\psi\rangle + \sum_{I=1}^{n+1}\langle \widehat{\nabla}_I\psi, \widehat{\nabla}_I\psi\rangle,
\end{aligned}
$$

where we have used the property 2.7 and 2.9 along with the fact that $\sum_{I=1}^{n+1}\widehat{\nabla}_I(\rho(e_I)\widehat{D}\psi) = \sum_{I=1}^{n+1}\rho(e_I)\widehat{\nabla}_I\widehat{D}\psi$. Now we integrate the identity

$$
\sum_{I=1}^{n+1}\widehat{\nabla}_I\langle \rho(e_I)\widehat{D}\psi,\psi\rangle + \widehat{\nabla}_I\langle \widehat{\nabla}_I\psi,\psi\rangle = \langle \widehat{D}^2\psi,\psi\rangle - \langle \widehat{D}\psi,\widehat{D}\psi\rangle + \langle \widehat{\nabla}^2\psi,\psi\rangle + \sum_{I=1}^{n+1}\langle \widehat{\nabla}_I\psi,\widehat{\nabla}_I\psi\rangle
$$

over M to obtain

$$
\begin{aligned}
&\int_M \left(\sum_{I=1}^{n+1}\widehat{\nabla}_I\langle c(e_I)\widehat{D}\psi,\psi\rangle + \sum_{I=1}^{n+1}\widehat{\nabla}_I\langle \widehat{\nabla}_I\psi,\psi\rangle\right)\mu_M \qquad (3.10)\\
&= \int_M \left(\langle \widehat{D}^2\psi,\psi\rangle - \langle \widehat{D}\psi,\widehat{D}\psi\rangle + \langle \widehat{\nabla}^2\psi,\psi\rangle + \sum_{I=1}^{n+1}\langle \widehat{\nabla}_I\psi,\widehat{\nabla}_I\psi\rangle\right)\mu_M
\end{aligned}
$$

which by Stokes's theorem yields

$$
-\int_\Sigma \left(\langle c(\nu)\widehat{D}\psi,\psi\rangle + \langle \widehat{\nabla}_\nu\psi,\psi\rangle\right)\mu_\Sigma = \int_M \left(\frac{1}{4}R[g]|\psi|^2 - |\widehat{D}\psi|^2 + |\widehat{\nabla}\psi|^2\right)\mu_M. \qquad (3.11)
$$

Using the proposition 2.1 relating the Dirac operator D on Σ and the Dirac operator $\widehat{D}$ on M, we obtain the desired identity

$$
\int_\Sigma \left(\langle \mathbf{D}\psi,\psi\rangle - \frac{1}{2}K|\psi|^2\right)\mu_\Sigma = \int_M \left(\frac{1}{4}R[g]|\psi|^2 - |\widehat{D}\psi|^2 + |\widehat{\nabla}\psi|^2\right)\mu_M. \qquad (3.12)
$$

An improved version of 3.12 can be obtained if we express $|\widehat{\nabla}\psi|^2$ in terms of the Penrose operator $\mathcal{Q}$ i.e.,

$$
|\widehat{\nabla}\psi|^2 = |\mathcal{Q}\psi|^2 + \frac{1}{n+1}|\widehat{D}\psi|^2, \qquad (3.13)
$$

where $\mathcal{Q}_X\psi := \widehat{\nabla}_X\psi + \frac{1}{n+1}\rho(X)\widehat{D}\psi$. This follows due to the following straightforward computations

$$
\begin{aligned}
|\mathcal{Q}\psi|^2 &= \sum_{I=1}^{n+1}\langle \widehat{\nabla}_{e_I}\psi + \frac{\rho(e_I)}{n+1}\widehat{D}\psi, \widehat{\nabla}_{e_I}\psi + \frac{\rho(e_I)}{n+1}\widehat{D}\psi\rangle \qquad (3.14)\\
&= |\widehat{\nabla}\psi|^2 + \frac{1}{n+1}\sum_{I=1}^{n+1}\langle \rho(e_I)\widehat{D}\psi, \widehat{\nabla}_{e_I}\psi\rangle + \frac{1}{n+1}\sum_{I=1}^{n+1}\langle \widehat{\nabla}_{e_I}\psi, \rho(e_I)\widehat{D}\psi\rangle \\
&\quad + \frac{1}{(n+1)^2}\sum_{I=1}^{n+1}\langle \rho(e_I)\widehat{D}\psi, \rho(e_I)\widehat{D}\psi\rangle \\
&= |\widehat{\nabla}\psi|^2 - \frac{1}{n+1}|\widehat{D}\psi|^2
\end{aligned}
$$

by the relation 2.9. This yields

$$\int_{\Sigma}\left(\langle \mathbf{D}\psi,\psi\rangle-\frac{1}{2}K|\psi|^2\right)\mu_{\Sigma}=\int_M\left(\frac{1}{4}R[g]|\psi|^2-\frac{n}{n+1}|\widehat{D}\psi|^2+|\mathcal{Q}\psi|^2\right)\mu_M. \tag{3.15}$$

Recall

$$\widetilde{D}^{+}:=\widehat{D}-\frac{n+1}{2},\ \ \widetilde{D}^{+}:=\widehat{D}+\frac{n+1}{2} \tag{3.16}$$

and note the following point-wise identity for the hermitian inner product on the spinor bundle

$$\begin{aligned}\langle \widetilde{D}^{+}\psi,\widetilde{D}^{-}\psi\rangle &= \langle(\widehat{D}-\frac{n+1}{2})\psi,(\widehat{D}+\frac{n+1}{2})\psi\rangle \\ &= |\widehat{D}\psi|^2+\frac{n+1}{2}\langle\widehat{D}\psi,\psi\rangle-\frac{n+1}{2}\langle\psi,\widehat{D}\psi\rangle-\frac{(n+1)^2}{4}|\psi|^2\end{aligned} \tag{3.17}$$

yields an expression for the Dirac operator

$$|\widehat{D}\psi|^2=\langle\widetilde{D}^{+}\psi,\widetilde{D}^{-}\psi\rangle-\frac{n+1}{2}\langle\widehat{D}\psi,\psi\rangle+\frac{n+1}{2}\langle\psi,\widehat{D}\psi\rangle+\frac{(n+1)^2}{4}|\psi|^2 \tag{3.18}$$

which while substituted into the Witten identity yields

$$\begin{aligned}\int_{\Sigma}\left(\langle \mathbf{D}\psi,\psi\rangle-\frac{1}{2}K|\psi|^2\right)\mu_{\sigma} &= \frac{1}{4}\int_M R[g]|\psi|^2\mu_M-\frac{n}{n+1}\int_M\Bigg(\langle\widetilde{D}^{+}\psi,\widetilde{D}^{-}\psi\rangle-\frac{n+1}{2}\langle\widehat{D}\psi,\psi\rangle \\ &\quad +\frac{n+1}{2}\langle\psi,\widehat{D}\psi\rangle+\frac{(n+1)^2}{4}|\psi|^2\Bigg)+\int_M|\mathcal{Q}\psi|^2\mu_M.\end{aligned} \tag{3.19}$$

Now recall the following identity as the failure of self-adjointness of $\widehat{D}$ on a manifold with boundary (follows by integration by parts and Stokes)

$$\int_M\langle\widehat{D}\psi,\psi\rangle=\int_M\langle\psi,\widehat{D}\psi\rangle+\int_{\Sigma}\langle\psi,\rho(\nu)\psi\rangle \tag{3.20}$$

which reduces the previous expression 3.19 to the following desired form

$$\begin{aligned}\int_{\Sigma}\left(\langle \widetilde{\mathbf{D}}^{+}\psi,\psi\rangle-\frac{1}{2}K|\psi|^2\right)\mu_{\sigma} &= \frac{1}{4}\int_M\big(R[g]-n(n+1)\big)|\psi|^2\mu_M-\frac{n}{n+1}\int_M\langle\widetilde{D}^{+}\psi,\widetilde{D}^{-}\psi\rangle \\ &\quad +\int_M|\mathcal{Q}\psi|^2\end{aligned} \tag{3.21}$$

with $\widetilde{\mathbf{D}}^{+}:=\mathbf{D}+\frac{n}{2}\rho(\nu)$. Similarly repeating the computations with

$$\begin{aligned}\langle \widetilde{D}^{-}\psi,\widetilde{D}^{+}\psi\rangle &= \langle(\widehat{D}+\frac{n+1}{2})\psi,(\widehat{D}-\frac{n+1}{2})\psi\rangle \\ &= |\widehat{D}\psi|^2-\frac{n+1}{2}\langle\widehat{D}\psi,\psi\rangle+\frac{n+1}{2}\langle\psi,\widehat{D}\psi\rangle-\frac{(n+1)^2}{4}|\psi|^2\end{aligned} \tag{3.22}$$

yields

$$\begin{aligned}\int_{\Sigma}\left(\langle \widetilde{\mathbf{D}}^{-}\psi,\psi\rangle-\frac{1}{2}K|\psi|^2\right)\mu_{\sigma} &= \frac{1}{4}\int_M\big(R[g]-n(n+1)\big)|\psi|^2\mu_M-\frac{n}{n+1}\int_M\langle\widetilde{D}^{-}\psi,\widetilde{D}^{+}\psi\rangle \\ &\quad +\int_M|\mathcal{Q}\psi|^2\end{aligned} \tag{3.23}$$

with $\widetilde{\mathbf{D}}^{-}:=\mathbf{D}-\frac{n}{2}\rho(\nu)$. This completes the proof of the lemma. □

Remark 4 *The eigenvalues of the modified operators $\widetilde{\mathbf{D}}^{\pm}$ can be imaginary in general since they are not self-adjoint. In particular, Adj$(\widetilde{\mathbf{D}}^{+}) = \widetilde{\mathbf{D}}^{-}$ and vice versa. However, it turns out if $\lambda(\mathbf{D})^2 \geq \frac{n^2}{4}$, then $\widetilde{\lambda}(\widetilde{\mathbf{D}}^{\pm})^2 \geq 0$, where λ denotes the eigenvalues of $\mathbf{D}$ and $\widetilde{\lambda}$ that of $\widetilde{\mathbf{D}}^{\pm}$. We prove this now.*

First, we prove an eigenvalue estimate for the intrinsic Dirac operator $\mathbf{D}$ on Σ assuming a curvature condition $R[\sigma] \geq (n-1)n$. This is well known [14].

Lemma 3.1 (Eigenvalue estimate of D on Σ) *Let the scalar curvature of a closed n dimensional smooth Riemannian spin manifold Σ verify $R[\sigma] \geq n(n-1)$. Then the first eigenvalue $\lambda_1(\mathbf{D})$ of the Dirac operator $\mathbf{D}$ on Σ verifies*

$$\lambda_1^2(\mathbf{D}) \geq \frac{n^2}{4}. \tag{3.24}$$

Proof. First, note the remark 3 about the relation between the induced Dirac operator $\mathbf{D}$ and the intrinsic Dirac operator D of Σ. At the level of square of the eigenvalue estimate, it therefore makes no difference to consider $\mathbf{D}$ instead of D on Σ. Also, recall the Witten identity for Σ. Since Σ is closed, then for any $\psi \in C^\infty(\Gamma(S^\Sigma))$

$$\int_\Sigma \left(\frac{1}{4}R[\sigma]|\psi|^2 - \frac{(n-1)|\mathbf{D}\psi|^2}{n}\right)\mu_\Sigma + \int_\Sigma |\mathcal{Q}\psi|^2 = 0 \tag{3.25}$$

Now let $\mathbf{D}\psi = \lambda_1(\mathbf{D})\psi$. λ_1 can't be zero since by the scalar curvature condition, (Σ, σ) does not admit any non-trivial harmonic spinor. In fact, the spectrum of $\mathbf{D}$ is unbounded discrete, and symmetric with respect to zero. Then

$$\int_\Sigma \left(\lambda_1(\mathbf{D})^2 - \frac{n}{4(n-1)}R[\sigma]\right)|\psi|^2 = \frac{n}{n-1}\int_\Sigma |\mathcal{Q}\psi|^2 \geq 0 \tag{3.26}$$

yielding $\lambda_1^2 \geq \frac{n}{4(n-1)} \inf_\Sigma R[\sigma] = \frac{n^2}{4}$. □

