Radiation estimates of the Minkowski space: coupled Einstein-Yang-Mills perturbations

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Abstract

Here we prove a global gauge-invariant radiation estimate for the perturbations of the $3+1$ dimensional Minkowski spacetime in the presence of Yang-Mills sources. In particular, we obtain a novel gauge invariant estimate for the Yang-Mills fields coupled to gravity in a double null framework in the Causal complement of a compact set of a Cauchy slice. A consequence of our result is the global exterior stability of the Minkowski space under coupled Yang-Mills perturbations. A special *null* structure present both in the Bianchi equations and the Yang-Mills equations is utilized crucially to obtain the dispersive estimates necessary to conclude the global existence property. Our result holds for any compact semi-simple gauge group.

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1. Introduction

A fundamental problem in the context of metric gravity is the existence of a stable ground state. For pure vacuum, the ground state with trivial topology is expected to be the Minkowski space. This is essentially motivated by the fact that the Minkowski space has zero ADM mass as a consequence of the positive mass theorem [1, 2]. Even similar results hold in a quasi-local sense due to [3, 4, 5]. More precisely, any Minkowski ball bounded by a space-like topological $2-$sphere contains zero gravitational energy. Naively, one would think if one were to perturb the Minkowski space by adding sufficiently small *energy* (defined to control appropriate norms), then such perturbations should disperse by losing energy to infinity through non-compact directions. Such a stability result for pure gravitational perturbations was first obtained by [7]. In particular, they proved that the Cauchy development of initial data sets that are close in an appropriate sense to the trivial Minkowski data lead to global, smooth, and geodesically complete solutions of the Einstein-vacuum equations that remain close in an appropriate, global sense to the Minkowski spacetime. In this study, the hyperbolic nature of the Bianchi equations was crucially utilized through the use of the Bel-Robinson stress-energy tensor. The special null structure present in these equations allowed the closure of the non-linear estimates. Later on, [8] provided simpler proof of the stability of Minkowski space in spacetime harmonic gauge. As such, it was believed earlier that the space-time harmonic gauge forms finite time coordinate singularity and therefore not suitable for long-time existence problem (notice that the spacetime harmonic gauge was first introduced by [10] to establish a local 'in time' wellposedness result for the vacuum Einstein's equations). However, the result of [9] suggests otherwise in the case of data close to the Minkowski trivial data in an appropriate sense. In particular, they cast Einstein's equations into a system of quasi-linear wave equations and utilized a weak null structure present in the equations in this particular gauge to control the non-linear terms. Suitable energies were constructed and controlled employing the conformal Killing fields of the background.
[11] provided an alternative proof by utilizing the energy estimates obtained from a less number of approximate conformal Killing fields (approximate inversion generator, approximate generator of time translation, and approximate scaling vector fields). In particular, she has avoided the use of the approximate rotation generators as commuting vector fields through the use of elliptic estimates. She also utilized a weaker fall-off condition on the data with zero ADM mass. In addition, this proof also required less regularity of the Weyl curvature (one derivative of the Weyl curvature). In addition to these studies of the classical initial value problem associated with the vacuum Einstein's equations where a combination of maximal slice and a null foliation (constructed utilizing an optical function) is utilized, [21] used a double null foliation to study the radiation problem associated with the vacuum Einstein's equations. In particular, their result implies the global stability of a suitably defined exterior domain of the Minkowski space under pure gravitational perturbations. In the stability problem of an exterior domain, the choice of double null foliation is ideal since one ultimately wants to investigate whether the gravitational radiation can escape to null infinity through the outgoing null direction or whether the associated non-linearities present in the equation can focus the radiation through the incoming null hypersurfaces to form *singularity* in a finite 'time'. Therefore the question of stability essentially turns into a competition be-

tween the geometric dispersion of energy and concentration of energy by the non-linearities present. In order for the dispersion to win at the level of small data, one, unfortunately, can not have an arbitrary quadratic non-linearity present in the relevant equations. In other words, the quadratic non-linearities must verify a particular structure known as the 'null' structure. Fortunately, the Bianchi equations while expressed in a double null coordinate precisely verify such a structure thereby affirming the dominance of dispersion phenomena. This in turn yields the stability result.

Moving one step further, one would like to understand the stability issues in the presence of suitable source terms. This is important since at the level of relativistic field theory, the Minkowski space never appears without additional fields (scalar fields, Maxwell fields or Yang-Mills fields) propagating on it. Therefore it is natural to ask the following question "*is Minkowski space stable under coupled gravitational and matter/radiation perturbations?*". [9] considered the small data perturbations of coupled gravity and mass-less scalar field and proved the global existence of the associated initial value problem in space-time harmonic gauge. The most interesting case of a coupled system is the study of Einstein-Maxwell system by [12]. In particular she extended the vacuum stability method of [7] including a Maxwell source term. The Maxwell equations share a remarkable similarity with that of the vacuum Bianchi equations and as such a null structure is present in these equations as well. [12] obtained gauge invariant estimate for the $U(1)$ curvature F together with the estimates for the Weyl curvature in a null-maximal framework which then yielded the global existence result. In addition the initial data she considered included a non-zero ADM mass. There have been several other studies as well in the direction of stability of the Minkowski space under coupled gravity-matter perturbations [13, 14, 15].

In this article, we want to include the Yang-Mills source terms in Einstein's equations and study the associated radiation problem. In other words, we study the global evolution of initial data that is specified on the complement of a large ball of an initial Cauchy hypersurface. To this end, we work in a double null framework to investigate the competition between dispersion and non-linearities. A fundamental difference between the $U(1)$ Maxwell theory and a non-abelian Yang-Mills theory is that the latter is no-linear. Therefore, in addition to investigating the non-linearities of the Bianchi equations, one ought to carefully treat the non-linearities of the Yang-Mills theory as well. However, since Yang-Mills theory is a gauge theory, one ought to work in a particular choice of gauge (or equivalently descend to the 'orbit space' of the theory). Unfortunately, there does not exist global gauge choice in Yang-Mills theory. The traditional choice of Lorentz gauge is known to develop finite time coordinate singularities in non-abelian theory contrary to the linear Maxwell theory where such a breakdown is absent. One may therefore choose a different gauge such as a 'generalized Coulomb gauge' to work with. However, the geometry of the orbit space of the theory (i.e., the space of connections modulo the bundle automorphisms) has a non-trivial consequence in the matter of gauge choice. The positivity of sectional curvature of the orbit space leads to the development of the so-called 'Gribov horizon' which essentially indicates the breakdown of the gauge choice. Therefore, we develop a novel gauge invariant framework for the Yang-Mills fields where we write down the Yang-Mills equations in the double null gauge in terms of the fully gauge covariant derivative. By the virtue of the compatibility of the fiber metrics with the fully gauge covariant derivative, we are able to obtain the necessary gauge-invariant energy estimates. The true non-linear characteristics of the Yang-Mills theory manifest themselves in the higher-order energy estimates when we commute the fully gauge covariant derivatives. The task, therefore, is reduced to establishing that this non-linearity is dominated by linear decay. As we describe in the appropriate section, the linear decay is just sufficient to overcome the non-linearities thereby closing the argument. This once again is a natural manifestation of the null structure that is present in the wave equation for the Yang-Mills curvature. Once we obtain the necessary gauge invariant estimates, we can work in a particular collection of gauge choices to yield a global existence result for the coupled theory. The latter part is standard and therefore we omit the discussion of such. The structure of the article is as follows. We start with writing down the Yang-Mills sourced Bianchi and Yang-Mills equations in the double null framework. We discuss the main theorem and its consequences along with a brief idea of the proof. Then we start estimating the relevant connection norms that are required for the energy estimates. This part is similar to the ideas of [21] and therefore we only estimate the non-trivial Yang-Mills coupling terms. Equipped with the connection estimates, we show that the energy (energy flux to be precise) for the curvature along the two families of the null hypersurfaces remain uniformly bounded in the exterior domain (causal complement of a large ball on the initial Cauchy slice) in terms of the data on the initial Cauchy slice. Through an application of Sobolev trace inequality, we immediately obtain the point-wise behavior of the Weyl and Yang-Mills curvatures. We conclude by discussing a few implications as well as the potential use of quasi-local energies (the one introduced by [3, 4]) to study the stability problems.

2. Einstein-Yang-Mills equations in double null foliation

Let the two null hypersurfaces H_{u_0} and $\underline{H}_{\underline{u}_0}$ (to be defined later) intersect at a topological $2-$sphere $S_{u_0,\underline{u}_0}$ in a globally hyperbolic spacetime M equipped with a Lorentzian metric g. The null hypersurfaces H and $\underline{H}$ are described by the level sets of the optical functions u and $\underline{u}$, respectively. In other words u and $\underline{u}$ satisfy the Eikonal equations

$$g^{\mu\nu}\partial_\mu u\partial_\nu u = 0,\ g^{\mu\nu}\partial_\mu \underline{u}\partial_\nu \underline{u} = 0. \tag{2.1}$$

Through the variation of u and $\underline{u}$, we can foliate a spacetime slab $\mathcal{D}_{u,\underline{u}}$ by these two family of null hypersurfaces. The geodesic generators of the double null foliation are the vector fields L and $\bar{L}$ given by

$$L := -g^{\mu\rho}\partial_\rho u\partial_\mu,\ \bar{L} := -g^{\mu\rho}\partial_\rho \underline{u}\partial_\mu \tag{2.2}$$

and manifestly they satisfy

$$\nabla_L L = 0 = \nabla_{\bar{L}}\bar{L}. \tag{2.3}$$

Whenever, we say H (resp. $\underline{H}$) we will always mean H_u (resp. $\underline{H}_{\underline{u}}$). In this notation, H_{u_0} and $\underline{H}_{\underline{u}_0}$ are the two initial null hypersurfaces corresponding to $u = u_0$ and $\underline{u} = \underline{u}_0$, respectively. Intersection of H and $\underline{H}$ is a topological $2-$ sphere $S_{u,\underline{u}}$. The spacetime metric in the double null gauge may be written as (in a local chart $(u, \underline{u}, \theta^1, \theta^2)$)

$$g := -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du), \tag{2.4}$$

where Ω is the null lapse function and $b := b^A\frac{\partial}{\partial\theta^A}$ is the null shift vector field. $\{\theta^A\}_{A=1}^2$ are the coordinates on the topological sphere $S_{u,\underline{u}}$. The induced metric on $S_{u,\underline{u}}$ is γ_{AB}. We can identify a normalized null frame (e_4, e_3, e_1, e_2) such that $g(e_4, e_4) = g(e_3, e_3) = 0 = g(e_A, e_4) = g(e_A, e_3) = 0$ and $g(e_4, e_3) = -2$, where (e_1, e_2) is an arbitrary frame on $S_{u,\underline{u}}$. We may identify e_4 and e_3 as follows

$$e_4 = \Omega^{-1}\frac{\partial}{\partial\underline{u}},\ e_3 = \Omega^{-1}(\frac{\partial}{\partial u} + b^A\frac{\partial}{\partial\theta^A}). \tag{2.5}$$

For the coordinate system, we may first define a coordinate chart $\mathcal{A}$ on $S_{u_0,\underline{u}_0}$ then drag it by the flow of the vector field e_4 along H_{u_0} and then drag it by the flow of e_3 to fill out the entire slab $\mathcal{D}_{u,\underline{u}}$. For more detailed information about the double null foliation of a spacetime, see [21, 19]. We define the areal radius of the topological $2-$ sphere $S_{u,\underline{u}}$ as follows

$$r(u,\underline{u}) := \sqrt{\frac{\text{Area}(S_{u,\underline{u}})}{4\pi}}. \tag{2.6}$$

In addition we also define $\tau_+ := \sqrt{1+\underline{u}^2}$ and $\tau_- := \sqrt{1+u^2}$. In the exterior region, $|u| \approx \tau_-$ and $r \approx \tau_+$.
Now we define a Yang-Mills theory on (M,g). We denote by $\mathfrak{P}$ a C^∞ principal bundle with base a $3+1$ dimensional Lorentzian manifold M and a Lie group G. We assume that G is compact semi-simple (for physical purposes) and therefore admits a positive definite non-degenerate bi-invariant metric. Its Lie algebra $\mathfrak{g}$ by construction admits an adjoint invariant, positive definite scalar product denoted by $\langle\ ,\ \rangle$ which enjoys the property: for $A,B,C \in \mathfrak{g}$,

$$\langle [A,B],C\rangle = \langle A,[B,C]\rangle. \tag{2.7}$$

as a consequence of adjoint invariance. A Yang-Mills connection is defined as a $1-$form w on $\mathfrak{P}$ with values in $\mathfrak{g}$ endowed with compatible properties. It's representative in a local trivialization of $\mathfrak{P}$ over $U \subset M$

$$\varphi : p \mapsto (x,a),\ p \in \mathfrak{P},\ x \in U,\ a \in G \tag{2.8}$$

is the $1-$form s^*w on U, where s is the local section of $\mathfrak{P}$ corresponding canonically to the local trivialization $s(x) = \varphi^{-1}(x,e)$, called a *gauge*. Let A_1 and A_2 be representatives of w in gauges s_1 and s_2 over $U_1 \subset M$ and $U_2 \in M$. In $U_1 \cap U_2$, one has

$$A_1 = Ad(u_{12}^{-1})A_2 + u_{12}\Theta_{MC}, \tag{2.9}$$

where Θ_{MC} is the Maurer-Cartan form on G, and $u_{12} : U_1 \cap U_2 \to G$ generates the transformation between the two local trivializations:

$$s_1 = R_{u_{12}} s_2, \tag{2.10}$$

$R_{u_{12}}$ is the right translation on $\mathfrak{P}$ by u_{12}. Given the principal bundle $\mathfrak{P} \to M$, a Yang-Mills potential A on M is a section of the fibered tensor product $T^*M \otimes_M \mathfrak{P}_{Affine,\mathfrak{g}}$ where $\mathfrak{P}_{Affine,\mathfrak{g}}$ is the affine bundle with base M and typical fibre $\mathfrak{g}$ associated to $\mathfrak{P}$ via relation (2.9). If $\widehat{A}$ is another Yang-Mills potential on M, then $A - \widehat{A}$ is a section of the tensor product of vector bundles $T^*M \otimes_M P_{Ad,\mathfrak{g}}$, where $\mathfrak{P}_{Ad,\mathfrak{g}} := \mathfrak{P} \times_{Ad} \mathfrak{g}$ is the vector bundle associated to $\mathfrak{P}$ by the adjoint representation of G on $\mathfrak{g}$. There is a inner product in the fibres of $\mathfrak{P}_{Ad,\mathfrak{g}}$, deduced from that on $\mathfrak{g}$. The curvature $\mathbf{C}$ of the connection w considered as a $1-$ form on $\mathfrak{P}$ is a $\mathfrak{g}$-valued $2-$form on $\mathfrak{P}$. Its representative in a gauge where w is represented by A is given by

$$F := dA + [A,A], \tag{2.11}$$

and relation between two representatives F_1 and F_2 on $U_1 \cap U_2$ is $F_1 = Ad(u_{12}^{-1})F_2$ and therefore F is a section of the vector bundle $\Lambda^2 T^*M \otimes_M \mathfrak{P}_{Ad,\mathfrak{g}}$. For a section $\mathfrak{O}$ of the vector bundle $\otimes^k T^*M \otimes_M \mathfrak{P}_{Ad,\mathfrak{g}}$, a natural covariant derivative is defined as follows

$$\widehat{D}\mathfrak{O} := D\mathfrak{O} + [A,\mathfrak{O}], \tag{2.12}$$

where D is the usual covariant derivative induced by the Lorentzian structure of M and by construction $\widehat{D}\mathfrak{D}$ is a section of the vector bundle $\otimes^{k+1}T^*M \otimes_M \mathfrak{P}_{Ad,\mathfrak{g}}$. The Yang-Mills coupling constant g_{YM} is set to 1.

The classical Yang-Mills equations (in the absence of sources) correspond to setting the natural (spacetime and gauge as defined in 2.12) covariant divergence of this curvature two-form F to zero. By virtue of its definition in terms of the connection, this curvature also satisfies the Bianchi identity that asserts the vanishing of its gauge covariant exterior derivative. Taken together these equations provide a geometrically natural nonlinear generalization of Maxwell's equations (when the latter are written in terms of a 'vector potential') and of course play a fundamental role in modern elementary particle physics. If nontrivial bundles are considered or nontrivial spacetime topologies are involved, then the foregoing so-called 'local trivializations' of the bundles in question must be patched together to give global descriptions but, by virtue of the covariance of the formalism, there is a natural way of carrying out this patching procedure at least over those regions of spacetime where the connections are well-defined.
Now that we have defined a Yang-Mills theory over the Lorentzian manifold M (which is to be sufficiently close to Minkowski spacetime in an appropriate sense), we may proceed to couple it with the Einstein's equations and write down the full set of equations in the double null coordinates. Before doing so we need to project the covariant derivatives (spacetime and gauge) onto the topological 2−spheres that are the leaves of the double null foliation. In this section, we explicitly define all the entities associated with the double null foliation and write down structure equations. The spacetime covariant derivative D admits the following usual decomposition in terms of its components parallel and orthogonal to the topological 2−sphere $S_{u,\underline{u}}$

$$D_{e_a}e_b = \nabla_{e_a}e_b - \frac{1}{2}\langle D_{e_a}e_b, e_3\rangle e_4 - \frac{1}{2}\langle D_{e_a}e_b, e_4\rangle e_3, \tag{2.13}$$

where ∇ is the $S_{u,\underline{u}}$-parallel covariant derivative. Similar to the spacetime covariant derivative, the spacetime gauge covariant derivative $\widehat{D}$ admits the following decomposition

$$\begin{aligned}\widehat{D}_{e_a}e_b &= \widehat{\nabla}_{e_a}e_b - \frac{1}{2}\langle \widehat{D}_{e_a}e_b, e_3\rangle e_4 - \frac{1}{2}\langle \widehat{D}_{e_a}e_b, e_4\rangle e_3 \\ &= \nabla_{e_a}e_b - \frac{1}{2}\langle \mathcal{D}_{e_a}e_b, e_3\rangle e_4 - \frac{1}{2}\langle \mathcal{D}_{e_a}e_b, e_4\rangle e_3\end{aligned} \tag{2.14}$$

simply because the basis $(e_3, e_4, e_a)_{a=1}^2$ are not sections of the gauge bundle and therefore gauge covariant derivative acts as ordinary covariant derivative. Now recalling the following definitions of the outgoing and incoming null second fundamental form of $S_{u,v}$

$$\chi_{ab} := \langle D_{e_a}e_4, e_b\rangle, \qquad \bar{\chi}_{ab} := \langle D_{e_a}e_3, e_b\rangle, \tag{2.15}$$

the spacetime covariant (also gauge covariant) derivative satisfies

$$D_{e_a}e_b = \nabla_{e_a}e_b + \frac{1}{2}\bar{\chi}_{ab}e_4 + \frac{1}{2}\chi_{ab}e_3. \tag{2.16}$$

Now let $K \in sections\{TS_{u,\underline{u}}\}$ and Z be a $\mathfrak{g}$−valued frame vector field on $S_{u,\underline{u}}$ that transforms as a tensor under gauge transformation, then using the definition of the gauge covariant derivative (2.9), the following holds for the gauge covariant derivative

$$\widehat{D}_K Z = \widehat{\nabla}_K Z + \frac{1}{2}\bar{\chi}(K,Z)e_4 + \frac{1}{2}\chi(K,Z)e_3, \tag{2.17}$$

where $\chi(K,Z) := \chi_{ab}K_a Z^P{}_{Qb}$ and $\bar{\chi}(K,Z) := \chi_{ab}K_a Z^P{}_{Qb}$. Now we recall the definitions of the remaining connection coefficients adapted to the double null framework

$$\eta_a := -\frac{1}{2}\langle D_{e_3}e_a, e_4\rangle,\ \omega := -\frac{1}{4}\langle D_{e_4}e_3, e_4\rangle = -\frac{1}{2}D_{e_4}\ln\Omega, \tag{2.18}$$
$$\bar{\eta}_a := -\frac{1}{2}\langle D_{e_4}e_a, e_3\rangle,\ \bar{\omega} := -\frac{1}{4}\langle D_{e_3}e_4, e_3\rangle = -\frac{1}{2}D_{e_3}\ln\Omega$$

and the torsion $\zeta_a := \frac{1}{2}\langle D_{e_a}e_4, e_3\rangle = \frac{1}{2}(\eta - \bar{\eta})$. Utilizing these definitions, let us write down the kinematical set of structural equations

$$D_{e_a}e_3 = \bar{\chi}_{ab}e_b + \zeta_a e_3,\ D_{e_3}e_a = \nabla_{e_3}e_a + \eta_a e_3, \tag{2.19}$$
$$D_{e_4}e_a = \nabla_{e_4}e_a + \bar{\eta}_a e_4,\ \nabla_{e_3}e_3 = -2\bar{\omega}e_3, \tag{2.20}$$
$$D_{e_4}e_4 = -2\omega e_4,\ \nabla_{e_4}e_3 = 2\omega e_3 + 2\bar{\eta}_a e_a, \tag{2.21}$$
$$D_{e_3}e_4 = 2\bar{\omega}e_4 + 2\eta_a e_a,\ D_{e_4}e_a = \nabla_{e_4}e_a + \bar{\eta}_a e_4, \tag{2.22}$$
$$D_{e_a}e_4 = \chi_{ab}e_b - \zeta_a e_4. \tag{2.23}$$

Also note $\eta_a = \zeta_a + \nabla_{e_a}\ln\Omega,\ \bar{\eta}_a = -\zeta_a + \nabla_{e_a}\ln\Omega$. Utilizing these kinematical structure equations, we obtain the dynamical set of structural equations suitable trace of which are nothing but the Einstein's equations sourced by Yang-Mills stress energy tensor and expressed in the double null framework. Before writing down such equations, let us recall the following decomposition of the spacetime Riemann curvature tensor

$$R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}) + \frac{1}{6}R(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}), \tag{2.24}$$

where W is the Weyl curvature tensor describing pure gravitational degrees of freedom. W is trace-free and enjoys the same algebraic symmetry of the Riemann curvature. Since the Ricci and the scalar curvatures are fixed by the Einstein's field equations, we ought to write down the equation for the null components of the Weyl curvature. These equations will constitute the null Bianchi equations. The components of the Weyl curvature are defined as follows

$$\alpha_{ab} := W(e_a, e_4, e_b, e_4),\ \bar{\alpha}_{ab} := W(e_a, e_3, e_b, e_3), \tag{2.25}$$
$$2\beta_a := W(e_4, e_3, e_4, e_a),\ 2\bar{\beta}_a := W(e_3, e_4, e_3, e_a), \tag{2.26}$$
$$\rho := \frac{1}{4}W(e_4, e_3, e_4, e_3),\ \sigma := \frac{1}{4}\,{}^*W(e_4, e_3, e_4, e_3), \tag{2.27}$$

where ${}^*W$ is the left Hodge dual of W defined as follows

$${}^*W_{\alpha\beta\gamma\delta} := \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta}, \tag{2.28}$$

where $\epsilon_{\alpha\beta\mu\nu}$ is the volume form on the spacetime M. In addition to the Weyl field, we also have the Yang-Mills curvature $F := \frac{1}{2}F^P{}_{Q\mu\nu}dx^\mu \wedge dx^\nu \in \Lambda^2 T^*M \otimes_M \mathfrak{P}_{Ad,\mathfrak{g}},\ P, Q = 1, 2, 3, ..., \dim(V)$, where V is the representation of the Lie algebra $\mathfrak{g}$. The components of the Yang-Mills curvature F are defined as follows

$$\alpha^F_a := F^P{}_Q(e_a, e_4),\ \bar{\alpha}^F_a := F^P{}_Q(e_a, e_3),\ \rho^F := \frac{1}{2}F^P{}_Q(e_3, e_4), \tag{2.29}$$
$$\sigma^F = \frac{1}{2}\,{}^*F^P{}_Q(e_3, e_4) = F^P{}_Q(e_1, e_2).$$

