

On a Rigidity Result in Positive Scalar Curvature Geometry

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Abstract

I prove a scalar curvature rigidity theorem for spheres. In particular, I prove that geodesic balls of radii strictly less than $\frac{\pi}{2}$ in $n + 1$ ($n \geq 2$) dimensional unit sphere are rigid under smooth perturbations that increase scalar curvature preserving the intrinsic and extrinsic geometry of the boundary, and such rigidity result fails for the hemisphere. The proof of this assertion requires the notion of a real Killing connection and solution of the boundary value problem associated with its Dirac operator. The result serves as the sharpest refinement of the now-disproven Min-Oo conjecture.

Contents

1	Introduction	1
2	Preliminaries	4
3	Spinors on positive scalar curvature geometry	7
4	Eigenvalue estimates	16
4.1	Proof of theorem 1.1	17
5	Rigidity Results	20
5.1	proof of theorem 1.2	21
	Bibliography	22

1. Introduction

The positive mass theorem is one of the most important results in differential geometry. This theorem states that any asymptotically flat Riemannian manifold (M, g) of dimension $n \leq 7$ of non-negative scalar curvature has a non-negative ADM mass. Moreover, the ADM mass for such a manifold is strictly positive unless (M, g) is isometric to the Euclidean space. Schoen-Yau [1, 2] proved this theorem first using minimal surface techniques. Later Witten [3], Taubes-Parker [4] proved it using the spinor method. These spinor methods remove the dimensional restriction but assume that the manifold is spin (which is a non-trivial condition and requires the vanishing of the second Steifel-Whitney class). Recently, Schoen-Yau [5] gave a proof of the positive mass theorem in any dimensions using the minimal surface technique where the spin condition is not required.

Theorem (Positive Mass Theorem) [1, 2, 5] *Assume that M is an asymptotically flat manifold with scalar curvature $R \geq 0$. We then have that the ADM mass is non-negative.*

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Furthermore, if the mass is zero, then M is isometric to $\mathbb{R}^n$ with the standard Euclidean metric.

Several rigidity results can be deduced by applying this theorem. The first observation was made by Miao [6] who pointed out that the positive mass theorem implies the following

Theorem [6] *Let D^n be a topological n -ball and g be a smooth metric on D^n that has the following properties*

- (a) *Scalar curvature of g is non-negative i.e., $R[g] \geq 0$,*
 - (b) *The induced metric on the boundary ∂D^n is the standard round metric on $\mathbb{S}^{n-1}$,*
 - (c) *The mean curvature of ∂D^n with respect to g is at least $n - 1$,*
- then g is isometric to the standard Euclidean metric on D^n .*

The sketch of the proof of this theorem can be stated as follows. Let us first remove the standard unit ball $\mathbb{B}^n$ from $\mathbb{R}^n$. The boundary is the unit sphere with the standard round metric. Now glue D^n to $\mathbb{R}^n - \mathbb{B}^n$ along the boundary sphere ∂D^n . Let the metric on this new manifold be denoted by $\tilde{g}$. The new manifold obtained thus verifies the scalar curvature condition $R[\tilde{g}] \geq 0$ and the unit sphere along which D^n is glued verifies the mean curvature condition as well. Now this is trivially asymptotically flat and in fact exactly Euclidean at infinity and therefore has zero ADM mass. Therefore by positive mass theorem, $\tilde{g}$ is isometric to the standard Euclidean metric on $\mathbb{R}^n$.

Later, Shi-Tam [7] proved a localized version of positive mass theorem. The following theorem was proved by Shi-Tam

Theorem [7] *Let Ω be a strictly convex domain in $\mathbb{R}^n$ with smooth boundary and a smooth Riemannian metric g verifying*

- (a) *The scalar curvature $R[g] \geq 0$,*
- (b) *The induced metric on the boundary $\partial\Omega$ by g is the same as the restriction of the standard Euclidean metric on $\partial\Omega$,*
- (c) *The mean curvature of $\partial\Omega$ with respect to g is strictly positive. Then the the Brown-York mass associated with $\partial\Omega$ is non-negative i.e.,*

$$\int_{\partial\Omega} (K_0 - K_g) \mu_{\sigma_g} \geq 0, \quad (1.1)$$

where K_0 is the mean curvature of $\partial\Omega$ with respect to the Euclidean metric whereas K_g is with respect to that of g . Moreover, if the equality holds, then g is isometric to the standard Euclidean metric.

Importantly note that the expression $\int_{\partial\Omega} (K_0 - K_g) \mu_{\sigma_g}$ approaches the ADM mass for (Ω, g) in the limit where $\partial\Omega$ approaches infinity. This entity is also called the Brown-York mass that arises in time-symmetric general relativistic settings. Analogous results have also been proven in the asymptotically hyperbolic framework where the manifold under study verifies a negative lower bound of $-n(n - 1)$ for the scalar curvature [8, 9, 10]. The other model geometry is the sphere with strictly positive curvature. Motivated by the relevant results on asymptotically flat and hyperbolic settings, Min-Oo [11] conjectured that an analogous should hold true for spheres. More precisely the Min-Oo conjecture states the following

Conjecture 1.1 (Min-Oo conjecture[11]) *Let g be a smooth metric on the upper hemisphere S_+^n with the following property*

- (a) *It has scalar curvature $R[g] \geq (n - 1)n$,*
- (b) *the induced metric on the boundary sphere ∂S_+^n is the same as the standard round metric*

on $\mathbb{S}^{n-1}$,

(c) The boundary ∂S_+^n is totally geodesic with respect to g . Then g is isometric to the standard round metric on $\mathbb{S}_+^n$.

This conjecture can be thought of as the positive mass theorem for the strictly positive scalar curvature geometry. There have been several results in special circumstances. This is true for the two-dimensional case by a theorem of Toponogov [12]. For $n \geq 3$, Hang and Wang [13] proved that this conjecture holds true for any metric conformal to the standard metric on the sphere. Huang and Hu [14] proved that the Min-Oo conjecture holds true for a class of manifolds that are graphs of hypersurfaces in $\mathbb{R}^{n+1}$. Finally, Brendle-Marques-Neves [15] proved that the conjecture is false generically for $n \geq 3$. In particular, they constructed counterexamples where the scalar curvature in the bulk can be increased keeping the boundary intact (i.e., totally geodesic and isometric to standard $n - 1$ sphere). They accomplish this in two steps. First, they perform a perturbation of the standard hemisphere to increase the scalar curvature. This process makes the mean curvature of the boundary strictly positive. In the next step, they perturb to make the boundary totally geodesic. This perturbation is supported close to the boundary and is not strong enough to reduce the scalar curvature below the strict lower bound $(n - 1)n$.

Motivated by these results, it is natural to ask whether the rigidity result holds for geodesic balls of radius strictly less than $\frac{\pi}{2}$ instead of the hemisphere. It turns out that indeed such a rigidity result holds true if one imposes certain additional conditions on the spin structures of the geodesic balls and their boundaries. This is accomplished in two steps. First, we consider a general positive scalar curvature connected spin $n + 1$ -dimensional Riemannian manifold with a mean convex boundary. Imposing a lower bound on the scalar curvature of the boundary, we prove a new and improved eigenvalue estimate for the boundary Dirac operator (see [16] for a review on the eigenvalue estimates of the Dirac operators). This step involves solving the Dirac equation associated with a real Killing connection using an APS type non-local boundary conditions [27, 28, 29]. The first theorem that we prove is

Theorem 1.1 (Eigenvalue Estimate) *Let (M, g) be an $n + 1$, $n \geq 2$ dimensional connected oriented Riemannian spin manifold with smooth spin boundary Σ . Let the induced metric on Σ by g is σ . Assume the following:*

- (a) *The scalar curvature of (M, g) $R[g] \geq n(n + 1)$,*
- (b) *The scalar curvature of (Σ, σ) $R[\sigma] \geq n(n - 1)$,*
- (c) *Σ is mean convex with respect to g i.e., its mean curvature $K > 0$, Then the first eigenvalue λ_1 of the Dirac operator $\mathbf{D}$ on (Σ, σ) verifies the estimate*

$$\lambda_1(\mathbf{D})^2 \geq \frac{1}{4} \inf_{\Sigma} K^2 + \frac{n^2}{4}. \quad (1.2)$$

This eigenvalue estimate is new (see [16] for an overview of the existing results; Montiel [17] attempted to obtain such an estimate but the inequality in theorem A of Montiel's article is impossible precisely because of the impossibility of prescribing a Dirichlet type boundary condition for Dirac equations on a compact manifold with boundary). In the second step, we use the eigenvalue estimate to prove a rigidity result for the geodesic balls of radius strictly less than $\frac{\pi}{2}$. In particular, we stress the fact that this rigidity fails for the geodesic ball of radius $\frac{\pi}{2}$ i.e., the upper hemisphere. The crucial assumption of the equality of the intrinsic and induced spin structure on the boundary Σ is verified automatically in dimensions greater than one. The lack of such a rigidity theorem for the hemisphere is due to the loss of surjectivity property of the Dirac operator associated with the real Killing connection. The rigidity theorem is stated as follows

Theorem 1.2 (Rigidity of Geodesic Balls) *Let M be a $n + 1$, $n \geq 2$ dimensional cap with boundary Σ and g be a smooth Riemannian metric on M that induces metric σ on Σ . Let the following conditions are verified by (M, g) and (Σ, σ)*

- (a) *Scalar curvature of (M, g) $R[g] \geq (n + 1)n$,*
 - (b) *The induced metric σ on Σ by g agrees with the metric σ_0 of the boundary of a geodesic ball of standard unit sphere of radius less than or equal to $l := \frac{\pi}{2} - \epsilon$ for ϵ greater than or equal to a small positive number let's say $\frac{1}{100}$,*
 - (c) *The mean curvature of Σ with respect to g and the standard round metric on unit sphere $\mathbb{S}^{n+1}$ coincide, constant, and strictly positive,*
- Then g is isometric to the standard round metric on the unit sphere $\mathbb{S}^{n+1}$. In particular, M is isometric to a geodesic ball of radius l in a standard unit sphere.*

This can be interpreted as the sharpest weaker version of the Min-Oo conjecture in a generic setting. The failure of rigidity of the hemisphere implies a lack of positive mass theorem in the strictly positive scalar curvature geometry. This has implications for the existence of black holes in de-Sitter spacetime in relativity.

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2. Preliminaries

We denote by (M, g) an $n + 1$, $n \geq 2$ dimensional smooth connected Riemannian spin manifold. Let (Σ, σ) be the smooth boundary of M and σ the induced Riemannian metric on it by g . Let the spin structure of M be denoted by Spin_M . Let $\{e_I\}_{I=1}^{n+1}$ be an orthonormal basis for the $SO(n + 1)$ -frame bundle of (M, g) . The last basis vector e_{n+1} is set to be the interior pointing normal vector to the boundary (Σ, σ) and is globally defined on (M, g) along (Σ, σ) and denote it by ν . This allows us to induce a spin structure on (Σ, σ) from Spin_M .