Now we construct eigenspinors of $\widetilde{\mathbf{D}}^{\pm}$. Let $\widetilde{\lambda}$ be an eigenvalue of $\widetilde{\mathbf{D}}$ i.e.,

$$\widetilde{\mathbf{D}}^{+}\Phi^{+} = \widetilde{\lambda}\Phi^{+}. \tag{3.27}$$

Lemma 3.2 *Let $\widetilde{\lambda}$ be an eigenvalue of $\widetilde{\mathbf{D}}^{+}$ with eigenspinor Φ^{+}. Define $\lambda^2 = \widetilde{\lambda}^2 + \frac{n^2}{4}$. Then*

$$\varphi := \left(\frac{n^2}{4} + (\widetilde{\lambda} - \lambda)^2\right)^{-1}\left[\frac{n}{2}\Phi^{+} + (\widetilde{\lambda} - \lambda)\rho(\nu)\Phi^{+}\right] \tag{3.28}$$

verifies the eigenvalue equation

$$\mathbf{D}\varphi = \lambda\varphi. \tag{3.29}$$

Proof. Consider a spinor $\psi \in C^\infty(\Gamma(S^\Sigma))$. First, note that $\frac{n^2}{4} + (\widetilde{\lambda} - \lambda)^2 \neq 0$ since $\lambda^2 = \widetilde{\lambda}^2 + \frac{n^2}{4}$. We prove the claim by computation and exploiting the following two properties of the boundary Dirac operator $\mathbf{D}$ and the boundary spin connection ∇ induced by (M, g). First, note that

$$[\nabla_X, \rho(\nu)] = 0 \;\forall\; X \in \Gamma(T\Sigma). \tag{3.30}$$

A proof of this may be as follows

$$\begin{aligned}
\nabla_X(\rho(\nu)\psi) &= (\widehat{\nabla}_X - \frac{1}{2}\rho(K(\cdot,X))\rho(\nu))\rho(\nu)\psi \\
&= \widehat{\nabla}_X(\rho(\nu)\varphi) - \frac{1}{2}\rho(K(\cdot,X))\rho(\nu)\rho(\nu)\psi \\
&= \rho(\widehat{\nabla}_X\nu)\psi + \rho(\nu)\widehat{\nabla}_X\varphi + \frac{1}{2}\rho(K(\cdot,X))\psi \\
&= -\rho(K(\cdot,X))\psi + \rho(\nu)\widehat{\nabla}_X\varphi + \frac{1}{2}\rho(K(\cdot,X))\psi \\
&= \rho(\nu)\widehat{\nabla}_X\psi - \frac{1}{2}\rho(K(\cdot,X))\psi \\
&= \rho(\nu)(\widehat{\nabla}_X - \frac{1}{2}\rho(K(\cdot,X))\rho(\nu))\psi \\
&= \rho(\nu)\nabla_X\psi
\end{aligned} \tag{3.31}$$

where we have used the definition of the second fundamental form $K(X,Y) := -g(\widehat{\nabla}_X\nu, Y)$, $X, Y \in \Gamma(T\Sigma)$ and the fact that $g(K(\cdot,X),\nu) = 0$ since the second fundamental form K is $\Sigma-$ tangential (more precisely a section of the bundle $T^*\Sigma \otimes T^*\Sigma$ and $X \in \Gamma(T\Sigma)$. This lets us utilize the Clifford relation $\rho(K(\cdot,X))\rho(\nu) + \rho(\nu)\rho(K(\cdot,X)) = 0$. Now we prove that the boundary Dirac operator $\mathbf{D}$ and $\rho(\nu)$ anti-commutes i.e.,

$$\{\mathbf{D}, \rho(\nu)\} = 0 \tag{3.32}$$

where $\{\cdot,\cdot\}$ denotes anti-commutation operator. The proof follows as a consequence of the following calculations

$$\mathbf{D}(\rho(\nu)\psi) := \sum_{I=1}^{n} \rho(e_I)\rho(\nu)\nabla_{e_I}(\rho(\nu)\psi),\ e_I \in \Gamma(T\Sigma), g(e_I, e_J) = \delta_{IJ} \tag{3.33}$$

which by means of the relation $[\nabla_X, \rho(\nu)] = 0\ \forall\ X \in \Gamma(T\Sigma)$ implies

$$\mathbf{D}(\rho(\nu)\psi) = \sum_{I=1}^{n} \rho(e_I)\rho(\nu)\rho(\nu)\nabla_{e_I}\psi = -\rho(\nu)\sum_{I=1}^{n} \rho(e_I)\rho(\nu)\nabla_{e_I}\psi = -\rho(\nu)\mathbf{D}\psi \tag{3.34}$$

where the Clifford relation $\rho(e_I)\rho(\nu) + \rho(\nu)\rho(e_I) = 0$ is used. The claim of the lemma now follows. Consider the following computations for φ as in the lemma and use the fact $\{\mathbf{D}, \rho(\nu)\} = 0$

$$\mathbf{D}\varphi = \left(\frac{n^2}{4} + (\widetilde{\lambda} - \lambda)^2\right)^{-1} \left[\frac{n}{2}\mathbf{D}\Phi^+ - (\widetilde{\lambda} - \lambda)\rho(\nu)\mathbf{D}\Phi^+\right]. \tag{3.35}$$

Note

$$\mathbf{D}\Phi^+ = (\mathbf{D} + \frac{n}{2}\rho(\nu) - \frac{n}{2}\rho(\nu))\Phi^+ = \widetilde{\mathbf{D}}^+\Phi^+ - \frac{n}{2}\rho(\nu)\Phi^+ = \widetilde{\lambda}\Phi^+ - \frac{n}{2}\rho(\nu)\Phi^+ \tag{3.36}$$

and a direct consequence of the expression $\varphi := \left(\frac{n^2}{4} + (\widetilde{\lambda} - \lambda)^2\right)^{-1} \left[\frac{n}{2}\Phi^+ + (\widetilde{\lambda} - \lambda)\rho(\nu)\Phi^+\right]$ yields

$$\Phi^+ = \frac{n}{2}\varphi - (\widetilde{\lambda} - \lambda)\rho(\nu)\varphi \tag{3.37}$$

and therefore

$$\begin{aligned}\mathbf{D}\Phi^{+} &= \widetilde{\lambda}\left(\frac{n}{2}\varphi - (\widetilde{\lambda}-\lambda)\rho(\nu)\varphi\right) - \frac{n}{2}\rho(\nu)\left(\frac{n}{2}\varphi - (\widetilde{\lambda}-\lambda)\rho(\nu)\varphi\right) \\ &= \frac{n\widetilde{\lambda}}{2}\varphi - \widetilde{\lambda}(\widetilde{\lambda}-\lambda)\rho(\nu)\varphi - \frac{n^2}{4}\rho(\nu)\varphi - \frac{n}{2}(\widetilde{\lambda}-\lambda)\varphi \\ &= \frac{n}{2}\lambda\varphi - (\widetilde{\lambda}^2 + \frac{n^2}{4})\rho(\nu)\varphi + \widetilde{\lambda}\lambda\rho(\nu)\varphi \\ &= \frac{n}{2}\lambda\varphi - \lambda(\lambda-\widetilde{\lambda})\rho(\nu)\varphi.\end{aligned} \tag{3.38}$$

Also, note

$$\begin{aligned}\frac{n}{2}\mathbf{D}\Phi^{+} - (\widetilde{\lambda}-\lambda)\rho(\nu)\mathbf{D}\Phi^{+} &= \frac{n^2}{4}\lambda\varphi - \frac{n}{2}\lambda(\lambda-\widetilde{\lambda})\rho(\nu)\varphi + \frac{n}{2}\lambda(\lambda-\widetilde{\lambda})\rho(\nu)\varphi + \lambda(\lambda-\widetilde{\lambda})^2\varphi \\ &= \lambda(\frac{n^2}{4} + (\lambda-\widetilde{\lambda})^2)\varphi\end{aligned}$$

implying

$$\mathbf{D}\varphi = \left(\frac{n^2}{4} + (\widetilde{\lambda}-\lambda)^2\right)^{-1}\left[\frac{n}{2}\mathbf{D}\Phi^{+} - (\widetilde{\lambda}-\lambda)\rho(\nu)\mathbf{D}\Phi^{+}\right] = \lambda\varphi. \tag{3.39}$$

This completes the proof of the lemma. □

Note $\frac{n^2}{4} + (\widetilde{\lambda}-\lambda)^2 = 2\lambda^2 - 2\lambda\widetilde{\lambda} \neq 0$. Similarly, an eigenspinor φ of $\mathbf{D}$ can be constructed from the first eigenspinor of $\widetilde{\mathbf{D}}^{-}$ as follows. Let $\widetilde{\Phi}$ be the first eigenspinor of $\widetilde{\mathbf{D}}^{-}$ i.e.,

$$\widetilde{\mathbf{D}}^{-}\widetilde{\Phi}^{+} = \widetilde{\mu}\widetilde{\Phi}^{+}. \tag{3.40}$$

Now define $\lambda^2 = \widetilde{\mu}^2 + \frac{n^2}{4}$ and note

$$\varphi := \left(\frac{n^2}{4} + (\widetilde{\mu}-\lambda)^2\right)^{-1}\left[\frac{n}{2}\widetilde{\Phi}^{+} - (\widetilde{\mu}-\lambda)\rho(\nu)\widetilde{\Phi}^{+}\right] \tag{3.41}$$

verifies

$$\mathbf{D}\varphi = \lambda\varphi. \tag{3.42}$$

Lemma 3.1 yields $\lambda \geq \frac{n}{2}$. Therefore $\widetilde{\lambda}^2, \widetilde{\mu}^2 \geq 0$ or the following claim follows as a consequence of lemma 3.1

Claim 1 *Let (Σ, σ) have scalar curvature $R[\sigma] \geq (n-1)n$ and assume that the intrinsic spin structure on (Σ, σ) coincides with the spin structure induced from (M, g). Then the first order Dirac type operators $\widetilde{\mathbf{D}}^{\pm}$ on (Σ, σ) induced by (M, g) have real eigenvalues.*

Proof. Let us assume that such a claim is not true. It suffices to consider the case when eigenvalues contain pure imaginary numbers. Let $\widetilde{\lambda} \in \mathbb{C}$ be the first eigenvalue of $\widetilde{\mathbf{D}}^{+}$ that is purely imaginary and the corresponding eigenspinor is Φ^{+}. Then if we define $\lambda^2 = \widetilde{\lambda}^2 + \frac{n^2}{4} < \frac{n^2}{4}$, then

$$\varphi := \left(\frac{n^2}{4} + (\widetilde{\lambda}-\lambda)^2\right)^{-1}\left[\frac{n}{2}\Phi^{+} + (\widetilde{\lambda}-\lambda)\rho(\nu)\Phi^{+}\right] \tag{3.43}$$

solves $\mathbf{D}\varphi = \lambda\varphi$. But $\lambda^2 \geq \frac{n^2}{4}$ by lemma 3.1 which is a contradiction. Therefore the claim is true. Similar argument works for $\widetilde{\mathbf{D}}^{-}$. □

More generally, one has the following proposition relating the eigenvalues of $\mathbf{D}$ and $\widetilde{\mathbf{D}}^{\pm}$ as a consequence of the claim 1

Proposition 3.2 *Let $E_{\mathbf{D}}$ and $E_{\widetilde{\mathbf{D}}^{\pm}}$ be the spaces of L^2-eigenspinors of the Dirac operators $\mathbf{D}$ and $\widetilde{\mathbf{D}}^{\pm}$ on Σ, respectively. Then under the curvature condition $R[\sigma] \geq n(n-1)$, we have the isomorphism*

$$E_{\mathbf{D}} \simeq E^{+}_{\widetilde{\mathbf{D}}^{\pm}} \cup E^{-}_{\widetilde{\mathbf{D}}^{\pm}} \cup 2E^{0}_{\widetilde{\mathbf{D}}^{\pm}}, \tag{3.44}$$

where superscripts $+, -$, and 0 denote positive, negative, and zero eigen spaces, respectively.