Also decompose the null second fundamental forms $\chi, \bar{\chi}$ into their trace and trace-free components

$$\chi = \hat{\chi} + \frac{1}{2}tr\chi\gamma,\ \bar{\chi} = \hat{\bar{\chi}} + \frac{1}{2}tr\bar{\chi}\gamma. \tag{2.30}$$

The Einstein's equations (with the choice of unit $8\pi G = c = 1$)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \mathfrak{T}_{\mu\nu} \tag{2.31}$$

in the double null framework reads

$$\nabla_4\mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2_\gamma - 2\omega\mathrm{tr}\chi - \mathfrak{T}_{44} \tag{2.32}$$

$$\nabla_4\hat{\chi} + \mathrm{tr}\chi\hat{\chi} = -2\omega\hat{\chi} - \alpha \tag{2.33}$$

$$\nabla_3 tr\underline{\chi} + \frac{1}{2}(tr\underline{\chi})^2 = -|\hat{\underline{\chi}}|^2_\gamma - 2\underline{\omega}tr\underline{\chi} - \mathfrak{T}_{33} \tag{2.34}$$

$$\nabla_3\hat{\underline{\chi}} + tr\underline{\chi}\hat{\underline{\chi}} = -2\underline{\omega}\hat{\underline{\chi}} - \underline{\alpha} \tag{2.35}$$

$$\nabla_4\eta_a = -\chi\cdot(\eta - \underline{\eta}) - \beta - \frac{1}{2}\mathfrak{T}_{a4} \tag{2.36}$$

$$\nabla_3\underline{\eta}_a = -\underline{\chi}\cdot(\underline{\eta} - \eta) + \underline{\beta} + \frac{1}{2}\mathfrak{T}_{a3} \tag{2.37}$$

$$\nabla_4\underline{\omega} = 2\omega\underline{\omega} + \frac{3}{4}|\eta - \underline{\eta}|^2 - \frac{1}{4}(\eta - \underline{\eta})\cdot(\eta + \underline{\eta}) - \frac{1}{8}|\eta + \underline{\eta}|^2 + \frac{1}{2}\rho + \frac{1}{4}\mathfrak{T}_{43}$$

$$\nabla_3\omega = 2\omega\underline{\omega} + \frac{3}{4}|\eta - \underline{\eta}|^2 + \frac{1}{4}(\eta - \underline{\eta})\cdot(\eta + \underline{\eta}) - \frac{1}{8}|\eta + \underline{\eta}|^2 + \frac{1}{2}\rho + \frac{1}{4}\mathfrak{T}_{43}$$

$$\nabla_4 tr\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi tr\underline{\chi} = 2\omega tr\underline{\chi} + 2\mathrm{div}\underline{\eta} + 2|\underline{\eta}|^2_\gamma + 2\rho - \hat{\chi}\cdot\hat{\underline{\chi}} \tag{2.38}$$

$$\nabla_3\mathrm{tr}\chi + \frac{1}{2}tr\underline{\chi}\mathrm{tr}\chi = 2\underline{\omega}\mathrm{tr}\chi + 2\mathrm{div}\eta + 2|\eta|^2 + 2\rho - \hat{\chi}\cdot\hat{\underline{\chi}} \tag{2.39}$$

$$\nabla_4\hat{\underline{\chi}} + \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} = \nabla\hat{\otimes}\underline{\eta} + 2\omega\hat{\underline{\chi}} - \frac{1}{2}tr\underline{\chi}\hat{\chi} + \underline{\eta}\hat{\otimes}\underline{\eta} + \hat{\mathfrak{T}}_{ab} \tag{2.40}$$

$$\nabla_3\hat{\chi} + \frac{1}{2}tr\underline{\chi}\hat{\chi} = \nabla\hat{\otimes}\eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} + \eta\hat{\otimes}\eta + \hat{\mathfrak{T}}_{ab} \tag{2.41}$$

$$\mathrm{div}\hat{\chi} = \frac{1}{2}\nabla\mathrm{tr}\chi - \frac{1}{2}(\eta - \underline{\eta})\cdot(\hat{\chi} - \frac{1}{2}\mathrm{tr}\chi\gamma_{ab}) - \beta + \frac{1}{2}\mathfrak{T}(e_4,\cdot) \tag{2.42}$$

$$\mathrm{div}\hat{\underline{\chi}} = \frac{1}{2}\nabla tr\underline{\chi} - \frac{1}{2}(\underline{\eta} - \eta)\cdot(\hat{\underline{\chi}} - \frac{1}{2}tr\underline{\chi}\gamma_{ab}) - \underline{\beta} + \frac{1}{2}\mathfrak{T}(e_3,\cdot) \tag{2.43}$$

$$\mathrm{curl}\eta = \hat{\underline{\chi}}\wedge\hat{\chi} + \sigma\epsilon = -\mathrm{curl}\underline{\eta} \tag{2.44}$$

$$K - \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} + \frac{1}{4}\mathrm{tr}\chi tr\underline{\chi} = -\rho + \frac{1}{4}\mathfrak{T}_{43}. \tag{2.45}$$

where $\mathfrak{T}_{\mu\nu} := \frac{1}{2}\left(F^P{}_{Q\mu\alpha}F^Q{}_{P\nu}{}^\alpha +^* F^P{}_{Q\mu\alpha}{}^*F^Q{}_{P\nu}{}^\alpha\right)$ is the Yang-Mills stress-energy tensor and K is the sectional curvature of the topological $2-$sphere $S_{u,\underline{u}}$ (or a constant multiple of Gauss curvature). Here equations (2.42-2.45) are the null-constraint equations. Now recall the Bianchi identities

$$D_\mu R_{\alpha\beta\nu\lambda} + D_\nu R_{\alpha\beta\lambda\mu} + D_\lambda R_{\alpha\beta\mu\nu} = 0 \tag{2.46}$$

which together with the decomposition (2.24) and the Einstein's equations yields the following Yang-Mills type equations for the Weyl curvature

$$D^\alpha W_{\alpha\beta\gamma\delta} = J[\mathfrak{T}]_{\beta\gamma\delta},\ D_{[\mu}W_{\gamma\delta]\alpha\beta} = \frac{1}{3}\epsilon_{\nu\mu\gamma\delta}J[\mathfrak{T}]^{*\nu}{}_{\alpha\beta}. \tag{2.47}$$

Here $J[\mathfrak{T}]$ is the source term determined fully by the Yang-Mills stress energy tensor $\mathfrak{T}$. After elementary algebraic manipulations, the differential equations for the Weyl curvature may be cast into the following double-null form [3]

$$\nabla_3\alpha_{ab} + \frac{1}{2}tr\underline{\chi}\alpha_{ab} = (\nabla\hat{\otimes}\beta)_{ab} + 4\underline{\omega}\alpha_{ab} - 3(\hat{\chi}\rho + {}^*\hat{\chi}\sigma)_{ab} + ((\zeta+4\eta)\hat{\otimes}\beta)_{ab} \tag{2.48}$$
$$+\frac{1}{2}(D_3\mathfrak{T}_{44} - D_4\mathfrak{T}_{43})\gamma_{ab} - D_4\mathfrak{T}_{ab} + \frac{1}{2}\left(D_b\mathfrak{T}_{4a} + D_a\mathfrak{T}_{4b}\right)$$
$$\mathfrak{D}_4\beta_a + 2\mathrm{tr}\chi\beta_a = (\mathrm{div}\alpha)_a - 2\omega\beta_a + (\eta\cdot\alpha)_a - \frac{1}{2}(D_a\mathfrak{T}_{44} - D_4\mathfrak{T}_{4a}) \tag{2.49}$$
$$\nabla_3\beta_a + tr\underline{\chi}\beta_a = \nabla_a\rho + {}^*\nabla_a\sigma + 2\underline{\omega}\beta_a + 2(\hat{\chi}\cdot\underline{\beta})_a + 3(\eta\rho + {}^*\eta\sigma)_a$$
$$+\frac{1}{2}(D_a\mathfrak{T}_{34} - D_4\mathfrak{T}_{3a})$$
$$\nabla_4\sigma + \frac{3}{2}\mathrm{tr}\chi\sigma = -\mathrm{div}\ {}^*\beta + \frac{1}{2}\underline{\hat{\chi}}\cdot\ {}^*\alpha - \zeta\cdot\ {}^*\beta - 2\underline{\eta}\cdot\ {}^*\beta - \frac{1}{4}(D_\mu\mathfrak{T}_{4\nu} - D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34}$$
$$\nabla_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma = -\mathrm{div}\ {}^*\underline{\beta} + \frac{1}{2}\hat{\chi}\cdot\ {}^*\underline{\alpha} - \zeta\cdot\ {}^*\underline{\beta} - 2\eta\cdot\ {}^*\underline{\beta} + \frac{1}{4}(D_\mu\mathfrak{T}_{3\nu} - D_\nu\mathfrak{T}_{3\mu})\epsilon^{\mu\nu}{}_{34}$$
$$\nabla_4\rho + \frac{3}{2}\mathrm{tr}\chi\rho = \mathrm{div}\beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha + \zeta\cdot\beta + 2\underline{\eta}\cdot\beta - \frac{1}{4}(D_3\mathfrak{T}_{44} - D_4\mathfrak{T}_{34}) \tag{2.50}$$
$$\nabla_3\rho + \frac{3}{2}tr\underline{\chi}\rho = -\mathrm{div}\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} + \zeta\cdot\underline{\beta} - 2\eta\cdot\underline{\beta} + \frac{1}{4}(D_3\mathfrak{T}_{34} - D_4\mathfrak{T}_{33}) \tag{2.51}$$
$$\nabla_4\underline{\beta}_a + \mathrm{tr}\chi\underline{\beta}_a = -\nabla_a\rho + {}^*\nabla_a\sigma + 2\omega\underline{\beta}_a + 2(\underline{\hat{\chi}}\cdot\beta)_a - 3(\underline{\eta}\rho - {}^*\underline{\eta}\sigma)_a$$
$$-\frac{1}{2}(D_a\mathfrak{T}_{43} - D_3\mathfrak{T}_{4a})$$
$$\nabla_3\underline{\beta}_a + 2tr\underline{\chi}\underline{\beta}_a = -(\mathrm{div}\underline{\alpha})_a - 2\underline{\omega}\underline{\beta}_a + (\underline{\eta}\cdot\underline{\alpha})_a + \frac{1}{2}(D_a\mathfrak{T}_{33} - D_3\mathfrak{T}_{3a}) \tag{2.52}$$
$$\nabla_4\underline{\alpha}_{ab} + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha} = -(\nabla\hat{\otimes}\underline{\beta})_{ab} + 4\omega\underline{\alpha}_{ab} - 3(\underline{\hat{\chi}}_{ab}\rho - {}^*\underline{\hat{\chi}}_{ab}\sigma) + ((\zeta - 4\underline{\eta})\hat{\otimes}\underline{\beta})_{ab} \tag{2.53}$$
$$+\frac{1}{2}(D_4\mathfrak{T}_{33} - D_3\mathfrak{T}_{34})\gamma_{ab} - D_3\mathfrak{T}_{ab} + \frac{1}{2}\left(D_a\mathfrak{T}_{3b} + D_b\mathfrak{T}_{3a}\right).$$

The null Yang-Mills equations are written as

$$\widehat{\nabla}_4\underline{\alpha}^F + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}^F = -\widehat{\nabla}\rho^F - {}^*\widehat{\nabla}\sigma^F - 2\ {}^*\underline{\eta}\sigma^F - 2\underline{\eta}\rho^F + 2\omega\underline{\alpha}^F - \underline{\hat{\chi}}\cdot\alpha^F \tag{2.54}$$
$$\widehat{\nabla}_3\alpha^F + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha^F = -\widehat{\nabla}\rho^F + {}^*\widehat{\nabla}\sigma^F - 2\ {}^*\eta\sigma^F + 2\eta\rho^F + 2\underline{\omega}\alpha^F - \hat{\chi}\cdot\underline{\alpha}^F \tag{2.55}$$
$$\widehat{\nabla}_4\rho^F + \mathrm{tr}\chi\rho^F = -\widehat{\mathrm{div}}\alpha^F - (\eta - \underline{\eta})\cdot\alpha^F \tag{2.56}$$
$$\widehat{\nabla}_4\sigma^F + \mathrm{tr}\chi\sigma^F = -\widehat{\mathrm{curl}}\alpha^F + (\eta - \underline{\eta})\cdot\ {}^*\alpha^F \tag{2.57}$$
$$\widehat{\nabla}_3\rho^F + \mathrm{tr}\underline{\chi}\rho^F = -\widehat{\mathrm{div}}\underline{\alpha}^F + (\eta - \underline{\eta})\cdot\underline{\alpha}^F \tag{2.58}$$
$$\widehat{\nabla}_3\sigma^F + \mathrm{tr}\underline{\chi}\sigma^F = -\widehat{\mathrm{curl}}\underline{\alpha}^F + (\eta - \underline{\eta})\cdot\ {}^*\underline{\alpha}^F. \tag{2.59}$$

We may write down explicitly the null components of the Yang-Mills stress energy tensor

$$\mathfrak{T}_{43} = \rho^F\cdot\rho^F + \sigma^F\cdot\sigma^F,\ \mathfrak{T}_{4A} = \alpha^F_A\cdot\rho^F + \epsilon_A{}^B\alpha^F_B\cdot\sigma^F, \tag{2.60}$$
$$\mathfrak{T}_{3A} = -\bar{\alpha}^F_A\cdot\rho^F + \epsilon_A{}^B\bar{\alpha}^F_B\cdot\sigma^F,\ \mathfrak{T}_{44} = \alpha^F_A\cdot\alpha^F_B\gamma^{AB},\ \mathfrak{T}_{33} = \bar{\alpha}^F_A\cdot\bar{\alpha}^F_B\gamma^{AB}, \tag{2.61}$$
$$\mathfrak{T}_{ab} = \frac{1}{2}(\rho^F\cdot\rho^F + \sigma^F\cdot\sigma^F)\gamma_{AB} - (\alpha^F_A\cdot\bar{\alpha}^F_B + \bar{\alpha}^F_A\cdot\alpha^F_B - \alpha^F_C\cdot\bar{\alpha}^F_D\gamma^{CD}\gamma_{AB}), \tag{2.62}$$

[3] $A\hat{\otimes}B := (A\otimes B + B\otimes A - A\cdot B\gamma)$, $(A\wedge B) := \epsilon^{ac}\gamma^{bd}A_{ab}B_{cd}$, $(\mathrm{curl}A)_{a_1\cdots a_n} := \epsilon^{cd}\nabla_c A_{da_1\cdots a_n}$

The Yang-Mills source terms that appear in the null Binachi equations are explicit computed as follows. Since we do not need the exact expressions for the source terms, we write these in the schematic form for convenience.

$$
\begin{aligned}
D_a\mathfrak{T}_{44} - D_4\mathfrak{T}_{4a} &\sim \langle\alpha^F, \widehat{\nabla}_b\alpha^F\rangle - \chi_{bc}\mathfrak{T}_{c4} + \eta_b\mathfrak{T}_{44} - 2\omega\mathfrak{T}_{4b} - \widehat{\nabla}_4(\alpha^F_b\cdot\rho^F - \alpha^F_b\cdot\sigma^F),\\
D_b\mathfrak{T}_{34} - D_4\mathfrak{T}_{3b} &\sim (\langle\widehat{\nabla}\rho^F, \rho^F\rangle + \langle\widehat{\nabla}\sigma^F, \sigma^F\rangle) - \underline{\chi}\alpha^F(\sigma^F + \rho^F) - \widehat{\nabla}_4\underline{\alpha}^F\cdot(\rho^F + \sigma^F)\\
&\quad -\underline{\alpha}^F\cdot(\widehat{\nabla}_4\rho^F + \widehat{\nabla}_4\sigma^F),\\
(D_\mu\mathfrak{T}_{4\nu} - D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34} &\sim \widehat{\nabla}(\alpha^F\cdot\rho^F - \alpha^F\cdot\sigma^F) - \chi(\rho^F\cdot\rho^F + \alpha^F\cdot\underline{\alpha}^F + \sigma^F\cdot\sigma^F) \qquad (2.63)\\
&\quad +(\eta - \underline{\eta})\alpha^F\cdot(\rho^F + \sigma^F)\\
D_3R_{44} - D_4R_{34} &\sim \langle\alpha^F, \widehat{\nabla}_3\alpha^F\rangle + \underline{\omega}|\alpha^F|^2 + \eta_A(\alpha^F\cdot\sigma^F + \alpha^F\cdot\rho^F) + \langle\rho^F, \widehat{\nabla}_4\rho^F\rangle\\
&\quad +\langle\sigma^F, \widehat{\nabla}_4\sigma^F\rangle + \underline{\eta}(\rho^F\underline{\alpha}^F + \rho^F\cdot\sigma^F),\\
D_bR_{43} - D_3R_{4b} &\sim \langle\rho^F, \widehat{\nabla}\rho^F\rangle + \langle\sigma^F, \widehat{\nabla}\sigma^F\rangle - \underline{\chi}\alpha^F(\rho^F + \sigma^F) + \chi\underline{\alpha}^F(\rho^F + \sigma^F), \qquad (2.64)\\
(D_\mu R_{3\nu} - D_\nu R_{3\mu})\epsilon^{\mu\nu}{}_{34} &\sim \widehat{\nabla}(\underline{\alpha}^F\rho^F - \underline{\alpha}^F\sigma^F) - \underline{\chi}(\rho^F\rho^F + \underline{\alpha}^F\alpha^F + \sigma^F\sigma^F) \qquad (2.65)\\
&\quad +(\eta + \underline{\eta})\underline{\alpha}^F(\rho^F + \sigma^F),\\
D_3R_{43} - D_4R_{33} &\sim 2\langle\rho^F, -\hat{\mathrm{div}}\underline{\alpha}^F + tr\underline{\chi}\rho^F + (\eta - \underline{\eta})\underline{\alpha}^F\rangle + 2\langle\sigma_F, -\hat{\mathrm{curl}}\underline{\alpha}^F - tr\underline{\chi}\sigma^F\\
&\quad +(\eta - \underline{\eta})\ {}^*\underline{\alpha}^F\rangle - 2\eta(\rho^F\underline{\alpha}^F + \rho^F\sigma^F) - \langle\underline{\alpha}^F, -\frac{1}{2}\mathrm{tr}\chi\underline{\alpha}^F - \widehat{\nabla}\rho^F - {}^*\widehat{\nabla}\sigma^F - 2\ {}^*\underline{\eta}\cdot\sigma^F\\
&\quad -2\underline{\eta}\cdot\rho^F + 2\omega\underline{\alpha}^F - \hat{\underline{\chi}}\cdot\alpha^F\rangle + 2\omega|\underline{\alpha}^F|^2 - 4\underline{\eta}_a(\underline{\alpha}^F\cdot\rho^F - \underline{\alpha}^F\cdot\sigma^F),\\
D_bR_{33} - D_3R_{3b} &\sim \langle\underline{\alpha}^F, \widehat{\nabla}\underline{\alpha}^F\rangle - \underline{\chi}\underline{\alpha}^F\cdot(\rho^F + \sigma^F) + \underline{\eta}|\underline{\alpha}^F|^2 - 2\underline{\omega}\underline{\alpha}^F\cdot(\rho^F + \sigma^F) \qquad (2.66)\\
&\quad -\widehat{\nabla}_3(\underline{\alpha}^F\cdot\rho^F + \underline{\alpha}^F\cdot\sigma^F).
\end{aligned}
$$

3. Main theorem and idea of the proof

In this section, we describe the main result of the article and sketch the main argument behind the proof. As discussed in [6], the Yang-Mills sourced null Bianchi equations and null Yang-Mills equations are manifestly hyperbolic contrary to the equations $d\Gamma + \Gamma\Gamma = R$ and $dA + [A, A] = F$ which are manifestly non-hyperbolic, where Γ, A, R, and F denote connection coefficients on the frame tangent bundle, connection on the principle $G-$bundle (or gauge bundle), curvature of frame tangent bundle, and curvature of the principle $G-$bundle, respectively. In the double null framework, $d\Gamma + \Gamma\Gamma = R$ reduces to transport and elliptic equations which may be utilized to control the connections in terms of the curvature. However, in order to utilize the equations $dA + [A, A] = F$, we need to make gauge choice for the Yang-Mills theory. This makes the global analysis tremendously complicated and non-trivial. The gauge choice for Yang-Mills equations is an extremely non-trivial one due to the presence of Gribov phenomena [16, 17]. The orbit space of the theory (space of connections mudulo the bundle automorphisms or the gauge transformations) has a positive definite sectional curvature and therefore the choice of a well known Coulomb gauge remains only valid in a small enough neighbourhood of a background connection. In fact this geometric property of the orbit space does not allow for any global gauge choice for the Yang-Mills theory [18]. Lorentz gauge for the Yang-Mills theory is also expected to be pathological in a sense that it can develop gauge singularity in finite time (although in the regime of small data such problem may be avoided through a careful analysis). Note that none of these problems are present in the Maxwell theory. In order to avoid these issues, we work in a fully gauge covariant framework where we *do not* split the gauge covariant covariant derivative into its spacetime (or spatial) part and a gauge part given by the Yang-Mills connection. The *true* non-linearity of the Yang-Mills theory shows up through the commutation of gauge covariant derivative. To our knowledge, this approach was not utilized previously except by [20] in their treatment of light cone parametrix of Yang-Mills equations on Minkowski background. Once we have obtained the necessary gauge invariant estimates, the remaining task to obtain a global solution is straightforward and therefore we do not discuss the detail in here. Let us now write down the main theorem of the article.