Let $\mathbb{C}l_n$ be the n dimensional complex Clifford algebra and its even part is denoted by $\mathbb{C}l_n^0$. Using ν we can induce an isomorphism between $\mathbb{C}l_n$ and $\mathbb{C}l_{n+1}^0$ as follows

$$\mathcal{I} : \mathbb{C}l_n \rightarrow \mathbb{C}l_{n+1}^0 \quad (2.1)$$

$$e_I \mapsto e_I \cdot \nu, \quad I = 1, \dots, n. \quad (2.2)$$

Now the principal $SO(n)$ bundle of oriented orthonormal frames on Σ can be identified as a sub-bundle of $SO(n + 1)$ bundle on M along Σ that leaves ν invariant. Let $\mathcal{J} : SO(n)_\Sigma \hookrightarrow SO(n + 1)_M|_\Sigma$ denote such an identification map. We can pull back the fiber bundle the spin bundle $\text{Spin}_M|_\Sigma$ on $SO(n + 1)_M|_\Sigma$ as a spin structure on Σ . Let us denote this induced spin bundle on Σ by Spin_Σ . Note the commutative diagram below.

$$\begin{array}{ccc} \text{Spin}_\Sigma & \xrightarrow{\mathcal{J}^*} & \text{Spin}_M|_\Sigma \\ \pi \downarrow & & \downarrow \pi \\ SO(n)_\Sigma & \xrightarrow{\mathcal{J}} & SO(n + 1)_M|_\Sigma \end{array}$$

Let S^M denote the spinor bundle of (M, g) . As a bundle S^M may be identified with the trivial bundle $\text{Spin}_M \times \Delta_{n+1} \rightarrow \Delta_{n+1}$, Δ_{n+1} is the space of spinors, of complex dimension $2^{\lfloor \frac{n+1}{2} \rfloor}$. Each tangent vector $X \in T_x M$ induces a skew-adjoint map $\rho(X) \in \text{End}(S_x^M)$. For the orthonormal basis $\{e_I\}_{I=1}^{n+1}$, Clifford relation holds

$$\rho(e_I)\rho(e_J) + \rho(e_J)\rho(e_I) = -2\delta_{IJ}\text{id} \quad (2.3)$$

for $i, j = 1, \dots, n+1$. For $n+1$ odd, the representation ρ is chosen such that the complex volume form acts as identity on Δ_{n+1} . Locally a section ψ of the spinor bundle S^M over an open set $M \subset M$ may be represented by the double

$$\psi = [s, \beta], \quad (2.4)$$

where $s : M \rightarrow \text{Spin}_M$ and $\beta : M \rightarrow \Delta_{n+1}$. For an element $A \in \text{Spin}(n+1)_M$, the gauge transformation is written as

$$s \mapsto sA, \quad \beta \mapsto \rho(A^{-1})\beta. \quad (2.5)$$

One may restrict the spinor bundle S^M on Σ by using the $\mathcal{I}$ map i.e., define the restricted bundle as $S^M|_\Sigma := \text{Spin}_\Sigma \times_{\rho \circ \mathcal{I}} \Delta$. The Clifford multiplication on Σ is defined as follows. For $X \in \Gamma(T\Sigma)$, the Clifford multiplication $\rho^\Sigma(X) = \rho(X) \circ \rho(\nu)$. The intrinsic spinor bundle of Σ $S^\Sigma = \text{Spin}_\Sigma \times_{\rho^\Sigma} \Delta_n$ can be identified with $S^M|_\Sigma$ if $n+1$ is odd. For $n+1$ even, one may obtain the identification $S^M|_\Sigma \simeq S^\Sigma \oplus S^\Sigma$. This implicitly uses the assumption that the intrinsic spin structure on Σ and that induced from (M, g) are the same. For details about the theory of induced spin structures, readers are referred to [19, 18].

Now we discuss the extrinsic Riemannian geometry of (Σ, σ) as a hypersurface in (M, g) . Let $\widehat{\nabla}$ be the metric compatible connection on (M, g) and ∇ is the induced connection on Σ . For two smooth vector fields $A, B \in T\Sigma$, $\widehat{\nabla}$ and ∇ are related by

$$\widehat{\nabla}_A B = \nabla_A B + \mathcal{H}(A, B)\nu, \quad (2.6)$$

where ν is interior directed unit normal vector to the boundary Σ and $\mathcal{H}$ is the second fundamental form of Σ in M .

On the spinor bundle S^M there is a natural Hermitian inner product $\langle \cdot, \cdot \rangle$ that is compatible with the connection $\widehat{\nabla}$ (we denote the Levi-Civita connection and its spin connection by the same symbol for convenience) and Clifford multiplication $\rho : Cl(M, \mathbb{C}) \rightarrow \text{End}(S^M)$. We denote a spinor as a section of the spin bundle by $\psi \in \Gamma(S^M)$. Naturally, its restriction to Σ is a section of S^Σ by previous discussion. The following relations holds $\forall X, Y \in \Gamma(TM)$

$$X\langle\psi_1, \psi_2\rangle = \langle\widehat{\nabla}_X\psi_1, \psi_2\rangle + \langle\psi_1, \widehat{\nabla}_X\psi_2\rangle \quad (2.7)$$

$$\widehat{\nabla}_X(\rho(Y)\psi) = \rho(\widehat{\nabla}_X Y)\psi + \rho(Y)\widehat{\nabla}_X\psi \quad (2.8)$$

$$\langle\rho(X)\psi_1, \psi_2\rangle = -\langle\psi_1, \rho(X)\psi_2\rangle \quad (2.9)$$

$$\langle\rho(X)\psi_1, \rho(X)\psi_2\rangle = g(X, X)\langle\psi_1, \psi_2\rangle. \quad (2.10)$$

These are the intrinsic identities that are verified by the geometric entities on (M, g) . Intrinsically, we can define the Dirac operator on M as the composition map

$$\Gamma(S^M) \xrightarrow{\widehat{\nabla}} \Gamma(T^*M \otimes S^M) \xrightarrow{\rho} \Gamma(S^M). \quad (2.11)$$

In local coordinates, the above decomposition gives the Dirac operator $\widehat{D}$

$$\widehat{D} := \sum_{I=1}^{n+1} \rho(e_I)\widehat{\nabla}_{e_I}. \quad (2.12)$$

The boundary map $i : \Sigma \hookrightarrow M$, induces a connection on Σ that verifies

$$\widehat{\nabla}_X Y = \nabla_X Y + \mathcal{H}(X, Y)\nu, \quad (2.13)$$

where $X, Y \in \Gamma(T\Sigma)$ and $\mathcal{H}$ is the second fundamental form of Σ while viewed as an embedded hypersurface in M . Now this split extends to the spinor bundles. The spin connection splits as follows

$$\widehat{\nabla}_X \psi = \nabla_X \psi + \frac{1}{2}\rho(\mathcal{H}(X, \cdot))\rho(\nu)\psi \quad (2.14)$$

for $X \in \Gamma(T\Sigma)$. With this split, we can write the hypersurface Dirac operator in terms of the bulk Dirac operator and the second fundamental form. We have the following proposition relating the Dirac operator of M and that of its boundary Σ

Proposition 2.1 *Let Σ be the smooth boundary of a smooth connected oriented $n+1$ dimensional Riemannian spin manifold (M, g) and H be the mean curvature of Σ in M . Under the identification $S^M|_\Sigma \simeq S^\Sigma$ or $S^M|_\Sigma \simeq S^\Sigma \oplus S^\Sigma$, The boundary Dirac operator $\mathbf{D}$ on Σ induced by the Riemannian structure of (Σ, σ) and the bulk Dirac operator $\widehat{D}$ on M verify*

$$\mathbf{D}\psi = -\rho(\nu)\widehat{D}\psi - \widehat{\nabla}_\nu \psi + \frac{1}{2}H\psi \quad (2.15)$$

for a C^∞ section ψ of the bundle S^M . Here H is the mean curvature (trace of the second fundamental form $\mathcal{H}$) of Σ in M defined with respect to ν .

Proof. The proof is a result of straightforward calculations. Let $\psi \in C^\infty(S^M)$. We denote the restriction of ψ to the boundary Σ by ψ as well for notational convenience. Consider the orthonormal frame $\{e_I\}_{I=1}^{n+1}$ on M along Σ such that $\{e_I\}_{I=1}^n$ are tangent to Σ while $e_{n+1} = \nu$ is the interior pointing normal to Σ . The hypersurface Dirac operator $\mathbf{D}$ is defined as follows

$$\begin{aligned} \mathbf{D}\psi &= \sum_{I=1}^n \rho(e_I)\rho(\nu)\nabla_{e_I}\psi = \sum_{I=1}^n \rho(e_I)\rho(\nu) \left(\widehat{\nabla}_{e_I}\psi - \frac{1}{2}\rho(\mathcal{H}(e_I, \cdot))\rho(\nu)\psi \right) \\ &= \sum_{I=1}^n \rho(e_I)\rho(\nu)\widehat{\nabla}_{e_I}\psi - \frac{1}{2}\sum_{I=1}^n \rho(e_I)\rho(\nu)\rho(\mathcal{H}(e_I, \cdot))\rho(\nu)\psi \end{aligned} \quad (2.16)$$

Now observe the following $(\mathcal{H}(e_I, \cdot) \in T^*\Sigma, I = 1, \dots, n)$ and therefore $g(\mathcal{H}(e_I, \cdot), \nu) = 0$ and therefore

$$\begin{aligned} \mathbf{D}\psi &= \sum_{I=1}^n c(e_I)c(\nu)\widehat{\nabla}_{e_I}\psi + \frac{1}{2}\sum_{I=1}^n \rho(e_I)\rho(\nu)\rho(\nu)\rho(\mathcal{H}(e_I, \cdot))\psi \\ &= \sum_{I=1}^n \rho(e_I)\rho(\nu)\widehat{\nabla}_{e_I}\psi + \frac{1}{2}H\psi = -\sum_{I=1}^n \rho(\nu)\rho(e_I)\widehat{\nabla}_{e_I}\psi + \frac{1}{2}H\psi \\ &= -\rho(\nu)\sum_{I=1}^{n+1} \rho(e_I)\widehat{\nabla}_{e_I}\psi + \rho(\nu)\rho(\nu)\widehat{\nabla}_\nu \psi + \frac{1}{2}H\psi \\ &= -\rho(\nu)\widehat{D}\psi - \widehat{\nabla}_\nu \psi + \frac{1}{2}H\psi, \end{aligned} \quad (2.17)$$

where we have used the Clifford algebra relation 2.3 and the mean curvature $H = \text{tr}\mathcal{H}$ and is defined with respect to the inward-pointing normal to Σ . $\square$