Proof. Proof of this proposition relies on the explicit construction of eigenspinors. Let $\varphi \in E^{+}_{\mathbf{D}}$ with positive eigenvalue λ ($\lambda^2 \geq \frac{n^2}{4}$) and $E^{+}_{\widetilde{\mathbf{D}}^{\pm}}$ be the spaces of eigenspinors of $\widetilde{\mathbf{D}}^{\pm}$ with positive eigenvalues. Then construct the maps

$$p_1 : E^{+}_{\mathbf{D}} \to E^{+}_{\widetilde{\mathbf{D}}^{+}} \cup E^{0}_{\widetilde{\mathbf{D}}^{+}} \tag{3.45}$$

$$\varphi \mapsto \Phi^{+} := \frac{n}{2}\varphi - (\widetilde{\lambda} - \lambda)\rho(\nu)\varphi,\ \widetilde{\lambda} = +\sqrt{\lambda^2 - \frac{n^2}{4}} \tag{3.46}$$

$$p_2 : E^{+}_{\mathbf{D}} \to E^{+}_{\widetilde{\mathbf{D}}^{-}} \cup E^{0}_{\widetilde{\mathbf{D}}^{-}} \tag{3.47}$$

$$\varphi \mapsto \widetilde{\Phi}^{+} := \frac{n}{2}\varphi + (\widetilde{\lambda} - \lambda)\rho(\nu)\varphi,\ \widetilde{\lambda} = +\sqrt{\lambda^2 - \frac{n^2}{4}} \tag{3.48}$$

Similarly, construct maps to the negative eigenspaces $E^{-}_{\widetilde{\mathbf{D}}^{\pm}}$

$$q_1 : E^{+}_{\mathbf{D}} \to E^{-}_{\widetilde{\mathbf{D}}^{+}} \cup E^{0}_{\widetilde{\mathbf{D}}^{+}} \tag{3.49}$$

$$\varphi \mapsto \Phi^{-} := (\widetilde{\lambda} - \lambda)\varphi - \frac{n}{2}\rho(\nu)\varphi,\ \widetilde{\lambda} = -\sqrt{\lambda^2 - \frac{n^2}{4}} \tag{3.50}$$

$$q_2 : E^{+}_{\mathbf{D}} \to E^{-}_{\widetilde{\mathbf{D}}^{-}} \cup E^{0}_{\widetilde{\mathbf{D}}^{-}} \tag{3.51}$$

$$\varphi \mapsto \widetilde{\Phi}^{-} = (\widetilde{\lambda} - \lambda)\varphi + \frac{n}{2}\rho(\nu)\varphi,\ \widetilde{\lambda} = -\sqrt{\lambda^2 - \frac{n^2}{4}} \tag{3.52}$$

Note that each of these maps are well defined. For example $\Phi^{+} = 0$ implies $\varphi = 0$ and the rest verifies the same too. The surjectivity follows since φ can be written explicitly in terms of Φ e.g., consider the map p_1. Let $\Phi^{+} \in E^{+}_{\widetilde{\mathbf{D}}^{+}}$. An explicit computation leads to

$$\varphi = \left(\frac{n^2}{4} + (\widetilde{\mu} - \lambda)^2\right)^{-1} \left[\frac{n}{2}\Phi^{+} - (\widetilde{\mu} - \lambda)\rho(\nu)\Phi^{+}\right] \in E^{+}_{\mathbf{D}}. \tag{3.53}$$

Similar calculations follow for the rest of the maps.
Now the spectrum of $\mathbf{D}$ is symmetric with respect to zero with finite multiplicities i.e., $E^{+}_{\mathbf{D}} \simeq E^{-}_{\mathbf{D}}$. Therefore the desired isomorphism $E_{\mathbf{D}} \simeq E^{+}_{\widetilde{\mathbf{D}}^{\pm}} \cup E^{-}_{\widetilde{\mathbf{D}}^{\pm}} \cup 2E^{0}_{\widetilde{\mathbf{D}}^{\pm}}$ follows. □

Remark 5 *if $R[\sigma] > (n-1)n$, then $\lambda_1(\mathbf{D})^2 > \frac{n^2}{4}$, and subsequently $E_{\mathbf{D}} \simeq E^{+}_{\widetilde{\mathbf{D}}^{\pm}} \cup E^{-}_{\widetilde{\mathbf{D}}^{\pm}}$.*

Corollary 3.1 *$E^{-}_{\widetilde{\mathbf{D}}^{-}} = \rho(\nu)(E^{+}_{\widetilde{\mathbf{D}}^{+}})$ and $E^{+}_{\widetilde{\mathbf{D}}^{-}} = \rho(\nu)(E^{-}_{\widetilde{\mathbf{D}}^{+}})$.*

Proof. Direct computation. □

4. Construction of L^2orthogonal eigen spinor bases

To implement a non-local boundary condition, we need to understand the spectral decomposition of a spinor $\psi \in C^\infty(\Gamma(S^\Sigma))$.

Proposition 4.1 *Φ^+ and $\widetilde{\Phi}^-$ are $L^2(\Sigma)$ orthogonal. Similarly Φ^- and $\widetilde{\Phi}^+$ are $L^2(\Sigma)$ orthogonal.*

Proof. The key point to note here is that we construct $\Phi^+, \widetilde{\Phi}^-$, the eigenspinors of $\widetilde{\mathbf{D}}^+$ and $\widetilde{\mathbf{D}}^-$ directly using the eigenspinors of the Dirac operator $\mathbf{D}$. First note if φ is an eigenspinor of the Diracl operator $\mathbf{D}$, then φ and $\rho(\nu)\varphi$ are L^2 orthogonal with respect the Hermitian inner product i.e.,

$$\int_\Sigma \langle \varphi, \rho(\nu)\varphi\rangle \mu_g = 0. \tag{4.1}$$

The proof of this claim follows by splitting φ into its chiral components $\varphi = \varphi_+ + \varphi_-$ where φ_+ and φ_- are point-wise orthogonal (more explicitly one may write $\varphi_+ := \frac{1}{2}(\varphi + \sqrt{-1}\rho(\nu)\varphi)$ and $\varphi_- := \frac{1}{2}(\varphi - \sqrt{-1}\rho(\nu)\varphi)$. If φ is an eigenspinor of $\mathbf{D}$ with eigenvalue λ, then the action of $\mathbf{D}$ interchanges φ_+ and φ_- due to the anti-commutation property $\{\mathbf{D}, \rho(\nu)\} = 0$

$$\mathbf{D}\varphi_+ = \lambda\varphi_-, \ \mathbf{D}\varphi_- = \lambda\varphi_+. \tag{4.2}$$

Now since Σ is closed, $\mathbf{D}$ is self-adjoint, we have

$$\int_\Sigma \langle \mathbf{D}\varphi_+, \varphi_-\rangle \mu_\sigma = \int_\Sigma \langle \varphi_+, \mathbf{D}\varphi_-\rangle \mu_\sigma \tag{4.3}$$

This implies for $\lambda \neq 0$

$$\int_{\widehat{\Sigma}} \langle \varphi_-, \varphi_-\rangle \mu_\sigma = \int_\Sigma \langle \varphi_+, \varphi_+\rangle \mu_\sigma. \tag{4.4}$$

Now $\langle \varphi, \rho(\nu)\varphi\rangle = \sqrt{-1}(|\varphi_-|^2 - |\varphi_+|^2)$ and therefore

$$\int_\Sigma \langle \varphi, \rho(\nu)\varphi\rangle \mu_g = \sqrt{-1}\int_\Sigma (|\varphi_-|^2 - |\varphi_+|^2)\mu_g = 0. \tag{4.5}$$

This essentially proves that the self-adjoint operator $\mathbf{D}$ has an orthonormal eigenspinor basis. In fact it does have a complete basis. Φ^+ and $\widetilde{\Phi}^-$ are constructed by using φ. We show using the L^2 orthogonality of φ and $\rho(\nu)\varphi$ that Φ^+ and $\widetilde{\Phi}^-$ are L^2 orthogonal with respect the same hermitian inner product. Explicit computation yields

$$\begin{aligned}
\langle \Phi^+, \widetilde{\Phi}^-\rangle_{L^2(\Sigma)} &= \int_\Sigma \langle \frac{n}{2}\varphi - (\widetilde{\lambda} - \lambda)\rho(\nu)\varphi, (\widetilde{\lambda} - \lambda)\varphi + \frac{n}{2}\rho(\nu)\varphi\rangle \mu_\sigma \\
&= \frac{n^2}{4}\int_\Sigma \langle \varphi, \rho(\nu)\varphi\rangle \mu_\sigma - (\widetilde{\lambda} - \lambda)^2 \int_\Sigma \langle \rho(\nu)\varphi, \varphi\rangle \mu_\sigma = 0.
\end{aligned} \tag{4.6}$$

A similar computation shows Φ^- and $\widetilde{\Phi}^+$ are L^2-orthogonal. This completes the proof. □

Again, recall that any spinor $\psi \in C^\infty(\Gamma(S^\Sigma))$ can be decomposed as sum of L^2-eigenspinors of the Dirac operator $\mathbf{D}$ since it is self-adjoint on closed Σ [15]. On the other hand, proposition 3.2 provides us with the explicit isomorphism between the space of eigenspinors of $\mathbf{D}$ and that of $\widetilde{\mathbf{D}}^{\pm}$. This in turn allows us to express $\psi \in \Gamma(S^\Sigma)$ as combinations of the eigenspinors of $\widetilde{\mathbf{D}}^{\pm}$. The only caveat is that positive and negative eigenspaces of $\widetilde{\mathbf{D}}^{+}$ are not orthogonal as expected from any generic non-self-adjoint operator. Therefore, we need to use the positive eigenspinors Φ^+ of $\widetilde{\mathbf{D}}^{+}$ and the negative eigenspinors of $\widetilde{\mathbf{D}}^{-}$. We denote the set of positive eigenspinors of $\widetilde{\mathbf{D}}^{+}$ by $\{\Phi^+_i\}$, set of negative eigenspinors of $\widetilde{\mathbf{D}}^{+}$ by $\{\Phi^-_i\}$, set of positive eigenspinors of $\widetilde{\mathbf{D}}^{-}$ by $\{\widetilde{\Phi}^+_i\}$, and the set of negative eigenspinors of $\widetilde{\mathbf{D}}^{-}$ by $\{\widetilde{\Phi}^-_i\}$. Corresponding eigenvalues are denoted by $\{\widetilde{\lambda}_i\}$.

Proposition 4.2 *Let $\psi \in C^\infty(\Gamma(S^\Sigma))$. Then there exists the following L^2-orthogonal decomposition in terms of the eigenspinors of $\widetilde{\mathbf{D}}^{\pm}$*

$$\psi = \sum_i a_i \Phi^+_i + \sum_j b_j \widetilde{\Phi}^-_j \tag{4.7}$$

$$\psi = \sum_i \widetilde{a}_i \Phi^-_i + \sum_j \widetilde{b}_j \widetilde{\Phi}^+_j \tag{4.8}$$

for $a_i, b_i, \widetilde{a}_i, \widetilde{b}_i \in \mathbb{C}\forall i$. This decomposition is unique.

Proof. First, note that ψ admits a unique decomposition in terms of the eigenspinors of the self-adjoint operator $\mathbf{D}$ (see [15] for a detailed exposition). The idea is to express the eigenspinors of $\mathbf{D}$ in terms of the eigenspinors of $\widetilde{\mathbf{D}}^{\pm}$ and use the existence and uniqueness theorem that works for the Dirac operator. A vital point to note here is that $\widetilde{D}^{\pm}$ are not self-adjoint and therefore they do not necessarily admit orthonormal eigenspinor basis separately. In fact, a direct computation shows that $\int_\Sigma \langle \Phi^+_i, \Phi^-_i \rangle \mu_\sigma \neq 0$. Let the positive eigenspace of $\mathbf{D}$ be denoted by $\{\varphi_i\}$. The negative eigenspace is automatically $\{\rho(\nu)\varphi_i\}$ (Under the assumption of the curvature $R[\sigma] \geq n(n-1)$, lemma 3.1 states that the spectra of $\mathbf{D}$ is strictly positive and negative. To this end, we use the proposition 3.2. First

$$\psi = \sum_i \alpha_i \varphi_i + \sum_i \beta_i \rho(\nu)\varphi_i, \;\; \alpha_i, \beta_i \in \mathbb{C}. \tag{4.9}$$

Now recall the following from proposition 3.2

$$\varphi_i = \left(\frac{n^2}{4} + (\widetilde{\lambda}_i - \lambda_i)^2\right)^{-1} \left[\frac{n}{2}\Phi^+_i + (\widetilde{\lambda}_i - \lambda_i)\rho(\nu)\Phi^+_i\right] \tag{4.10}$$

which imply by the relations $\rho(\nu)\Phi^+_i = \widetilde{\Phi}^-_i$ and $\rho(\nu)\widetilde{\Phi}^-_i = -\Phi^+_i$ from corollary 3.1

$$\varphi_i = \left(\frac{n^2}{4} + (\widetilde{\lambda}_i - \lambda_i)^2\right)^{-1} \left[\frac{n}{2}\Phi^+_i + (\widetilde{\lambda}_i - \lambda_i)\widetilde{\Phi}^-_i\right], \tag{4.11}$$

$$\rho(\nu)\varphi_i = \left(\frac{n^2}{4} + (\widetilde{\lambda}_i - \lambda_i)^2\right)^{-1} \left[\frac{n}{2}\widetilde{\Phi}^-_i - (\widetilde{\lambda}_i - \lambda_i)\Phi^+_i\right] \tag{4.12}$$

which upon substitution in 4.9 yields

$$\psi = \sum_i \frac{\alpha_i}{\frac{n^2}{4} + (\widetilde{\lambda}_i - \lambda_i)^2}\left(\frac{n}{2}\Phi^+_i + (\widetilde{\lambda}_i - \lambda_i)\widetilde{\Phi}^-_i\right) + \sum_i \frac{\beta_i}{\frac{n^2}{4} + (\widetilde{\lambda}_i - \lambda_i)^2}\left(\frac{n}{2}\widetilde{\Phi}^-_i - (\widetilde{\lambda}_i - \lambda_i)\Phi^+_i\right)$$

$$= \sum_i a_i\Phi_i^+ + \sum_i d_i\widetilde{\Phi}_i^-$$

where

$$a_i := \frac{1}{\frac{n^2}{4}+(\widetilde{\lambda}_i-\lambda_i)^2}\left(\frac{n\alpha_i}{2}-(\widetilde{\lambda}_i-\lambda_i)\beta_i\right)\in\mathbb{C} \tag{4.13}$$

$$d_i := \frac{1}{\frac{n^2}{4}+(\widetilde{\lambda}_i-\lambda_i)^2}\left(\frac{n\beta_i}{2}+(\widetilde{\lambda}_i-\lambda_i)\alpha_i\right)\in\mathbb{C}. \tag{4.14}$$