Theorem 3.1. *Let (M,g) be a $3+1$ dimensional Lorentzian globally hyperbolic manifold diffeomorphic to $\mathbb{R}^{1+3}$. Let $S_{u_0,\underline{u}_0}$ be a space-like topological $2-$sphere obtained by the intersection of an incoming and outgoing null cone $\underline{H}_{\underline{u}_0}$ and H_{u_0} such that $u_0+\underline{u}_0=0$. Let Σ_0 be a space-like Cauchy hypersurface diffeomorphic to $\mathbb{R}^3$ and foliated by the topological $2-$spheres $S_{u,\underline{u}}$ such that $u+\underline{u}=0$ and such spheres constitute the initial spheres for the canonical double null foliation of the causal future of Σ_0. Let us also consider the space-like topological ball $B_{u_0,\underline{u}_0}\subset\Sigma_0$ bounded by the sphere $S_{u_0,\underline{u}_0}$ and denote the exterior initial Cauchy hypersurface $\Sigma_0-B_{u_0,\underline{u}_0}$ by $\widehat{\Sigma}_0$ ($\widehat{\Sigma}_0$ is diffeomorphic to the complement of a ball in $\mathbb{R}^3$(. Let us define the initial data norm as follows*

$$\begin{aligned}
\mathbf{W}_0 :=& \int_{\widehat{\Sigma}_0} r^4(|\alpha|^2+|\beta|^2)+\int_{\widehat{\Sigma}_0}\tau_-^4|\underline{\beta}|^2+\int_{\widehat{\Sigma}_0}\tau_-^2r^2(\rho^2+\sigma^2) \qquad (3.1)\\
&+\int_{\widehat{\Sigma}_0}(\tau_-^4|\underline{\alpha}|^2+r^4|\beta|^2+r^2\tau_-^2|\underline{\beta}|^2+r^4|\rho|^2+r^4|\sigma|^2),\\
&+\int_{\widehat{\Sigma}_0} r^4(|r\nabla\alpha|^2+|r\nabla\beta|^2)+\int_{\widehat{\Sigma}_0}\tau_-^4|r\nabla\underline{\beta}|^2+\int_{\widehat{\Sigma}_0}\tau_-^2r^2(|r\nabla\rho|^2+|r\nabla\sigma|^2)\\
&+\int_{\widehat{\Sigma}_0}(\tau_-^4|r\nabla\underline{\alpha}|^2+r^4|r\nabla\beta|^2+r^2\tau_-^2|r\nabla\underline{\beta}|^2+r^4|r\nabla\rho|^2+r^4|r\nabla\sigma|^2)\\
&+\int_{\widehat{\Sigma}_0} r^4|rD_4\alpha|^2+\int_{\widehat{\Sigma}_0}\tau_-^4|\tau_-D_3\underline{\alpha}|^2\\
&+\int_{\widehat{\Sigma}_0} r^4(|r^2\nabla^2\alpha|^2+|r^2\nabla^2\beta|^2)+\int_{\widehat{\Sigma}_0}\tau_-^4|r^2\nabla^2\underline{\beta}|^2+\int_{\widehat{\Sigma}_0}\tau_-^2r^2(|r^2\nabla^2\rho|^2+|r^2\nabla^2\sigma|^2)\\
&+\int_{\widehat{\Sigma}_0}(\tau_-^4|r^2\nabla^2\underline{\alpha}|^2+r^4|r^2\nabla^2\beta|^2+r^2\tau_-^2|r^2\nabla^2\underline{\beta}|^2+r^4|r^2\nabla^2\rho|^2+r^4|r^2\nabla^2\sigma|^2)\\
&+\int_{\widehat{\Sigma}_0} r^4|r^2\nabla D_4\alpha|^2+\int_{\widehat{\Sigma}_0}\tau_-^4|\tau_-r\nabla D_3\underline{\alpha}|^2
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Y}_0 :=& \int_{\widehat{\Sigma}_0}(r^2|\alpha^F|^2+\tau_-^2|\rho^F|^2+\tau_-^2|\sigma^F|^2)+\int_{\widehat{\Sigma}_0}(\tau_-^2|\underline{\alpha}^F|^2+r^2|\rho^F|^2+r^2|\sigma^F|^2), \qquad (3.2)\\
&+\sum_{I=1}^{3}\int_{\widehat{\Sigma}_0}(r^2|r^I\widehat{\nabla}^I\alpha^F|^2+\tau_-^2|r^I\widehat{\nabla}^I\rho^F|^2+\tau_-^2|r^I\widehat{\nabla}^I\sigma^F|^2)\\
&+\int_{\widehat{\Sigma}_0}(\tau_-^2|r^I\widehat{\nabla}^I\underline{\alpha}^F|^2+r^2|r^I\widehat{\nabla}^I\rho^F|^2+r^2|r^I\widehat{\nabla}^I\sigma^F|^2)\\
&+\int_{\widehat{\Sigma}_0} r^2|r^I\widehat{\nabla}^{I-1}\widehat{D}_4\alpha^F|^2+\int_{\widehat{\Sigma}_0}\tau_-^2|r^{I-1}\tau_-\widehat{\nabla}^{I-1}\widehat{D}_3\underline{\alpha}^F|^2+\int_{\widehat{\Sigma}_0}r^6|\widehat{D}_4^2\alpha^F|^2+\int_{\widehat{\Sigma}_0}\tau_-^6|\widehat{D}_3^2\underline{\alpha}^F|^2.
\end{aligned}$$

Under the assumption that $W_0+Y_0\leq\epsilon_0$ for a sufficiently small $\epsilon_0>0$, a solution to the coupled Einstein-Yang-Mills equations exists in the whole causal complement of the ball $B_{u_0,\underline{u}_0}$ and such causal complement is foliated by two families of incoming and outgoing null *hypersurfaces. Moroever, the following norms of Weyl and Yang-Mills curvatures remain uniformly bounded*

$$
\begin{aligned}
\mathbf{W} := &\int_H r^4(|\alpha|^2 + |\beta|^2) + \int_H \tau_-^4|\underline{\beta}|^2 + \int_H \tau_-^2 r^2(\rho^2 + \sigma^2) \qquad (3.3)\\
&+ \int_{\underline{H}} (\tau_-^4|\underline{\alpha}|^2 + r^4|\beta|^2 + r^2\tau_-^2|\underline{\beta}|^2 + r^4|\rho|^2 + r^4|\sigma|^2),\\
&+ \int_H r^4(|r\nabla\alpha|^2 + |r\nabla\beta|^2) + \int_H \tau_-^4|r\nabla\underline{\beta}|^2 + \int_H \tau_-^2 r^2(|r\nabla\rho|^2 + |r\nabla\sigma|^2)\\
&+ \int_{\underline{H}} (\tau_-^4|r\nabla\underline{\alpha}|^2 + r^4|r\nabla\beta|^2 + r^2\tau_-^2|r\nabla\underline{\beta}|^2 + r^4|r\nabla\rho|^2 + r^4|r\nabla\sigma|^2)\\
&+ \int_H r^4|r\mathfrak{D}_4\alpha|^2 + \int_{\underline{H}} \tau_-^4|\tau_- D_3\underline{\alpha}|^2 +\\
&+ \int_H r^4(|r^2\nabla^2\alpha|^2 + |r^2\nabla^2\beta|^2) + \int_H \tau_-^4|r^2\nabla^2\underline{\beta}|^2 + \int_H \tau_-^2 r^2(|r^2\nabla^2\rho|^2 + |r^2\nabla^2\sigma|^2)\\
&+ \int_{\underline{H}} (\tau_-^4|r^2\nabla^2\underline{\alpha}|^2 + r^4|r^2\nabla^2\beta|^2 + r^2\tau_-^2|r^2\nabla^2\underline{\beta}|^2 + r^4|r^2\nabla^2\rho|^2 + r^4|r^2\nabla^2\sigma|^2)\\
&+ \int_H r^4|r^2\nabla\mathfrak{D}_4\alpha|^2 + \int_{\underline{H}} \tau_-^4|\tau_- r\nabla D_3\underline{\alpha}|^2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Y} := &\int_H (r^2|\alpha^F|^2 + \tau_-^2|\rho^F|^2 + \tau_-^2|\sigma^F|^2) + \int_{\underline{H}} (\tau_-^2|\underline{\alpha}^F|^2 + r^2|\rho^F|^2 + r^2|\sigma^F|^2), \qquad (3.4)\\
&+ \sum_{I=1}^{3} \int_H (r^2|r^I\widehat{\nabla}^I\alpha^F|^2 + \tau_-^2|r^I\widehat{\nabla}^I\rho^F|^2 + \tau_-^2|r^I\widehat{\nabla}^I\sigma^F|^2) \qquad (3.5)\\
&+ \int_{\underline{H}} (\tau_-^2|r^I\widehat{\nabla}^I\underline{\alpha}^F|^2 + r^2|r^I\widehat{\nabla}^I\rho^F|^2 + r^2|r^I\widehat{\nabla}^I\sigma^F|^2)\\
&+ \int_H r^2|r^I\widehat{\nabla}^{I-1}\widehat{D}_4\alpha^F|^2 + \int_{\underline{H}} \tau_-^2|r^{I-1}\tau_-\widehat{\nabla}^{I-1}\widehat{D}_3\underline{\alpha}^F|^2 + \int_H r^6|\widehat{D}_4^2\alpha^F|^2 + \int_{\underline{H}} \tau_-^6|\widehat{D}_3^2\underline{\alpha}^F|^2.
\end{aligned}
$$

More precisely, there exists $\delta = \delta(\epsilon_0) > 0$ such that

$$\mathbf{W} + \mathbf{Y} \leq \delta(\epsilon_0) \qquad (3.6)$$

uniformly in u and $\underline{u}$. Here $H \equiv H_u$ and $\underline{H} \equiv \underline{H}_{\underline{u}}$.

As a consequence, we immediately note the following global exterior stability theorem for the Minkowski space under coupled Einstein-Yang-Mills behaviour.

Remark 1. ***Exterior Stability Theorem:*** *Consider a strongly asymptotically flat initial data set $(\widehat{\Sigma}_0 - B_{u_0,\underline{u}_0}, g, k, F)$ i.e., in the neighbourhood of infinity, the data exhibits the following asymptotic behaviour*

$$g_{ij} = \delta_{ij} + o_4(r^{-\frac{3}{2}}), \qquad (3.7)$$

$$k_{ij} = o_3(r^{-\frac{5}{2}}), \qquad (3.8)$$

$$F = o_3(r^{-\frac{5}{2}}). \qquad (3.9)$$

Furthermore, assume the following smallness condition on the initial data

$$\mathbf{W}_0 + \mathbf{Y}_0 < \epsilon_0 \qquad (3.10)$$

for ϵ_0 sufficiently small, then the following exterior stability result holds for the Minkowski space as the vacuum solution of the coupled Einstein-Yang-Mills equations
(a) $\mathfrak{M} = \mathfrak{M}^+ \cup \mathfrak{M}^-$, where $\mathfrak{M}^+$ (resp. $\mathfrak{M}^-$) is the compliment of the future (resp. past) of $\widehat{\Sigma}_0 - B_{u_0,\underline{u}_0}$ is globally hyperbolic.
(b) $\mathfrak{M}^+$ can be foliated by a canonical double null foliation $H_u, \underline{H}_u$ whose outgoing leaves are complete for all $|u| > |u_0|$.
(c) The Weyl and Yang-Mills curvature components verify the following decay property

$$|\alpha| \lesssim r^{-\frac{7}{2}},\ |\underline{\alpha}| \lesssim r^{-1}|u|^{-\frac{5}{2}},\ |\beta| \lesssim r^{-\frac{7}{2}},\ |\rho,\sigma| \lesssim r^{-3}|u|^{-\frac{1}{2}}, \tag{3.11}$$

$$|\alpha^F| \lesssim r^{-\frac{5}{2}},\ |\underline{\alpha}^F| \lesssim r^{-1}|u|^{-\frac{3}{2}},\ |\rho^F,\sigma^F| \lesssim r^{-2}|u|^{-\frac{1}{2}}, \tag{3.12}$$

where the norms of the Yang-Mills curvature components are gauge invariant. The point-wise decay is obtained by the application of global Sobolev inequalities.

Now we briefly sketch the idea of the proof of the main theorem (3.1). First, we assume that the connection and the curvature norms (both Weyl and Yang-Mills) are bounded from above by a small number ϵ. Under this bootstrap assumption, we can integrate the transport equations for the connections as well as utilize the estimates from the elliptic equations to show that the suitable connection norms can be controlled by the Weyl and Yang-Mills curvature norms. Now we utilize the symmetric hyperbolic characteristic of the null Bianchi and null Yang-Mills equations to control the Weyl and Yang-Mills curvature energies. The null structure present in the Bianchi and Yang-Mills equations plays a vital role in controlling the curvature energies along the null cones solely in terms of the initial data uniformly in $u$and $\underline{u}$ in the exterior region. Now choosing the initial data norm sufficiently small we can make curvature and connection norms (since connections are estimated by curvature) less than $\epsilon/2$ thereby closing the boot-strap. As it turns out, the expected linear decay of the Yang-Mills fields is just sufficient to tame the associated non-linearities.

4. Important Inequalities

One of the most important features of our study is the utilization of the fully gauge-covariant structure of the Yang-Mills equations. In other words, we would like work with the Yang-Mills curvature and the fully gauge covariant derivative directly instead of the working with Yang-Mills connection. To that end, we need all the necessary inequalities in a fully gauge-invariant form. Let us now extend all the Sobolev and trace inequalities to the fibres of the vector bundle $\otimes^k T^*M \otimes_M \mathfrak{P}_{Ad,\mathfrak{g}}$. Let us write both the degenerate and the non-degenerate (no loss in decay along $\underline{u}$) versions. Assume $\sup_{\mathcal{K}} |\mathrm{tr}\chi - \frac{2}{r}| \leq \epsilon,\ \sup_{\mathcal{D}} |\mathrm{tr}\underline{\chi} + \frac{2}{r}| \leq \epsilon$ for sufficiently small ϵ.

Lemma 4.1. *Let F is an arbitrary $S_{r,t}-$tangent section of the associated bundle $^k\otimes T^*S \otimes \mathfrak{P}_{Ad,\rho}$. The following Sobolev embedding holds for F*

$$\sup_{S_{u,\underline{u}}} r|F|_{\gamma,\delta} \leq C_{sob} \left(\int_{S_{u,\underline{u}}} |F|^2_{\gamma,\delta} + r^2|\widehat{\nabla} F|^2_{\gamma,\delta} + r^4|\widehat{\nabla}^2 F|^2_{\gamma,\delta} \right)^{\frac{1}{2}} \tag{4.1}$$

Proof. Write $f = \sqrt{F^a{}_{bIJ}F^b{}_{aKL}\gamma^{IK}\gamma^{JL} + \delta}$ and since f is gauge invariant $\nabla f = \widehat{\nabla} f = \frac{1}{2f}\langle F, \widehat{\nabla} F\rangle_{\gamma,\delta}$. Now use the standard Sobolev embedding for f and take the limit $\delta \to 0$ to yield the result. ☐

Lemma 4.2. *The following gauge-invariant Sobolev inequality holds in the context of Cauchy-double-null frame-work*

$$\sup_{S_{u,\underline{u}}} r^{\frac{3}{2}}|F| \leq C\left[\left(\int_{S_{u,\underline{u}_0}} r^4|F|^4_{\gamma,\delta}\right)^{\frac{1}{4}} + \left(\int_{S_{u,\underline{u}_0}} r^4|r\widehat{\nabla}F|^4_{\gamma,\delta}\right)^{\frac{1}{4}}\right. \tag{4.2}$$

$$\left. + \left(\int_{H_u\cap\mathcal{K}} |F|^2_{\gamma,\delta} + r^2|\widehat{\nabla}F|^2_{\gamma,\delta} + r^2|\widehat{\nabla}_4F|^2_{\gamma,\delta} + r^4|\widehat{\nabla}^2F|^2_{\gamma,\delta} + r^4|\widehat{\nabla}\widehat{\nabla}_4F|^2_{\gamma,\delta}\right)^{\frac{1}{2}}\right].$$

The following degenerate version holds too

$$\sup_{S_{u,\underline{u}}}(r\tau_-^{\frac{1}{2}}|F|) \leq C\left[\left(\int_{S_{u,\underline{u}_0}} r^2\tau_-^2|F|^4_{\gamma,\delta}\right)^{\frac{1}{4}} + \left(\int_{S_{u,\underline{u}_0}} r^2\tau_-^2|r\widehat{\nabla}F|^4_{\gamma,\delta}\right)^{\frac{1}{4}}\right. \tag{4.3}$$

$$\left. + \left(\int_{H_u\cap\mathcal{K}} |F|^2_{\gamma,\delta} + r^2|\widehat{\nabla}F|^2_{\gamma,\delta} + \tau_-^2|\widehat{\nabla}_4F|^2 + r^4|\widehat{\nabla}^2F|^2_{\gamma,\delta} + r^2\tau_-^2|\widehat{\nabla}\widehat{\nabla}_4F|^2_{\gamma,\delta}\right)^{\frac{1}{2}}\right].$$

Analogous inequality holds on $\underline{H}$ too i.e.,

$$\sup_{S_{u,\underline{u}}} r^{\frac{3}{2}}|F| \leq C\left[\left(\int_{S_{u_0,\underline{u}}} r^4|F|^4_{\gamma,\delta}\right)^{\frac{1}{4}} + \left(\int_{S_{u_0,\underline{u}}} r^4|r\widehat{\nabla}F|^4_{\gamma,\delta}\right)^{\frac{1}{4}}\right. \tag{4.4}$$

$$\left. + \left(\int_{\underline{H}_u\cap\mathcal{K}} |F|^2_{\gamma,\delta} + r^2|\widehat{\nabla}F|^2_{\gamma,\delta} + r^2|\widehat{\nabla}_3F|^2_{\gamma,\delta} + r^4|\widehat{\nabla}^2F|^2_{\gamma,\delta} + r^4|\widehat{\nabla}\widehat{\nabla}_3F|^2_{\gamma,\delta}\right)^{\frac{1}{2}}\right],$$

and

$$\sup_{S_{u,\underline{u}}}(r\tau_-^{\frac{1}{2}}|F|) \leq C\left[\left(\int_{S_{u_0,\underline{u}}} r^2\tau_-^2|F|^4_{\gamma,\delta}\right)^{\frac{1}{4}} + \left(\int_{S_{u_0,\underline{u}}} r^2\tau_-^2|r\widehat{\nabla}F|^4_{\gamma,\delta}\right)^{\frac{1}{4}}\right. \tag{4.5}$$

$$\left. + \left(\int_{\underline{H}_u\cap\mathcal{K}} |F|^2_{\gamma,\delta} + r^2|\widehat{\nabla}F|^2_{\gamma,\delta} + \tau_-^2|\widehat{\nabla}_3F|^2 + r^4|\widehat{\nabla}^2F|^2_{\gamma,\delta} + r^2\tau_-^2|\widehat{\nabla}\widehat{\nabla}_3F|^2_{\gamma,\delta}\right)^{\frac{1}{2}}\right].$$

Proof. The proof relies on the following two inequalities

$$\sup_{S_{u,\underline{u}}}|G| \leq Cr^{-\frac{1}{2}}\left(\int_{S_{u,\underline{u}}} |G|^4 + r^4|\nabla G|^4\right)^{\frac{1}{4}}, \tag{4.6}$$

$$\int_{S_{u,\underline{u}}} r^4|G|^4 \leq \int_{S_{u,\underline{u}_0}} r^4|G|^4 + C\left(\int_{H_u\cap\mathcal{K}} r^6|G|^6\right)^{\frac{1}{2}}\left(\int_{H_u\cap\mathcal{K}} r^2|\nabla_4G|^2\right)^{\frac{1}{2}}$$

Now for a section of the associated bundle, both of these two inequalities hold with ∇ replaced by $\widehat{\nabla}$ and ∇_4 replaced by $\widehat{\nabla}_4$. This follows in an exact similar way as the previous case i.e., set $G = \sqrt{F^a{}_{bA_1A_2A_3\cdots A_n} F^b{}_{aB_1B_2B_3\cdots B_n}\gamma^{A_1B_1}\gamma^{A_2B_2}\gamma^{A_3B_3}\cdots\gamma^{A_nB_n}+\delta}$, apply the standard Sobolev inequality, and pass to the limit $\delta \to 0$ i.e.,

$$|\nabla G|^2 = |\nabla\sqrt{F^a{}_{bA_1A_2A_3\cdots A_n} F^b{}_{aB_1B_2B_3\cdots B_n}\gamma^{A_1B_1}\gamma^{A_2B_2}\gamma^{A_3B_3}\cdots\gamma^{A_nB_n}+\delta}|^2$$
$$= |\frac{\langle F, \widehat{\nabla} F\rangle}{2G}|^2 \leq \frac{1}{4}|\widehat{\nabla} F|^2_{\gamma,\delta}.$$

Similar argument yields

$$|\nabla_4 F|^2 \leq \frac{1}{4}|\widehat{\nabla}_4 F|^2_{\gamma,\delta}. \tag{4.7}$$

Therefore first we obtain

$$\sup_{S_{u,\underline{u}}} |F| \leq Cr^{-\frac{1}{2}}\left(\int_{S_{u,\underline{u}}} |F|^4_{\gamma,\delta} + r^4|\widehat{\nabla} F|^4_{\gamma,\delta}\right)^{\frac{1}{4}}. \tag{4.8}$$

Similarly we obtain

$$\int_{S_{u,\underline{u}}} r^4|G|^4_{\gamma,\delta} \leq \int_{S_{u,\underline{u}_0}} r^4|G|^4_{\gamma,\delta} + C\left(\int_{H_u\cap\mathcal{K}} r^6|G|^6_{\gamma,\delta}\right)^{\frac{1}{2}}\left(\int_{H_u\cap\mathcal{K}} r^2|\widehat{\nabla}_4 G|^2_{\gamma,\delta}\right)^{\frac{1}{2}}$$

Now we can treat $\widehat{\nabla} F$ as a new section J of the bundle ${}^{k+1}\otimes T^*S \otimes \mathfrak{P}_{Ad,\mathfrak{g}}$ and apply the previous two inequalities to conclude the result. $\square$

In addition to the gauge invariant Sobolev inequalities in null cones, we also have the co-dimension-1 trace inequalities and usual Sobolev embedding on the topological 2−spheres $S_{u,\underline{u}}$ ([26], section 5, inequalities 2 and 5 after appropriate scaling).In addition, one also has the following gauge-invariant transport inequalities that are useful for estimating connections in terms of the Weyl and Yang-Mills curvature.