In addition to the relationship between the extrinsic and intrinsic Dirac operators, we have the usual Schrödinger-Lichnerowicz formula for the Dirac operator $\widehat{D}$ on M

$$\widehat{D}^2 = -\widehat{\nabla}^2 + \frac{1}{4}R[g], \quad (2.18)$$

where $R[g]$ is the scalar curvature of (M, g) . As an obvious consequence, one observes that a closed spin manifold with strictly positive scalar curvature can not have non-trivial harmonic spinors. We explicitly work with L^2 Sobolev spaces on M defined with respect to the Riemannian metric g . We denote by $W^{s,2}(S^M)$ the Sobolev space of spinors on M whose first s weak derivatives are in $L^2(M)$. $W^{s,2}$ completes C_c^∞ i.e., space of compactly supported sections of S^M with respect to the norm

$$\|\psi\|_{W^{s,2}} := \|\psi\|_{L^2(M)} + \sum_{I=1}^s \|\widehat{\nabla}^I \psi\|_{L^2(M)}, \quad (2.19)$$

where

$$\begin{aligned} \|\psi\|_{L^2(M)}^2 &:= \int_M \langle \psi, \psi \rangle \mu_g, \\ \|\widehat{\nabla}^I \psi\|_{L^2(M)}^2 &:= \int_M g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} \dots g^{i_I j_I} \langle \widehat{\nabla}_{i_1} \widehat{\nabla}_{i_2} \widehat{\nabla}_{i_3} \dots \widehat{\nabla}_{i_I} \psi, \widehat{\nabla}_{j_1} \widehat{\nabla}_{j_2} \widehat{\nabla}_{j_3} \dots \widehat{\nabla}_{j_I} \psi \rangle \mu_g. \end{aligned} \quad (2.20)$$

For a manifold with a boundary such as M , we also need to be concerned about the boundary regularity of the spinor. First, recall that all functions belonging to $W^{s,2}$, $s \in \mathbb{N}$ have traces on smooth boundary Σ . This passes to the spinors since one may think of $|\psi|^2 := \langle \psi, \psi \rangle$ as a function on (M, g) and the spin connection $\widehat{\nabla}$ is compatible with the Hermitian inner product $\langle \cdot, \cdot \rangle$. More precisely there is a trace-map

$$\text{Tr} : W^{s,2}(M) \rightarrow W^{s-\frac{1}{2},2}(\Sigma), \quad s \in \mathbb{N} \quad (2.21)$$

that is bounded and surjective. We will mostly be concerned with spinors that are in $W^{s,2}(M) \cap W^{s-\frac{1}{2},2}(\Sigma)$, $s > \frac{n+1}{2} + 1$ such that the Dirac operator makes sense in a point-wise manner. The background geometry is always assumed to be smooth.

3. Spinors on positive scalar curvature geometry

A spinor field $\psi \in \Gamma(S^M)$ is called a Killing spinor with Killing constant α if for all vectors X tangent to M , the following equation is verified

$$\widehat{\nabla}_X \psi = \alpha \rho(X) \psi. \quad (3.1)$$

If (M, g) carries a Killing spinor, the first integrability condition of 3.1 forces (M, g) to be an Einstein manifold with Ricci curvature $\text{Ric}[g] = 4n\alpha^2 g$. We have three cases: α can be real and non-zero, then M is compact and ψ is called a *real* Killing spinor, α can be purely imaginary, then M is non-compact and ψ is called imaginary Killing spinor, and α can be 0 in which case α is called a parallel spinor. The manifolds with real Killing spinors have been classified by Bär [20]. A prototypical example is the unit sphere with the standard round metric. Hitchin [22] showed that manifolds with parallel spinor fields can be characterized by their holonomy groups. Baum [23] classified the manifolds with imaginary Killing spinors while Rademacher later extended the classification to include generalized imaginary Killing spinors where α is allowed to be purely imaginary functions.

In this article, we are concerned with real Killing spinors with Killing constant $\alpha = \pm \frac{1}{2}$. On (M, g) , let us define a modified connection called Killing connection á la Bär [20]

$$\tilde{\nabla}_X \psi = \hat{\nabla}_X \psi + \alpha \rho(X) \psi, \quad (3.2)$$

for $X \in C^\infty(\Gamma(TM))$, $\psi \in C^\infty(\Gamma(S^M))$. The corresponding Dirac operator $\tilde{D}$ reads

$$\tilde{D} = \hat{D} - (n+1)\alpha. \quad (3.3)$$

From now on we fix $\alpha = \pm \frac{1}{2}$. Define $\tilde{D}^\pm$ to be

$$\tilde{D}^\pm = \hat{D} \mp \frac{n+1}{2}. \quad (3.4)$$

We have the following proposition

Proposition 3.1 *Let (M, g) be a $n+1$, $n \geq 2$ dimensional smooth connected oriented Riemannian spin manifold and $\psi \in C^\infty(\Gamma(S^M))$. Let $R[g]$ be the scalar curvature of (M, g) and K be the mean curvature of the boundary (Σ, σ) of (M, g) with respect to the interior pointing unit normal vector ν . Then the following Witten-type identity is verified by ψ*

$$\begin{aligned} \int_\Sigma \left(\langle \tilde{\mathbf{D}}^\pm \psi, \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_\sigma &= \frac{1}{4} \int_M (R[g] - n(n+1)) |\psi|^2 \mu_M - \frac{n}{n+1} \int_M \langle \tilde{D}^\pm \psi, \tilde{D}^\mp \psi \rangle \\ &\quad + \int_M |\mathcal{Q}\psi|^2, \end{aligned}$$

where $\mathcal{Q}_X$ is the Penrose operator defined as

$$\mathcal{Q}_X := \hat{\nabla}_X + \frac{1}{n+1} \rho(X) \hat{D} \quad (3.5)$$

for $X \in C^\infty(\Gamma(S^M))$ and the modified hypersurface Dirac operator $\tilde{\mathbf{D}}^\pm$ is defined as

$$\tilde{\mathbf{D}}^\pm := \mathbf{D} \pm \frac{n}{2} \rho(\nu). \quad (3.6)$$

Proof. We prove every identity assuming the data are in C^∞ . First, we prove the following well-known identity of the written type. This is standard, nevertheless, we present it for completeness

$$\int_\Sigma \left(\langle \mathbf{D}\psi, \psi \rangle - \frac{1}{2} H |\psi|^2 \right) \mu_\Sigma = \int_M \left(\frac{1}{4} R[g] |\psi|^2 - |\hat{D}\psi|^2 + |\hat{\nabla}\psi|^2 \right) \mu_M. \quad (3.7)$$

First, recall the Schrödinger-Lichnerowicz identity on M

$$\hat{D}^2 \psi = -\hat{\nabla}^2 \psi + \frac{1}{4} R[g] \psi, \quad (3.8)$$

for a spinor $\psi \in C^\infty(\Gamma(S^M))$. Let us consider an orthonormal frame $\{e_I\}_{I=1}^{n+1}$ on M with e_{n+1} being the unit inside pointing normal ν of the boundary surface $\partial M = \Sigma$. Now we evaluate the following using the compatibility property of the Hermitian inner product $\langle \cdot, \cdot \rangle$ on the spin bundle S^M

$$\sum_{I=1}^{n+1} \hat{\nabla}_I \langle \rho(e_I) \hat{D}\psi, \psi \rangle + \sum_{I=1}^{n+1} \hat{\nabla}_I \langle \hat{\nabla}_I \psi, \psi \rangle \quad (3.9)$$

$$\begin{aligned}
&= \sum_{I=1}^{n+1} \langle \rho(e_I) \widehat{\nabla}_I \widehat{D}\psi, \psi \rangle + \sum_{I=1}^{n+1} \langle \rho(e_I) \widehat{D}\psi, \widehat{\nabla}_I \psi \rangle + \langle \widehat{\nabla}^2 \psi, \psi \rangle + \sum_{I=1}^{n+1} \langle \widehat{\nabla}_I \psi, \widehat{\nabla}_I \psi \rangle \\
&= \langle \widehat{D}^2 \psi, \psi \rangle - \sum_{I=1}^{n+1} \langle \widehat{D}\psi, \rho(e_I) \widehat{\nabla}_I \psi \rangle + \langle \widehat{\nabla}^2 \psi, \psi \rangle + \sum_{I=1}^{n+1} \langle \widehat{\nabla}_I \psi, \widehat{\nabla}_I \psi \rangle \\
&= \langle \widehat{D}^2 \psi, \psi \rangle - \langle \widehat{D}\psi, \widehat{D}\psi \rangle + \langle \widehat{\nabla}^2 \psi, \psi \rangle + \sum_{I=1}^{n+1} \langle \widehat{\nabla}_I \psi, \widehat{\nabla}_I \psi \rangle,
\end{aligned}$$

where we have used the property 2.7 and 2.9 along with the fact that $\sum_{I=1}^{n+1} \widehat{\nabla}_I(\rho(e_I) \widehat{D}\psi) = \sum_{I=1}^{n+1} \rho(e_I) \widehat{\nabla}_I \widehat{D}\psi$. Now we integrate the identity

$$\sum_{I=1}^{n+1} \widehat{\nabla}_I \langle \rho(e_I) \widehat{D}\psi, \psi \rangle + \widehat{\nabla}_I \langle \widehat{\nabla}_I \psi, \psi \rangle = \langle \widehat{D}^2 \psi, \psi \rangle - \langle \widehat{D}\psi, \widehat{D}\psi \rangle + \langle \widehat{\nabla}^2 \psi, \psi \rangle + \sum_{I=1}^{n+1} \langle \widehat{\nabla}_I \psi, \widehat{\nabla}_I \psi \rangle$$

over M to obtain

$$\begin{aligned}
&\int_M \left(\sum_{I=1}^{n+1} \widehat{\nabla}_I \langle \rho(e_I) \widehat{D}\psi, \psi \rangle + \sum_{I=1}^{n+1} \widehat{\nabla}_I \langle \widehat{\nabla}_I \psi, \psi \rangle \right) \mu_M \\
&= \int_M \left(\langle \widehat{D}^2 \psi, \psi \rangle - \langle \widehat{D}\psi, \widehat{D}\psi \rangle + \langle \widehat{\nabla}^2 \psi, \psi \rangle + \sum_{I=1}^{n+1} \langle \widehat{\nabla}_I \psi, \widehat{\nabla}_I \psi \rangle \right) \mu_M
\end{aligned} \tag{3.10}$$

which by Stokes's theorem yields

$$- \int_{\Sigma} \left(\langle c(\nu) \widehat{D}\psi, \psi \rangle + \langle \widehat{\nabla}_{\nu} \psi, \psi \rangle \right) \mu_{\Sigma} = \int_M \left(\frac{1}{4} R[g] |\psi|^2 - |\widehat{D}\psi|^2 + |\widehat{\nabla}\psi|^2 \right) \mu_M. \tag{3.11}$$