Decomposition in terms of Φ_i^- and $\widetilde{\Phi}_i^+$ follows in a similar way. One may normalize Φ_i^+ and $\widetilde{\Phi}_i^-$ to have $L^2(\Sigma)$ norm 1. This completes the proof of the proposition. $\square$

Now we can start solving the boundary value problem for the operators $\widetilde{D}^{\pm}$ on M. We will essentially work with an APS-type boundary condition. First, consider the following spaces. On the boundary Σ, $L^2(\Gamma(S^\Sigma))$ splits into two orthogonal subspaces. This is as follows. Let D be the Dirac operator on Σ. Consider the eigenvalue equation for $\widetilde{\mathbf{D}}^+$ and $\widetilde{\mathbf{D}}^-$

$$\widetilde{\mathbf{D}}^+\psi=\widetilde{\lambda}\psi,\ \widetilde{\mathbf{D}}^-\varphi=\widetilde{\lambda}\varphi,\ \widetilde{\lambda}\in\mathbb{R}. \tag{4.15}$$

The spectrum $\{\widetilde{\lambda}\in\mathbb{R}\}$ is symmetric with respect to zero by claim 1 and proposition 3.2. A generic L^2 section ψ of S^Σ is written as in proposition 4.2. One can split $L^2(\Gamma(S^\Sigma))$ into $L^2_{\widetilde{\mathbf{D}}^+}(\Gamma(S^\Sigma))^+$ and $L^2_{\widetilde{\mathbf{D}}^-}(\Gamma(S^\Sigma))^-$ as follows

$$L^2_{\widetilde{\mathbf{D}}^+}(\Gamma(S^\Sigma))^+ := \left\{\psi\in L^2(\Gamma(S^\Sigma))|\psi=\sum_i a_i\Phi_i^+,\ \widetilde{\mathbf{D}}^+\Phi_i^+=\widetilde{\lambda}_i\Phi_i^+,\widetilde{\lambda}_i\geq 0, a_i\in\mathbb{C}\right\}, \tag{4.16}$$

$$L^2_{\widetilde{\mathbf{D}}^-}(\Gamma(S^\Sigma))^- := \left\{\psi\in L^2(\Gamma(S^\Sigma))|\psi=\sum_i d_i\widetilde{\Phi}_i^-,\ \widetilde{\mathbf{D}}^-\widetilde{\Phi}_i^-=\widetilde{\lambda}_i\widetilde{\Phi}_i^-,\widetilde{\lambda}_i\leq 0, b_i\in\mathbb{C}\right\}. \tag{4.17}$$

Naturally $L^2_{\widetilde{\mathbf{D}}^+}(\Gamma(S^\Sigma))^+$ and $L^2_{\widetilde{\mathbf{D}}^-}(\Gamma(S^\Sigma))^-$ are L^2 orthogonal by construction (proposition 4.1). Let us denote by $\widetilde{P}^+_{\geq 0},\widetilde{P}^-_{\leq 0}$ the projection operators

$$\widetilde{P}^+_{\geq 0}: L^2(\Gamma(S^\Sigma))\to L^2_{\widetilde{\mathbf{D}}^+}(\Gamma(S^\Sigma))^+, \tag{4.18}$$

$$\widetilde{P}^-_{\leq 0}: L^2(\Gamma(S^\Sigma))\to L^2_{\widetilde{\mathbf{D}}^-}(\Gamma(S^\Sigma))^-. \tag{4.19}$$

Similarly, we can perform decomposition of $L^2(\Gamma(S^\Sigma))$ by the non-negative eigenspinors of $\widetilde{\mathbf{D}}^-$ and non-positive eigenspinors of $\widetilde{\mathbf{D}}^+$. Define similarly

$$L^2_{\widetilde{\mathbf{D}}^-}(\Gamma(S^\Sigma))^+ := \left\{\psi\in L^2(\Gamma(S^\Sigma))|\psi=\sum_i b_i\widetilde{\Phi}_i^+,\ \widetilde{\mathbf{D}}^-\widetilde{\Phi}_i^+=\widetilde{\lambda}_i\widetilde{\Phi}_i^+,\widetilde{\lambda}_i\geq 0\right\}, \tag{4.20}$$

$$L^2_{\widetilde{\mathbf{D}}^+}(\Gamma(S^\Sigma))^- := \left\{\psi\in L^2(\Gamma(S^\Sigma))|\psi=\sum_i c_i\Phi_i^-,\ \widetilde{\mathbf{D}}^+\Phi_i^-=\widetilde{\lambda}_i\Phi_i^-,\widetilde{\lambda}_i\leq 0\right\}. \tag{4.21}$$

Naturally $L^2_{\widetilde{\mathbf{D}}^-}(\Gamma(S^\Sigma))^+$ and $L^2_{\widetilde{\mathbf{D}}^+}(\Gamma(S^\Sigma))^-$ are L^2 orthogonal again by construction (proposition 4.1). Let us denote by $\widetilde{P}^-_{\geq 0},\widetilde{P}^+_{\leq 0}$ the projection operators

$$\widetilde{P}^-_{\geq 0}: L^2(\Gamma(S^\Sigma))\to L^2_{\widetilde{\mathbf{D}}^-}(\Gamma(S^\Sigma))^+, \tag{4.22}$$

$$\widetilde{P}^+_{\leq 0}: L^2(\Gamma(S^\Sigma))\to L^2_{\widetilde{\mathbf{D}}^+}(\Gamma(S^\Sigma))^-. \tag{4.23}$$

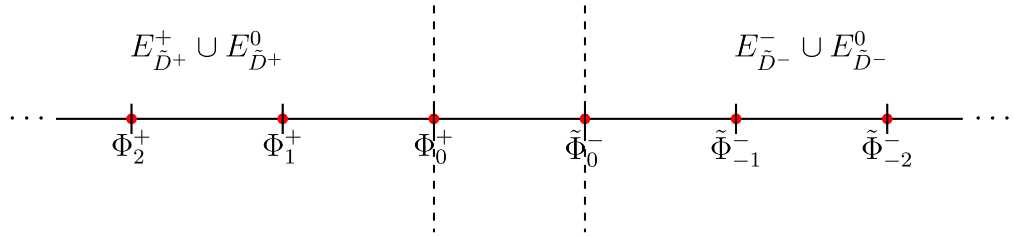

Figure 1: Note that $E^+_{\tilde{D}^+} \cup E^-_{\tilde{D}^-} \cup E^0_{\tilde{D}^+} \cup E^0_{\tilde{D}^-}$ contain the complete eigenspace information of $\psi|_{\Sigma}$. The non-local boundary condition implemented here specifies one half of it i.e., the eigenspace information contained in $E^+_{\tilde{D}^+} \cup E^0_{\tilde{D}^+}$. Also, note that the union of the eigenspace information contained in the boundary conditions for the forward Killing Dirac equation and the adjoint Killing Dirac equation should have the cardinality as the complete eigenspace $E^+_{\tilde{D}^+} \cup E^-_{\tilde{D}^-} \cup E^0_{\tilde{D}^+} \cup E^0_{\tilde{D}^-}$

4.1. Proof of theorem 1.1

Let us recall the statement of the theorem 1.1

Theorem *Let $\Phi \in L^2(\Gamma(S^M))$ and $\alpha \in H^{\frac{1}{2}}(\Gamma(S^\Sigma))$. Let the scalar curvature of a $n+1$ dimensional smooth connected oriented Riemannian spin manifold (M,g) verify $R[g] \geq (n+1)n$ and the mean curvature K of its boundary (Σ,σ) with respect to the inward-pointing normal vector be strictly positive i.e., Σ is strictly mean convex in M and its scalar curvature verifies $R[\sigma] \geq (n-1)n$. Then the following boundary value problem*

$$\widetilde{D}^+\psi := (\widehat{D} - \frac{n+1}{2})\psi = \Phi, \tag{4.24}$$

$$\widetilde{P}^+_{\geq 0}\psi = \widetilde{P}^+_{\geq 0}\alpha, \tag{4.25}$$

has a unique solution $\psi \in H^1(\Gamma(S^M)) \cap H^{\frac{1}{2}}(\Gamma(S^\Sigma))$ that verifies

$$||\psi||_{H^1(\Gamma(S^M))\cap H^{\frac{1}{2}}(\Gamma(S^\Sigma))} \leq C(n)\left(||\Phi||_{L^2(\Gamma(S^M))} + ||\widetilde{P}^+_{\geq 0}\alpha||_{H^{\frac{1}{2}}(\Gamma(S^\Sigma))}\right), \tag{4.26}$$

for a constant $C(n)$ depending on the geometry of (M,g) and (Σ,σ) and the dimension n.

First, recall that for a differential operator $\widehat{D}^+ : C^\infty_c(\Gamma(S^M)) \to C^\infty_c(\Gamma(S^M))$ and its L^2 adjoint $(\widetilde{D}^+)^* : C^\infty_c(\Gamma(S^M)) \to C^\infty_c(\Gamma(S^M))$, one has the splitting property if $\widetilde{D}^+$ and $(\widetilde{D}^+)^*$ has injective symbols. In particular $H^s(\Gamma(S^\Sigma)) = \text{Range}(\widetilde{D}^+) \oplus \text{Ker}((\widetilde{D}^+)^*)$. Here $\widetilde{D}^+$ can be regarded as a map $\widetilde{D}^+ : H^{s+1}(\Gamma(S^\Sigma)) \to H^s(\Gamma(S^\Sigma))$ and $(\widetilde{D}^+)^* : H^s(S^\Sigma) \to H^{s-1}(S^\Sigma)$. When both $\widetilde{D}^+$ and $(\widetilde{D}^+)^*$ have injective symbols (which is the case here), the range of D is closed. This follows from the fact that in such case $\text{Range}(\widetilde{D}^+) = \text{Range}(\widetilde{D}^+(\widetilde{D}^+)^*)$ and as $\widetilde{D}^+(\widetilde{D}^+)^*$ is elliptic, this is closed. We prove the invertibility of the operator $\widehat{D} - \frac{n+1}{2}$ between the appropriate Sobolev spaces. We will prove that with this non-local boundary condition,

$$\widehat{D} - \frac{n+1}{2} : H^s(\Gamma(S^M)) \cap H^{s-\frac{1}{2}}(\Gamma(S^\Sigma))_+ \to H^{s-1}(\Gamma(S^M)) \cap H^{s-\frac{3}{2}}(\Gamma(S^\Sigma)),\ s \geq 1 \tag{4.27}$$

is an isomorphism which obviously implies $\text{ind}(\widetilde{D}^+) = 0$. The operator $\widetilde{D}^+$ is formally self-adjoint. One needs to be careful about the boundary conditions. We prove the desired isomorphism in two parts.

Triviality of the Kernel: Consider the kernel with the non-local boundary condition prescribed in the theorem

$$\widetilde{D}^+\psi := (D - \frac{n+1}{2})\psi = 0 \text{ on } M \tag{4.28}$$

$$\widetilde{P}^{+}_{\geq 0}\psi = 0 \ on \ \Sigma. \tag{4.29}$$

We prove that this equation has no nontrivial solution. Recall the Witten identity 3.1

$$\begin{aligned}\int_{\Sigma}\left(\langle \widetilde{\mathbf{D}}^{+}\psi,\psi\rangle - \frac{1}{2}K|\psi|^2\right)\mu_{\sigma} &= \frac{1}{4}\int_M \big(R[g]-n(n+1)\big)|\psi|^2\mu_M - \frac{n}{n+1}\int_M\langle \widetilde{D}^{+}\psi,\widetilde{D}^{-}\psi\rangle \\ &+ \int_M |\mathcal{Q}\psi|^2,\end{aligned}$$

and substitute $\widetilde{D}^{+}\psi = 0$. This yields

$$\begin{aligned}\int_{\Sigma}\left(\langle \widetilde{\mathbf{D}}^{+}\widetilde{P}^{+}_{\geq 0}\psi,\widetilde{P}^{+}_{\geq 0}\psi\rangle + \langle \widetilde{\mathbf{D}}^{+}\widetilde{P}^{-}_{\leq 0}\psi,\widetilde{P}^{-}_{\leq 0}\psi\rangle - \frac{1}{2}K|\psi|^2\right)\mu_{\sigma} &= \frac{1}{4}\int_M \big(R[g]-n(n+1)\big)|\psi|^2\mu_M \\ &+ \int_M |\mathcal{Q}\psi|^2.\end{aligned}$$

The boundary condition $\widetilde{P}^{+}_{\geq 0}\psi = 0 \ on \ \Sigma$ together with the fact that $\int_{\Sigma}\langle \widetilde{\mathbf{D}}^{+}\widetilde{P}^{-}_{\leq 0}\psi,\widetilde{P}^{-}_{\leq 0}\psi\rangle\mu_{\sigma} = \int_{\Sigma}\langle \widetilde{P}^{-}_{\leq 0}\psi,\widetilde{\mathbf{D}}^{-}\widetilde{P}^{-}_{\leq 0}\psi\rangle\mu_{\sigma}$ implies

$$\begin{aligned}&\int_{\Sigma}\left(\langle \widetilde{P}^{-}_{\leq 0}\psi,\widetilde{\mathbf{D}}^{-}\widetilde{P}^{-}_{\leq 0}\psi\rangle - \frac{1}{2}K|\widetilde{P}^{-}_{\leq 0}\psi|^2\right)\mu_{\sigma} \\ &= \frac{1}{4}\int_M \big(R[g]-n(n+1)\big)|\psi|^2\mu_M + \int_M|\mathcal{Q}\psi|^2 \geq 0.\end{aligned} \tag{4.30}$$