Lemma 4.3. *Consider a globally hyperbolic portion of the spacetime $\mathcal{K}$ foliated by a double null foliation. Let us assume that $|\Omega tr\chi - \overline{\Omega tr\chi}| \leq \frac{\epsilon}{r^2}$. Also an $S-$tangential tensor field $\mathcal{G}$ satisfies the following transport equation*

$$\widehat{\nabla}_4\mathcal{G} + \lambda_0 tr\chi\mathcal{G} = \Omega^{-1}\mathcal{H}. \tag{4.9}$$

The following gauge-invariant inequality holds for $\mathcal{G}$

$$|r^{\lambda_1}\mathcal{G}|_{p,S}(u,\underline{u}) \lesssim \left(|r^{\lambda_1}\mathcal{G}|_{p,S}(u,\underline{u}_0) + \int_{\underline{u}_0}^{\underline{u}} |r^{\lambda_1}\mathcal{H}|_{p,S}(u,\underline{u}^{'})d\underline{u}^{'}\right) \tag{4.10}$$

for $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$.

Lemma 4.4. *Assume that for $\epsilon > 0$ sufficiently small, $|\Omega tr\underline{\chi} - \overline{\Omega tr\underline{\chi}}| \lesssim \epsilon r^{-1}\tau_-^{-1}$ and let an $S-$tangential tensor field $\mathcal{G}$ satisfies the following transport equation*

$$\widehat{\nabla}_3\mathcal{G} + \lambda_0 tr\underline{\chi}\mathcal{G} = \Omega^{-1}\underline{\mathcal{H}}. \tag{4.11}$$

The following gauge-invariant inequality holds for $\mathcal{G}$

$$|r^{\lambda_1}\mathcal{G}|_{p,S}(u,\underline{u}) \lesssim \left(|r^{\lambda_1}\mathcal{G}|_{p,S}(u_0,\underline{u}) + \int_{u_0}^{u} |r^{\lambda_1}\mathcal{H}|_{p,S}(u,\underline{u}^{'})d\underline{u}^{'}\right) \tag{4.12}$$

for $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$.

5. Commutation Formulae

In order to obtain the higher order energy estimates, we need the commutation formulae $[\widehat{\nabla}^I, \widehat{\nabla}_4]$ and $[\widehat{\nabla}^I, \widehat{\nabla}_3]$. For a $S-$tangential tensor field that is a section of the bundle ${}^k\otimes T^*S$, the gauge covariant derivative $\widehat{\nabla}$ and the ordinary covariant derivative ∇ coincide. These commutators are important since, the true non-linear characteristics of the Yang-Mills fields manifest itself through these commutators in the higher order energy estimates.

Lemma 5.1. *Suppose $\mathcal{G}$ is a section of the product vector bundle ${}^k\otimes T^*S\otimes\mathfrak{P}_{Ad,\rho}$, $k \geq 1$, that satisfies $\widehat{\nabla}_4\mathcal{G} = \mathcal{F}_1$ and $\widehat{\nabla}_4\widehat{\nabla}^I\mathcal{G} = \mathcal{F}_1^I$, then $\mathcal{F}_1^I$ verifies the following schematic expression*

$$
\begin{aligned}
\mathcal{F}_1^I \sim & \sum_{J_1+J_2+J_3+J_4=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\nabla^{J_3}\beta\widehat{\nabla}^{J_4}\mathcal{G} \\
& + \sum_{J_1+J_2+J_3+J_4+J_5=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\alpha^F\widehat{\nabla}^{J_4}(\rho^F,\sigma^F)\widehat{\nabla}^{J_5}\mathcal{G} \\
& + \underbrace{\sum_{J_1+J_2+J_3+J_4=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\alpha^F\widehat{\nabla}^{J_4}\mathcal{G}}_{C_1} + \sum_{J_1+J_2+J_3=I} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\mathcal{F}_1 \\
& + \sum_{J_1+J_2+J_3+J_4=I} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\chi\widehat{\nabla}^{J_4}\mathcal{G}.
\end{aligned} \tag{5.1}
$$

Similarly, for $\widehat{\nabla}_3\mathcal{G} = \mathcal{F}_2$, and $\widehat{\nabla}_3\widehat{\nabla}^I\mathcal{G} = \mathcal{F}_2^I$,

$$
\begin{aligned}
\mathcal{F}_2^I \sim & \sum_{J_1+J_2+J_3+J_4=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\nabla^{J_3}\underline{\beta}\widehat{\nabla}^{J_4}\mathcal{G} \\
& + \sum_{J_1+J_2+J_3+J_4+J_5=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\underline{\alpha}^F\widehat{\nabla}^{J_4}(\rho^F,\sigma^F)\widehat{\nabla}^{J_5}\mathcal{G} \\
& + \underbrace{\sum_{J_1+J_2+J_3+J_4=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\underline{\alpha}^F\widehat{\nabla}^{J_4}\mathcal{G}}_{C_2} + \sum_{J_1+J_2+J_3=I} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\mathcal{F}_2 \\
& + \sum_{J_1+J_2+J_3+J_4=I} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\underline{\chi}\widehat{\nabla}^{J_4}\mathcal{G}.
\end{aligned} \tag{5.2}
$$

By moving the top derivatives of $\mathcal{G}$ multiplied by $tr\chi$ and $tr\underline{\chi}$ from the right hand side to the left hand side, one may also obtain

$$
\begin{aligned}
\mathcal{F}_1^I + \frac{I}{2}tr\chi\widehat{\nabla}^I\mathcal{G} \sim & \sum_{J_1+J_2+J_3+J_4=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\nabla^{J_3}\beta\widehat{\nabla}^{J_4}\mathcal{G} \\
& + \sum_{J_1+J_2+J_3+J_4+J_5=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\alpha^F\widehat{\nabla}^{J_4}(\rho^F,\sigma^F)\widehat{\nabla}^{J_5}\mathcal{G} \\
& + \sum_{J_1+J_2+J_3+J_4=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\alpha^F\widehat{\nabla}^{J_4}\mathcal{G} + \sum_{J_1+J_2+J_3=I} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\mathcal{F}_1 \\
& + \sum_{J_1+J_2+J_3+J_4=I} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\hat{\chi}\widehat{\nabla}^{J_4}\mathcal{G} + \sum_{J_1+J_2+J_3+J_4=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2+1}\widehat{\nabla}^{J_3}tr\chi\widehat{\nabla}^{J_4}\mathcal{G}.
\end{aligned} \tag{5.3}
$$

$$
\mathcal{F}_2^I + \frac{I}{2}tr\underline{\chi}\widehat{\nabla}^I\mathcal{G} \sim \sum_{J_1+J_2+J_3+J_4=I-1} \nabla^{J_1}(\eta+\underline{\eta})^{J_2}\nabla^{J_3}\underline{\beta}^R\widehat{\nabla}^{J_4}\mathcal{G}
$$

$$+\sum_{J_1+J_2+J_3+J_4=I-1}\nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\underline{\alpha}^F\widehat{\nabla}^{J_4}\mathcal{G} \tag{5.4}$$

$$+\sum_{J_1+J_2+J_3=I}\nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\mathcal{F}_2+\sum_{J_1+J_2+J_3+J_4=I}\nabla^{J_1}(\eta+\underline{\eta})^{J_2}\widehat{\nabla}^{J_3}\underline{\hat{\chi}}\widehat{\nabla}^{J_4}\mathcal{G}$$

$$+\sum_{J_1+J_2+J_3+J_4=I-1}\nabla^{J_1}(\eta+\underline{\eta})^{J_2+1}\widehat{\nabla}^{J_3}tr\underline{\chi}\widehat{\nabla}^{J_4}\mathcal{G}.$$

This form of the commutation formulae will be very useful since we need to keep track of $tr\chi$ and $tr\underline{\chi}$ which behave like $2/r$ and $-2/r$ that are not integrable.

Proof. Recall the following gauge-covariant commutation rule

$$\begin{aligned}
[\widehat{\nabla}_4,\widehat{\nabla}_B]\mathcal{G}^P{}_{QA_1A_2A_3\cdots A_n} &= [\widehat{D}_4,\widehat{D}_B]\mathcal{G}^P{}_{QA_1A_2A_3\cdots A_n}+(\nabla_B\log\Omega)\widehat{\nabla}_4\mathcal{G}^P{}_{QA_1A_2A_3\cdots A_n}\\
&-\gamma^{CD}\chi_{BD}\widehat{\nabla}_C\mathcal{G}^P{}_{QA_1A_2A_3\cdots A_n}-\sum_{i=1}^n\gamma^{CD}\chi_{BD}\bar{\eta}_{A_i}\mathcal{G}^P{}_{QA_1A_2A_3\cdot\cdot\hat{A}_iC\cdot\cdot A_n}\\
&+\sum_{i=1}^n\gamma^{CD}\chi_{A_iB}\bar{\eta}_D\mathcal{G}^P{}_{QA_1A_2A_3\cdot\cdot\hat{A}_iC\cdot\cdot A_n}
\end{aligned}$$

and

$$\begin{aligned}
[\widehat{D}_4,\widehat{D}_A]\mathcal{G}^P{}_{QA_1A_2\cdots A_n} &= (\nabla_A\log\Omega)\widehat{\nabla}_4\mathcal{G}^P{}_{QA_1A_2\cdots A_n}-\sum_i R(e_C,e_{A_i},e_4,e_A)\mathcal{G}^P{}_{QA_1\cdot\cdot\hat{A}_i,\cdot\cdot A_n} \\
&+F^P{}_{R4A}\mathcal{G}^R{}_{QA_1A_2\cdots A_n}-F^R{}_{Q4A}\mathcal{G}^P{}_{RA_1A_2\cdots A_n}.
\end{aligned}\tag{5.5}$$

In schematic form the commutation formula reads

$$[\widehat{\nabla}_4,\widehat{\nabla}_B]\mathcal{G}\sim(\beta+\alpha^F\cdot(\rho^F+\sigma^F))\mathcal{G}+\alpha^F\mathcal{G}+(\eta+\bar{\eta})\hat{\nabla}_4\mathcal{G}-\chi\hat{\nabla}\mathcal{G}+\chi\bar{\eta}\mathcal{G}. \tag{5.6}$$

Similarly for $[\widehat{\nabla}_3,\widehat{\nabla}_B]$, we have

$$[\widehat{\nabla}_3,\widehat{\nabla}_B]\mathcal{G}\sim(\bar{\beta}+\bar{\alpha}^F\cdot(\rho^F+\sigma^F))\mathcal{G}+\bar{\alpha}^F\mathcal{G}+(\eta+\bar{\eta})\hat{\nabla}_3\mathcal{G}-\bar{\chi}\nabla\mathcal{G}+\bar{\chi}\eta\mathcal{G}. \tag{5.7}$$

The commutation formula for general $\widehat{\nabla}^I$ may be obtained through an iteration argument (see [19**?**]). ☐

Remark 2. *Note that the commutation terms C_1 and C_2 are produced because of the non-abelian (and hence non-linear) characteristic of the Yang-Mills theory. For any section of the bundle $\otimes^kT^*S$, C_1 and C_2 drop out.*

6. Norms

$3+1$ dimensional Minkowski space is equipped with a 15 dimensional conformal group. Out of these 15 Killing and conformal Killing vector fields, one may utilize the time-like ones to construct conserved positive definite energies for a theory that posses a stress-energy tensor. However, if we perturb the Minkowski space, the isometry is no longer present and one can not hope to construct the conserved energies. Nevertheless, since we are interested in the small data problem, one might hope that the perturbed space-time does have vector fields that are approximately Killing or conforml Killing in a suitable sense. In other words, appropriate norms of the deformation tensors (or the trace-free part of the deformation tensor) are sufficiently small. In the regime of such smallness, one may hope to utilize the usual energy argument to establish a stability result. With this hope, we define the approximate Killing and conformal vector field in the same forms as their Minkowski ones

$$\widehat{\mathbf{n}} := \frac{1}{2}(e_4 + e_3),\ K = \frac{1}{2}(\underline{u}^2 e_4 + u^2 e_3),\ S = \frac{1}{2}(\underline{u} e_4 + u e_3). \tag{6.1}$$

On the Minkowski space, $\widehat{\mathbf{n}}$, K, and S are the generators of time translation, inversion, and scaling vector field, respectively. In this current context, we endow them with the characteristics of 'approximate' Killing and conformal Killing vector fields. We need to construct suitable scaled norms for the Weyl curvature components as well as that of Yang-Mills curvature components. Since the Weyl curvature scales as length^{-2} and Yang-Mills curvature scales as length^{-1}, an appropriate norm would scale as length$^2 *$ weyl + length $*$ Yang-Mills. This simple scaling essentially tells us that in order to establish the existence of a solution of the coupled Einstein-Yang-Mills theory, we need one extra regularity for the Yang-Mills curvature. In the current context, however, there are two length scales present $\underline{u}$ ($\sim r(u, \underline{u})$) and u and it is crucial to separate out these length scales since fields do not decay in an isotropic way. In order to determine the scaling behaviour of different Weyl and Yang-Mills curvature components, we contract the respective stress tensors with the generator of approximate inversion and that of approximate time translations. The connection norms follow from the null transport equations (while estimating energies for the curvature we need the connection norms that can be estimated a priori by means of the null evolution and constraint equations).

6.1. Curvature norms

The 'true' gravitational degrees of freedom is encoded in the Weyl curvature of the spacetime metric. Therefore a natural expectation that the total energy of the Einstein-Yang-Mill system associated to characteristics may be described in terms of suitable norms of the Weyl and Yang-Mills curvature (notice that one doe not have the concept of local energy density associated to pure gravity due to equivalence principle as such one may define a quasi-local notion [3, 4]). One may utilize the Bel-Robianson tensor associated to a Weyl field to construct suitable norms for the pure gravitational sector. Let us recall the Bel-Robinson tensor associated to the Weyl tensor

$$Q(W)_{\alpha\beta\gamma\delta} = W_{\alpha\rho\gamma\sigma} W_{\beta\ \ \delta}^{\ \rho\ \ \sigma} + {}^*W_{\alpha\rho\gamma\sigma}\, {}^*W_{\beta\ \ \delta}^{\ \rho\ \ \sigma}. \tag{6.2}$$

In addition we will also have the Yang-Mills stress energy tensor

$$\mathfrak{T}_{\alpha\beta} = F^a{}_{b\alpha\mu} F^b{}_{a\beta}{}^{\mu} + {}^*F^a{}_{b\alpha\mu}\, {}^*F^b{}_{a\beta}{}^{\mu}. \tag{6.3}$$

Here ${}^*W_{\alpha\beta\gamma\delta} := \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu} W_{\mu\nu\gamma\delta}$ and ${}^*F_{\alpha\beta} := \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}$. We note ${}^*({}^*F, W) = -F, W$ and therefore $\langle F, F\rangle_{g,\delta} + \langle {}^*F,\ {}^*F\rangle_{g,\delta} = 0$. Let us define the Bel-Robinson energy associated with the exterior Cauchy hypersurface $\widehat{\Sigma}$

$$\mathcal{Q}_0^{in} := \int_{\widehat{\Sigma}_0} Q(W)(K, K, \hat{\mathbf{n}}, \hat{\mathbf{n}}) \tag{6.4}$$

At the level of curvature flux through the null hyper-surface, this would correspond to the following entities that we are primarily interested in controlling

$$\mathcal{F}_0^W = \int_H Q(W)(K, K, \hat{\mathbf{n}}, e_4) + \int_{\underline{H}} Q(W)(K, K, \hat{\mathbf{n}}, e_3). \tag{6.5}$$

For the Yang-Mills fields, we define the energy by means of the Yang-Mills stress energy tensor

$$\mathcal{T}_0^{in} = \int_{\widehat{\Sigma}_0} \mathfrak{T}[F](K, \hat{\mathbf{n}}). \tag{6.6}$$

and the associated flux

$$\mathcal{F}_0^F = \int_H \mathfrak{T}[F](K, e_4) + \int_{\underline{H}} \mathfrak{T}[F](K, e_3). \tag{6.7}$$

Now utilizing the relations below

$$Q(W)(e_4, e_4, e_4, e_4) = 2|\alpha|^2,\ Q(W)(e_4, e_4, e_4, e_3) = 4|\beta(W)|^2, \tag{6.8}$$

$$Q(W)(e_4, e_4, e_3, e_3) = 4(\rho(\mathcal{W})^2 + \sigma(W)^2),\ Q(W)(e_4, e_3, e_3, e_3) = 4|\underline{\beta}(W)|^2,$$

$$Q(W)(e_3, e_3, e_3, e_3) = 2|\underline{\alpha}(W)|^2, \tag{6.9}$$

we compute

$$Q(K, K, \hat{\mathbf{n}}, e_4) = \frac{1}{4}\underline{u}^4|\alpha|^2 + \frac{1}{2}u^4|\underline{\beta}|^2 + \frac{(\underline{u}^4 + 2u^2\underline{u}^2)}{2}|\beta|^2 + \frac{(u^4 + 2u^2\underline{u}^2)}{2}(\rho^2 + \sigma^2), \tag{6.10}$$

$$Q(K, K, \hat{\mathbf{n}}, e_3) = \frac{1}{4}u^4|\underline{\alpha}|^2 + \frac{1}{2}\underline{u}^4|\beta|^2 + \frac{\underline{u}^4 + 2\underline{u}^2u^2}{2}(\rho^2 + \sigma^2) + \frac{u^4 + 2u^2\underline{u}^2}{2}|\underline{\beta}|^2, \tag{6.11}$$

$$\mathfrak{T}(\mathcal{F})(K, e_4) = \frac{1}{2}\underline{u}^2|\alpha^F|^2 + \frac{1}{2}u^2(|\rho^F|^2 + |\sigma^F|^2), \tag{6.12}$$

$$\mathfrak{T}(\mathcal{F})(K, e_3) = \frac{1}{2}u^2|\underline{\alpha}^F|^2 + \frac{1}{2}\underline{u}^2(|\rho^F|^2 + |\sigma^F|^2) \tag{6.13}$$

which automatically provides us with the scaling behaviour. With the help of the scaling behaviour (6.10-6.13) dictated by the approximate Killing fields, we define the appropriate norms of the Weyl and Yang-Mills curvature that need control. Notice that sphere derivative ∇ (and its gauge covariant counterpart $\widehat{\nabla}$) and e_4 null derivative ∇_4 (and its gauge covariant counterpart) scale as r^{-1} while the null derivative ∇_3 scales as $|u|^{-1}$.

$$\begin{aligned}
\mathcal{W}_0 :=& \int_H r^4(|\alpha|^2 + |\beta|^2) + \int_H \tau_-^4|\underline{\beta}|^2 + \int_H \tau_-^2r^2(\rho^2 + \sigma^2) \quad (6.14)\\
&+ \int_{\underline{H}} (\tau_-^4|\underline{\alpha}|^2 + r^4|\beta|^2 + r^2\tau_-^2|\underline{\beta}|^2 + r^4|\rho|^2 + r^4|\sigma|^2),
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_1 =& \int_H r^4(|r\nabla\alpha|^2 + |r\nabla\beta|^2) + \int_H \tau_-^4|r\nabla\underline{\beta}|^2 + \int_H \tau_-^2r^2(|r\nabla\rho|^2 + |r\nabla\sigma|^2)\\
&+ \int_{\underline{H}} (\tau_-^4|r\nabla\underline{\alpha}|^2 + r^4|r\nabla\beta|^2 + r^2\tau_-^2|r\nabla\underline{\beta}|^2 + r^4|r\nabla\rho|^2 + r^4|r\nabla\sigma|^2),\\
&+ \int_H r^4|r\mathfrak{D}_4\alpha|^2 + \int_{\underline{H}} \tau_-^4|\tau_-D_3\underline{\alpha}|^2 + \int_{\underline{H}} r^6|\mathfrak{D}_4\beta|^2 + \int_H \tau_-^6|\mathfrak{D}_3\underline{\beta}|^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_2 =& \int_H r^4(|r^2\nabla^2\alpha|^2 + |r^2\nabla^2\beta|^2) + \int_H \tau_-^4|r^2\nabla^2\underline{\beta}|^2 + \int_H \tau_-^2r^2(|r^2\nabla^2\rho|^2 + |r^2\nabla^2\sigma|^2)\\
&+ \int_{\underline{H}} (\tau_-^4|r^2\nabla^2\underline{\alpha}|^2 + r^4|r^2\nabla^2\beta|^2 + r^2\tau_-^2|r^2\nabla^2\underline{\beta}|^2 + r^4|r^2\nabla^2\rho|^2 + r^4|r^2\nabla^2\sigma|^2)\\
&+ \int_H r^4|r^2\nabla\mathfrak{D}_4\alpha|^2 + \int_{\underline{H}} \tau_-^4|\tau_-r\nabla\mathfrak{D}_3\underline{\alpha}|^2 + \int_{\underline{H}} r^8|\nabla\mathfrak{D}_4\beta|^2 + \int_H u^6r^2|\nabla\mathfrak{D}_3\underline{\beta}|^2
\end{aligned}$$

and

$$\mathcal{Y}_0 := \int_H (r^2|\alpha^F|^2 + \tau_-^2|\rho^F|^2 + \tau_-^2|\sigma^F|^2) + \int_{\underline{H}} (\tau_-^2|\underline{\alpha}^F|^2 + r^2|\rho^F|^2 + r^2|\sigma^F|^2), \quad (6.15)$$

$$\mathcal{Y}_{1,2,3} = \sum_{I=1}^{3} \int_H (r^2|r^I\widehat{\nabla}^I\alpha^F|^2 + \tau_-^2|r^I\widehat{\nabla}^I\rho^F|^2 + \tau_-^2|r^I\widehat{\nabla}^I\sigma^F|^2) \quad (6.16)$$

$$+ \int_{\underline{H}} (\tau_-^2|r^I\widehat{\nabla}^I\underline{\alpha}^F|^2 + r^2|r^I\widehat{\nabla}^I\rho^F|^2 + r^2|r^I\widehat{\nabla}^I\sigma^F|^2)$$

$$+ \int_H r^2|r^I\widehat{\nabla}^{I-1}\widehat{D}_4\alpha^F|^2 + \int_{\underline{H}} \tau_-^2|r^{I-1}\tau_-\widehat{\nabla}^{I-1}\widehat{D}_3\underline{\alpha}^F|^2 + \int_H r^6|\widehat{D}_4^2\alpha^F|^2 + \int_{\underline{H}} \tau_-^6|\widehat{D}_3^2\underline{\alpha}^F|^2$$

6.2. Connection Norms

Let us now define the norm of the spin connections. Once again we need to weight by suitable factors of r and τ_- to respect the scaling

$$\begin{aligned}\mathcal{O}_{p,q}(u,\underline{u}) &:= ||r^{2+q-\frac{2}{p}}\nabla^q(\text{tr}\chi - \underline{\text{tr}\chi})||_{L^p(S_{u,\underline{u}})} + ||r^{2+q-\frac{2}{p}}\nabla^q(\text{tr}\underline{\chi} - \underline{\text{tr}\underline{\chi}})||_{L^p(S_{u,\underline{u}})} \\ &+||r^{2+q-\frac{2}{p}}\nabla^q(\eta,\underline{\eta})||_{L^p(S_{u,\underline{u}})} + ||r^{2+q-\frac{2}{p}}\nabla^q\omega||_{L^p(S_{u,\underline{u}})} + ||r^{1+q-\frac{2}{p}}\tau_-\nabla^q\underline{\omega}||_{L^p(S_{u,\underline{u}})} \\ &+||r^{2+q-\frac{2}{p}}\nabla^q\hat{\chi}||_{L^p(S_{u,\underline{u}})} + ||r^{1+q-\frac{2}{p}}\tau_-\nabla^q\underline{\hat{\chi}}||_{L^p(S_{u,\underline{u}})} + ||r^{3+q-\frac{2}{p}}\nabla^q\nabla_4\omega||_{L^p(S_{u,\underline{u}})} \\ &+||r^{1+q-\frac{2}{p}}\nabla^q\nabla_3\underline{\omega}||_{L^p(S_{u,\underline{u}})}\end{aligned} \quad (6.17)$$

and we also need the following null norm

$$\begin{aligned}\mathcal{O}_{\underline{H}} &:= r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\hat{\chi}||_{L^2(\underline{H})} + r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\text{tr}\chi||_{L^2(\underline{H})}r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\eta||_{L^2(\underline{H})} \\ &+r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\underline{\eta}||_{L^2(\underline{H})} + r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\underline{\hat{\chi}}||_{L^2(\underline{H})} + r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\text{tr}\underline{\chi}||_{L^2(\underline{H})} \\ &+r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\omega||_{L^2(\underline{H})} + r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\underline{\omega}||_{L^2(\underline{H})}.\end{aligned} \quad (6.18)$$

The total connection norm up to the second derivative is defined as follows

$$\mathcal{O} := \sup_{u,\underline{u}} \mathcal{O}_{p,q} + \sup_{\mathcal{K}} O_{\underline{H}} \quad (6.19)$$

We are interested in $0 \leq q \leq 2$ and $2 \leq p \leq 4$. Notice that the point-wise norms of the connection coefficients are controlled by $\mathcal{O}_{p,q}(u,\underline{u})$ by means of the Sobolev inequalities on the topological $2-$sphere.