Using the proposition 2.1 relating the Dirac operator D on Σ and the Dirac operator $\widehat{D}$ on M , we obtain the desired identity

$$\int_{\Sigma} \left(\langle \mathbf{D}\psi, \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_{\Sigma} = \int_M \left(\frac{1}{4} R[g] |\psi|^2 - |\widehat{D}\psi|^2 + |\widehat{\nabla}\psi|^2 \right) \mu_M. \tag{3.12}$$

An improved version of 3.12 can be obtained if we express $|\widehat{\nabla}\psi|^2$ in terms of the Penrose operator $\mathcal{Q}$ i.e.,

$$|\widehat{\nabla}\psi|^2 = |\mathcal{Q}\psi|^2 + \frac{1}{n+1} |\widehat{D}\psi|^2, \tag{3.13}$$

where $\mathcal{Q}_X \psi := \widehat{\nabla}_X \psi + \frac{1}{n+1} \rho(X) \widehat{D}\psi$. This yields

$$\int_{\Sigma} \left(\langle \mathbf{D}\psi, \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_{\Sigma} = \int_M \left(\frac{1}{4} R[g] |\psi|^2 - \frac{n}{n+1} |\widehat{D}\psi|^2 + |\mathcal{Q}\psi|^2 \right) \mu_M. \tag{3.14}$$

Recall

$$\widetilde{D}^+ := \widehat{D} - \frac{n+1}{2}, \quad \widetilde{D}^- := \widehat{D} + \frac{n+1}{2} \tag{3.15}$$

and note the following point-wise identity

$$\langle \widetilde{D}^+ \psi, \widetilde{D}^- \psi \rangle = \langle (\widehat{D} - \frac{n+1}{2}) \psi, (\widehat{D} + \frac{n+1}{2}) \psi \rangle \tag{3.16}$$

$$= |\widehat{D}\psi|^2 + \frac{n+1}{2}\langle \widehat{D}\psi, \psi \rangle - \frac{n+1}{2}\langle \psi, \widehat{D}\psi \rangle - \frac{(n+1)^2}{4}|\psi|^2$$

yields an expression for the Dirac operator

$$|\widehat{D}\psi|^2 = \langle \widetilde{D}^+\psi, \widetilde{D}^-\psi \rangle - \frac{n+1}{2}\langle \widehat{D}\psi, \psi \rangle + \frac{n+1}{2}\langle \psi, \widehat{D}\psi \rangle + \frac{(n+1)^2}{4}|\psi|^2 \quad (3.17)$$

which while substituted into the Witten identity yields

$$\begin{aligned} \int_{\Sigma} \left(\langle \mathbf{D}\psi, \psi \rangle - \frac{1}{2}K|\psi|^2 \right) \mu_{\sigma} &= \frac{1}{4} \int_M R[g]|\psi|^2 \mu_M - \frac{n}{n+1} \int_M \left(\langle \widetilde{D}^+\psi, \widetilde{D}^-\psi \rangle - \frac{n+1}{2}\langle \widehat{D}\psi, \psi \rangle \right. \\ &\quad \left. + \frac{n+1}{2}\langle \psi, \widehat{D}\psi \rangle + \frac{(n+1)^2}{4}|\psi|^2 \right) + \int_M |\mathcal{Q}\psi|^2 \mu_M. \end{aligned} \quad (3.18)$$

Now recall the following identity as the failure of self-adjointness of $\widehat{D}$ on a manifold with boundary (follows by integration by parts and Stokes)

$$\int_M \langle \widehat{D}\psi, \psi \rangle = \int_M \langle \psi, \widehat{D}\psi \rangle + \int_{\Sigma} \langle \psi, \rho(\nu)\psi \rangle \quad (3.19)$$

which reduces the previous expression 3.18 to the following desired form

$$\begin{aligned} \int_{\Sigma} \left(\langle \widetilde{\mathbf{D}}^+\psi, \psi \rangle - \frac{1}{2}K|\psi|^2 \right) \mu_{\sigma} &= \frac{1}{4} \int_M (R[g] - n(n+1)) |\psi|^2 \mu_M - \frac{n}{n+1} \int_M \langle \widetilde{D}^+\psi, \widetilde{D}^-\psi \rangle \\ &\quad + \int_M |\mathcal{Q}\psi|^2 \end{aligned} \quad (3.20)$$

with $\widetilde{\mathbf{D}}^+ := \mathbf{D} + \frac{n}{2}\rho(\nu)$. Similarly repeating the computations with

$$\begin{aligned} \langle \widetilde{D}^-\psi, \widetilde{D}^+\psi \rangle &= \langle (\widehat{D} + \frac{n+1}{2})\psi, (\widehat{D} - \frac{n+1}{2})\psi \rangle \\ &= |\widehat{D}\psi|^2 - \frac{n+1}{2}\langle \widehat{D}\psi, \psi \rangle + \frac{n+1}{2}\langle \psi, \widehat{D}\psi \rangle - \frac{(n+1)^2}{4}|\psi|^2 \end{aligned} \quad (3.21)$$

yields

$$\begin{aligned} \int_{\Sigma} \left(\langle \widetilde{\mathbf{D}}^-\psi, \psi \rangle - \frac{1}{2}K|\psi|^2 \right) \mu_{\sigma} &= \frac{1}{4} \int_M (R[g] - n(n+1)) |\psi|^2 \mu_M - \frac{n}{n+1} \int_M \langle \widetilde{D}^-\psi, \widetilde{D}^+\psi \rangle \\ &\quad + \int_M |\mathcal{Q}\psi|^2 \end{aligned} \quad (3.22)$$

with $\widetilde{\mathbf{D}}^- := \mathbf{D} - \frac{n}{2}\rho(\nu)$. This completes the proof of the lemma. $\square$

Remark 1 *The spectra of the modified operators $\widetilde{\mathbf{D}}^{\pm}$ can be imaginary in general since they are not self-adjoint. In particular, $\text{Adj}(\widetilde{\mathbf{D}}^+) = \widetilde{\mathbf{D}}^-$ and vice versa. However, it turns out if $\lambda(\mathbf{D}) \geq \frac{n}{2}$, then $\widetilde{\lambda}(\widetilde{\mathbf{D}}^{\pm})^2 \geq 0$, where λ denotes the spectra of $\mathbf{D}$ and $\widetilde{\lambda}$ that of $\widetilde{\mathbf{D}}^{\pm}$. We prove this now.*

First, we prove an eigenvalue estimate for the intrinsic Dirac operator $\mathbf{D}$ on Σ assuming a curvature condition $R[\sigma] \geq (n-1)n$. This is well known [21].

Lemma 3.1 (Eigenvalue estimate of $\mathbf{D}$ on Σ) *Let the scalar curvature of a closed n dimensional smooth Riemannian spin manifold Σ verify $R[\sigma] \geq n(n-1)$. Then the first eigenvalue $\lambda_1(\mathbf{D})$ of the Dirac operator $\mathbf{D}$ on Σ verifies*

$$\lambda_1^2(\mathbf{D}) \geq \frac{n^2}{4}. \quad (3.23)$$

Proof. First, recall the Witten identity for Σ . Since Σ is closed, then for any $\psi \in C^\infty(\Gamma(S^\Sigma))$

$$\int_\Sigma \left(\frac{1}{4} R[\sigma] |\psi|^2 - \frac{(n-1)|\mathbf{D}\psi|^2}{n} \right) \mu_\Sigma + \int_\Sigma |\mathcal{Q}\psi|^2 = 0 \quad (3.24)$$

Now let $\mathbf{D}\psi = \lambda_1(\mathbf{D})\psi$. λ_1 can't be zero since by the scalar curvature condition, (Σ, σ) does not admit any non-trivial harmonic spinor. In fact, the spectrum of $\mathbf{D}$ is unbounded discrete, and symmetric with respect to zero. Then

$$\int_\Sigma \left(\lambda_1(\mathbf{D})^2 - \frac{n}{4(n-1)} R[\sigma] \right) |\psi|^2 = \frac{n}{n-1} \int_\Sigma |\mathcal{Q}\psi|^2 \geq 0 \quad (3.25)$$

yielding $\lambda_1^2 \geq \frac{n}{4(n-1)} \inf_\Sigma R[\sigma] = \frac{n^2}{4}$. $\square$

Now we construct eigenspinors of $\tilde{\mathbf{D}}^\pm$. Let $\tilde{\lambda}$ be the first eigenvalue of $\tilde{\mathbf{D}}$ i.e.,

$$\tilde{\mathbf{D}}^+ \Phi^+ = \tilde{\lambda} \Phi^+. \quad (3.26)$$

Define $\lambda^2 = \tilde{\lambda}^2 + \frac{n^2}{4}$ and observe

$$\varphi := \left(\frac{n^2}{4} + (\tilde{\lambda} - \lambda)^2 \right)^{-1} \left[\frac{n}{2} \Phi^+ + (\tilde{\lambda} - \lambda) \rho(\nu) \Phi^+ \right] \quad (3.27)$$

verifies the eigenvalue equation

$$\mathbf{D}\varphi = \lambda\varphi. \quad (3.28)$$

Note $\frac{n^2}{4} + (\tilde{\lambda} - \lambda)^2 = 2\lambda^2 - 2\lambda\tilde{\lambda} \neq 0$ under the teleological assumption $\lambda^2 \geq \frac{n^2}{4}$. Similarly, an eigenspinor φ of $\mathbf{D}$ can be constructed from the first eigenspinor of $\tilde{\mathbf{D}}^-$ as follows. Let $\tilde{\Phi}$ be the first eigenspinor of $\tilde{\mathbf{D}}^-$ i.e.,

$$\tilde{\mathbf{D}}^- \tilde{\Phi}^+ = \tilde{\mu} \tilde{\Phi}^+. \quad (3.29)$$

Now define $\lambda^2 = \tilde{\mu}^2 + \frac{n^2}{4}$ and note

$$\varphi := \left(\frac{n^2}{4} + (\tilde{\mu} - \lambda)^2 \right)^{-1} \left[\frac{n}{2} \tilde{\Phi}^+ - (\tilde{\mu} - \lambda) \rho(\nu) \tilde{\Phi}^+ \right] \quad (3.30)$$

verifies

$$\mathbf{D}\varphi = \lambda\varphi. \quad (3.31)$$

Lemma 3.1 yields $\lambda \geq \frac{n}{2}$. Therefore $\tilde{\lambda}^2, \tilde{\mu}^2 \geq 0$ or the following claim follows as a consequence of lemma 3.1

Claim 1 Let (Σ, σ) have scalar curvature $R[\sigma] \geq (n-1)n$ and assume that the intrinsic spin structure on (σ, σ) coincides with the spin structure induced from (M, g) . Then the first order Dirac type operators $\tilde{\mathbf{D}}^\pm$ on (Σ, σ) induced by (M, g) have real spectra.