Using the positive mean curvature condition $K>0$ (note that strict positivity is crucial since $\widetilde{\mathbf{D}}^{-}$ may have non-trivial kernel which would constitute the kernel of $\widetilde{D}^{+}$ if K is allowed to vanish)

$$\int_{\Sigma}\left(\langle \widetilde{P}^{-}_{\leq 0}\psi,\widetilde{\mathbf{D}}^{-}\widetilde{P}^{-}_{\leq 0}\psi\rangle - \frac{1}{2}K|\widetilde{P}^{-}_{\leq 0}\psi|^2\right)\mu_{\sigma} \leq 0 \tag{4.31}$$

which yields by the bulk curvature condition $R[g] \geq (n+1)n$

$$0 \leq \int_{\Sigma}\left(\langle \widetilde{P}^{-}_{\leq 0}\psi,\widetilde{\mathbf{D}}^{-}\widetilde{P}^{-}_{\leq 0}\psi\rangle - \frac{1}{2}K|\widetilde{P}^{-}_{\leq 0}\psi|^2\right)\mu_{\sigma} \leq 0 \tag{4.32}$$

since $\int_{\Sigma}\langle \widetilde{P}^{-}_{\leq 0}\psi,\widetilde{\mathbf{D}}^{-}\widetilde{P}^{-}_{\leq 0}\psi\rangle\mu_{\sigma} \leq 0$ implying $\widetilde{P}^{+}_{\leq 0}\psi = 0$ on Σ i.e.,

$$\psi = 0 \ on \ \Sigma. \tag{4.33}$$

Now unique continuation of the homogeneous first-order elliptic partial differential equation then implies

$$\psi = 0 \ on \ M. \tag{4.34}$$

One may also obtain this directly from the inequality 4.30. Note that $K>0$ is essential since for $K=0$ case, there can be non-trivial $\widetilde{\mathbf{D}}^{-}-$harmonic spinor on Σ and consequently the Kernel of $\widetilde{D}^{+}$ would be non-trivial finite dimensional.

Triviality of the co-kernel: The range of $\widetilde{D}^{+}$ is closed since the symbols of $\widetilde{D}^{+}$ and its L^2-adjoint $(\widetilde{D}^{+})^{*}$ are injective. For an element $\varphi^{'}$ is the co-kernel of $\widetilde{D}^{+}$ and $\forall\varphi$ in the domain of $\widetilde{D}^{+}$ such that $(\widehat{D}-\frac{n+1}{2})\varphi = \Psi \in \text{range}(\widehat{D}-\frac{n+1}{2})$,

$$\int_M \langle (D-\frac{n+1}{2})\varphi,\varphi^{'}\rangle\mu_M = 0 \tag{4.35}$$

The kernel is defined by the boundary condition

$$\widetilde{P}^{+}_{\geq 0}\varphi = 0. \tag{4.36}$$

Integration by parts yields

$$\int_M \langle \varphi, (D - \frac{n+1}{2})\varphi^{'}\rangle \mu_M + \int_\Sigma \langle \varphi, \rho(\nu)\varphi^{'}\rangle \mu_\sigma = 0 \tag{4.37}$$

Now $\varphi = \widetilde{P}^{+}_{\geq 0}\varphi + \widetilde{P}^{-}_{\leq 0}\varphi$ and the boundary condition implies $\widetilde{P}^{+}_{\geq 0}\varphi = 0$. Therefore, we have

$$\int_M \langle \varphi, (D - \frac{n+1}{2})\varphi^{'}\rangle \mu_M + \int_\Sigma \langle \widetilde{P}^{-}_{\leq 0}\varphi, \widetilde{P}^{+}_{\geq 0}(\rho(\nu)\varphi^{'}) + \widetilde{P}^{-}_{\leq 0}(\rho(\nu)\varphi^{'})\rangle \mu_\sigma = 0 \tag{4.38}$$

or

$$\int_M \langle \varphi, (D - \frac{n+1}{2})\varphi^{'}\rangle \mu_M + \int_M \langle \widetilde{P}^{-}_{\leq 0}\varphi, \widetilde{P}^{-}_{\leq 0}(\rho(\nu)\varphi^{'})\rangle \mu_\sigma = 0 \tag{4.39}$$

$\forall \varphi$ in the domain of $\widetilde{D}^{+}$. This yields the boundary condition to define co-kernel to be the following

$$\widetilde{P}^{-}_{\leq 0}(\rho(\nu)\varphi^{'}) = 0 \ on \ \Sigma. \tag{4.40}$$

Let us simplify that. Now consider the decomposition of $\varphi^{'}$ in terms of the eigenspinors of $\widetilde{\mathbf{D}}^{+}$ and $\widetilde{\mathbf{D}^{-}}$

$$\varphi^{'} = \widetilde{P}^{+}_{\geq 0}\varphi^{'} + \widetilde{P}^{-}_{\leq 0}\varphi^{'} = \sum_i a_i\Phi^{+}_i + \sum_i d_i\widetilde{\Phi}^{-}_i, \ a_i, d_i \in \mathbb{C}. \tag{4.41}$$

Action of $\rho(\nu)$ on $\varphi^{'}$ yields by 3.1 ($\rho(\nu)(\Phi^{+}_i) = \widetilde{\Phi}^{-}_i$)

$$\rho(\nu)\varphi^{'} = \sum_i a_i\widetilde{\Phi}^{-}_i - \sum_i d_i\Phi^{+}_i \tag{4.42}$$

and consequently $\widetilde{P}^{-}_{\leq 0}(\rho(\nu)\varphi^{'}) = \sum_i a_i\Phi^{-}_i$. Therefore $\widetilde{P}^{-}_{\leq 0}(\rho(\nu)\varphi^{'}) = 0$ implies

$$a_i = 0 \ \forall i \implies \widetilde{P}^{+}_{\geq 0}\varphi^{'} = 0. \tag{4.43}$$

Therefore for $\varphi^{'} \in$ Co-Ker$(D - \frac{n+1}{2})$, the co-kernel is defined as

$$\widetilde{D}^{+}\varphi^{'} = (D - \frac{n+1}{2})\varphi^{'} = 0 \ on \ M, \ \widetilde{P}^{+}_{\geq 0}\varphi^{'} = 0 \ on \ \Sigma. \tag{4.44}$$

Therefore $\widetilde{D}^{+}$ is actually self-adjoint with respect to this new non-local boundary condition. Following exact similar computations as those of the kernel case, we obtain

$$0 \leq \int_\Sigma \left(\langle \widetilde{P}^{-}_{\leq 0}\psi, \widetilde{\mathbf{D}}^{-}\widetilde{P}^{-}_{\leq 0}\psi\rangle - \frac{1}{2}K|\widetilde{P}^{-}_{\leq 0}\psi|^2\right)\mu_\sigma \leq 0 \tag{4.45}$$

since $\int_\Sigma \langle \widetilde{P}^{-}_{\leq 0}\psi, \widetilde{\mathbf{D}}^{-}\widetilde{P}^{-}_{\leq 0}\psi\rangle \mu_\sigma \leq 0$ implying $\widetilde{P}^{+}_{\leq 0}\psi = 0$ on Σ i.e.,

$$\psi = 0 \ on \ \Sigma. \tag{4.46}$$

Now unique continuation of homogeneous first order elliptic partial differential equation then implies

$$\psi = 0 \ on \ M. \tag{4.47}$$

Therefore the desired co-kernel is trivial. Therefore

$$\text{Kernel}(\widetilde{D}^{+}) := \Big\{\psi \in H^{s}(\Gamma(S^{M})) \cap H^{s-\frac{1}{2}}(\Gamma(S^{\Sigma}))|\widetilde{D}^{+}\psi = (D - \frac{n+1}{2})\psi = 0 \ on \ M,$$

$$\widetilde{P}^{+}_{\geq 0}\psi = 0 \ on \ \Sigma\Big\} = \{0\}$$

$$\text{Co-Kernel}(\widetilde{D}^{+}) := \Big\{\psi \in H^{s-1}(\Gamma(S^{M})) \cap H^{s-\frac{3}{2}}(\Gamma(S^{\Sigma}))|\widetilde{D}^{+}\psi = (D - \frac{n+1}{2})\psi = 0 \ on \ M,$$

$$\widetilde{P}^{+}_{\geq 0}\psi = 0 \ on \ \Sigma\Big\} = \text{Kernel}(\widetilde{D}^{+}) = \{0\}$$

The estimates follow trivially as a consequence of the isomorphism property.

Remark 6 *The condition $K > 0$ is vital. Without this condition, the theorem 1.1 breaks down. This condition will turn out to be vital again when proving the rigidity property of spherical caps. In particular, we shall see that the rigidity argument can never be extended to the hemisphere since the equator of the sphere will correspond to $K = 0$ condition where the isomorphism property fails because of 4.31.*

5. Eigenvalue estimates

Under the curvature condition $R[\sigma] \geq (n-1)n$, the intrinsic Dirac operator $\mathbf{D}$ already verifies an absolute lower eigenvaluesl bound $\frac{n}{2}$. This can be further improved in terms of the mean curvature information if (Σ, σ) is realized as an embedded hypersurface in (M, g), in particular bounding a domain. For this, we first obtain the eigenvalue estimates for the operators $\widetilde{\mathbf{D}}^{\pm}$. Let $\widetilde{\lambda}_1 >$ be the first positive eigenvalue of $\widetilde{\mathbf{D}}^{+}$ with eigenspinor ξ i.e.,

$$\widetilde{\mathbf{D}}^{+}\xi = \widetilde{\lambda}_1\xi, \widetilde{\lambda}_1 > 0, \ on \ \Sigma. \tag{5.1}$$

We can use ξ as the boundary condition for the boundary value problem associated with the operator $\widetilde{D}^{+} := \widehat{D} - \frac{n+1}{2}$ on (M, g), that is, for ψ

$$\widehat{D}\psi - \frac{n+1}{2}\psi = 0, \ on \ M \tag{5.2}$$

$$\widetilde{P}^{+}_{\geq 0}\psi = \xi \ on \ \Sigma. \tag{5.3}$$

This has a unique solution ψ according to the theorem 1.1. Now we prove theorem 1.2.

5.1. Proof of theorem 1.2

Let us recall the statement of the theorem 1.2

Theorem (Theorem 1.1) *Let (M, g) be an $n+1-$ dimensional smooth connected oriented Riemannian spin manifold with smooth spin boundary Σ for $n \geq 2$. Let the induced metric on Σ by g is σ. Assume the following:*
(a) The scalar curvature of (M, g) $R[g] \geq n(n+1)$,
(b) The scalar curvature of (Σ, σ) $R[\sigma] \geq n(n-1)$,
(c) Σ is mean convex with respect to g i.e., its mean curvature $K > 0$ if $\mathcal{D}^{+}$ is non-empty and the bound 1.11 if $\mathcal{D}^{+}$ is empty.
Then the first eigenvalue λ_1 of the Dirac operator $\mathbf{D}$ on (Σ, σ) verifies the estimate

$$\lambda_1(\mathbf{D})^2 \geq \frac{1}{4}\inf_{\Sigma} K^2 + \frac{n^2}{4}. \tag{5.4}$$

First, elliptic regularity of the first order Dirac type operator implies the existence of a smooth solution. The standard procedure to this conclusion is to observe that the solution to the boundary value problem 5.2 exists in $H^s(M)$ given $H^{s-1}(M)$ data for every $s \geq 1$ with associated trace on Σ. Then $C^\infty = \cap H^s$. The regularity of the background metrics g and σ is assumed to be C^∞. First, recall that ψ admits the following decomposition by proposition 4.2

$$\psi = \sum_{i=0}^{\infty} a_i \Phi_i^+ + \sum_{i=0}^{\infty} d_i \widetilde{\Phi}_i^-. \tag{5.5}$$

The Witten identity for $\psi \in \mathcal{S}$

$$\int_\Sigma \left(\langle \widetilde{\mathbf{D}}^+ \psi, \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_\sigma = \frac{1}{4} \int_M \left(R^M - n(n+1) \right) |\psi|^2 \mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0.$$

yields via spectral decomposition 4.2

$$\int_\Sigma \left(\langle \widetilde{\mathbf{D}}^+ \widetilde{P}^+_{\geq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle + \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^-_{\leq 0}\psi, \widetilde{P}^-_{\leq 0}\psi \rangle + \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^-_{\leq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle + \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^+_{\geq 0}\psi, \widetilde{P}^-_{\leq 0}\psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_\sigma$$
$$\geq \frac{1}{4} \int_M \left(R^M - n(n+1) \right) |\psi|^2 \mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0.$$