Remark 3. *Bootstrap Assumption: Assume that $\mathcal{O} \leq \epsilon$ and $\mathbf{W} + \mathbf{Y} := \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 \leq \epsilon$ for a sufficiently small $\epsilon > 0$.*

7. Norm estimates using the equations of motions

In this section, we want to estimate the connection coefficients in terms of the Weyl and Yang-Mills curvatures by means of the available null-transport and constraint equations. Null transport equations enables us to make use of the weighted transport inequalities whereas the constraint equations yield elliptic estimates. The technology of such coupled estimates were described in full detail by [21] for the vacuum case. In the current context, we have the additional Yang-Mills terms. Since the procedure is exactly the same as [21], we only present estimating one connection component in detail. In addition, the at the level of connection estimates, the true nonlinear characteristics of the Yang-Mills theory (the feature that separates it from the usual $U(1)-$ Maxwell theory) does not manifest itself in a crucial way (contrary to the energy estimates where it does).

7.1. Connection estimates

Proposition 7.1. *Assume that the boot-strap assumption (remark 2) holds. The Ricci coefficient $\hat{\chi}$ verifies the following estimate*

$$\|r^{3-\frac{2}{p}}\nabla\hat{\chi}\|_{L^p(S_{u,\underline{u}})} \lesssim (\mathcal{I}_0+\mathcal{W}_0),\ \ \|r^{2-\frac{2}{p}}\hat{\chi}\|_{L^p(S_{u,\underline{u}})} \lesssim (\mathcal{I}_0+\mathcal{W}_0), \tag{7.1}$$

for $p \in [2,4]$.

Proof. Following [21], we introduce the tensor $U := \Omega^{-1}\nabla\mathrm{tr}\chi + \Omega^{-1}\mathrm{tr}\chi\zeta$ and write down its evolution equation

$$\nabla_4 U + \frac{3}{2}\mathrm{tr}\chi U = \mathcal{F}, \tag{7.2}$$

where $\mathcal{F}$ is obtained through the use of the null-transport equation after commuting with ∇. In order to obtain this evolution equation, first we need to obtain an expression for $\nabla_4\nabla\mathrm{tr}\chi$. Recall the equation for $\mathrm{tr}\chi$

$$\nabla_4\mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2 - 2\omega\mathrm{tr}\chi - \mathfrak{T}_{44}, \tag{7.3}$$

where $\mathfrak{T}_{44} = \alpha^F\cdot\alpha^F$. Explicit computation yields

$$\nabla_4\nabla\mathrm{tr}\chi = -\mathrm{tr}\chi\nabla\mathrm{tr}\chi - 2\hat{\chi}\nabla\hat{\chi} - 2\nabla\omega\mathrm{tr}\chi - 2\omega\nabla\mathrm{tr}\chi - 2\alpha^F\cdot\widehat{\nabla}\alpha^F + [\nabla_4,\nabla]\mathrm{tr}\chi,$$

where we have utilized the fact that $\mathfrak{T}_{44}$ is a gauge-invariant object together with the metric compatibility of the gauge covariant projected connection $\widehat{\nabla}$. Now compute $[\nabla,\nabla_4]\mathrm{tr}\chi$

$$[\nabla,\nabla_4]\mathrm{tr}\chi = \chi\nabla\mathrm{tr}\chi - (\zeta+\eta)\nabla_4\mathrm{tr}\chi = \chi\nabla\mathrm{tr}\chi - (\zeta+\eta)(-\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2_\gamma - 2\omega\mathrm{tr}\chi - \alpha^F\cdot\alpha^F).$$

Explicit calculations yield

$$\nabla_4 U = 2\Omega^{-1}\omega\nabla\mathrm{tr}\chi + \Omega^{-1}\nabla_4\nabla\mathrm{tr}\chi + 2\Omega^{-1}\omega\mathrm{tr}\chi\zeta + \Omega^{-1}\zeta\nabla_4\mathrm{tr}\chi + \Omega^{-1}\mathrm{tr}\chi\nabla_4\zeta \tag{7.4}$$

which upon substitution of the transport equation for ζ

$$\nabla_4\zeta = 2\nabla\omega + \chi(\underline{\eta}-\zeta) + 2\omega(\zeta+\underline{\eta}) - \beta - \sigma^F\epsilon\cdot\underline{\alpha}^F - \rho^F\cdot\underline{\alpha}^F \tag{7.5}$$

yields

$$\begin{aligned}\mathcal{F} \sim -\hat{\chi}U - \Omega^{-1}\hat{\chi}\nabla\hat{\chi} - \Omega^{-1}\eta|\hat{\chi}|^2 + \Omega^{-1}\mathrm{tr}\chi\hat{\chi}\underline{\eta} - \Omega^{-1}\mathrm{tr}\chi\beta + \Omega^{-1}\alpha^F\cdot\widehat{\nabla}\alpha^F \\ +\Omega^{-1}\mathrm{tr}\chi(\sigma^F,\rho^F)\cdot\underline{\alpha}^F + \Omega^{-1}(\zeta+\eta)\alpha^F\cdot\alpha^F\end{aligned} \tag{7.6}$$

An application of the transport inequality yields

$$||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}})} \lesssim ||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}_0})} + \int_{\underline{u}_0}^{\underline{u}} ||r^{3-\frac{2}{p}}\Omega\mathcal{F}||_{L^p(S_{u,\underline{u}^{'}})}d\underline{u}^{'}. \tag{7.7}$$

Now we explicitly estimate each term of the error term $\mathcal{F}$

$$\begin{aligned}||r^{3-\frac{2}{p}}\Omega\mathcal{F}||_{L^p(S_{u,\underline{u}^{'}})} \lesssim ||r^{3-\frac{2}{p}}\Omega\hat{\chi}U||_{L^p(S_{u,\underline{u}^{'}})} + ||r^{3-\frac{2}{p}}\hat{\chi}\nabla\hat{\chi}||_{L^p(S_{u,\underline{u}^{'}})} + ||r^{3-\frac{2}{p}}\eta|\hat{\chi}|^2||_{L^p(S_{u,\underline{u}^{'}})} \\ +||r^{3-\frac{2}{p}}\underline{\eta}\cdot\hat{\chi}\mathrm{tr}\chi||_{L^p(S_{u,\underline{u}^{'}})} + ||r^{3-\frac{2}{p}}\mathrm{tr}\chi\beta||_{L^p(S_{u,\underline{u}^{'}})} + ||r^{3-\frac{2}{p}}\alpha^F\cdot\widehat{\nabla}\alpha^F||_{L^p(S_{u,\underline{u}^{'}})}\end{aligned} \tag{7.8}$$

$$+||r^{3-\frac{2}{p}}\mathrm{tr}\chi(\rho^F,\sigma^F)\cdot\alpha^F||_{L^p(S_{u,\underline{u}'})}+||r^{3-\frac{2}{p}}(\zeta+\eta)\alpha^F\cdot\alpha^F||_{L^p(S_{u,\underline{u}'})}.$$

We can estimate each term separately

$$||r^{3-\frac{2}{p}}\Omega\hat{\chi}U||_{L^p(S_{u,\underline{u}'})}\lesssim\epsilon\frac{1}{r^2}||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}'})},\ ||r^{3-\frac{2}{p}}\hat{\chi}\nabla\hat{\chi}||_{L^p(S_{u,\underline{u}'})}\lesssim\epsilon\frac{1}{r^2}||r^{3-\frac{2}{p}}\nabla\hat{\chi}||_{L^p(S_{u,\underline{u}'})},\tag{7.9}$$

$$||r^{3-\frac{2}{p}}\eta|\hat{\chi}|^2||_{L^p(S_{u,\underline{u}'})}\lesssim\epsilon^3\frac{1}{r^3},\ ||r^{3-\frac{2}{p}}\underline{\eta}\cdot\hat{\chi}\mathrm{tr}\chi||_{L^p(S_{u,\underline{u}'})}\lesssim\epsilon^2\frac{1}{r^2},\ ||r^{3-\frac{2}{p}}\mathrm{tr}\chi\beta||_{L^p(S_{u,\underline{u}'})}\lesssim\mathcal{W}_0\frac{1}{r^{\frac{3}{2}}},$$

$$||r^{3-\frac{2}{p}}\alpha^F\cdot\widehat{\nabla}\alpha^F||_{L^p(S_{u,\underline{u}'})}\lesssim\mathcal{Y}^2\frac{1}{r^3},||r^{3-\frac{2}{p}}\mathrm{tr}\chi(\rho^F,\sigma^F)\cdot\alpha^F||_{L^p(S_{u,\underline{u}'})}\lesssim\mathcal{Y}^2\frac{|u|^{-\frac{1}{2}}}{r^{\frac{5}{2}}},$$

$$||r^{3-\frac{2}{p}}(\zeta+\eta)\alpha^F\cdot\alpha^F||_{L^p(S_{u,\underline{u}'})}\lesssim\epsilon\mathcal{Y}^2\frac{1}{r^4}.$$

Collecting all the terms together we obtain

$$\begin{aligned}||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}})}&\lesssim||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}_0})}+\int_{\underline{u}_0}^{\underline{u}}\Big(\epsilon\frac{1}{r^2}||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}'})}+\epsilon\frac{1}{r^2}||r^{3-\frac{2}{p}}\nabla\hat{\chi}||_{L^p(S_{u,\underline{u}'})}\\&\quad+\epsilon^3\frac{1}{r^3}+\epsilon^2\frac{1}{r^2}+\mathcal{W}\frac{1}{r^{\frac{3}{2}}}+\mathcal{Y}^2\frac{1}{r^3}+\mathcal{Y}^2\frac{|u|^{-\frac{1}{2}}}{r^{\frac{5}{2}}}+\epsilon\mathcal{Y}^2\frac{1}{r^4}\Big)d\underline{u}'\\&\lesssim\mathcal{I}_0+(\mathcal{W}+\mathcal{Y}^2)+\epsilon\int_{\underline{u}_0}^{\underline{u}}\frac{1}{r^2}||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}'})}d\underline{u}'+\int_{\underline{u}_0}^{\underline{u}}\frac{1}{r^2}||r^{3-\frac{2}{p}}\nabla\hat{\chi}||_{L^p(S_{u,\underline{u}'})}d\underline{u}'.\end{aligned}$$

Now in order to estimate $||r^{3-\frac{2}{p}}\nabla\hat{\chi}||_{L^p(S_{u,\underline{u}'})}$, we invoke the following constraint equation

$$\begin{aligned}\mathrm{div}\hat{\chi}&=\frac{1}{2}\nabla\mathrm{tr}\chi-\frac{1}{2}(\eta-\underline{\eta})\cdot(\hat{\chi}-\frac{1}{2}\mathrm{tr}\chi\gamma)-\beta+\frac{1}{2}\mathfrak{T}(e_4,\cdot)\\&=-\zeta\hat{\chi}+\frac{1}{2}\Omega U-\beta+\frac{1}{2}\alpha^F\cdot(\rho^F+\sigma^F).\end{aligned}\tag{7.10}$$

An application of elliptic estimates yields

$$\begin{aligned}||r^{3-\frac{2}{p}}\nabla\hat{\chi}||_{L^p(S_{u,\underline{u}'})}&\lesssim||r^{3-\frac{2}{p}}\Omega U||_{L^p(S_{u,\underline{u}'})}+||r^{3-\frac{2}{p}}(\eta,\underline{\eta})\hat{\chi}||_{L^p(S_{u,\underline{u}'})}+||r^{3-\frac{2}{p}}\beta||_{L^p(S_{u,\underline{u}'})}\\&\quad+||r^{3-\frac{2}{p}}\alpha^F\cdot(\rho^F,\sigma^F)||_{L^p(S_{u,\underline{u}'})}\\&\lesssim||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}'})}+\frac{\epsilon^2}{r}+\frac{\mathcal{W}}{r^{\frac{1}{2}}}+\frac{\mathcal{Y}^2}{|u|^{\frac{1}{2}}r^{\frac{3}{2}}}.\end{aligned}\tag{7.11}$$

Therefore

$$||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}})}\lesssim\mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2+\epsilon\int_{\underline{u}_0}^{\underline{u}}\frac{1}{r^2}||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}'})}d\underline{u}'\tag{7.12}$$

and an application of Grönwall's inequality yields

$$||r^{3-\frac{2}{p}}U||_{L^p(S_{u,\underline{u}})}\lesssim\mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2\tag{7.13}$$

and thereore

$$||r^{3-\frac{2}{p}}\nabla\hat{\chi}||_{L^p(S_{u,\underline{u}})}\lesssim\mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2\tag{7.14}$$

as well as

$$||r^{2-\frac{2}{p}}\hat{\chi}||_{L^p(S_{u,\underline{u}})}\lesssim\mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2.\tag{7.15}$$

This concludes the proof. $\square$

Remark 4. *Notice that the Yang-Mills source terms appear as (gauge-invariant) components of the stress energy tensor that is quadratic in the Yang-Mills field strength. In addition note that the evolution equation for trχ does not contain Weyl curvature component which allows us to obtain estimates without losing a derivative of Weyl curvature. Since Yang-Mills field strength has one higher order of differentiability, we could close the argument.*

Proposition 7.2. *Assume that the boot-strap assumption (remark 2) holds. The Ricci coefficient $\hat{\underline{\chi}}$ verifies the following estimate*

$$||r^{2-\frac{2}{p}}|u|\nabla\hat{\underline{\chi}}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2,\ ||r^{1-\frac{2}{p}}|u|\hat{\underline{\chi}}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2 \tag{7.16}$$

for $p \in [2, 4]$.

Proof. The proof follow exactly the same way as that of $\hat{\chi}$. Construct a new entity $\underline{u} := \Omega^{-1}\nabla\mathrm{tr}\underline{\chi} - \Omega^{-1}\mathrm{tr}\underline{\chi}\zeta$ and apply the transport inequality in the e_3 direction. This together with the elliptic estimate from equation

$$\mathrm{div}\hat{\underline{\chi}} = \frac{1}{2}\nabla\mathrm{tr}\underline{\chi} - \frac{1}{2}(\underline{\eta} - \eta)\cdot(\hat{\underline{\chi}} - \frac{1}{2}\mathrm{tr}\underline{\chi}\gamma) - \underline{\beta} + \frac{1}{2}\mathfrak{T}(e_3, \cdot) \tag{7.17}$$

yields the result. Here we encounter the Yang-Mills source terms of the following type

$$T_1 = \int_{u_0}^{u} ||r^{3-\frac{2}{p}}\underline{\alpha}^F\cdot\widehat{\nabla}\underline{\alpha}^F||_{L^p(S_{u^{'},\underline{u}})}du^{'} \lesssim \mathcal{Y}^2\int_{u_0}^{u}|u^{'}|^{-3}du^{'} \lesssim \mathcal{Y}^2, \tag{7.18}$$

$$T_2 = \int_{u_0}^{u} ||r^{3-\frac{2}{p}}\mathrm{tr}\underline{\chi}(\rho^F,\sigma^F)\cdot\underline{\alpha}^F||_{L^p(S_{u^{'},\underline{u}})}du^{'} \lesssim \mathcal{Y}^2\int_{u_0}^{u}r^{-1}|u^{'}|^{-2}du^{'} \lesssim \mathcal{Y}^2, \tag{7.19}$$

$$T_3 = \int_{u_0}^{u} ||r^{3-\frac{2}{p}}(\zeta-\underline{\eta})\underline{\alpha}^F\cdot\underline{\alpha}^F||_{L^p(S_{u^{'},\underline{u}})}du^{'} \lesssim \epsilon\mathcal{Y}^2\int_{u_0}^{u}r^{-1}|u^{'}|^{-3}du^{'} \lesssim \mathcal{Y}^2, \tag{7.20}$$

$$T_4 = \int_{u_0}^{u} ||r^{3-\frac{2}{p}}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)||_{L^p(S_{u^{'},\underline{u}})}du^{'} \lesssim \mathcal{Y}^2\int_{u_0}^{u}r^{-1}|u^{'}|^{-1}du^{'} \lesssim \mathcal{Y}^2. \tag{7.21}$$

Here the Yang-Mills source terms are controlled by means of global Sobolev inequalities on null hypersurfaces. Collecting all the terms and application of Grönwall's inequality yields the result. □

Proposition 7.3. *Assume that the boot-strap assumption (remark 2) holds. The Ricci coefficients $\eta, \underline{\eta}$ verify the following estimate*

$$||r^{3-\frac{2}{p}}\nabla\eta||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2, ||r^{2-\frac{2}{p}}\eta||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2, \tag{7.22}$$

$$||r^{3-\frac{2}{p}}\nabla\underline{\eta}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2, ||r^{2-\frac{2}{p}}\underline{\eta}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2 \tag{7.23}$$

Proof. The proof is obtained by introducing the mass aspect functions [21]

$$\mu = -\mathrm{div}\eta + \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} - \rho,\ \underline{\mu} = -\mathrm{div}\underline{\eta} + \frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi} - \rho \tag{7.24}$$

together with the Hodge system

$$\mathrm{div}\eta = -\mu + \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} - \rho, \mathrm{curl}\eta = \hat{\underline{\chi}}\wedge\hat{\chi} + \sigma\epsilon, \tag{7.25}$$

$$\mathrm{div}\underline{\eta} = -\underline{\mu} + \frac{1}{2}\underline{\hat{\chi}}\cdot\hat{\chi} - \rho,\ \mathrm{curl}\underline{\eta} = \underline{\hat{\chi}}\wedge\hat{\chi} - \sigma\epsilon. \tag{7.26}$$

These mass aspect functions are introduced to avoid he loss of derivatives of the Weyl curvature components β and $\underline{\beta}$ since the null evolution equations for η and $\underline{\eta}$ contains β and $\underline{\beta}$, respectively. Therefore, a naive application of the transport inequality on $\nabla_4\nabla\eta$ ($\nabla_3\nabla\underline{\eta}$) would cause a regularity non-closure problem. In our previous article [6] of semi-global estimates for the characteristic initial value problem associated with the Einstein-Yang-Mills equations, we have utilized a slightly different mass aspect functions ($\mu = -\mathrm{div}\eta - \rho$, $\underline{\mu} = -\mathrm{div}\underline{\eta} - \rho$). Here once again, we focus on controlling the Yang-Mills coupling terms that appear since the remaining procedure is exactly similar to [21]. The Yang-Mills terms that appear in the evolution equation for μ are as follows

$$\begin{aligned} YM \sim \alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F) + \widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F) + \mathrm{tr}\underline{\chi}|\alpha^F|^2 + \underline{\omega}|\alpha^F|^2 + \eta(\alpha^F\cdot\sigma^F + \alpha^F\cdot\rho^F) \\ +\mathrm{tr}\chi\rho^F\cdot\sigma^F + (\eta,\underline{\eta})\rho^F\cdot\sigma^F + \hat{\chi}\alpha\cdot\underline{\alpha}^F. \end{aligned} \tag{7.27}$$

Following the method of [21], the norms that we need to control are following

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{2-\frac{2}{p}}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)||_{L^p(S_{u,\underline{u}''})}d\underline{u}' \lesssim \mathcal{Y}^2\int_{\underline{u}}^{\underline{u}^*}|u|^{-\frac{1}{2}}r^{-\frac{7}{2}}d\underline{u}' \lesssim \mathcal{Y}^2\frac{1}{r^{\frac{5}{2}}}, \tag{7.28}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{2-\frac{2}{p}}\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)||_{L^p(S_{u,\underline{u}''})}d\underline{u}' \lesssim \mathcal{Y}^2\int_{\underline{u}}^{\underline{u}^*}|u|^{-\frac{1}{2}}r^{-\frac{7}{2}}d\underline{u}' \lesssim \mathcal{Y}^2\frac{1}{r^{\frac{5}{2}}}, \tag{7.29}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{2-\frac{2}{p}}\mathrm{tr}\underline{\chi}|\alpha^F|^2||_{L^p(S_{u,\underline{u}''})}d\underline{u}' \lesssim \mathcal{Y}^2\int_{\underline{u}}^{\underline{u}^*}r^{-4}d\underline{u}' \lesssim \mathcal{Y}^2\frac{1}{r^3}, \tag{7.30}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{2-\frac{2}{p}}\underline{\omega}|\alpha^F|^2||_{L^p(S_{u,\underline{u}''})}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\int_{\underline{u}}^{\underline{u}^*}|u|^{-1}r^{-4}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\frac{1}{r^3}, \tag{7.31}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{2-\frac{2}{p}}\eta\alpha^F\cdot(\rho^F,\sigma^F)||_{L^p(S_{u,\underline{u}''})}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\int_{\underline{u}}^{\underline{u}^*}|u|^{-\frac{1}{2}}r^{-\frac{9}{2}}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\frac{1}{r^{\frac{7}{2}}}, \tag{7.32}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{2-\frac{2}{p}}\mathrm{tr}\chi\rho^F\cdot\sigma^F||_{L^p(S_{u,\underline{u}''})}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\int_{\underline{u}}^{\underline{u}^*}|u|^{-1}r^{-3}d\underline{u}' \lesssim \mathcal{Y}^2\frac{1}{r^2}, \tag{7.33}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{2-\frac{2}{p}}(\eta,\underline{\eta})\rho^F\cdot\sigma^F||_{L^p(S_{u,\underline{u}''})}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\int_{\underline{u}}^{\underline{u}^*}|u|^{-1}r^{-4}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\frac{1}{r^3}, \tag{7.34}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{2-\frac{2}{p}}\hat{\chi}\alpha^F\cdot\underline{\alpha}^F||_{L^p(S_{u,\underline{u}''})}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\int_{\underline{u}}^{\underline{u}^*}|u|^{-\frac{3}{2}}r^{-\frac{7}{2}}d\underline{u}' \lesssim \epsilon\mathcal{Y}^2\frac{1}{r^{\frac{5}{2}}}. \tag{7.35}$$

Collecting all the terms and following the steps of [21], one obtains the result. Similar procedure follows for $\underline{\eta}$ as well. □

The remaining connection estimates follow in a similar way. Once again, the only modification one needs to worry about is the Yang-Mills source terms.