More generally, one has the following proposition relating the spectra of $\mathbf{D}$ and $\tilde{\mathbf{D}}^\pm$ as a consequence of the claim 1

Proposition 3.2 Let $E_{\mathbf{D}}$ and $E_{\tilde{\mathbf{D}}^\pm}$ be the spaces of L^2 -eigenspinors of the Dirac operators $\mathbf{D}$ and $\tilde{\mathbf{D}}^\pm$ on Σ , respectively. Then under the curvature condition $R[\sigma] \geq n(n-1)$, we have the isomorphism

$$E_{\mathbf{D}} \simeq E_{\tilde{\mathbf{D}}^\pm}^+ \cup E_{\tilde{\mathbf{D}}^\pm}^- \cup 2E_{\tilde{\mathbf{D}}^\pm}^0, \quad (3.32)$$

where superscripts $+$, $-$, and 0 denote positive, negative, and zero eigen spaces, respectively.

Proof. Proof of this proposition relies on the explicit construction of eigenspinors. Let $\varphi \in E_{\mathbf{D}}^+$ with positive eigenvalue λ ($\lambda^2 \geq \frac{n^2}{4}$) and $E_{\tilde{\mathbf{D}}^\pm}^+$ be the spaces of eigenspinors of $\tilde{\mathbf{D}}^\pm$ with positive eigenvalues. Then construct the maps

$$p_1 : E_{\mathbf{D}}^+ \rightarrow E_{\tilde{\mathbf{D}}^+}^+ \quad (3.33)$$

$$\varphi \mapsto \Phi^+ := \frac{n}{2}\varphi - (\tilde{\lambda} - \lambda)\rho(\nu)\varphi, \quad \tilde{\lambda} = +\sqrt{\lambda^2 - \frac{n^2}{4}} \quad (3.34)$$

$$p_2 : E_{\mathbf{D}}^+ \rightarrow E_{\tilde{\mathbf{D}}^-}^+ \quad (3.35)$$

$$\varphi \mapsto \tilde{\Phi}^+ := \frac{n}{2}\varphi + (\tilde{\lambda} - \lambda)\rho(\nu)\varphi, \quad \tilde{\lambda} = +\sqrt{\lambda^2 - \frac{n^2}{4}} \quad (3.36)$$

Similarly, construct maps to the negative eigenspaces

$$q_1 : E_{\mathbf{D}}^+ \rightarrow E_{\tilde{\mathbf{D}}^+}^- \quad (3.37)$$

$$\varphi \mapsto \Phi^- := (\tilde{\lambda} - \lambda)\varphi - \frac{n}{2}\rho(\nu)\varphi, \quad \tilde{\lambda} = -\sqrt{\lambda^2 - \frac{n^2}{4}} \quad (3.38)$$

$$q_2 : E_{\mathbf{D}}^+ \rightarrow E_{\tilde{\mathbf{D}}^-}^- \quad (3.39)$$

$$\varphi \mapsto \tilde{\Phi}^- = (\tilde{\lambda} - \lambda)\varphi + \frac{n}{2}\rho(\nu)\varphi, \quad \tilde{\lambda} = -\sqrt{\lambda^2 - \frac{n^2}{4}} \quad (3.40)$$

Note that each of these maps are well defined. For example $\Phi^+ = 0$ implies $\varphi = 0$ and the rest verifies the same too. The surjectivity follows since φ can be written explicitly in terms of Φ e.g., consider the map p_1 . Let $\Phi^+ \in E_{\tilde{\mathbf{D}}^+}^+$. An explicit computation leads to

$$\varphi = \left(\frac{n^2}{4} + (\tilde{\mu} - \lambda)^2 \right)^{-1} \left[\frac{n}{2}\Phi^+ - (\tilde{\mu} - \lambda)\rho(\nu)\Phi^+ \right] \in E_{\mathbf{D}}^+. \quad (3.41)$$

Similar calculations follow for the rest of the maps.

Now the spectrum of $\mathbf{D}$ is symmetric with respect to zero with finite multiplicities i.e., $E_{\mathbf{D}}^+ \simeq E_{\mathbf{D}}^-$. Therefore the desired isomorphism $E_{\mathbf{D}} \simeq E_{\tilde{\mathbf{D}}^\pm}^+ \cup E_{\tilde{\mathbf{D}}^\pm}^- \cup 2E_{\tilde{\mathbf{D}}^\pm}^0$ follows. $\square$

Corollary 3.1 $E_{\tilde{\mathbf{D}}^-}^- = \rho(\nu)(E_{\tilde{\mathbf{D}}^+}^+)$ and $E_{\tilde{\mathbf{D}}^-}^+ = \rho(\nu)(E_{\tilde{\mathbf{D}}^+}^-)$.

Proof. Direct computation. $\square$

Now we can start solving the boundary value problem for the operators $\tilde{D}^\pm$ on M . We will essentially work with an APS-type boundary condition. First, consider the following spaces. On the boundary Σ , $L^2(\Gamma(S^\Sigma))$ splits into two orthogonal subspaces. This is as follows. Let D be the Dirac operator on Σ . Consider the eigenvalue equation for $\tilde{D}^+$

$$\tilde{D}^+\psi = \tilde{\lambda}\psi, \quad \tilde{\lambda} \in \mathbb{R}. \quad (3.42)$$

The spectrum $\{\tilde{\lambda} \in \mathbb{R}\}$ is symmetric with respect to zero by claim 1 and proposition 3.2. A generic L^2 section ψ of S^Σ is written as $\sum_i a_i \Phi_i$, $a_i \in \mathbb{C}$ and $\tilde{D}^+\Phi_i = \tilde{\lambda}_i \Phi_i$, $\tilde{\lambda}_i \in \mathbb{R}$. One can split $L^2(\Gamma(S^\Sigma))$ into $L^2_{\tilde{D}^+}(\Gamma(S^\Sigma))^+$ and $L^2_{\tilde{D}^+}(\Gamma(S^\Sigma))^-$ as follows

$$L^2_{\tilde{D}^+}(\Gamma(S^\Sigma))^+ := \left\{ \psi \in L^2(\Gamma(S^\Sigma)) \mid \psi = \sum_i a_i \Phi_i^+, \tilde{D}^+\Phi_i^+ = \tilde{\lambda}_i \Phi_i^+, \tilde{\lambda}_i \geq 0 \right\}, \quad (3.43)$$

$$L^2_{\tilde{D}^+}(\Gamma(S^\Sigma))^- := \left\{ \psi \in L^2(\Gamma(S^\Sigma)) \mid \psi = \sum_i a_i \Phi_i^-, \tilde{D}^+\Phi_i^- = \tilde{\lambda}_i \Phi_i^-, \tilde{\lambda}_i < 0 \right\}. \quad (3.44)$$

Naturally $L^2_{\tilde{D}^+}(\Gamma(S^\Sigma))^+$ and $L^2_{\tilde{D}^+}(\Gamma(S^\Sigma))^-$ are L^2 orthogonal. Let us denote by $\tilde{P}_{\geq 0}^+, \tilde{P}_{< 0}^+$ the projection operators

$$\tilde{P}_{\geq 0}^+ : L^2(\Gamma(S^\Sigma)) \rightarrow L^2_{\tilde{D}^+}(\Gamma(S^\Sigma))^+, \quad (3.45)$$

$$\tilde{P}_{< 0}^+ : L^2(\Gamma(S^\Sigma)) \rightarrow L^2_{\tilde{D}^+}(\Gamma(S^\Sigma))^- . \quad (3.46)$$

Similarly, we can perform decomposition of $L^2(\Gamma(S^\Sigma))$ by the eigenspinors of $\tilde{D}^-$. Define similarly

$$L^2_{\tilde{D}^-}(\Gamma(S^\Sigma))^+ := \left\{ \psi \in L^2(\Gamma(S^\Sigma)) \mid \psi = \sum_i a_i \tilde{\Phi}_i^+, \tilde{D}^-\tilde{\Phi}_i^+ = \tilde{\lambda}_i \tilde{\Phi}_i^+, \tilde{\lambda}_i \geq 0 \right\}, \quad (3.47)$$

$$L^2_{\tilde{D}^-}(\Gamma(S^\Sigma))^- := \left\{ \psi \in L^2(\Gamma(S^\Sigma)) \mid \psi = \sum_i a_i \tilde{\Phi}_i^-, \tilde{D}^-\tilde{\Phi}_i^- = \tilde{\lambda}_i \tilde{\Phi}_i^-, \tilde{\lambda}_i < 0 \right\}. \quad (3.48)$$

Naturally $L^2_{\tilde{D}^-}(\Gamma(S^\Sigma))^+$ and $L^2_{\tilde{D}^-}(\Gamma(S^\Sigma))^-$ are L^2 orthogonal. Let us denote by $\tilde{P}_{\geq 0}^-, \tilde{P}_{< 0}^-$ the projection operators

$$\tilde{P}_{\geq 0}^- : L^2(\Gamma(S^\Sigma)) \rightarrow L^2_{\tilde{D}^-}(\Gamma(S^\Sigma))^+, \quad (3.49)$$

$$\tilde{P}_{< 0}^- : L^2(\Gamma(S^\Sigma)) \rightarrow L^2_{\tilde{D}^-}(\Gamma(S^\Sigma))^- . \quad (3.50)$$

Theorem 3.1 *Let $\Phi \in H^{s-1}(\Gamma(S^M))$ and $\alpha \in H^{s-\frac{1}{2}}(\Gamma(S^\Sigma))$, $s > \frac{n+1}{2} + 1$ and $n \geq 2$. Let the scalar curvature of a $n+1$ dimensional smooth connected oriented Riemannian spin manifold (M, g) verify $R[g] \geq (n+1)n$ and the mean curvature K of its boundary (Σ, σ) with respect to the inward-pointing normal vector be strictly positive i.e., Σ is strictly mean convex in M and its scalar curvature verifies $R[\sigma] \geq (n-1)n$. Then the following boundary value problem*

$$\tilde{D}^+\psi := (\hat{D} - \frac{n+1}{2})\psi = \Phi, \quad (3.51)$$

$$\tilde{P}_{\geq 0}^+\psi = \tilde{P}_{\geq 0}^+\alpha, \quad (3.52)$$

has a unique solution $\psi \in H^s(\Gamma(S^M)) \cap H^{s-\frac{1}{2}}(\Gamma(S^\Sigma))$ that verifies

$$\|\psi\|_{H^s(\Gamma(S^M)) \cap H^{s-\frac{1}{2}}(\Gamma(S^\Sigma))} \leq C \left(\|\Phi\|_{H^{s-1}(\Gamma(S^M))} + \|\tilde{P}_{\geq 0}^+ \alpha\|_{H^{s-\frac{1}{2}}(\Gamma(S^\Sigma))} \right), \quad (3.53)$$

for a constant C depending on the geometry of (M, g) and (Σ, σ) .