Now notice $\int_\Sigma \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^-_{\leq 0}\psi, \widetilde{P}^-_{\leq 0}\psi \rangle \mu_\sigma = \int_\Sigma \langle \widetilde{P}^-_{\leq 0}\psi, \widetilde{\mathbf{D}}^- \widetilde{P}^-_{\leq 0}\psi \rangle \mu_\sigma \leq 0$, $\int_\Sigma \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^+_{\geq 0}\psi, \widetilde{P}^-_{\leq 0}\psi \rangle = 0$ due to L^2-orthogonality of the Eigenspaces $E^+_{\widetilde{\mathbf{D}}^+}$ and $E^-_{\widetilde{\mathbf{D}}^-}$ along with $\widetilde{\mathbf{D}}^+ E^+_{\widetilde{\mathbf{D}}^+} \subset E^+_{\widetilde{\mathbf{D}}^+}$. Therefore the inequality reduces to

$$\int_\Sigma \left(\langle \widetilde{\mathbf{D}}^+ \widetilde{P}^+_{\geq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle + \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^-_{\leq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_\sigma$$
$$\geq \frac{1}{4} \int_M \left(R^M - n(n+1) \right) |\psi|^2 \mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0. \tag{5.6}$$

Now note the problematic term $\int_\Sigma \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^-_{\leq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle \mu_\sigma$ that does not have a good sign. But this term can be reduced to a lower lower-order term. In fact,

$$\int_\Sigma \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^-_{\leq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle \mu_\sigma = n \int_\Sigma \langle \rho(\nu) \widetilde{P}^-_{\leq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle \mu_\sigma \tag{5.7}$$
$$= n \int_\Sigma \langle \rho(\nu) \widetilde{P}^-_{\leq 0}\psi, \xi \rangle \mu_\sigma = -n \int_\Sigma \langle \widetilde{P}^-_{\leq 0}\psi, \rho(\nu)\xi \rangle \mu_\sigma,$$

where the identity $\widetilde{\mathbf{D}}^+ = \widetilde{\mathbf{D}}^- + n\rho(\nu)$ along with the fact that $\widetilde{\mathbf{D}}^- E^-_{\widetilde{\mathbf{D}}^-} \subset E^-_{\widetilde{\mathbf{D}}^-}$ is used. Now explicitly write out $\widetilde{P}^-_{\leq 0}\psi$ following the decomposition 5.5

$$\widetilde{P}^-_{\leq 0}\psi = \sum_i d_i \widetilde{\Phi}_i^- \tag{5.8}$$

and $\rho(\nu)\xi = \rho(\nu)(\Phi_1^+) = \widetilde{\Phi}_1^-$ due to corollary 3.1. Therefore, simple computation yields

$$\int_\Sigma \langle \widetilde{\mathbf{D}}^+ \widetilde{P}^-_{\leq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle \mu_\sigma = -n\bar{d}_1 ||\widetilde{\Phi}_1^-||^2_{L^2(\Sigma)}. \tag{5.9}$$

The inequality 5.6 reads

$$\int_\Sigma \left(\langle \widetilde{\mathbf{D}}^+ \widetilde{P}^+_{\geq 0}\psi, \widetilde{P}^+_{\geq 0}\psi \rangle - \frac{1}{2} K |\psi|^2 - n\bar{d}_1 |\widetilde{\Phi}_1^-|^2 \right) \mu_\sigma \tag{5.10}$$

$$\geq \frac{1}{4}\int_M \Big(R^M - n(n+1)\Big)|\psi|^2\mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0.$$

The boundary condition 5.3 implies together with the eigenvalue equation 5.1

$$\begin{aligned}
\int_\Sigma \Big(\langle \widetilde{\mathbf{D}}^+\xi,\xi\rangle - \frac{1}{2}K|\psi|^2 - n\bar{d}_1|\widetilde{\Phi}_1^-|^2\Big)\mu_\sigma &\geq \int_\Sigma \Big(\langle \widetilde{\mathbf{D}}^+\widetilde{P}^+_{\geq 0}\psi,\widetilde{P}^+_{\geq 0}\psi\rangle - \frac{1}{2}K|\psi|^2 - n\bar{d}_1|\widetilde{\Phi}_1^-|^2\Big)\mu_\sigma \\
&\geq \frac{1}{4}\int_M \Big(R^M - n(n+1)\Big)|\psi|^2\mu_M + \int_M |\mathcal{Q}\psi|^2 \\
&\geq 0
\end{aligned}$$

or

$$\begin{aligned}
\int_\Sigma \Big(\widetilde{\lambda}_1|\xi|^2 - \frac{1}{2}K|\psi|^2 - n\bar{d}_1|\widetilde{\Phi}_1^-|^2\Big)\mu_\sigma &\geq \frac{1}{4}\int_M \Big(R^M - n(n+1)\Big)|\psi|^2\mu_M + \int_M |\mathcal{Q}\psi|^2 \\
&\geq 0
\end{aligned}$$

Now note $||\xi||^2_{L^2(\Sigma)} = ||\psi||^2_{L^2} - \sum_i |d_i|^2||\widetilde{\Phi}_i^-||^2_{L^2(\Sigma)}$ by the decomposition 5.5 (proposition 4.2). This yields

$$\int_\Sigma(\widetilde{\lambda}_1 - \frac{1}{2}\inf_\Sigma K)|\xi|^2\mu_g - \underbrace{\int_\Sigma(\frac{1}{2}\inf_\Sigma K|d_1|^2 + n\bar{d}_1)|\widetilde{\Phi}_1^-|^2\mu_\sigma}_{I} - \underbrace{\frac{1}{2}\inf_\Sigma K\sum_{i\neq 1}|d_i|^2||\widetilde{\Phi}_i^-||^2_{L^2(\Sigma)}}_{II} \tag{5.11}$$
$$\geq \frac{1}{4}\int_M \Big(R^M - n(n+1)\Big)|\psi|^2\mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0.$$

First, note that the left and right-hand sides of the previous inequality are both real and therefore $\Im(d_1) = 0$. The inequality 5.11 becomes

$$\int_\Sigma(\widetilde{\lambda}_1 - \frac{1}{2}\inf_\Sigma K)|\xi|^2\mu_g - \underbrace{\int_\Sigma(\frac{1}{2}\inf_\Sigma K|\Re(d_1)|^2 + n\Re(d_1))|\widetilde{\Phi}_1^-|^2\mu_\sigma}_{I} - \underbrace{\frac{1}{2}\inf_\Sigma K\sum_{i\neq 1}|d_i|^2||\widetilde{\Phi}_i^-||^2_{L^2(\Sigma)}}_{II}$$
$$\geq \frac{1}{4}\int_M \Big(R^M - n(n+1)\Big)|\psi|^2\mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0.$$

Notice that the term $II \leq 0$. If $I \leq 0$, then we have the desired result $\widetilde{\lambda}_1 \geq \frac{1}{2}\inf_\Sigma K$. But non-positivity of the term I is not so obvious since $\Re(d_1)$ can be negative. Now consider two cases

Case I: $\Re(d_1) \geq 0$

$\Re(d_1) \geq 0$ implies $I \leq 0$ and therefore

$$\int_\Sigma(\widetilde{\lambda}_1 - \frac{1}{2}K)|\xi|^2\mu_g \geq \frac{1}{4}\int_M \Big(R^M - n(n+1)\Big)|\psi|^2\mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0$$

implying

$$\widetilde{\lambda}_1 \geq \frac{1}{2}\inf_\Sigma K. \tag{5.12}$$

Case II: $\Re(d_1) < 0$

We want $I \leq 0$ which implies

$$\frac{1}{2}\inf_\Sigma K|\Re(d_1)|^2 - n|\Re(d_1)| \geq 0 \textit{ or } \inf_\Sigma K \geq \frac{2n}{|\Re(d_1)|} = \frac{2n}{|d_1|} \tag{5.13}$$

since $\Im(d_1) = 0$. Then similarly we have

$$\int_\Sigma (\widetilde{\lambda}_1 - \frac{1}{2}K)|\xi|^2\mu_g \geq \frac{1}{4}\int_M \left(R^M - n(n+1)\right)|\psi|^2\mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0$$

implying

$$\widetilde{\lambda}_1 \geq \frac{1}{2}\inf_\Sigma K. \tag{5.14}$$

Therefore, in either case, we have

$$\lambda_1^2 = \widetilde{\lambda}_1^2 + \frac{n^2}{4} \geq \frac{1}{4}\inf_\Sigma K^2 + \frac{n^2}{4} \tag{5.15}$$

where λ_1 is the first positive eigenvalue of the Dirac operator $\mathbf{D}$ on Σ. This completes the proof of the theorem 1.2.
Let us define the following spaces

$$\mathcal{D}^+ := \{\varphi \in H^1(\Gamma(S^M)) \cap H^{\frac{1}{2}}(\Gamma(S^\Sigma)) | \widetilde{D}^+\varphi = 0, on\ M, \widetilde{P}^+_{\geq 0}\psi = \xi, on\ \Sigma, \Re(\int_\Sigma \langle \varphi|_\Sigma, \widetilde{\Phi}^-_1\rangle\mu_\sigma) \geq 0\}$$

$$\mathcal{D}^- := \{\varphi \in H^1(\Gamma(S^M)) \cap H^{\frac{1}{2}}(\Gamma(S^\Sigma)) | \widetilde{D}^+\varphi = 0, on\ M, \widetilde{P}^+_{\geq 0}\psi = \xi, on\ \Sigma, \Re(\int_\Sigma \langle \varphi|_\Sigma, \widetilde{\Phi}^-_1\rangle\mu_\sigma) < 0\}.$$

Obviously, any $\psi \in \Gamma(S^M)$ solving the Dirac equation for the real Killing connection with the new non-local boundary condition

$$\widetilde{D}\psi = 0\ on\ M, \tag{5.16}$$

$$\widetilde{P}^+_{\geq 0}\psi = \xi,\ on\ \Sigma \tag{5.17}$$

lies in $\mathcal{D}^+ \cup \mathcal{D}^-$. Clearly $\mathcal{D}^+ \cup \mathcal{D}^-$ is non-empty due to theorem 1.1 and $\mathbb{R} = \mathbb{R}_{\geq 0} \cup \mathbb{R}_{<0}$. If $\mathcal{D}^+$ is non-empty then the eigenvalue estimate $\lambda_1^2 \geq \frac{1}{4}\inf_\Sigma K^2 + \frac{n^2}{4}$ is unconditional i.e., does not depend on the specific lower bound on the mean curvature K of the boundary Σ (as long as $K > 0$ is verified this works). However, if $\mathcal{D}^+$ is empty, then one requires a more precise lower bound on K. Let us understand this in more detail. First, consider for the restriction of $\psi|_\Sigma$ (which we also denote by ψ) one has by the decomposition 4.2

$$\psi = \sum_{i=0}^\infty a_i\Phi^+_i + \sum_{i=0}^\infty d_i\widetilde{\Phi}^-_i \tag{5.18}$$

and for a $\psi \in \mathcal{D}^-$,

$$\psi = \xi + \sum_{d_i} d_i\widetilde{\Phi}^-_i,\ \Re(d_1) < 0. \tag{5.19}$$

For convenience, let us consider the eigenspinors to have L^2-norm 1. The following identities hold (note that for $\psi \in \mathcal{D}^+ \cup \mathcal{D}^-$, $\Im(d_1) = 0$)

$$||\psi||^2_{L^2(\Sigma)} = 1 + d_1^2 + \sum_{i\neq 1}|d_i|^2, \tag{5.20}$$

$$\Re(\int_\Sigma \langle \psi, \widetilde{\Phi}^-_1\rangle\mu_\sigma) = d_1. \tag{5.21}$$

These yield for $\psi \in \mathcal{D}^-$,

$$|\Re(\int_\Sigma \langle \psi, \widetilde{\Phi}_1^- \rangle \mu_\sigma)|^2 \leq ||\psi||^2_{L^2(\Sigma)} - 1. \tag{5.22}$$

Therefore we can obtain the lower bound of the mean curvature K in terms of the following geometric entity by 5.13 in case $\mathcal{D}^+$ is empty and the solutions only lie in $\mathcal{D}^-$

$$\inf_\Sigma K \geq \frac{2n}{\sqrt{\sup_{\psi \in \mathcal{D}^-} ||\psi||^2_{L^2(\Sigma)} - 1}} \tag{5.23}$$

Note that $||\psi||_{L^2(\Sigma)} > 1$ since $\psi \in \mathcal{D}^-$ i.e., $d_1 \neq 0$ and $||\psi||_{L^2(\Sigma)} < \infty$ sine $\psi \in H^1(M) \cap H^{\frac{1}{2}}(\Sigma)$.
Let us understand the equality case in detail. To this end, we first, invoke the following result by [5] on the classification of manifolds with real Killing spinors

Theorem 5.1 *[5] Let M be compact and simply connected.*
(a) Let (M, g) be a complete simply connected Riemannian spin $n+1-$manifold carrying a non-trivial real Killing spinor with Killing constant $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$, then if $n+1$ is even and $n+1 \neq 6$, then (M, g) is isometric to the standard sphere.
(b) Let (M, g) be a complete simply connected Riemannian spin manifold of dimension $n+1$ with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. If $n+1 = 2m-1,\ m \geq 3$ odd, then there are two possibilities
(b1) (M, g) is isometric to the standard sphere
(b2) (M, g) is of type $(1, 1)$ and M is an Einstein-Sasaki manifold,

(c) Let (M, g) be a complete simply connected Riemannian spin manifold of dimension $n+1$ with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. If $n+1 = 4m-1,\ m \geq 3$, then there are two possibilities
(c1) (M, g) is isometric to the standard sphere
(c2) (M, g) is of type $(2, 0)$ and M is an Einstein-Sasaki manifold, but does not carry a Sasaki-3-structure
(c3) M is of type $(m+1, 0)$ and M carries a Sasaki-3-structure,

(d) Let (M, g) be a $7-$dimensional complete simply connected Riemannian spin manifold with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. Then there are four possibilities
(d1) (M,g) is isometric to $\mathbb{S}^7$
(d2) M is of type $(1, 0)$ and M carries a nice $3-$form ϕ with $\nabla\phi =^* \phi$ but not a Sasaki structure
(d3) M is of type $(2, 0)$ and M carries a Sasaki structure, but not a Sasaki-3-structure
(d4) M is of type $(3, 0)$ and M carries a Sasaki-3-structure.