Proposition 7.4. *Assume that the boot-strap assumption (remark 2) holds. The Ricci coefficients verifies the following estimate*

$$||r^{3-\frac{2}{p}}\nabla tr\chi||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2,\ ||r^{2-\frac{2}{p}}(tr\chi - \overline{tr\chi})||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2, \tag{7.36}$$

$$||r^{3-\frac{2}{p}}\nabla tr\underline{\chi}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2,\ ||r^{2-\frac{2}{p}}(tr\underline{\chi} - \overline{tr\underline{\chi}})||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2, \tag{7.37}$$

$$||r^{2-\frac{2}{p}}\omega||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2,\ ||r^{1-\frac{2}{p}}|u|\underline{\omega}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2, \tag{7.38}$$

$$||r^{3-\frac{2}{p}}\nabla\omega||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2,\ ||r^{2-\frac{2}{p}}|u|\nabla\underline{\omega}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0 + \mathcal{W} + \mathcal{Y}^2. \tag{7.39}$$

Proof. Once again, we only sketch the estimates for the Yang-Mills coupling terms. The Yang-Mills coupling terms that appear while executing the procedure of [21] to obtain the relevant estimates are as follow

$$\int_{\underline{u}_0}^{\underline{u}} ||r^{3-\frac{2}{p}}\alpha^F \cdot \widehat{\nabla}\alpha^F||_{L^p(S_{u,\underline{u}^{'}})}d\underline{u}^{'} \lesssim \mathcal{Y}^2 \int_{\underline{u}_0}^{\underline{u}} r^{-3}d\underline{u}^{'} \lesssim \mathcal{Y}^2, \tag{7.40}$$

$$\int_{u_0}^{u} ||r^{3-\frac{2}{p}}\underline{\alpha}^F \cdot \widehat{\nabla}\underline{\alpha}^F||_{L^p(S_{u^{'},\underline{u}})}du^{'} \lesssim \mathcal{Y}^2 \int_{u_0}^{u} |u^{'}|^{-3}du^{'} \lesssim \mathcal{Y}^2, \tag{7.41}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{-\frac{2}{p}}\rho^F \cdot \sigma^F||_{L^p(S_{u,\underline{u}^{'}})}d\underline{u}^{'} \lesssim \mathcal{Y}^2 \int_{\underline{u}}^{\underline{u}^*} |u|^{-1}r^{-4}d\underline{u}^{'} \lesssim \mathcal{Y}^3\frac{1}{r^3}, \tag{7.42}$$

$$\int_{u_0}^{u} ||r^{-\frac{2}{p}}\rho^F \cdot \sigma^F||_{L^p(S_{u^{'},\underline{u}})}du^{'} \lesssim \mathcal{Y}^2 \int_{u_0}^{u} |u|^{-1}r^{-4}du^{'} \lesssim \mathcal{Y}^2\frac{1}{r^2}, \tag{7.43}$$

$$\int_{\underline{u}}^{\underline{u}^*} ||r^{1-\frac{2}{p}}\widehat{\nabla}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)||_{L^p(S_{u,\underline{u}^{'}})}d\underline{u}^{'} \lesssim \mathcal{Y}^2 \int_{\underline{u}}^{\underline{u}^*} |u|^{-1}r^{-4}d\underline{u}^{'} \lesssim \mathcal{Y}^3\frac{1}{r^3}, \tag{7.44}$$

$$\int_{u_0}^{u} ||r^{1-\frac{2}{p}}\widehat{\nabla}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)||_{L^p(S_{u^{'},\underline{u}})}du^{'} \lesssim \mathcal{Y}^2 \int_{u_0}^{u} |u|^{-1}r^{-4}du^{'} \lesssim \mathcal{Y}^2\frac{1}{r^2} \tag{7.45}$$

□

The second derivative estimates follow exactly in a similar way and we omit the proof. We refer the reader to [21] for exact procedure.

Proposition 7.5. *Assume that the boot-strap assumption (remark 2) holds. The Ricci coefficients verify the following second derivative estimates*

$$||r^{4-\frac{2}{p}}\nabla^2\hat{\chi}||_{L^p(S_{u,\underline{u}})} \lesssim (\mathcal{I}_0+\mathcal{W}_0+\mathcal{Y}^2),\ ||r^{3-\frac{2}{p}}|u|\nabla^2\underline{\hat{\chi}}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, \tag{7.46}$$

$$||r^{4-\frac{2}{p}}\nabla^2\eta||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2,\ ||r^{4-\frac{2}{p}}\nabla^2\underline{\eta}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, \tag{7.47}$$

$$||r^{4-\frac{2}{p}}\nabla^2 tr\chi||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, ||r^{4-\frac{2}{p}}\nabla^2 tr\underline{\chi}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, \tag{7.48}$$

$$||r^{4-\frac{2}{p}}\nabla^2\omega||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2,\ ||r^{3-\frac{2}{p}}|u|\nabla^2\underline{\omega}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, \tag{7.49}$$

$$||r^{4-\frac{2}{p}}\nabla\nabla_4\omega||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2,\ ||r^{2-\frac{2}{p}}|u|^2\nabla\nabla_3\underline{\omega}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, \tag{7.50}$$

$$||r^{5-\frac{2}{p}}\nabla^2\nabla_4\omega||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2,\ ||r^{3-\frac{2}{p}}|u|^2\nabla^2\nabla_3\underline{\omega}||_{L^p(S_{u,\underline{u}})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2 \tag{7.51}$$

Proof. The proof of this proposition is essentially same as the first derivative estimates. This is nothing but a consequence of scaling. □

Proposition 7.6. *The following third derivative estimates hold for the connection coefficients*

$$\begin{aligned}
r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\hat{\chi}||_{L^2(\underline{H})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3 tr\chi||_{L^2(\underline{H})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2,\\
r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\eta||_{L^2(\underline{H})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\underline{\eta}||_{L^2(\underline{H})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2,\\
r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\underline{\hat{\chi}}||_{L^2(\underline{H})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2, r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3 tr\underline{\chi}||_{L^2(\underline{H})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2,\\
r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla^3\underline{\omega}||_{L^2(\underline{H})} \lesssim \mathcal{I}_0+\mathcal{W}+\mathcal{Y}^2.
\end{aligned} \tag{7.52}$$

Proof. The proof is essentially the same as presented in [21] for the vacuum case. There here we only estimate the Yang-Mills source terms. Note that the Riemann curvature components in [21] becomes Weyl curvature in the current context. See [6] for the Einstein-Yang-Mills case with the regularity level one lower than the present case. However, such argument can be extended easily to the higher derivative case since the relative regularity is preserved (i.e., the regularity level of Yang-Mills curvature is one order higher than that of the Weyl curvature). □

7.2. Energy estimate for the Weyl and Yang-Mills curvature

The true non-linear effect of the Yang-Mills coupling terms appears at the level of energy estimates. Null Bianchi equations and the null Yang-Mills equations are manifestly hyperbolic. This is essentially as a consequence of Einstein's and Yang-Mills equations being derivable from a Lagrangian. This can also be interpreted as the existence of the Bel-Robinson tensor for gravity and a canonical Yang-Mills stress energy tensor (see [6]). We can explicitly utilize this manifestly hyperbolic character to estimate the pure gravitational and Yang-Mills energy. The following integration lemma turns out to be crucial.

Lemma 7.1. *Let $f : M \to \mathbb{R}$ be a gauge invariant object on the spacetime. The following integration by parts identities holds for f* [4]

$$\int_{D_{u\underline{u}}} \nabla_4 f = \int_{\underline{H}_{\underline{u}}} f - \int_{\widetilde{\Sigma}_0} f + \int_{D_{u\underline{u}}} (2\omega - tr\chi) f \tag{7.53}$$

and

$$\int_{D_{u\underline{u}}} \nabla_3 f = \int_{H_u} f - \int_{\widetilde{\Sigma}_0} f + \int_{D_{u\underline{u}}} (2\underline{\omega} - tr\underline{\chi}) f, \tag{7.54}$$

Proof.

$$\int_{D_{u\underline{u}}} \nabla_4 f = \int_{u_0}^{u} du^{'} \int_{\underline{u}_0(u)}^{\underline{u}} \left(\int_{S_{u^{'}\underline{u}^{'}}} \frac{\partial f}{\partial \underline{u}^{'}} \Omega \mu_\gamma \right) d\underline{u}^{'} \tag{7.55}$$

$$= \int_{u_0}^{u} du^{'} \int_{\underline{u}_0(u^{'})}^{\underline{u}} \left\{ \frac{d}{d\underline{u}^{'}} \int_{S_{u^{'}\underline{u}^{'}}} f \Omega \mu_\gamma - \int_{S_{u^{'}\underline{u}^{'}}} f \left(\frac{\partial \Omega}{\partial \underline{u}^{'}} + \frac{\Omega}{2} \mathrm{tr}_\gamma \partial_u \gamma \right) \mu_\gamma \right\} d\underline{u}^{'}$$

$$= \int_{u_0}^{u} du^{'} \int_{S_{u^{'}\underline{u}}} f \Omega \mu_\gamma - \int_{u_0}^{u} \int_{S_{u^{'}\underline{u}_0}} f \Omega \mu_\gamma + \int_{D_{u\underline{u}}} f (2\omega - \mathrm{tr}\chi)$$

$$= \int_{\underline{H}_{\underline{u}}} f - \int_{\widetilde{\Sigma}_0} f + \int_{D_{u\underline{u}}} (2\omega - \mathrm{tr}\chi) f.$$

The other part follows in a similar fashion. □

Utilizing this integration lemma, we may proceed to obtain the energy estimates in the exterior domain. Let us now identify the Bianchi pairs: (α, β), $(\underline{\alpha}, \underline{\beta})$, $(\beta, \{\rho, \sigma\})$, $(\underline{\beta}, \{\rho, \sigma\})$, $(\alpha^F, \ \{\rho^F, \sigma^F\})$, $(\underline{\alpha}^F, \{\rho^F, \sigma^F\})$.

[4] Note that instead of the canonical foliation and space-time harmonic gauge, if one were to choose a maximal slice, then it would be impossible to connect the leaves of the null foliation H_u to that of the space-like maximal foliation. In such a case one would require an additional oscillation lemma.

7.2.1. Lowest order Weyl energy estimates
Utilizing the symmetric hyperbolic characteristic of the Yang-Mills sourced Bianchi equations, we will obtain the necessary energy estimates for the Weyl curvature. To this end lemma (6.1) play an extremely crucial role. First consider the pair (α, β). Define $f := r^4|\alpha|^2$ and $g := r^4|\beta|^2$ and apply the integration lemma

$$\int_{D_{u,\underline{u}}} \nabla_3(r^4|\alpha|^2) + \int_{D_{u,\underline{u}}} \nabla_4(r^4|\beta|^2) = \int_{H_u} r^4|\alpha|^2 + \int_{\underline{H}_{\underline{u}}} r^4|\beta|^2 - \int_{\widehat{\Sigma}_0} r^4|\alpha|^2$$
$$- \int_{\widehat{\Sigma}_0} r^4|\beta|^2 + \int_{D_{u,\underline{u}}} (2\underline{\omega} - \mathrm{tr}\underline{\chi})r^4|\alpha|^2 + \int_{D_{u,\underline{u}}} (2\omega - \mathrm{tr}\chi)r^4|\beta|^2$$

Now explicitly evaluate

$$\nabla_3(r^4|\alpha|^2) = 4r^3\Omega^{-1}\frac{dr}{du}|\alpha|^2 + 2r^4\langle\alpha, \nabla_3\alpha\rangle, \tag{7.56}$$
$$\nabla_4(r^4|\beta|^2) = 4r^3\Omega^{-1}\frac{dr}{d\underline{u}}|\beta|^2 + 2r^4\langle\beta, \mathfrak{D}_4\beta\rangle \tag{7.57}$$

Recall the expressions

$$\frac{d}{du}\int \mu_\gamma = \int \mathrm{tr}\underline{\chi}\Omega\mu_\gamma, \ \frac{d}{d\underline{u}}\int \mu_\gamma = \int \mathrm{tr}\chi\Omega\mu_\gamma \tag{7.58}$$

and evaluate

$$\frac{dr}{du} = \frac{r}{2}\overline{\Omega\mathrm{tr}\underline{\chi}}, \ \frac{dr}{d\underline{u}} = \frac{r}{2}\overline{\Omega\mathrm{tr}\chi}. \tag{7.59}$$

Explicit calculations yield

$$\int_{D_{u,\underline{u}}} \nabla_3(r^4|\alpha|^2) + \int_{D_{u,\underline{u}}} \nabla_4(r^4|\beta|^2) = \int_{D_{u,\underline{u}}} \Big(2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\alpha|^2 \tag{7.60}$$
$$+2r^4\langle\alpha, -\frac{1}{2}\mathrm{tr}\underline{\chi}\alpha + \nabla\widehat{\otimes}\beta + 4\underline{\omega}\alpha_{ab} - 3(\hat{\chi}\rho + {}^*\hat{\chi}\sigma)_{ab} + ((\zeta + 4\eta)\hat{\otimes}\beta)_{ab}$$
$$+\frac{1}{2}(D_3\mathfrak{T}_{44} - D_4\mathfrak{T}_{43})\gamma_{ab}\rangle + 2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\beta|^2 + 2r^4\langle\beta, -2\mathrm{tr}\chi\beta + \mathrm{div}\alpha$$
$$-2\omega\beta_a + (\eta\cdot\alpha)_a - \frac{1}{2}(D_a\mathfrak{T}_{44} - D_4\mathfrak{T}_{4a})\rangle\Big)$$

which after cancellation of the principal terms via integration by parts reduces to

$$\int_{H_u} r^4|\alpha|^2 + \int_{\underline{H}_{\underline{u}}} r^4|\beta|^2 + \int_{D_{u,\underline{u}}} (3r^4\mathrm{tr}\chi - 2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\chi})|\beta|^2 - \int_{D_{u,\underline{u}}} 2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\alpha|^2$$
$$= \int_{\widehat{\Sigma}_0} r^4|\alpha|^2 + \int_{\widehat{\Sigma}_0} r^4|\beta|^2 + \mathcal{E}_{\alpha,\beta},$$

where the error term $\mathcal{E}_{\alpha,\beta}$ verifies

$$\mathcal{E}_{\alpha,\beta} = \int_{D_{u,\underline{u}}} \Big(2r^4\langle\alpha, 4\underline{\omega}\alpha_{ab} - 3(\hat{\chi}\rho + {}^*\hat{\chi}\sigma)_{ab} + ((\zeta + 4\eta)\hat{\otimes}\beta)_{ab} \tag{7.61}$$
$$-D_4\mathfrak{T}_{ab} + \frac{1}{2}\left(D_b\mathfrak{T}_{4a} + D_a\mathfrak{T}_{4b}\right)\rangle + 2r^4\langle\beta, -2\omega\beta_a + (\eta\cdot\alpha)_a - \frac{1}{2}(D_a\mathfrak{T}_{44} - D_4\mathfrak{T}_{4a})\rangle\Big),$$

where note that $(D_3\mathfrak{T}_{44} - D_4\mathfrak{T}_{43})\gamma_{ab}$ drops out due to the trace-less property of α.

Proposition 7.7. *The error term $\mathcal{E}_{\alpha,\beta}$ verifies the following estimates*

$$|\mathcal{E}_{\alpha,\beta}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1) \tag{7.62}$$

Proof. We start estimating the error terms individually. In the global exterior stability problem of Minkowski space, there are two competing forces present. The geometric dispersion leads to decay of energy in the outward directions (the rate of decay in the different directions will be different) while the non-linearities tend to concentrate energy. Global existence holds true if the decay is sufficient to over the energy concentration by nonlinearities. This will manifest itself as certain integrability condition.

$$|\int_{D_{u,\underline{u}}} 2r^4\langle\alpha, 4\underline{\omega}\alpha\rangle| \lesssim \left(\int_{u_0}^{u} \frac{1}{r^2}du^{'}\right) \sup_{\mathcal{K}} ||r^2\omega||_{L^\infty}(\int_H r^4|\alpha|^2) \lesssim \epsilon\mathcal{W}_0, \tag{7.63}$$

where by $\int_H r^4|\alpha|^2$ we mean $\sup_u \int_{H_u} r^4|\alpha|^2$. Similar notation follows for $\underline{H}$ too i.e., $\int_{\underline{H}} |f|^2$ for some f will be used to denote $\sup_{\underline{u}} \int_{\underline{H}_{\underline{u}}} |f|^2$. The remaining estimates read

$$\begin{aligned}
|\int_{D_{u,\underline{u}}} r^4\langle\alpha, \widehat{\chi}\rho\rangle| &\lesssim \left(\int_{\underline{u}_0}^{\underline{u}} \frac{1}{r^2}d\underline{u}^{'}\right) \sup_{\mathcal{K}} ||r^2\widehat{\chi}||_{L^\infty}(\int_{\underline{H}} r^4|\rho|^2) \\
&+ \left(\int_{u_0}^{u} \frac{1}{r^2}d\underline{u}^{'}\right) \sup_{\mathcal{K}} ||r^2\widehat{\chi}||_{L^\infty}(\int_H r^4|\alpha|^2) \lesssim \epsilon\mathcal{W}_0, \\
|\int_{D_{u,\underline{u}}} r^4\langle\alpha, {}^*\widehat{\chi}\sigma\rangle| &\lesssim \left(\int_{\underline{u}_0}^{\underline{u}} \frac{1}{r^2}d\underline{u}^{'}\right) \sup_{\mathcal{K}} ||r^2\widehat{\chi}||_{L^\infty}(\int_{\underline{H}} r^4|\sigma|^2) \\
&+ \left(\int_{u_0}^{u} \frac{1}{r^2}d\underline{u}^{'}\right) \sup_{\mathcal{K}} ||r^2\widehat{\chi}||_{L^\infty}(\int_H r^4|\alpha|^2) \lesssim \epsilon\mathcal{W}_0, \\
|\int_{D_{u,\underline{u}}} r^4\langle\alpha, (\eta+\underline{\eta})\otimes\beta\rangle| &\lesssim |\left(\int_{u_0}^{u} \frac{1}{r^2}du^{'}\right)| \sup_{\mathcal{K}} |r^2(\eta+\underline{\eta})| \int_H r^4|\alpha|^2 \\
&+ |\left(\int_{u_0}^{u} \frac{1}{r^2}du^{'}\right)| \sup_{\mathcal{K}} |r^2(\eta+\underline{\eta})| \int_H r^4|\beta|^2 \\
&\lesssim \epsilon\mathcal{W}_0, \\
|\int_{D_{u,\underline{u}}} r^4\langle\beta, \omega\beta\rangle| &\lesssim |\left(\int_{u_0}^{u} \frac{1}{r^2}du^{'}\right)| \sup_{\mathcal{K}} |r^2\omega| \int_H r^4|\beta|^2 \lesssim \epsilon\mathcal{W}_0, \\
|\int_{D_{u,\underline{u}}} r^4\langle\beta, \eta\cdot\alpha\rangle| &\lesssim |\left(\int_{u_0}^{u} \frac{1}{r^2}du^{'}\right)| \sup_{\mathcal{K}} |r^2\eta| \int_H r^4|\alpha|^2 \\
&+ |\left(\int_{u_0}^{u} \frac{1}{r^2}du^{'}\right)| \sup_{\mathcal{K}} |r^2\eta| \int_H r^4|\beta|^2 \lesssim \epsilon\mathcal{W}_0.
\end{aligned} \tag{7.64}$$

In all the previous terms, we observed that the decay factors are integrable yielding a uniform estimate. Now we estimate the coupled terms

$$|\int_{D_{u,\underline{u}}} r^4\alpha_{AB}\nabla_4\mathfrak{T}_{AB}| \tag{7.65}$$

Explicitly compute and write schematically

$$\begin{aligned}\nabla_4\mathfrak{T}_{AB} &= D_4\mathfrak{T}_{AB} \\ \sim \langle\rho^F, \widehat{\nabla}\alpha^F + \mathrm{tr}\chi\rho^F + (\eta+\underline{\eta})\alpha^F\rangle &+ \langle\sigma^F, \widehat{\nabla}\alpha^F + \mathrm{tr}\chi\sigma^F + (\eta+\underline{\eta}\alpha^F\rangle \\ +\langle\underline{\alpha}^F, \widehat{\nabla}_4\alpha^F\rangle + \langle\alpha^F, \mathrm{tr}\chi\underline{\alpha}^F &+ \widehat{\nabla}(\rho^F,\sigma^F) + \underline{\eta}(\sigma^F,\rho^F) + \omega\underline{\alpha}^F - \widehat{\chi}\alpha^F\rangle.\end{aligned} \tag{7.66}$$