Proof. We prove the Fredholm property of the operator $\hat{D} - \frac{n+1}{2}$. In fact we will prove that with this APS-type boundary condition,

$$\hat{D} - \frac{n+1}{2} : H^s(\Gamma(S^M)) \cap H^{s-\frac{1}{2}}(\Gamma(S^\Sigma))_+ \rightarrow H^{s-1}(\Gamma(S^M)) \cap H^{s-\frac{3}{2}}(\Gamma(S^\Sigma))_+ \quad (3.54)$$

is an isomorphism. First we prove that the Kernel is trivial i.e.,

$$\tilde{D}^+ \psi := (D - \frac{n+1}{2})\psi = 0 \text{ on } M \quad (3.55)$$

$$\tilde{P}_{\geq 0}^+ \psi = 0 \text{ on } \Sigma \quad (3.56)$$

has no-nontrivial solution. Recall the Witten identity 3.1

$$\begin{aligned} \int_{\Sigma} \left(\langle \tilde{D}^+ \psi, \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_{\sigma} &= \frac{1}{4} \int_M (R[g] - n(n+1)) |\psi|^2 \mu_M - \frac{n}{n+1} \int_M \langle \tilde{D}^+ \psi, \tilde{D}^- \psi \rangle \\ &\quad + \int_M |\mathcal{Q}\psi|^2, \end{aligned}$$

and substitute $\tilde{D}^+ \psi = 0$. This yields

$$\begin{aligned} \int_{\Sigma} \left(\langle \tilde{D}^+ \tilde{P}_{\geq 0}^+ \psi, \tilde{P}_{\geq 0}^+ \psi \rangle + \langle \tilde{D}^+ \tilde{P}_{< 0}^+ \psi, \tilde{P}_{< 0}^+ \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_{\sigma} &= \frac{1}{4} \int_M (R[g] - n(n+1)) |\psi|^2 \mu_M \\ &\quad + \int_M |\mathcal{Q}\psi|^2. \end{aligned}$$

The boundary condition $\tilde{P}_{\geq 0}^+ \psi = 0$ on Σ implies

$$\int_{\Sigma} \left(\langle \tilde{D}^+ \tilde{P}_{< 0}^+ \psi, \tilde{P}_{< 0}^+ \psi \rangle - \frac{1}{2} K |\tilde{P}_{< 0}^+ \psi|^2 \right) \mu_{\sigma} = \frac{1}{4} \int_M (R[g] - n(n+1)) |\psi|^2 \mu_M + \int_M |\mathcal{Q}\psi|^2 \geq 0$$

Using the non-negative mean curvature condition $K \geq 0$

$$\int_{\Sigma} \left(\langle \tilde{D}^+ \tilde{P}_{< 0}^+ \psi, \tilde{P}_{< 0}^+ \psi \rangle - \frac{1}{2} K |\tilde{P}_{< 0}^+ \psi|^2 \right) \mu_{\sigma} \leq 0 \quad (3.57)$$

which yields by the bulk curvature condition $R[g] \geq (n-1)n$

$$0 \leq \int_{\Sigma} \left(\langle \tilde{D}^+ \tilde{P}_{< 0}^+ \psi, \tilde{P}_{< 0}^+ \psi \rangle - \frac{1}{2} K |\tilde{P}_{< 0}^+ \psi|^2 \right) \mu_{\sigma} \leq 0 \quad (3.58)$$

implying $\tilde{P}_{< 0}^+ \psi = 0$ on Σ i.e.,

$$\psi = 0 \text{ on } \Sigma. \quad (3.59)$$

Now unique continuation of homogeneous first order elliptic partial differential equation then implies

$$\psi = 0 \text{ on } M. \quad (3.60)$$

Co-kernel needs to be obtained. Let $\varphi' \in \text{Co-Kernel}(\widehat{D} - \frac{n+1}{2})$. Then for $(\widehat{D} - \frac{n+1}{2})\varphi = \Psi \in \text{range}(\widehat{D} - \frac{n+1}{2})$, we have the trivial L^2 orthogonal relation

$$\int_M \langle (D - \frac{n}{2})\varphi, \varphi' \rangle \mu_M = 0 \quad (3.61)$$

which after integration by parts yields

$$\int_M \langle \varphi, (D - \frac{n}{2})\varphi' \rangle \mu_M + \int_\Sigma \langle \varphi, \rho(\nu)\varphi' \rangle \mu_\sigma = 0 \quad (3.62)$$

Now $\varphi = \tilde{P}_{\geq 0}^+ \varphi + \tilde{P}_{< 0}^+ \varphi$ and the boundary condition implies $\tilde{P}_{\geq 0}^+ \varphi = 0$. Therefore, we have

$$\int_M \langle \varphi, (D - \frac{n}{2})\varphi' \rangle \mu_M + \int_\Sigma \langle \tilde{P}_{< 0}^+ \varphi, \tilde{P}_{\geq 0}^+ \rho(\nu)\varphi' + \tilde{P}_{< 0}^+ (\rho(\nu)\varphi') \rangle \mu_\sigma = 0 \quad (3.63)$$

or

$$\int_M \langle \varphi, (D - \frac{n}{2})\varphi' \rangle \mu_M + \int_\Sigma \langle \tilde{P}_{< 0}^+ \varphi, \tilde{P}_{< 0}^+ (\rho(\nu)\varphi') \rangle \mu_\sigma = 0 \quad (3.64)$$

yielding the boundary condition to define co-kernel to be

$$\tilde{P}_{< 0}^+ (\rho(\nu)\varphi') = 0 \text{ on } \Sigma. \quad (3.65)$$

Let us simplify that. Now consider the decomposition of φ' in terms of the eigenspinors of $\tilde{\mathbf{D}}^-$

$$\varphi' = \tilde{P}_{\geq 0}^- \varphi' + \tilde{P}_{< 0}^- \varphi' = \sum_i a_i \tilde{\Phi}_i^+ + \sum_i b_i \tilde{\Phi}_i^-, \quad a_i, b_i \in \mathbb{C}. \quad (3.66)$$

Action of $\rho(\nu)$ on φ' yields by 3.1

$$\rho(\nu)\varphi' = \sum_i a_i \Phi_i^- + \sum_i b_i \Phi_i^+ \quad (3.67)$$

and consequently $\tilde{P}_{< 0}^+ (\rho(\nu)\varphi') = \sum_i a_i \Phi_i^-$. Therefore $\tilde{P}_{< 0}^+ (\rho(\nu)\varphi') = 0$ yields

$$a_i = 0 \quad \forall i - \{0\} \implies \tilde{P}_{> 0}^- \varphi' = 0. \quad (3.68)$$

Therefore for $\varphi' \in \text{Co-Ker}(D - \frac{n+1}{2})$, the co-kernel is defined as

$$\tilde{D}^+ \varphi' = (D - \frac{n+1}{2})\varphi' = 0 \text{ on } M, \quad \tilde{P}_{> 0}^- \varphi' = 0 \text{ on } \Sigma. \quad (3.69)$$

Now we prove the triviality of the co-kernel. We can use the different forms of the Witten identity (recall that 3.1 has two forms of the Witten identity in terms of $\tilde{\mathbf{D}}^\pm$. Consider the second form

$$\begin{aligned} \int_\Sigma \left(\langle \tilde{\mathbf{D}}^- \psi, \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_\sigma &= \frac{1}{4} \int_M (R[g] - n(n+1)) |\psi|^2 \mu_M - \frac{n}{n+1} \int_M \langle \tilde{D}^- \psi, \tilde{D}^+ \psi \rangle \\ &\quad + \int_M |\mathcal{Q}\psi|^2 \end{aligned}$$

which yields for $\psi = \varphi' \in \text{co-kernel}(\tilde{D}^+)$

$$\int_\Sigma \left(\langle \tilde{\mathbf{D}}^- \varphi', \varphi' \rangle - \frac{1}{2} K |\varphi'|^2 \right) \mu_\sigma = \frac{1}{4} \int_M (R[g] - n(n+1)) |\varphi'|^2 \mu_M + \int_M |\mathcal{Q}\varphi'|^2 \geq 0.$$

Imposing boundary condition yields

$$\int_{\Sigma} \left(\langle \tilde{\mathbf{D}}^- \tilde{P}_{\leq 0}^- \varphi', \tilde{P}_{\leq 0}^- \varphi' \rangle - \frac{1}{2} K |\tilde{P}_{\leq 0}^- \varphi'|^2 \right) \mu_{\sigma} = \frac{1}{4} \int_M (R[g] - n(n+1)) |\varphi'|^2 \mu_M + \int_M |\mathcal{Q}\varphi'|^2 \geq 0.$$

and the left hand side verifies

$$\int_{\Sigma} \left(\langle \tilde{\mathbf{D}}^- \tilde{P}_{\leq 0}^- \varphi', \tilde{P}_{\leq 0}^- \varphi' \rangle - \frac{1}{2} K |\tilde{P}_{\leq 0}^- \varphi'|^2 \right) \mu_{\sigma} \leq 0 \quad (3.70)$$

and therefore

$$\int_{\Sigma} \left(\langle \tilde{\mathbf{D}}^- \tilde{P}_{\leq 0}^- \varphi', \tilde{P}_{\leq 0}^- \varphi' \rangle - \frac{1}{2} K |\tilde{P}_{\leq 0}^- \varphi'|^2 \right) \mu_{\sigma} = 0 \quad (3.71)$$

Using the assumption $K > 0$, we obtain $\tilde{P}_{\leq 0}^- \varphi' = 0$ on Σ i.e.,

$$\varphi' = 0 \text{ on } \Sigma. \quad (3.72)$$

Invoking the unique continuation property of homogeneous first-order elliptic partial differential equations yields

$$\varphi' = 0 \text{ on } M. \quad (3.73)$$

Therefore the desired co-kernel is trivial. Therefore

$$\begin{aligned} \text{Kernel}(\tilde{D}^+) := \left\{ \psi \in H^s(\Gamma(S^M)) \cap H^{s-\frac{1}{2}}(\Gamma(S^{\Sigma})) \mid \tilde{D}^+ \psi = (D - \frac{n+1}{2})\psi = 0 \text{ on } M, \right. \\ \left. \tilde{P}_{\geq 0}^+ \psi = 0 \text{ on } \Sigma \right\} = \{0\} \end{aligned}$$

$$\begin{aligned} \text{Co-Kernel}(\tilde{D}^+) := \left\{ \psi \in H^s(\Gamma(S^M)) \cap H^{s-\frac{1}{2}}(\Gamma(S^{\Sigma})) \mid \tilde{D}^+ \psi = (D - \frac{n+1}{2})\psi = 0 \text{ on } M, \right. \\ \left. \tilde{P}_{> 0}^- \psi = 0 \text{ on } \Sigma \right\} = \{0\} \end{aligned}$$