For the equality case, let us consider the constant K case first. Let the mean curvature of (Σ, σ) be a constant $K > 0$ if $\mathcal{D}^+$ is non-empty and verifies the appropriate lower bound if $\mathcal{D}^+$ is empty. Then according to theorem 1.2, the eigenvalue estimate reads

$$\lambda_1(\mathbf{D}) \geq \frac{1}{4}K^2 + \frac{n^2}{4}. \tag{5.24}$$

When the equality holds, the inequality 5.14 yields

$$R[g] = n(n+1),\ \mathcal{Q}\psi = 0 \tag{5.25}$$

which implies together with $\widetilde{D}^+\psi = (D - \frac{n+1}{2})\psi = 0$ yields

$$\widehat{\nabla}_X\psi = -\frac{1}{2}\rho(X)\psi,\ X \in \Gamma(TM) \tag{5.26}$$

i.e., ψ is a *real* Killing spinor with Killing constant $\alpha = \frac{1}{2}$ (relation 3.1). As a consequence of being Killing it has constant norm i.e., $\nabla_X|\psi|^2 = \frac{1}{2}\langle\rho(X)\psi,\psi\rangle + \frac{1}{2}\langle\psi,\rho(X)\psi\rangle = 0$. Therefore, One may set $|\psi| = 1$ without loss of generality. Now, one could perform all the foregoing analysis by solving the boundary value problem associated with $\widetilde{D}^-$ i.e.,

$$(D + \frac{n+1}{2})\psi = 0 \ on\ M \tag{5.27}$$

$$\widetilde{P}^-_{\geq 0}\psi = \widetilde{\Phi}^+_1 \ on\ \Sigma. \tag{5.28}$$

In such case, we will have the same estimate for the eigenvalue of $\mathbf{D}$ and the equality case would correspond to ψ verifying the Killing equation with $\alpha = \frac{1}{2}$ i.e.,

$$\widehat{\nabla}_X\psi = -\frac{1}{2}\rho(X)\psi,\ X \in \Gamma(TM). \tag{5.29}$$

Therefore, we have the following corollary of theorem 1.2

Corollary 5.1 *Let (Σ,σ) be the constant mean curvature boundary of a $n+1$ dimensional simply connected oriented Riemannian spin manifold (M,g) and let the mean curvature be a constant $K > 0$ if $\mathcal{D}^+$ is non-empty and the bound 5.23 if $\mathcal{D}^+$ is empty. Also, assume the scalar curvatures $R[g] \geq (n+1)n$ and $R[\sigma] \geq n(n-1)$. Then the intrinsic Dirac operator $\mathbf{D}$ associated with the Riemannian metric σ on (Σ,σ) has the lowest eigenvalue $\lambda_1(\mathbf{D})^2 \geq \frac{1}{2}K^2 + \frac{n^2}{4}$. Moreover, $\lambda_1(\mathbf{D})^2 = \frac{1}{2}K^2 + \frac{n^2}{4}$ if and only if (M,g) is positive Einstein with Ricci curvature verifying Ric $= ng$ and admits real Killing spinors-specifically one of the manifolds classified by Bär in theorem 5.1.*

Proof. First, once again note that $\mathcal{D}^+ \cup \mathcal{D}^-$ is nonempty by theorem 1.1. In addition, if $\mathcal{D}^+$ is nonempty, then $K > 0$ while the bound 5.23 is assumed if empty. The 'if' part of the equality case $\lambda_1(\mathbf{D})$ is straightforward and discussed above. We prove the reverse direction now. Suppose (Σ,σ) is the constant mean curvature boundary of (M,g) which is a simply connected $n+1$ dimensional Riemannian spin manifold that is Einstein and admits real Killing spinors with Killing constant $\alpha = \pm\frac{1}{2}$ (i.e., manifolds that fall under Bär's classification). If ψ is such a Killing spinor with $|\psi| = 1$, then the following inequality 5.14

$$\int_\Sigma \left(\widetilde{\lambda}_1|\xi|^2 - \frac{1}{2}K|\xi|^2\right)\mu_\sigma \geq \frac{1}{4}\int_M \left(R^M - n(n+1)\right)|\psi|^2\mu_M + \int_M |\mathcal{Q}\psi|^2 = 0$$

yields

$$\widetilde{\lambda}_1 \geq \frac{1}{2}K \implies \lambda_1(\mathbf{D})^2 \geq \frac{K^2}{4} + \frac{n^2}{4} \tag{5.30}$$

which is of course true by the preceding analysis. Now as the equality is verified, one has

$$\widehat{\nabla}_X\psi = \pm\frac{1}{2}\rho(X)\psi \tag{5.31}$$

and as a consequence of the proposition 2.1,

$$\mathbf{D}\psi = \frac{1}{2}K\psi \pm \frac{n}{2}\rho(\nu)\psi \ on\ \Sigma. \tag{5.32}$$

Now compute the Rayleigh quotient of $\mathbf{D}$

$$\lambda_1^2(\mathbf{D}) \leq \frac{\int_\Sigma \langle \mathbf{D}\psi, \mathbf{D}\psi\rangle \mu_\sigma}{\int_\Sigma |\psi|^2 \mu_\sigma} = \frac{K^2}{4} + \frac{n^2}{4}. \tag{5.33}$$

Therefore $\lambda_1(\mathbf{D})^2 = \frac{K^2}{4} + \frac{n^2}{4}$. Therefore equality holds for the eigenvalue and the reverse direction follows. $\square$

When the mean curvature is not constant, then we have the following bound for $\lambda_1(\mathbf{D})$ on the boundary of simply connected spin manifolds admitting real Killing spinors (Bar's [5] classification)

$$\frac{1}{4}\inf_\Sigma K^2 + \frac{n^2}{4} \leq \lambda_1(\mathbf{D})^2 \leq \frac{1}{4}\sup_\Sigma K^2 + \frac{n^2}{4}. \tag{5.34}$$

Remark 7 *We can never prove the case for $K = 0$ since that would result in losing the isomorphism property of $\widetilde{D}^\pm$ and failure of theorem 1.1. Of course, without theorem 1.1, the rest of the preceding analysis fails.*

6. Rigidity Results

Using the result of the theorem 1.2, we will prove the theorem 1.3. Let us recall the statement of the theorem 1.3

Theorem (Rigidity of Geodesic Balls) *Let M be a $n+1$, $n \geq 2$ dimensional cap with boundary Σ and g be a smooth Riemannian metric on M that induces metric σ on Σ. Let the following conditions are verified by (M, g) and (Σ, σ)*
(a) Scalar curvature of (M, g) $R[g] \geq (n+1)n$,
(b) The induced metric σ on Σ by g agrees with the metric σ_0 of the boundary of a geodesic ball of standard unit sphere of radius less than or equal to $l := \frac{\pi}{2} - \epsilon$ for any $\epsilon > 0$ if $\mathcal{D}^+$ is non-empty and $\epsilon \geq \sin^{-1}\sqrt{\frac{K_0^2}{n^2+K_0^2}}$ if $\mathcal{D}^+$ is empty.
(c) The mean curvature of Σ with respect to g and the standard round metric on unit sphere $\mathbb{S}^{n+1}$ coincide, constant, and strictly positive,
Then g is isometric to the standard round metric on the unit sphere $\mathbb{S}^{n+1}$. In particular, M is isometric to a geodesic ball of radius l in a standard unit sphere.

We prove this theorem through eigenvalue comparison. First, note that $\mathcal{D}^+ \cup \mathcal{D}^-$ is non-empty on geodesic balls of mean curvature strictly positive on $\mathbb{S}^{n+1}$. But first, consider the elementary calculations regarding a geodesic ball (M_0, g_0) of radius strictly less than $\frac{\pi}{2}$ in $\mathbb{S}^{n+1}$ i.e., g_0 is isometric to the standard round metric on $\mathbb{S}^{n+1}$. To be more explicit, if $\mathbb{S}^{n+1}$ is a unit sphere in $\mathbb{R}^{n+2}$ written as $\sum_{i=1}^{n+2} x_i^2 = 1$, then (M_0, g_0) corresponds to the totally geodesic subset of $\mathbb{S}^{n+1}$ with $x^{n+2} \geq c$ with $0 < \sin\epsilon \leq c \leq 1$ if $\mathcal{D}^+$ is non-empty and $\sin\epsilon \geq \sqrt{\frac{K_0^2}{n^2+K_0^2}}$ if empty. Note that the set of geodesic balls with the bound $K \geq K_0$ is nonempty since the range of the mean curvature of geodesic balls of radius strictly less than $\frac{\pi}{2}$ is $(0, \infty)$ and $1 < ||\psi||_{L^2(\Sigma)} < \infty$ since $\psi \in H^{\frac{1}{2}}(\Sigma)$. Here K_0 is the corresponding lower bound on the mean curvature of the boundary of (M, g). The boundary of (M_0, g_0) corresponds to the sphere $\mathbb{S}^n_r$ of radius $r = \sqrt{1-c^2}$. This sphere has constant mean curvature $\widetilde{K}$. In usual spherical coordinate, the metric g_0 reads

$$g_0 = d\theta^2 + \sin^2\theta (d\mathbb{S}^n)^2,\ \theta \in [0, \sin^{-1}\sqrt{1-c^2}]. \tag{6.1}$$

The second fundamental form of the coordinate spheres ($\theta =$ constant) w.r.t g_0 reads

$$K_{AB} = \langle \nabla_A \partial_\theta, B \rangle = \sin\theta \cos\theta (d\mathbb{S}^n)^2(A, B),\ \ A, B \in \Gamma(T\mathbb{S}^n). \tag{6.2}$$

The mean curvature of the θ =constant spheres with respect to the inward-pointing normal field $-\partial_\theta$ reads

$$K = \mathrm{tr}_{\sin^2\theta (d\mathbb{S}^n)^2}(\frac{1}{2}\partial_\theta(\sin^2\theta (d\mathbb{S}^n)^2)) = n\cot\theta \tag{6.3}$$

Therefore (Σ, σ) has the mean curvature

$$\widetilde{K} = \frac{nc}{\sqrt{1-c^2}} > 0. \tag{6.4}$$

and moreover $\widetilde{K} \geq K_0$ if $\mathcal{D}^+$ is empty. The induced metric σ on Σ reads

$$\sigma = (1-c^2)(d\mathbb{S}^n)^2. \tag{6.5}$$

6.1. proof of theorem 1.3

(M, g) is simply connected and its boundary is (Σ, σ). The scalar curvature of (Σ, σ) is

$$R[\sigma] = \frac{(n-1)n}{1-c^2} \geq (n-1)n \tag{6.6}$$

and therefore verifies the hypothesis of the foregoing analysis. By the hypothesis of the theorem, the intrinsic geometry and the mean curvature of Σ are the same in (M, g). In particular, mean curvature of (Σ, σ) in (M, g)

$$K = \frac{nc}{\sqrt{1-c^2}} = \widetilde{K}. \tag{6.7}$$

Now the intrinsic Dirac operator $\mathbf{D}$ on (Σ, σ) has the lowest eigenvalue [6]

$$\lambda_1(\mathbf{D})^2 = \frac{n^2}{4(1-c^2)} = \frac{1}{4}\frac{n^2c^2}{1-c^2} + \frac{n^2}{4} = \frac{K^2}{4} + \frac{n^2}{4} \tag{6.8}$$

and therefore equality holds for the lowest eigenvalue. Therefore by Corollary 5.1, (M, g) has non-trivial real Killing spinor fields with Killing constant $\pm\frac{1}{2}$, and therefore it is Einstein i.e., $\mathrm{Ric}[g] = ng$. It is simply connected and therefore falls under the classification of [5]. We will prove that g is actually isometric to the standard spherical metric g_0. First, note the Gauss equation

$$R[\sigma] = R[g] - 2\mathrm{Ric}[g](\nu, \nu) + K^2 - K_{ij}K^{ij} \tag{6.9}$$

Now recall the decomposition of the second fundamental form $K_{ij} = \widehat{K}_{ij} + \frac{1}{n}K\sigma_{ij}$ and obtain

$$R[\sigma] = R[g] - 2\mathrm{Ric}[g](\nu, \nu) + (1 - \frac{1}{n})K^2 - \widehat{K}_{ij}\widehat{K}^{ij} \implies \widehat{K}_{ij}\widehat{K}^{ij} = 0 \tag{6.10}$$

Therefore (Σ, σ) is totally umbilic in (M, g). Now recall the following theorem of [21] regarding the existence of totally umbilic hypersurfaces in an Einstein manifold.