Now we estimate each term separately

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^4\alpha\rho^F\widehat{\nabla}\alpha^F| &\lesssim \int_{D_{u,\underline{u}}}(r^2\alpha)(r^2u^{\frac{1}{2}}\rho^F)u^{-\frac{1}{2}}(r^2|\widehat{\nabla}\alpha^F|)r^{-2} \\ &\lesssim \sup_{\mathcal{K}}(r^2u^{\frac{1}{2}}|\rho^F|)\left(\int_H r^4|\alpha|^2 + \int_H r^4|\widehat{\nabla}\alpha^F|^2\right) \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1)\end{aligned} \tag{7.67}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^4\alpha\rho^F\mathrm{tr}\chi\rho^F| = |\int_{D_{u,\underline{u}}} r^4\alpha\rho^F(\mathrm{tr}\chi - \frac{2}{r} + \frac{2}{r})\rho^F| \lesssim |\int_{D_{u,\underline{u}}} r^4\alpha\rho^F(\mathrm{tr}\chi - \frac{2}{r})\rho^F| \\ +|\int_{D_{u,\underline{u}}} r^3\alpha\rho^F\cdot\rho^F| \lesssim \sup_{\mathcal{K}}(r^2|\mathrm{tr}\chi - \frac{2}{r}|)\sup_K(r^2u^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^u \frac{1}{r^2u^{\frac{3}{2}}})\left(\int_H r^4|\alpha|^2 + \int_H u^2|\rho^F|^2\right) \\ +\sup_{\mathcal{K}}(r^2u^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^u\frac{1}{ru^{\frac{3}{2}}})\left(\int_H r^4|\alpha|^2 + \int_H u^2|\rho^F|^2\right) \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0), \\ |\int_{D_{u,\underline{u}}} r^4\alpha\rho^F\cdot\alpha^F(\eta+\underline{\eta})| \lesssim \sup_{\mathcal{K}}(r^2|\eta+\underline{\eta}|)\sup_K(r^2u^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^u\frac{1}{r^2u^{\frac{3}{2}}})\left(\int_H r^4|\alpha|^2 + \int_H r^2|\alpha^F|^2\right) \\ \lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0)\end{aligned}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^4\alpha\underline{\alpha}^F\cdot\widehat{\nabla}_4\alpha^F| \lesssim \sup_{\mathcal{K}}(u^{\frac{5}{2}}|\underline{\alpha}^F|)(\int_{u_0}^u u^{-\frac{5}{2}})\left(\int_H r^4|\alpha|^2 + \int_H r^2|\widehat{\nabla}_4\alpha^F|^2\right) \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1), \\ |\int_{D_{u,\underline{u}}} r^4\alpha\alpha^F\cdot\underline{\alpha}^F\mathrm{tr}\chi| = |\int_{D_{u,\underline{u}}} r^4\alpha\alpha^F\cdot\underline{\alpha}^F(\mathrm{tr}\chi - \frac{2}{r} + \frac{2}{r})| \\ \lesssim \sup_{\mathcal{K}} r^2|\mathrm{tr}\chi - \frac{2}{r}|\sup_{\mathcal{K}}|u|^{\frac{3}{2}}|r\underline{\alpha}^F|\left(\int_H r^4|\alpha|^2 + r^2|\alpha^F|^2\right)(\int_{u_0}^u |u'|^{-\frac{3}{2}}r^{-2}du') \\ +\sup_{\mathcal{K}}|u|^{\frac{3}{2}}r|\underline{\alpha}^F|\left(\int_H r^4|\alpha|^2 + r^2|\alpha^F|^2\right)(\int_{u_0}^u |u'|^{-\frac{3}{2}}r^{-1}du') \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0), \\ |\int_{D_{u,\underline{u}}} r^4\alpha\langle\alpha^F, \widehat{\nabla}(\rho^F,\sigma^F)\rangle| \lesssim \sup_{\mathcal{K}}|r^{\frac{5}{2}}\alpha^F|(\int_{u_0}^u r^{-\frac{3}{2}}u^{-1})(\int_H r^4|\alpha|^2 + r^2u^2|\widehat{\nabla}(\rho^F,\sigma^F)|^2) \\ \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1), \\ |\int_{D_{u,\underline{u}}} r^4\alpha\alpha^F\cdot(\sigma^F,\rho^F)\eta| \lesssim \sup_{\mathcal{K}}|r^2u^{\frac{1}{2}}(\sigma^F,\rho^F)|\sup_{\mathcal{K}}|r^2\eta|(\int_{u_0}^u r^{-3}u^{-\frac{1}{2}})(\int_H r^4|\alpha|^2 + r^2|\alpha^F|^2) \\ \lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0), \\ |\int_{D_{u,\underline{u}}} r^4\alpha\alpha^F\cdot\underline{\alpha}^F\omega| \lesssim \sup_{\mathcal{K}}|u^{\frac{3}{2}}r\underline{\alpha}^F|\sup_{\mathcal{K}} r^2|\omega|(\int_{u_0}^u u^{-\frac{3}{2}}r^{-2})\sup_{u,\underline{u}}(\int_H (r^4|\alpha|^2 + r^2|\alpha^F|^2) \\ \lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0), \\ |\int_{D_{u,\underline{u}}} r^4\alpha\alpha^F\cdot\alpha^F\widehat{\chi}| \lesssim \sup_{\mathcal{K}} r^{\frac{5}{2}}|\alpha^F|\sup_{\mathcal{K}} r^2|\widehat{\chi}|(\int_{u_0}^u r^{-\frac{7}{2}})(\int_H r^4|\alpha|^2 + r^2|\alpha^F|^2) \\ \lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0).\end{aligned}$$

Here we heavily make use of the gauge-invariant Sobolev inequalities (lemma 4.2) to estimate the point-wise norms for the curvature components. Collecting all the terms together, we obtain

$$|\int_{D_{u,\underline{u}}} r^4\alpha_{AB}\nabla_4\mathfrak{T}_{AB}| \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.68}$$

Next we have to estimate

$$|\int_{D_{u,\underline{u}}} r^4\alpha\nabla T_{4A}|. \tag{7.69}$$

An explicit calculation yields

$$\nabla T_{4A} \sim \widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)+\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+(\eta+\underline{\eta})(\rho^F,\sigma^F)\cdot\alpha^F+\chi(\rho^F\cdot\rho^F+\sigma^F\cdot\sigma^F+\alpha^F\cdot\underline{\alpha}^F).$$

Similar to the previous case, we estimate each of these coupling terms separately

$$|\int_{D_{u,\underline{u}}} r^4\alpha\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)| \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^4\alpha\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)| \lesssim \sup_{\mathcal{K}} r^{\frac{5}{2}}|\alpha^F|(\int_{u_0}^{u}|u^{'}|^{-1}r^{-\frac{3}{2}}du^{'})\left(\int_H r^4|\alpha|^2+\int_H u^2r^2|\widehat{\nabla}(\rho^F,\sigma^F)|^2\right)$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^4\alpha(\eta+\underline{\eta})(\rho^F,\sigma^F)\cdot\alpha^F|$$
$$\lesssim (\sup_{\mathcal{K}}|r^2(\eta+\underline{\eta})|)(\sup_{\mathcal{K}} r^{\frac{5}{2}}|\alpha^F|)(\int_{u_0}^{u}|u^{'}|^{-1}r^{-\frac{5}{2}}du^{'})\left(\int_H r^4|\alpha|^2+\int_H u^2|(\rho^F,\sigma^F)|^2\right)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\alpha\widehat{\chi}\rho^F\cdot\rho^F| \lesssim$$
$$\sup_{\mathcal{K}} r^2|\widehat{\chi}|\sup_{\mathcal{K}} r^2|u|^{\frac{1}{2}}|\rho^F|(\int_{u_0}^{u} r^{-2}|u^{'}|^{-\frac{3}{2}}du^{'})\left(\int_H r^4|\alpha|^2+\int_H |u|^2|\rho^F|^2\right)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\alpha\widehat{\chi}(\alpha^F\cdot\underline{\alpha}^F)|$$
$$\lesssim \sup_{\mathcal{K}} r^2|\widehat{\chi}|\sup_{\mathcal{K}} r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{u_0}^{u} r^{-2}|u^{'}|^{-\frac{3}{2}}du^{'})\left(\int_H r^4|\alpha|^2+\int_H r^2|\alpha^F|^2\right)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\alpha\mathrm{tr}\chi\rho^F\cdot\rho^F| \leq |\int_{D_{u,\underline{u}}} r^4\alpha(\mathrm{tr}\chi-\frac{2}{r})\rho^F\cdot\rho^F|+|\int_{D_{u,\underline{u}}} 2r^3\alpha\rho^F\cdot\rho^F|$$
$$\lesssim \sup_{\mathcal{K}} r^2|\mathrm{tr}\chi-\frac{2}{r}|\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^{u} r^{-2}|u^{'}|^{-\frac{3}{2}}du^{'})\left(\int_H r^4|\alpha|^2+|u|^2|\rho^F|^2\right)$$
$$+\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^{u} r^{-1}|u^{'}|^{-\frac{3}{2}}du^{'})\left(\int_H r^4|\alpha|^2+|u|^2|\rho^F|^2\right)$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\alpha\mathrm{tr}\chi\underline{\alpha}^F\cdot\alpha^F| \leq |\int_{D_{u,\underline{u}}} r^4\alpha(\mathrm{tr}\chi-\frac{2}{r})\alpha^F\cdot\underline{\alpha}^F|+|\int_{D_{u,\underline{u}}} 2r^3\alpha\alpha^F\cdot\underline{\alpha}^F|$$

$$\lesssim \sup_{\mathcal{K}} r^2|\mathrm{tr}\chi - \frac{2}{r}| \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{u_0}^{u} r^{-2}|u'|^{-\frac{3}{2}}du')\left(\int_H r^4|\alpha|^2 + r^2|\alpha^F|^2\right)$$

$$+|\sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{u_0}^{u} r^{-1}|u'|^{-\frac{3}{2}}du')\left(\int_H r^4|\alpha|^2 + r^2|\alpha^F|^2\right)$$

$$\lesssim \epsilon^2(\mathcal{W}_0 + \mathcal{Y}_0) + \epsilon(\mathcal{W}_0 + \mathcal{Y}_0) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0).$$

Collecting all the terms, we obtain

$$|\int_{D_{u,\underline{u}}} r^4\alpha\nabla\mathfrak{T}_{4A}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1). \tag{7.70}$$

The remaining terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^4|\beta|^2\omega| \lesssim \sup_{\mathcal{K}} r^2|\omega|(\int_{u_0}^{u} r^{-2}du')\int_H r^4|\beta|^2 \lesssim \epsilon\mathcal{W}_0, \tag{7.71}$$

$$|\int_{D_{u,\underline{u}}} r^4\beta\alpha\eta| \lesssim \sup_{\mathcal{K}} r^2|\eta|(\int_{u_0}^{u} r^{-2}du')\int_H (r^4|\beta|^2 + r^4|\alpha|^2) \lesssim \epsilon\mathcal{W}_0. \tag{7.72}$$

Now we need to estimate

$$|\int_{D_{u,\underline{u}}} r^4\beta(D_4\mathfrak{T}_{4A} - D_A\mathfrak{T}_{44})|. \tag{7.73}$$

Explicit calculations yield

$$D_a\mathfrak{T}_{44} - D_4\mathfrak{T}_{4a} \sim \langle\alpha^F, \widehat{\nabla}_b\alpha^F\rangle - \chi_{bc}\mathfrak{T}_{c4} + \eta_b\mathfrak{T}_{44} - 2\omega\mathfrak{T}_{4b} - \widehat{\nabla}_4(\alpha^F_b\cdot\rho^F - \alpha^F_b\cdot\sigma^F),$$

and recall

$$\mathfrak{T}_{44} \sim \alpha^F\cdot\alpha^F,\ \mathfrak{T}_{4A} \sim \alpha^F\cdot\rho^F + \alpha^F\cdot\sigma^F, \tag{7.74}$$

$$\widehat{\nabla}_4\rho^F \sim \widehat{\nabla}\alpha^F + \mathrm{tr}\chi\rho^F + (\eta - \underline{\eta})\alpha^F \tag{7.75}$$

We estimate each term separately

$$|\int_{D_{u,\underline{u}}} r^4\beta\langle\alpha^F, \widehat{\nabla}\alpha^F\rangle| \lesssim \sup_{\mathcal{K}} r^{\frac{5}{2}}|\alpha^F|(\int_{u_0}^{u} r^{-\frac{5}{2}}du')\left(\int_H r^4|\beta|^2 + \int_H r^4|\widehat{\nabla}\alpha^F|^2\right) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1), \tag{7.76}$$

$$|\int_{D_{u,\underline{u}}} r^4\beta\langle\widehat{\nabla}_4\alpha^F, \rho^F\rangle| \lesssim \sup_{\mathcal{K}} r^2|u|^{\frac{1}{2}}|\rho^F|(\int_{u_0}^{u} r^{-2}|u|^{-\frac{1}{2}})\left(\int_H r^4|\beta|^2 + \int_H r^4|\widehat{\nabla}_4\alpha^F|^2\right) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^4\beta\langle\alpha^F, \widehat{\nabla}\alpha^F\rangle| \lesssim \sup_{\mathcal{K}} r^{\frac{5}{2}}|\alpha^F|\int_H (r^4|\beta|^2 + r^4|\widehat{\nabla}\alpha^F|^2)(\int_{u_0}^{u} r^{-\frac{5}{2}}du') \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^4\beta\alpha^F\cdot\rho^F\mathrm{tr}\chi| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\beta(\eta - \underline{\eta})\alpha^F\cdot\alpha^F| \lesssim \epsilon^2(\mathcal{W}_0 + \mathcal{Y}_0).$$

Similar estimates hold if one replaces ρ^F by σ^F. Collecting all the terms, we obtain

$$|\int_{D_{u,\underline{u}}} r^4\beta(D_4\mathfrak{T}_{4A} - D_A\mathfrak{T}_{44})| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1). \tag{7.77}$$

Therefore we obtain the desired estimate $|\mathcal{E}_{\alpha,\beta}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1)$. □

Utilizing these estimates, we obtain the following lemma is the exterior domain.

Lemma 7.2. *The following estimate holds*

$$\int_{H_u} r^4|\alpha|^2 + \int_{\underline{H}_{\underline{u}}} r^4|\beta|^2 \lesssim \int_{\widehat{\Sigma}_0} r^4|\alpha|^2 + \int_{\widehat{\Sigma}_0} r^4|\beta|^2 + \epsilon(\mathbf{W}+\mathbf{Y}). \tag{7.78}$$

Proof. The proof is a consequence of the previous proposition (6.7) and the fact that $3\mathrm{tr}\chi - 2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi} = \frac{2}{r} + \frac{C\epsilon}{r^2}$ and $2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi} = -\frac{4}{r} + \frac{C\epsilon}{r^2}$. The last two estimates follow from the connection estimates (section 7.1) □

Now consider the pair $\underline{\alpha}$ and $\underline{\beta}$ and write

$$\begin{aligned}\int_{D_{u,\underline{u}}} \nabla_4(u^4|\underline{\alpha}|^2) + \int_{D_{u,\underline{u}}} \nabla_3(u^4|\underline{\beta}|^2) &= \int_{H_u} u^4|\underline{\beta}|^2 + \int_{\underline{H}_{\underline{u}}} u^4|\underline{\alpha}|^2 - \int_{\widehat{\Sigma}_0} u^4|\underline{\alpha}|^2 \\ &- \int_{\widehat{\Sigma}_0} u^4|\underline{\beta}|^2 + \int_{D_{u,\underline{u}}} (2\underline{\omega} - \mathrm{tr}\underline{\chi})u^4|\underline{\beta}|^2 + \int_{D_{u,\underline{u}}} (2\omega - \mathrm{tr}\chi)u^4|\underline{\alpha}|^2\end{aligned}$$

Now explicitly compute the following

$$\int_{D_{u,\underline{u}}} \nabla_4(u^4|\underline{\alpha}|^2) + \int_{D_{u,\underline{u}}} \nabla_3(u^4|\underline{\beta}|^2) \tag{7.79}$$

$$\begin{aligned}= \int_{D_{u,\underline{u}}} \Big(& 2u^4\langle \underline{\alpha}, -\frac{1}{2}\mathrm{tr}\chi\underline{\alpha} - s(\nabla\hat{\otimes}\underline{\beta})_{ab} + 4\omega\underline{\alpha}_{ab} - 3(\underline{\hat{\chi}}_{ab}\rho - {}^*\underline{\hat{\chi}}_{ab}\sigma) + ((\zeta - 4\underline{\eta})\hat{\otimes}\underline{\beta})_{ab} \\ & + \frac{1}{2}(D_4\mathfrak{T}_{33} - D_3\mathfrak{T}_{34})\gamma_{ab} - D_3\mathfrak{T}_{ab} + \frac{1}{2}(D_a\mathfrak{T}_{3b} + D_b\mathfrak{T}_{3a})\rangle + 4\Omega^{-1}u^3|\underline{\beta}|^2 \\ & + 2u^4\langle \underline{\beta}, -2\mathrm{tr}\underline{\chi}\underline{\beta} - (\mathrm{div}\underline{\alpha})_a - 2\underline{\omega}\underline{\beta}_a + (\underline{\eta}\cdot\underline{\alpha})_a + \frac{1}{2}(D_a\mathfrak{T}_{33} - D_3\mathfrak{T}_{3a})\rangle \Big).\end{aligned}$$

Explicit calculations yield

$$\begin{aligned}\int_{H_u} u^4|\underline{\beta}|^2 + \int_{\underline{H}_{\underline{u}}} u^4|\underline{\alpha}|^2 - \int_{D_{u,\underline{u}}} (4\Omega^{-1}u^3 - 3\mathrm{tr}\underline{\chi}u^4)|\underline{\beta}|^2 = \int_{\widehat{\Sigma}_0} u^4|\underline{\alpha}|^2 + \int_{\widehat{\Sigma}_0} u^4|\underline{\beta}|^2 \\ + \mathcal{E}_{\underline{\alpha},\underline{\beta}},\end{aligned}$$

where the error term $\mathcal{E}_{\underline{\alpha},\underline{\beta}}$ satisfies

$$\begin{aligned}\mathcal{E}_{\underline{\alpha},\underline{\beta}} = \int_{D_{u,\underline{u}}} \Big(& 2u^4\langle \underline{\alpha}, 4\omega\underline{\alpha}_{ab} - 3(\underline{\hat{\chi}}_{ab}\rho - {}^*\underline{\hat{\chi}}_{ab}\sigma) + ((\zeta - 4\underline{\eta})\hat{\otimes}\underline{\beta})_{ab} \\ & - D_3\mathfrak{T}_{ab} + \frac{1}{2}(D_a\mathfrak{T}_{3b} + D_b\mathfrak{T}_{3a})\rangle \\ & + 2u^4\langle \underline{\beta}, -2\underline{\omega}\underline{\beta}_a + (\underline{\eta}\cdot\underline{\alpha})_a + \frac{1}{2}(D_a\mathfrak{T}_{33} - D_3\mathfrak{T}_{3a})\rangle \Big)\end{aligned} \tag{7.80}$$

Proposition 7.8. *The error term $\mathcal{E}_{\underline{\alpha},\underline{\beta}}$ verifies*

$$|\mathcal{E}_{\underline{\alpha},\underline{\beta}}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1). \tag{7.81}$$

Proof. Note the appearance of the special structure in the null Bianchi equations. The faster decaying connection ω appears with $\underline{\alpha}$[5]

$$|\int_{D_{u,\underline{u}}} u^4\omega|\underline{\alpha}|^2| \lesssim \sup_{\mathcal{K}} r^2|\omega|(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}') \int_{\underline{H}} u^4|\underline{\alpha}|^2 \lesssim \epsilon\mathcal{W}_0. \tag{7.82}$$

[5]There are several places where we observe that the slow decaying connection coefficient $\underline{\omega}$ appears with the good curvature components i.e., the ones that decay faster in r

The remaining terms are estimated as

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}\widehat{\chi}\rho| \lesssim \sup_{\mathcal{K}} |u|r|\widehat{\chi}|(\int_{\underline{u}_0}^{\underline{u}} |u|r^{-3}d\underline{u}^{'})(\int_{\underline{H}} u^4|\underline{\alpha}|^2 + \int_{\underline{H}} r^4|\rho|^2) \lesssim \epsilon\mathcal{W}_0, \tag{7.83}$$

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}(\eta,\underline{\eta})\underline{\beta}| \lesssim \sup_{\mathcal{K}} r^2|\eta,\underline{\eta}|(\int_{\underline{u}_0}^{\underline{u}} |u|r^{-3}d\underline{u}^{'})(\int_{\underline{H}} u^4|\underline{\alpha}|^2 + \int_{\underline{H}} u^2r^2|\underline{\beta}|^2) \lesssim \epsilon\mathcal{W}_0.$$

Now we want to estimate

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}D_3\mathfrak{T}_{AB}| \tag{7.84}$$

$$\nabla_3\mathfrak{T}_{AB} \sim \langle(\rho^F,\sigma^F), -\widehat{\nabla}\underline{\alpha}^F - tr\underline{\chi}(\rho^F,\sigma^F) + (\eta-\underline{\eta})\cdot\underline{\alpha}^F\rangle \tag{7.85}$$
$$+\langle\alpha^F,\widehat{\nabla}_3\underline{\alpha}^F\rangle + \langle\underline{\alpha}^F, -\frac{1}{2}tr\underline{\chi}\alpha^F - \widehat{\nabla}\rho^F + {}^*\widehat{\nabla}\sigma^F - 2\,{}^*\eta\sigma^F + 2\eta\rho^F + 2\underline{\omega}\alpha^F - \hat{\chi}\cdot\underline{\alpha}^F\rangle$$

Now estimate each term separately

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}\langle\rho^F,\widehat{\nabla}\underline{\alpha}^F\rangle| \tag{7.86}$$
$$\lesssim \sup_{\mathcal{K}} r^2|u|^{\frac{1}{2}}|\rho^F|(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}(\mathrm{tr}\underline{\chi}+\frac{2}{r})\rho^F\cdot\rho^F|$$
$$\lesssim \sup_{\mathcal{K}} r^2|\mathrm{tr}\underline{\chi}+\frac{2}{r}|\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{3}{2}}r^{-5}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + r^2|\rho^F|^2) \lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} u^4r^{-1}\underline{\alpha}\rho^F\cdot\rho^F| \lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{3}{2}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + r^2|\rho^F|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}(\eta-\underline{\eta})\rho^F\cdot\rho^F|$$
$$\lesssim \sup_{\mathcal{K}}(r^2|\eta,\underline{\eta}|)\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}}r^{-4}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + |u|^2|\underline{\alpha}^F|^2) \lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}\alpha^F\cdot\widehat{\nabla}_3\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{\underline{u}_0}^{\underline{u}} r^{-\frac{5}{2}}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + u^4|\widehat{\nabla}_3\underline{\alpha}^F|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}\underline{\alpha}^F\cdot\widehat{\nabla}\rho^F| \lesssim \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + r^4|\widehat{\nabla}\rho^F|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} u^4\underline{\alpha}\underline{\alpha}^F\cdot\underline{\alpha}^F\widehat{\chi}|$$
$$\lesssim \sup_{\mathcal{K}}(r^2|\widehat{\chi}|)\sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{-\frac{1}{2}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + u^2|\underline{\alpha}^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0).$$