The estimates follow trivially as a consequence of the isomorphism property. $\square$

Remark 2 *The condition $K > 0$ is vital. Without this condition, the theorem 3.1 breaks down. This condition will turn out to be vital again when proving the rigidity property of spherical caps. In particular, we shall see that the rigidity argument can never be extended to the hemisphere since the equator of the sphere will correspond to $K = 0$ condition where the isomorphism property fails because of 3.71.*

4. Eigenvalue estimates

Under the curvature condition $R[\sigma] \geq (n-1)n$, the intrinsic Dirac operator $\mathbf{D}$ already verifies an absolute lower spectral bound $\frac{n}{2}$. This can be further improved in terms of the mean curvature information if (Σ, σ) is realized as an embedded hypersurface in (M, g) , in particular bounding a domain. For this, we first obtain the eigenvalue estimates for the operators $\tilde{\mathbf{D}}^{\pm}$. Let $\tilde{\lambda}_1 > 0$ be the first positive eigenvalue of $\tilde{\mathbf{D}}^+$ with eigenspinor ξ i.e.,

$$\tilde{\mathbf{D}}^+ \xi = \tilde{\lambda}_1 \xi, \tilde{\lambda}_1 > 0, \text{ on } \Sigma. \quad (4.1)$$

We can use ξ as the boundary condition for the boundary value problem associated with the operator $\tilde{D}^+ := \hat{D} - \frac{n+1}{2}$ on (M, g) , that is,

$$\hat{D}\psi - \frac{n+1}{2}\psi = 0, \text{ on } \Omega \quad (4.2)$$

$$\tilde{P}_{\geq 0}^+ \psi = \xi \text{ on } \Sigma. \quad (4.3)$$

This has a unique solution ψ according to the theorem 3.1. Now we prove theorem 1.1

4.1. Proof of theorem 1.1

Let us recall the statement of the theorem 1.1

Theorem (Theorem 1.1) *Let (M, g) be an $n+1$ - dimensional smooth connected oriented Riemannian spin manifold with smooth spin boundary Σ for $n \geq 2$. Let the induced metric on Σ by g is σ . Assume the following:*

- (a) *The scalar curvature of (M, g) $R[g] \geq n(n+1)$,*
- (b) *The scalar curvature of (Σ, σ) $R[\sigma] \geq n(n-1)$,*
- (c) *Σ is mean convex with respect to g i.e., its mean curvature $K > 0$.*

Then the first eigenvalue λ_1 of the Dirac operator $\mathbf{D}$ on (Σ, σ) verifies the estimate

$$\lambda_1(\mathbf{D})^2 \geq \frac{1}{4} \inf_{\Sigma} K^2 + \frac{n^2}{4}. \quad (4.4)$$

First, elliptic regularity of the first order Dirac type operator implies the existence of a smooth solution. The standard procedure to this conclusion is to make the observation that the solution to the boundary value problem 4.2 exists in $H^s(M)$ given $H^{s-1}(M)$ data for every $s \geq 1$ with associated trace on Σ . Then $C^\infty = \cap H^s$. The regularity of the background metrics g and σ is assumed to be C^∞ . The Witten identity

$$\int_{\Sigma} \left(\langle \tilde{\mathbf{D}}^+ \psi, \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_{\sigma} = \frac{1}{4} \int_{\Omega} \left(R^{\Omega} - n(n+1) \right) |\psi|^2 \mu_{\Omega} + \int_{\Omega} |\mathcal{Q}\psi|^2 \geq 0.$$

yields

$$\int_{\Sigma} \left(\langle \tilde{\mathbf{D}}^+ \tilde{P}_{\geq 0}^+ \psi, \tilde{P}_{\geq 0}^+ \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_{\sigma} \geq \frac{1}{4} \int_{\Omega} \left(R^{\Omega} - n(n+1) \right) |\psi|^2 \mu_{\Omega} + \int_{\Omega} |\mathcal{Q}\psi|^2 \geq 0.$$

since $\int_{\Sigma} \langle \tilde{\mathbf{D}}^+ \tilde{P}_{< 0}^+ \psi, \tilde{P}_{< 0}^+ \psi \rangle \leq 0$. The boundary condition 4.3 implies together with the eigenvalue equation 4.1

$$\begin{aligned} \int_{\Sigma} \left(\langle \tilde{\mathbf{D}}^+ \xi, \xi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_{\sigma} &= \int_{\Sigma} \left(\langle \tilde{\mathbf{D}}^+ \tilde{P}_{\geq 0}^+ \psi, \tilde{P}_{\geq 0}^+ \psi \rangle - \frac{1}{2} K |\psi|^2 \right) \mu_{\sigma} \\ &\geq \frac{1}{4} \int_{\Omega} \left(R^{\Omega} - n(n+1) \right) |\psi|^2 \mu_{\Omega} + \int_{\Omega} |\mathcal{Q}\psi|^2 \geq 0 \end{aligned} \quad (4.5)$$

or

$$\begin{aligned} \int_{\Sigma} \left(\tilde{\lambda}_1 |\psi|^2 - \frac{1}{2} K |\psi|^2 \right) \mu_{\sigma} &\geq \int_{\Sigma} \left(\tilde{\lambda} |\xi|^2 - \frac{1}{2} K |\psi|^2 \right) \mu_{\sigma} \\ &\geq \frac{1}{4} \int_{\Omega} \left(R^{\Omega} - n(n+1) \right) |\psi|^2 \mu_{\Omega} + \int_{\Omega} |\mathcal{Q}\psi|^2 \geq 0 \end{aligned} \quad (4.6)$$

since $\|\psi\|_{L^2(\Sigma)} \geq \|\xi\|_{L^2(\Sigma)}$. This yields

$$\tilde{\lambda}_1 \geq \frac{1}{2} \inf_{\Sigma} K \quad (4.7)$$

which yields

$$\lambda_1^2 = \tilde{\lambda}_1^2 + \frac{n^2}{4} \geq \frac{1}{4} \inf_{\Sigma} K^2 + \frac{n^2}{4} \quad (4.8)$$

where λ_1 is the first positive eigenvalue of the Dirac operator $\mathbf{D}$ on Σ . This completes the proof of the theorem 1.1.

Let us understand the equality case in detail. To this end, we first, invoke the following result by [20] on the classification of manifolds with real Killing spinors

Theorem 4.1 [20] *Let M be compact and simply connected.*

(a) *If the holonomy group $\text{Hol}(\widetilde{M})$ is reducible, then $\widetilde{M}$ is flat and therefore M is isometric to the standard sphere.*

(b) *Let (M, g) be a complete simply connected Riemannian spin $n+1$ -manifold carrying a non-trivial real Killing spinor with Killing constant $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$, then if $n+1$ is even and $n+1 \neq 6$, then (M, g) is isometric to the standard sphere.*

(c) *Let (M, g) be a complete simply connected Riemannian spin manifold of dimension $n+1$ with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. If $n+1 = 2m-1$, $m \geq 3$ odd, then there are two possibilities*

(c1) *(M, g) is isometric to the standard sphere*

(c2) *(M, g) is of type $(1, 1)$ and M is an Einstein-Sasaki manifold,*

(d) *Let (M, g) be a complete simply connected Riemannian spin manifold of dimension $n+1$ with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. If $n+1 = 4m-1$, $m \geq 3$, then there are two possibilities*

(d1) *(M, g) is isometric to the standard sphere*

(d2) *(M, g) is of type $(2, 0)$ and M is an Einstein-Sasaki manifold, but does not carry a Sasaki-3-structure*

(d3) *M is of type $(m+1, 0)$ and M carries a Sasaki-3-structure,*

(e) *Let (M, g) be a 7-dimensional complete simply connected Riemannian spin manifold with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. Then there are four possibilities*

(e1) *(M, g) is isometric to $\mathbb{S}^7$*

(e2) *M is of type $(1, 0)$ and M carries a nice 3-form ϕ with $\nabla\phi = *\phi$ but not a Sasaki structure*

(e3) *M is of type $(2, 0)$ and M carries a Sasaki structure, but not a Sasaki-3-structure*

(e4) *M is of type $(3, 0)$ and M carries a Sasaki-3-structure.*

For the equality case, let us consider the constant K case first. Let the mean curvature of (Σ, σ) be a constant $K > 0$. Then according to theorem 1.1, the eigenvalue estimate reads

$$\lambda_1(\mathbf{D}) \geq \frac{1}{4}K^2 + \frac{n^2}{4}. \quad (4.9)$$

When the equality holds, the inequality 4.6 yields

$$R[g] = n(n+1), \quad \mathcal{Q}\psi = 0 \quad (4.10)$$

which implies together with $\widetilde{D}^+\psi = (D - \frac{n+1}{2})\psi = 0$ yields

$$\widehat{\nabla}_X\psi = \frac{1}{2}\rho(X)\psi, \quad X \in \Gamma(TM) \quad (4.11)$$

i.e., ψ is a real Killing spinor with Killing constant $\alpha = \frac{1}{2}$ (relation 3.1). As a consequence of being Killing it has constant norm i.e., $\nabla_X|\psi|^2 = \frac{1}{2}\langle\rho(X)\psi, \psi\rangle + \frac{1}{2}\langle\psi, \rho(X)\psi\rangle = 0$. Therefore, One may set $|\psi| = 1$ without loss of generality. Now, one could perform all the foregoing analysis by solving the boundary value problem associated with $\widetilde{D}^-$ i.e.,

$$(D + \frac{n+1}{2})\psi = 0 \text{ on } M \quad (4.12)$$

$$\widetilde{P}_{\geq 0}^-\psi = 0 \text{ on } \Sigma. \quad (4.13)$$

In such case, we will have the same estimate for the eigenvalue of $\mathbf{D}$ and the equality case would correspond to ψ verifying the Killing equation with $\alpha = -\frac{1}{2}$ i.e.,

$$\widehat{\nabla}_X \psi = -\frac{1}{2}\rho(X)\psi, \quad X \in \Gamma(TM). \quad (4.14)$$