Theorem 6.1 ([21]) *Let (Σ, σ) be a totally umbilical Einstein hypersurface in a complete Einstein manifold. Then if σ has positive Ricci curvature, then both g and σ have constant sectional curvature.*

This theorem applies in our context. This yields, (M,g) is a locally symmetric space. Now we use the second integrability condition on the existence of a Killing spinor by [31]

$$\nabla_{e_I} C_{e_J e_K e_L e_N}[\rho(e_L),\rho(e_N)]\psi \mp C_{e_J e_K e_I e_L}\rho(e_L)\psi = 0, \tag{6.11}$$

where C is the Weyl curvature of (M,g). The constancy of the sectional curvature yields

$$C = 0. \tag{6.12}$$

Therefore, (M,g) is a maximally symmetric space and more precisely isometric to the standard sphere metric g_0 (topologically can be either a sphere or its quotient by the discrete subgroups of the isometry group $SO(n+2)$ acting freely and properly discontinuously on the sphere). Since (M,g) is simply connected and (Σ,σ) is its boundary sphere, (M,g) is the geodesic ball (M_0,g_0). This proves the rigidity theorem 1.3.

6.2. Proof of the mass theorem 1.4

Let us recall the statement of theorem 1.4

Theorem (Mass theorem) *Let (M,g) be an $n+1$, $n\geq 2$ dimensional connected oriented Riemannian spin manifold with smooth spin boundary Σ. Let the induced metric on Σ by g be σ. Assume the following:*
(a) The scalar curvature of (M,g) $R[g]\geq n(n+1)$,
(b) The scalar curvature of (Σ,σ) $R[\sigma]\geq n(n-1)$,
(c) Σ is strictly mean convex with respect to g. Then the mass m of (M,g) is defined as

$$m := \inf_{\varphi\in\mathcal{D}^{\pm}\cup\mathcal{D}^{-}} \int_{\Sigma} \langle \widetilde{\mathbf{D}}^{+}\varphi - \frac{1}{2}K\varphi,\varphi\rangle\mu_{\sigma} \tag{6.13}$$

verifies

$$m\geq 0 \tag{6.14}$$

and $m=0$ if and only if (M,g) is an Einstein manifold that admits a real Killing spinor i.e., belongs to the classification of Bär [5].

Proof of the theorem 1.4 is a straightforward consequence of the Witten identity and Bär's classification of manifolds with real Killing spinor [5]. Recall the Witten identity

$$\begin{aligned}\int_{\Sigma}\left(\langle\widetilde{\mathbf{D}}^{\pm}\psi,\psi\rangle - \frac{1}{2}K|\psi|^2\right)\mu_{\sigma} &= \frac{1}{4}\int_M \left(R[g]-n(n+1)\right)|\psi|^2\mu_M - \frac{n}{n+1}\int_M\langle\widetilde{D}^{\pm}\psi,\widetilde{D}^{\mp}\psi\rangle \\ &+ \int_M |\mathcal{Q}\psi|^2.\end{aligned}$$

Now solve the following PDE

$$\widetilde{D}^{\pm}\psi = \widehat{D}\psi \pm \frac{n+1}{2}\psi = 0, \text{ on } M \tag{6.15}$$

$$\widetilde{P}^{\pm}_{\geq 0}\psi = \xi \text{ on } \Sigma. \tag{6.16}$$

for ξ belonging to $L^2_{\widetilde{\mathbf{D}}^{\pm}}(\Gamma(S^{\Sigma}))^{\pm}$ (as defined in 4.16). According to the theorem 1.1, ψ exists if (M,g) and (Σ,σ) verify the desired curvature and mean curvature conditions as stated in the hypothesis of the theorem. Consequently the Witten identity yields

$$\int_{\Sigma}\left(\langle\widetilde{\mathbf{D}}^{\pm}\psi,\psi\rangle - \frac{1}{2}K|\psi|^2\right)\mu_{\sigma} = \frac{1}{4}\int_M\left(R[g]-n(n+1)\right)|\psi|^2\mu_M + \int_M|\mathcal{Q}\psi|^2 \geq 0$$

leading to

$$m := \inf_{\varphi \in \mathcal{D}^+ \cup \mathcal{D}^-} \int_\Sigma \left(\langle \widetilde{\mathbf{D}}^\pm \psi, \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_\sigma \geq 0. \tag{6.17}$$

Notice that the zero spinor does not lie in $\mathcal{D}^+ \cup \mathcal{D}^-$ since that would violate the boundary condition $\widetilde{P}^\pm_{\geq 0}\psi = \xi$ *on* Σ by unique continuation property of homogeneous first order differential equations (note ξ is the first positive non-trivial L^2-normalized eigenspinor of $\widetilde{\mathbf{D}}^\pm$ and therefore can not be identically 0 everywhere on the boundary). Now if $m = 0$ is attained by $\varphi \in \mathcal{D}^+ \cup \mathcal{D}^-$, then Witten identity 6.17 implies

$$R[g] = n(n+1),\ \mathcal{Q}_{e_I}\varphi = \widehat{\nabla}_{e_I}\varphi + \frac{1}{n+1}\rho(e_I)\widehat{D}\varphi = 0\ \forall\ I = 1, ..., n+1. \tag{6.18}$$

This together with the fact that $\varphi \in \mathcal{D}^+ \cup \mathcal{D}^-$ yields

$$\widehat{\nabla}_X\varphi \pm \frac{1}{2}\rho(X)\varphi = 0,\ \forall X \in \Gamma(TM). \tag{6.19}$$

Therefore φ is a real Killing spinor with Killing constant $-\frac{1}{2}$ and the identity $[\nabla_X, \widehat{D}] = \frac{1}{2}\rho(\mathrm{Ric}(\cdot, X))\varphi$ implies (M, g) is Einstein

$$\mathrm{Ric}[g] = ng. \tag{6.20}$$

Therefore (M, g) is an Einstein manifold that admits real Killing spinor i.e., belongs to the set constructed by Bär [5]. Now for the other direction i.e., if (M, g) is an Einstein manifold that admits real Killing spinor i.e., belongs to the set constructed by Bär. Then there exists a Killing spinor φ with Killing constant $-\frac{1}{2}$ (or $\frac{1}{2}$) i.e.,

$$\widehat{\nabla}_{e_I}\varphi \pm \frac{1}{2}\rho(e_I)\varphi = 0\ \forall\ I = 1, 2, 3, ..., n+1 \tag{6.21}$$

which upon Clifford multiplication by $\rho(e_I)$ and summing over $I = 1, 2, 3, ..., n$ yields

$$\widehat{D}\varphi \mp \frac{n+1}{2}\varphi = 0. \tag{6.22}$$

Proposition 2.1 therefore yields

$$\mathbf{D}\varphi = \frac{K}{2}\varphi \mp \frac{n}{2}\rho(\nu)\varphi \ or\ \widetilde{\mathbf{D}}^\pm\varphi = \frac{1}{2}K\varphi \tag{6.23}$$

which implies

$$\langle \widetilde{\mathbf{D}}^\pm\varphi, \varphi \rangle - \frac{1}{2}K|\varphi|^2 = 0 \implies m = 0. \tag{6.24}$$

This completes the proof of theorem 1.4

Bibliography

[1] L. Andersson, M. Dahl, Scalar curvature rigidity for asymptotically locally hyperbolic manifolds, *Annals of Global Analysis and Geometry*, vol. 16, 1-27, 1998.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer, eigenvaluesl asymmetry and Riemannian geometry. I, *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 77, 43-69, 1975.

[3] M.F. Atiyah, V.K. Patodi, I.M. Singer, eigenvaluesl asymmetry and Riemannian geometry. II, *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 78, 405-432, 1975.

[4] M.F. Atiyah, V.K. Patodi, I.M. Singer, eigenvaluesl asymmetry and Riemannian geometry. III, *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 79, 71-99, 1976.

[5] C. Bär, Real Killing spinors and holonomy, *Communications in Mathematical Physics*, vol. 154, 509-521, 1993.

[6] C. Bär, The Dirac operator on space forms of positive curvature, *Journal of the Mathematical Society of Japan*, vol. 48, 69-83, 1996.

[7] C. Bär, W. Ballmann, Guide to elliptic boundary value problems for Dirac-type operators, Springer, 2016.

[8] H. Baum, Complete Riemannian manifolds with imaginary Killing spinors, *Annals of Global Analysis and Geometry*, vol. 7, 205-226, 1989.

[9] S. Brendle, F. Marques, A. Neves, Deformations of the hemisphere that increase scalar curvature, *Inventiones Mathematicae*, vol. 185, 175-197, 2011.

[10] S. Brendle, F.C. Marques, Scalar curvature rigidity of geodesic balls in $\mathbb{S}^n$, *Journal of Differential Geometry*, vol. 88, 379-394, 2011.

[11] P.T. Chruściel, M. Herzlich, The mass of asymptotically hyperbolic Riemannian manifolds, *Pacific Journal of Mathematics*, 231-264, 2003.

[12] P.T. Chruściel, E. Delay, The hyperbolic positive energy theorem, arXiv:1901.05263, 2019.

[13] G. Cox, P. Miao, F.-F. Tam, Remarks on a scalar curvature rigidity theorem of Brendle and Marques, *Asian Journal of Mathematics*, vol. 17, 457-470, 2013.

[14] T. Friedrich, Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung, *Mathematische Nachrichten*, vol. 97, 117-146, 1980.

[15] T. Friedrich, Dirac Operators in Riemannian Geometry, Graduate Studies in Mathematics, vol. 25, 2000.

[16] N. Ginoux, *The Dirac Spectrum*, vol. 1976, Springer, 2009.

[17] F. Hang, X. Wang, Rigidity and non-rigidity results on the sphere, *Communications in Analysis and Geometry*, vol. 14, 91-106, 2006.

[18] O. Hijazi, S. Montiel, A. Roldán, Eigenvalue boundary problems for the Dirac operator, *Communications in Mathematical Physics*, vol. 231, 375-390, 2002.

[19] N. Hitchin, Harmonic spinors, *Advances in Mathematics*, vol. 14, 1-55, 1974.

[20] L.-H. Huang, D. Wu, Rigidity theorems on hemispheres in non-positive space forms, *Communications in Analysis and Geometry*, vol. 18, 339-363, 2010.

[21] N. Koiso, Hypersurfaces of Einstein manifolds, *Annales Scientifiques de l'École Normale Supérieure*, vol. 14, 433-443, 1981.

[22] P. Miao, Positive mass theorem on manifolds admitting corners along a hypersurface, *Advances in Theoretical and Mathematical Physics*, vol. 6, 1163-1182, 2002.

[23] M. Min-Oo, Scalar curvature rigidity of certain symmetric spaces, in *Geometry, Topology, and Dynamics* (Montreal, 1995), 127-137, 1998.

[24] S. Montiel, A compact approach to the positivity of Brown-York's mass and its relation with the Min-Oo conjecture, Yau's Problem #100 and rigidity of hypersurfaces, arXiv:2409.17170, 2024.

[25] T. Parker, C.H. Taubes, On Witten's proof of the positive energy theorem, *Communications in Mathematical Physics*, vol. 84, 223-238, 1982.

[26] R. Schoen, S.T. Yau, On the proof of the positive mass conjecture in general relativity, *Communications in Mathematical Physics*, vol. 65, 45-76, 1979.

[27] R. Schoen, S.T. Yau, Proof of the positive mass theorem. II, *Communications in Mathematical Physics*, vol. 79, 231-260, 1981.

[28] R. Schoen, S.T. Yau, Positive Scalar Curvature and Minimal Hypersurface Singularities, *Surveys in Differential Geometry*, XXIV, 2019.

[29] Y. Shi, L-F. Tam, Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature, *Journal of Differential Geometry*, vol. 62, 79-125, 2002.

[30] V. Toponogov, Evaluation of the length of a closed geodesic on a convex surface, *Doklady Akademii Nauk SSSR*, vol. 124, 282-284, 1959.

[31] P. Van Nieuwenhuizen, P.N. Warner, Integrability conditions for Killing spinors, *Communications in Mathematical Physics*, vol. 93, 277-284, 1984.

[32] E. Witten, A new proof of the positive energy theorem, *Communications in Mathematical Physics*, vol. 80, 381-402, 1981.