Therefore, we have the following corollary of theorem 1.1

Corollary 4.1 *Let (Σ, σ) be the constant mean curvature boundary of a $n+1$ dimensional simply connected oriented Riemannian spin manifold (M, g) and let the mean curvature be a constant $K > 0$. Also, assume the scalar curvatures $R[g] \geq (n+1)n$ and $R[\sigma] \geq n(n-1)$. Then the intrinsic Dirac operator $\mathbf{D}$ associated with the Riemannian metric σ on (Σ, σ) has the lowest eigenvalue $\lambda_1(\mathbf{D})^2 \geq \frac{1}{2}K^2 + \frac{n^2}{4}$. Moreover, $\lambda_1(\mathbf{D})^2 = \frac{1}{2}K^2 + \frac{n^2}{4}$ if and only if (M, g) is positive Einstein with Ricci curvature verifying $\text{Ric} = ng$ and admits real Killing spinors-specifically one of the manifolds classified by Bär in theorem 4.1.*

Proof. The ‘if’ part of the equality case $\lambda_1(\mathbf{D})$ is straightforward and discussed above. We prove the reverse direction now. Suppose (Σ, σ) is the constant mean curvature boundary of (M, g) which is a simply connected $n+1$ dimensional Riemannian spin manifold that is Einstein and admits real Killing spinors with Killing constant $\alpha = \pm\frac{1}{2}$ (i.e., manifolds that fall under Bär’s classification). If ψ is such a Killing spinor with $|\psi| = 1$, then the following inequality 4.6

$$\int_{\Sigma} \left(\widetilde{\lambda}_1 |\psi|^2 - \frac{1}{2}K |\psi|^2 \right) \mu_{\sigma} \geq \frac{1}{4} \int_{\Omega} \left(R^{\Omega} - n(n+1) \right) |\psi|^2 \mu_{\Omega} + \int_{\Omega} |\mathcal{Q}\psi|^2 = 0$$

yields

$$\widetilde{\lambda}_1 \geq \frac{1}{2}K \implies \lambda_1(\mathbf{D})^2 \geq \frac{K^2}{4} + \frac{n^2}{4} \quad (4.15)$$

which is of course true by the preceding analysis. Now as the equality is verified, one has

$$\widetilde{\nabla}_X \psi = \pm \frac{1}{2}\rho(X)\psi \quad (4.16)$$

and as a consequence of the proposition 2.1,

$$\mathbf{D}\psi = \frac{1}{2}K\psi \pm \frac{n}{2}\rho(\nu)\psi \text{ on } \Sigma. \quad (4.17)$$

Now compute the Raleigh quotient of $\mathbf{D}$

$$\lambda_1^2(\mathbf{D}) \leq \frac{\int_{\Sigma} \langle \mathbf{D}\psi, \mathbf{D}\psi \rangle \mu_{\sigma}}{\int_{\Sigma} |\psi|^2 \mu_{\sigma}} = \frac{K^2}{4} + \frac{n^2}{4}. \quad (4.18)$$

Therefore $\lambda_1(\mathbf{D})^2 = \frac{K^2}{4} + \frac{n^2}{4}$. Therefore equality holds for the eigenvalue and the reverse direction follows. $\square$

When the mean curvature is not constant, then we have the following bound for $\lambda_1(\mathbf{D})$ on the boundary of simply connected spin manifolds admitting real Killing spinors (Bar's [20] classification)

$$\frac{1}{4} \inf_{\Sigma} K^2 + \frac{n^2}{4} \leq \lambda_1(\mathbf{D})^2 \leq \frac{1}{4} \sup_{\Sigma} K^2 + \frac{n^2}{4}. \quad (4.19)$$

Remark 3 *We can not prove the case for $K = 0$ since that would result in losing the isomorphism property of $\tilde{D}^{\pm}$ and failure of theorem 3.1. Of course, without theorem 3.1, the rest of the preceding analysis fails. In fact, $K = 0$ case is not expected to hold due to Brendle-Marques-Neves [15].*

Remark 4 *Instead of the condition $K > 0$, one may impose that $\text{Ker}(\tilde{\mathbf{D}}^-) = \{0\}$ in 3.71 of theorem 3.1. But then the equality case 4.17 would not have any solution since $\mathbf{D}\psi = \frac{n}{2}\rho(\nu)\psi$ is precisely $\text{Ker}(\tilde{\mathbf{D}}^-)$ thereby making the argument void. Therefor, either way $K = 0$ case fails.*

5. Rigidity Results

Using the result of the theorem 1.1, we will prove the theorem 1.2. Let us recall the statement of the theorem 1.2

Theorem (Rigidity of Geodesic Balls) *Let M be a $n + 1$, $n \geq 2$ dimensional cap with boundary Σ and g be a smooth Riemannian metric on M that induces metric σ on Σ . Let the following conditions are verified by (M, g) and (Σ, σ)*

- (a) *Scalar curvature of (M, g) $R[g] \geq (n + 1)n$,*
- (b) *The induced metric σ on Σ by g agrees with the metric σ_0 of the boundary of a geodesic ball of standard unit sphere of radius less than or equal to $l := \frac{\pi}{2} - \epsilon$ for ϵ greater than or equal to a small positive number let's say $\frac{1}{100}$,*
- (c) *The mean curvature of Σ with respect to g and the standard round metric on unit sphere $\mathbb{S}^{n+1}$ coincide, constant, and strictly positive,*

Then g is isometric to the standard round metric on the unit sphere $\mathbb{S}^{n+1}$. In particular, M is isometric to a geodesic ball of radius l in a standard unit sphere.

We prove this theorem through eigenvalue comparison. But first, consider the elementary calculations regarding a geodesic ball (M_0, g_0) of radius strictly less than $\frac{\pi}{2}$ in $\mathbb{S}^{n+1}$ i.e., g_0 is isometric to the standard round metric on $\mathbb{S}^{n+1}$. To be more explicit, if $\mathbb{S}^{n+1}$ is a unit sphere in $\mathbb{R}^{n+2}$ written as $\sum_{i=1}^{n+2} x_i^2 = 1$, then (M_0, g_0) corresponds to the subset of $\mathbb{S}^{n+1}$ with $x^{n+2} \geq c$ with $\sin(\frac{1}{100}) < \sin \epsilon \leq c \leq 1$. The boundary of (M_0, g_0) corresponds to the sphere $\mathbb{S}_r^n$ of radius $r = \sqrt{1 - c^2}$. This sphere has constant mean curvature $\tilde{K}$. In usual spherical coordinate, the metric g_0 reads

$$g_0 = d\theta^2 + \sin^2 \theta (d\mathbb{S}^n)^2, \quad \theta \in [0, \sin^{-1} \sqrt{1 - c^2}]. \quad (5.1)$$

The second fundamental form of the coordinate spheres ($\theta = \text{constant}$) w.r.t g_0 reads

$$K_{AB} = \langle \nabla_A \partial_\theta, B \rangle = \sin \theta \cos \theta (d\mathbb{S}^n)^2(A, B), \quad A, B \in \Gamma(T\mathbb{S}^n). \quad (5.2)$$

The mean curvature of the $\theta = \text{constant}$ spheres with respect to the inward-pointing normal field $-\partial_\theta$ reads

$$K = \text{tr}_{\sin^2 \theta (d\mathbb{S}^n)^2} \left(\frac{1}{2} \partial_\theta (\sin^2 \theta (d\mathbb{S}^n)^2) \right) = n \cot \theta \quad (5.3)$$

Therefore (Σ, σ) has the mean curvature

$$\tilde{K} = \frac{nc}{\sqrt{1-c^2}} > 0. \quad (5.4)$$

The induced metric σ on Σ reads

$$\sigma = (1 - c^2)(d\mathbb{S}^n)^2. \quad (5.5)$$

5.1. proof of theorem 1.2

(M, g) is simply connected and its boundary is (Σ, σ) . The scalar curvature of (Σ, σ) is

$$R[\sigma] = \frac{(n-1)n}{1-c^2} \geq (n-1)n \quad (5.6)$$

and therefore verifies the hypothesis of the foregoing analysis. By the hypothesis of the theorem, the intrinsic geometry and the mean curvature of Σ are the same in (M, g) . In particular, mean curvature of (Σ, σ) in (M, g)

$$K = \frac{nc}{\sqrt{1-c^2}}. \quad (5.7)$$

Now the intrinsic Dirac operator $\mathbf{D}$ on (Σ, σ) has the lowest eigenvalue [24]

$$\lambda_1(\mathbf{D})^2 = \frac{n^2}{4(1-c^2)} = \frac{1}{4} \frac{n^2 c^2}{1-c^2} + \frac{n^2}{4} = \frac{K^2}{4} + \frac{n^2}{4} \quad (5.8)$$

and therefore equality holds for the lowest eigenvalue. Therefore by Corollary 4.1, (M, g) has non-trivial real Killing spinor fields with Killing constant $\pm \frac{1}{2}$, and therefore it is Einstein i.e., $\text{Ric}[g] = ng$. It is simply connected and therefore falls under the classification of [20]. We will prove that g is actually isometric to the standard spherical metric g_0 . First, note the Gauss equation

$$R[\sigma] = R[g] - 2\text{Ric}[g](\nu, \nu) + K^2 - K_{ij}K^{ij} \quad (5.9)$$

Now recall the decomposition of the second fundamental form $K_{ij} = \hat{K}_{ij} + \frac{1}{n}K\sigma_{ij}$ and obtain

$$R[\sigma] = R[g] - 2\text{Ric}[g](\nu, \nu) + (1 - \frac{1}{n})K^2 - \hat{K}_{ij}\hat{K}^{ij} \implies \hat{K}_{ij}\hat{K}^{ij} = 0 \quad (5.10)$$

Therefore (Σ, σ) is totally umbilic in (M, g) . Now recall the following theorem of [25] regarding the existence of totally umbilic hypersurfaces in an Einstein manifold.

Theorem 5.1 ([25]) *Let (Σ, σ) be a totally umbilical Einstein hypersurface in a complete Einstein manifold. Then if σ has positive Ricci curvature, then both g and σ have constant sectional curvature.*

This theorem applies in our context. This yields, (M, g) is a locally symmetric space. Now we use the second integrability condition on the existence of a Killing spinor by [26]

$$\nabla_{e_I} C_{e_J e_K e_L e_N} [\rho(e_L), \rho(e_N)] \psi \mp C_{e_J e_K e_L e_N} \rho(e_L) \psi = 0, \quad (5.11)$$

where C is the Weyl curvature of (M, g) . The constancy of the sectional curvature yields

$$C = 0. \quad (5.12)$$

Therefore, (M, g) is a maximally symmetric space and more precisely isometric to the standard sphere metric g_0 (topologically can be either a sphere or its quotient by the discrete subgroups of the isometry group $SO(n+2)$ acting freely and properly discontinuously on the sphere). Since (M, g) is simply connected and (Σ, σ) is its boundary sphere, (M, g) is the geodesic ball (M_0, g_0) . This proves the rigidity theorem 1.2.

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