Next we need to estimate

$$|\int_{D_{u,\underline{u}}} u^4 \underline{\alpha} \nabla \mathfrak{T}_{3A}| \tag{7.87}$$

An explicit calculations yield

$$\nabla \mathfrak{T}_{3A} \sim \widehat{\nabla} \underline{\alpha}^F \cdot (\rho^F, \sigma^F) + \underline{\alpha}^F \cdot \widehat{\nabla}(\rho^F, \sigma^F) + (\eta + \underline{\eta})(\rho^F, \sigma^F) \cdot \underline{\alpha}^F + \underline{\chi}(\rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F + \alpha^F \cdot \underline{\alpha}^F)$$

We estimate each term separately

$$|\int_{D_{u,\underline{u}}} u^4 \underline{\alpha} \widehat{\nabla} \underline{\alpha}^F \cdot (\rho^F, \sigma^F)| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1) \tag{7.88}$$

$$|\int_{D_{u,\underline{u}}} u^4 \underline{\alpha} \underline{\alpha}^F \cdot \widehat{\nabla}(\rho^F, \sigma^F)| \lesssim \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}})(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}} r^{-3} d\underline{u}^{'}) \int_{\underline{H}} (u^4|\underline{\alpha}|^2 + r^4|\widehat{\nabla}(\rho^F, \sigma^F)|^2)$$
$$\lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1),$$
$$|\int_{D_{u,\underline{u}}} u^4 \underline{\alpha} \mathrm{tr}\underline{\chi} \alpha^F \cdot \underline{\alpha}^F|$$
$$\lesssim \left(\sup_{\mathcal{K}}(r^2|\mathrm{tr}\underline{\chi} + \frac{2}{r}|) \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{\underline{u}_0}^{\underline{u}} \frac{|u|}{r^{\frac{9}{2}}} d\underline{u}^{'}) + \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{\underline{u}_0}^{\underline{u}} \frac{|u|}{r^{\frac{7}{2}}} d\underline{u}^{'}) \right) \int_{\underline{H}} (u^4|\underline{\alpha}|^2 + u^2|\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} u^4 \underline{\alpha} \widehat{\underline{\chi}} \alpha^F \cdot \underline{\alpha}^F| \lesssim \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F| \sup_{\mathcal{K}}(r|u||\widehat{\underline{\chi}}|)(\int_{\underline{u}_0}^{\underline{u}} \frac{1}{r^{\frac{7}{2}}} d\underline{u}^{'}) \int_{\underline{H}} (u^4|\underline{\alpha}|^2 + u^2|\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0 + \mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} u^4 \underline{\alpha} \widehat{\underline{\chi}} \rho^F \cdot \rho^F| \lesssim \sup_{\mathcal{K}}(r|u|\widehat{\chi}) \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{\underline{u}_0}^{\underline{u}} \frac{|u|^{\frac{1}{2}}}{r^3} d\underline{u}^{'})(\int_{\underline{H}} (u^4|\underline{\alpha}|^2 + r^2|\rho^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0 + \mathcal{Y}_0).$$

Therefore collecting all the terms together yields

$$|\int_{D_{u,\underline{u}}} u^4 \underline{\alpha} \nabla \mathfrak{T}_{3A}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1) \tag{7.89}$$

$$|\int_{D_{u,\underline{u}}} u^4 |\underline{\beta}|^2 \underline{\omega}| \lesssim \sup_{\mathcal{K}}(r|u||\underline{\omega}|)(\int_{u_0}^{u} |u^{'}|^{-1} r^{-1} du^{'}) \int_H u^4 |\underline{\beta}|^2 \lesssim \epsilon \mathcal{W}_0, \tag{7.90}$$
$$|\int_{D_{u,\underline{u}}} u^4 \underline{\beta} \underline{\alpha} \underline{\eta}| \lesssim \sup_{\mathcal{K}}(r^2|\underline{\eta}|)(\int_{\underline{u}_0}^{\underline{u}} \frac{|u|}{r^3} d\underline{u}^{'}) \int_{\underline{H}} (r^2 u^2 |\underline{\beta}|^2 + u^4 |\underline{\alpha}|^2) \lesssim \epsilon \mathcal{W}_0.$$

Now we have to estimate

$$|\int_{D_{u,\underline{u}}} u^4 \underline{\beta}(D_A \mathfrak{T}_{33} - D_3 \mathfrak{T}_{3A})| \tag{7.91}$$

Recall the following explicit calculations

$$D_A \mathfrak{T}_{33} - D_3 \mathfrak{T}_{3A} \sim \langle \underline{\alpha}^F, \widehat{\nabla} \underline{\alpha}^F \rangle - \underline{\chi} \underline{\alpha}^F \cdot (\rho^F + \sigma^F) + \underline{\eta} |\underline{\alpha}^F|^2 - 2\underline{\omega} \underline{\alpha}^F \cdot (\rho^F + \sigma^F) \tag{7.92}$$

$$-\widehat{\nabla}_3(\underline{\alpha}^F \cdot \rho^F + \underline{\alpha}^F \cdot \sigma^F)$$

$$|\int_{D_{u,\underline{u}}} u^4 \underline{\beta}\underline{\alpha}^F \cdot \widehat{\nabla}\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{\underline{u}_0}^{\underline{u}} \frac{|u|^{\frac{1}{2}}}{r^3} d\underline{u}^{'}) \int_{\underline{H}} (u^2 r^2 |\underline{\beta}|^2 + r^2 u^2 |\widehat{\nabla}\underline{\alpha}^F|^2) \quad (7.93)$$
$$\lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1),$$
$$|\int_{D_{u,\underline{u}}} u^4 \underline{\beta}\mathrm{tr}\underline{\chi}\underline{\alpha}^F \cdot \rho^F| \lesssim \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{u_0}^{u} |u^{'}|^{-\frac{1}{2}} r^{-2} du^{'}) \int_{H} (u^4 |\underline{\beta}|^2 + u^2 |\rho^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} u^4 \underline{\beta}\widehat{\nabla}_3\underline{\alpha}^F \cdot \rho^F| \lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}} r^{-3} d\underline{u}^{'}) \int_{\underline{H}} (u^2 r^2 |\underline{\beta}|^2 + u^4 |\widehat{\nabla}_3\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1),$$
$$|\int_{D_{u,\underline{u}}} u^4 \underline{\beta}(\eta - \underline{\eta})\underline{\alpha}^F \cdot \underline{\alpha}^F|$$
$$\lesssim \sup_{\mathcal{K}}(r^2|\eta, \underline{\eta}|) \sup_{\mathcal{K}} |u|^{\frac{3}{2}} r |\underline{\alpha}^F| (\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}} r^{-4} d\underline{u}^{'}) \int_{\underline{H}} (u^2 r^2 |\underline{\beta}|^2 + u^2 |\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0 + \mathcal{Y}_0).$$

Collecting all the terms together, we obtain

$$|\int_{D_{u,\underline{u}}} u^4 \underline{\beta}(D_A \mathfrak{T}_{33} - D_3 \mathfrak{T}_{3A})| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1) \quad (7.94)$$

and therefore

$$|\mathcal{E}_{\underline{\alpha},\underline{\beta}}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1). \quad (7.95)$$

□

Lemma 7.3. *The following estimate hold in the exterior region*

$$\int_{H_u} u^4 |\underline{\beta}|^2 + \int_{\underline{H}_{\underline{u}}} u^4 |\underline{\alpha}|^2 \lesssim \int_{\widehat{\Sigma}_0} u^4 |\underline{\alpha}|^2 + \int_{\widehat{\Sigma}_0} u^4 |\underline{\beta}|^2 + \epsilon(\mathbf{W} + \mathbf{Y}) \quad (7.96)$$

Proof. The proof is a consequence of the proposition (6.8) and the fact that $4\Omega^{-1}u^{-1} - 3\mathrm{tr}\underline{\chi} < 0$ modulo $\frac{C\epsilon}{r^2}$ terms [6]. □

Now consider the pair $||r^2\beta||_H$ and $||r^2(\rho, \sigma)||_{\underline{H}_{\underline{u}}}$

$$\int_{D_{u,\underline{u}}} \nabla_3(r^4|\beta|^2) + \int_{D_{u,\underline{u}}} \nabla_4(r^4(\rho^2 + \sigma^2)) = \int_H r^4|\beta|^2 + \int_{\underline{H}} r^4(\rho^2 + \sigma^2) \quad (7.97)$$
$$- \int_{\widehat{\Sigma}_0} r^4|\beta|^2 - \int_{\widehat{\Sigma}_0} r^4(\rho^2 + \sigma^2) + \int_{D_{u,\underline{u}}} (2\omega - \mathrm{tr}\chi) r^4(\rho^2 + \sigma^2) + \int_{D_{u,\underline{u}}} (2\underline{\omega} - \mathrm{tr}\underline{\chi}) r^4 |\beta|^2$$

Explicitly compute the following

$$\int_{D_{u,\underline{u}}} \nabla_3(r^4|\beta|^2) + \int_{D_{u,\underline{u}}} \nabla_4(r^4(\rho^2 + \sigma^2)) \quad (7.98)$$

[6] Notice that one has $|2r - (\underline{u} - u)| \lesssim \mathbf{W} + \mathbf{Y}$ which can be proven by the exact method of [7]

$$= \int_{D_{u,\underline{u}}} 2\Omega^{-1}r^4\overline{\Omega\mathrm{tr}\underline{\chi}}|\beta|^2 + 2\int_{D_{u,\underline{u}}} r^4\langle\beta, \nabla_3\beta\rangle + \int_{D_{u,\underline{u}}} 2\Omega^{-1}r^4\overline{\Omega\mathrm{tr}\chi}(\rho^2+\sigma^2)$$
$$+2\int_{D_{u,\underline{u}}} r^4\left(\langle\rho,\nabla_4\rho\rangle + \langle\sigma,\nabla_4\sigma\rangle\right)$$
$$= \int_{D_{u,\underline{u}}} 2\Omega^{-1}r^4\overline{\Omega\mathrm{tr}\underline{\chi}}|\beta|^2 + \int_{D_{u,\underline{u}}} 2\Omega^{-1}r^4\overline{\Omega\mathrm{tr}\chi}(\rho^2+\sigma^2)$$
$$+2\int_{D_{u,\underline{u}}}\langle\beta, -tr\underline{\chi}\beta_a + \nabla_a\rho + {}^*\nabla_a\sigma + 2\underline{\omega}\beta_a + 2(\hat{\chi}\cdot\underline{\beta})_a + 3(\eta\rho + {}^*\eta\sigma)_a$$
$$+\frac{1}{2}(D_a\mathfrak{T}_{34} - D_4\mathfrak{T}_{3a})\rangle + 2\int_{D_{u,\underline{u}}} r^4\langle\rho, -\frac{3}{2}\mathrm{tr}\chi\rho + \mathrm{div}\beta - \frac{1}{2}\hat{\underline{\chi}}\cdot\alpha + \zeta\cdot\beta + 2\underline{\eta}\cdot\beta$$
$$-\frac{1}{4}(D_3\mathfrak{T}_{44} - D_4\mathfrak{T}_{34})\rangle + 2\int_{D_{u,\underline{u}}} r^4\langle\sigma, -\frac{3}{2}\mathrm{tr}\chi\sigma - \mathrm{div}\,{}^*\beta + \frac{1}{2}\hat{\underline{\chi}}\cdot{}^*\alpha - \zeta\cdot$$
$${}^*\beta - 2\underline{\eta}\cdot{}^*\beta - \frac{1}{4}(D_\mu\mathfrak{T}_{4\nu} - D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34}\rangle$$

which after integration by parts simplifies to

$$\int_H r^4|\beta|^2 + \int_{\underline{H}} r^4(\rho^2+\sigma^2) - \int_{D_{u,\underline{u}}}(2\Omega^{-1}r^4\overline{\Omega\mathrm{tr}\underline{\chi}} - \mathrm{tr}\underline{\chi})|\beta|^2 \tag{7.99}$$
$$+2\int_{D_{u,\underline{u}}} r^4(\mathrm{tr}\chi - \Omega^{-1}\overline{\Omega\mathrm{tr}\chi}) = \int_{\widehat{\Sigma}_0} r^4|\beta|^2 + \int_{\widehat{\Sigma}_0} r^4(\rho^2+\sigma^2) + \mathcal{E}_{\beta,\rho,\sigma},$$

where $\mathcal{E}_{\beta,\rho,\sigma}$ verifies the following schematic expression

$$\mathcal{E}_{\beta,\rho,\sigma} \sim 2\int_{D_{u,\underline{u}}} r^4\langle\beta, 2\underline{\omega}\beta_a + 2(\hat{\chi}\cdot\underline{\beta})_a + 3(\eta\rho + {}^*\eta\sigma)_a + \frac{1}{2}(D_a\mathfrak{T}_{34} - D_4\mathfrak{T}_{3a})\rangle$$
$$+2\int_{D_{u,\underline{u}}} r^4\langle\rho, -\frac{1}{2}\hat{\underline{\chi}}\cdot\alpha + \zeta\cdot\beta + 2\underline{\eta}\cdot\beta - \frac{1}{4}(D_3\mathfrak{T}_{44} - D_4\mathfrak{T}_{34})\rangle$$
$$+2\int_{D_{u,\underline{u}}} r^4\langle\sigma, \frac{1}{2}\hat{\underline{\chi}}\cdot{}^*\alpha - \zeta\cdot{}^*\beta - 2\underline{\eta}\cdot{}^*\beta - \frac{1}{4}(D_\mu\mathfrak{T}_{4\nu} - D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34}\rangle.$$

Proposition 7.9. *The error term $\mathcal{E}_{\beta,\rho,\sigma}$ verifies the following estimate*

$$|\mathcal{E}_{\beta,\rho,\sigma}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1) \tag{7.100}$$

Proof. Once again, we estimate the most important terms. These include the coupling terms.

$$|\int_{D_{u,\underline{u}}} r^4\beta\underline{\omega}\beta| \lesssim \sup_{\mathcal{K}}(r|u||\underline{\omega}|)(\int_{u_0}^{u}|u'|^{-1}r^{-1}du')\int_H r^4|\beta|^2 \lesssim \epsilon\mathcal{W}_0, \tag{7.101}$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\widehat{\chi}\underline{\beta}| \lesssim \sup_{\mathcal{K}}(r^2|\widehat{\chi}|)(\int_{u_0}^{u}|u'|^{-2}du')\int_H (r^4|\beta|^2 + u^4|\underline{\beta}|^2) \lesssim \epsilon\mathcal{W}_0,$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\eta\rho| \lesssim \sup_{\mathcal{K}}(r^2|\eta|)(\int_{u_0}^{u}|u'|^{-1}r^{-1}du')\int_H (r^4|\beta|^2 + r^2u^2|\rho|^2) \lesssim \epsilon\mathcal{W}_0,$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\eta\sigma| \lesssim \sup_{\mathcal{K}}(r^2|\eta|)(\int_{u_0}^{u}|u'|^{-1}r^{-1}du')\int_H (r^4|\beta|^2 + r^2u^2|\sigma|^2) \lesssim \epsilon\mathcal{W}_0.$$

Now we have to estimate the coupling terms

$$|\int_{D_{u,\underline{u}}} r^4\langle\beta, D_A\mathfrak{T}_{34} - D_4\mathfrak{T}_{A3}\rangle| \tag{7.102}$$

Explicit computations yield

$$D_A\mathfrak{T}_{34} - D_4\mathfrak{T}_{3A} \sim (\langle\widehat{\nabla}\rho^F,\rho^F\rangle + \langle\widehat{\nabla}\sigma^F,\sigma^F\rangle) - \underline{\chi}\alpha^F(\sigma^F+\rho^F) - \widehat{\nabla}_4\underline{\alpha}^F\cdot(\rho^F+\sigma^F)$$
$$-\underline{\alpha}^F\cdot(\widehat{\nabla}_4\rho^F + \widehat{\nabla}_4\sigma^F)$$

and

$$\widehat{\nabla}_4\underline{\alpha}^F + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}^F = -\widehat{\nabla}\rho^F - {}^*\widehat{\nabla}\sigma^F - 2\,{}^*\underline{\eta}\sigma^F - 2\underline{\eta}\rho^F + 2\omega\underline{\alpha}^F - \hat{\underline{\chi}}\cdot\alpha^F, \tag{7.103}$$
$$\widehat{\nabla}_4\rho^F = -\hat{\mathrm{div}}\alpha^F - \mathrm{tr}\chi\rho^F - (\eta-\underline{\eta})\cdot\alpha^F \tag{7.104}$$
$$\widehat{\nabla}_4\sigma^F = -\hat{\mathrm{curl}}\alpha^F - \mathrm{tr}\chi\sigma^F + (\eta-\underline{\eta})\cdot{}^*\alpha^F \tag{7.105}$$

We estimate term by term as follows

$$|\int_{D_{u,\underline{u}}} r^4\beta\widehat{\nabla}\rho^F\cdot\rho^F| \lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\rho^F|)(\int_{u_0}^u |u'|^{-\frac{3}{2}}r^{-1}du')\int_H(r^4|\beta|^2 + r^2u^2|\widehat{\nabla}\rho^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1),$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\widehat{\nabla}\rho^F\cdot\sigma^F| \lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\sigma^F|)(\int_{u_0}^u |u'|^{-\frac{3}{2}}r^{-1}du')\int_H(r^4|\beta|^2 + r^2u^2|\widehat{\nabla}\sigma^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1),$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\mathrm{tr}\underline{\chi}\alpha^F\cdot\rho^F| \lesssim \sup_{\mathcal{K}}(r^2|\mathrm{tr}\underline{\chi}+\frac{2}{r}|)\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^u r^{-3}|u'|^{-\frac{1}{2}}du')\int_H(r^4|\beta|^2 + r^2|\alpha^F|^2)$$
$$+\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^u r^{-2}|u'|^{-\frac{1}{2}}du')\int_H(r^4|\beta|^2 + r^2|\alpha^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\widehat{\underline{\chi}}\alpha^F\cdot\rho^F| \lesssim \sup_{\mathcal{K}}(r|u|\widehat{\underline{\chi}}|)\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^u r^{-2}|u'|^{-\frac{3}{2}}du')\int_H(r^4|\beta|^2 + r^2|\alpha^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\mathrm{tr}\underline{\chi}\underline{\alpha}^F\cdot\rho^F|$$
$$\lesssim \left(\sup_{\mathcal{K}}(r^2|\mathrm{tr}\underline{\chi}+\frac{2}{r}|)\sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)\int_{u_0}^u r^{-1}|u'|^{-\frac{5}{2}}du' + \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)\int_{u_0}^u |u'|^{-\frac{5}{2}}du')\right)$$
$$(\int_H r^4|\beta|^2 + u^2|\rho^F|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\underline{\eta}\rho^F\cdot\rho^F| \lesssim \sup_{\mathcal{K}}(r^2|\underline{\eta}|)\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^u r^{-2}|u'|^{-\frac{3}{2}}du')\int_H(r^4|\beta|^2 + u^2|\rho^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\omega\rho^F\cdot\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}}(r^2|\omega|)\sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{u_0}^u r^{-1}|u'|^{-\frac{5}{2}}du')\int_H(r^4|\beta|^2 + u^2|\rho^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\widehat{\underline{\chi}}\alpha^F\cdot\rho^F| \lesssim \sup_{\mathcal{K}}(r|u||\widehat{\underline{\chi}}|)\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^u r^{-2}|u'|^{-\frac{3}{2}}du')\int_H(r^4|\beta|^2 + r^2|\alpha^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} r^4\beta\underline{\alpha}^F\cdot\widehat{\nabla}\alpha^F| \lesssim \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{u_0}^u r^{-1}|u'|^{-\frac{3}{2}}du')\int_H(r^4|\beta|^2 + r^4|\widehat{\nabla}\alpha^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^4\beta\underline{\alpha}^F \mathrm{tr}\chi\rho^F| \lesssim \left(\sup_{\mathcal{K}} r^2|\mathrm{tr}\chi + \frac{2}{r}|)(\int_{u_0}^{u} r^{-1}|u^{'}|^{-\frac{5}{2}}du^{'}) + (\int_{u_0}^{u} |u^{'}|^{-\frac{5}{2}}du^{'})\right) \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)$$

$$\int_H (r^4|\beta|^2 + u^2|\rho^F|^2) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\beta\underline{\alpha}^F(\eta,\underline{\eta})\alpha^F| \lesssim \sup_{\mathcal{K}}(r^2|\eta,\underline{\eta}|) \sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{u_0}^{u} r^{-2}|u^{'}|^{-\frac{3}{2}}du^{'}) \int_H (r^4|\beta|^2 + r^2|\alpha^F|^2)$$

$$\lesssim \epsilon^2(\mathcal{W}_0 + \mathcal{Y}_0).$$

Collecting all the terms together yields

$$|\int_{D_{u,\underline{u}}} r^4\langle\beta, D_A\mathfrak{T}_{34} - D_4\mathfrak{T}_{A3}\rangle| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1). \tag{7.106}$$

The remaining terms verify

$$|\int_{D_{u,\underline{u}}} r^4\rho\underline{\widehat{\chi}}\alpha| \lesssim \sup_{\mathcal{K}}(r|u||\underline{\widehat{\chi}}|)(\int_{u_0}^{u} |u^{'}|^{-2}du^{'}) \int_H (u^2r^2|\rho|^2 + r^4|\alpha|^2) \lesssim \epsilon\mathcal{W}_0, \tag{7.107}$$

$$|\int_{D_{u,\underline{u}}} r^4\rho(\eta,\underline{\eta})\beta| \lesssim \sup_{\mathcal{K}}(r^2|\eta,\underline{\eta}|)(\int_{u_0}^{u} r^{-1}|u^{'}|^{-1}du^{'}) \int_H (u^2r^2\rho^2 + r^4|\beta|^2) \lesssim \epsilon\mathcal{W}_0.$$

Now we ought to estimate the coupling term

$$|\int_{D_{u,\underline{u}}} r^4\rho(D_3\mathfrak{T}_{44} - D_4\mathfrak{T}_{43})| \tag{7.108}$$

Explicit calculations yield

$$\begin{aligned} D_3\mathfrak{T}_{44} - D_4\mathfrak{T}_{34} \sim \langle\alpha^F, \widehat{\nabla}_3\alpha^F\rangle + \underline{\omega}|\alpha^F|^2 + \eta_A(\alpha^F\cdot\sigma^F + \alpha^F\cdot\rho^F) + \langle\rho^F, \widehat{\nabla}_4\rho^F\rangle \\ + \langle\sigma^F, \widehat{\nabla}_4\sigma^F\rangle + \underline{\eta}(\rho^F\underline{\alpha}^F + \rho^F\cdot\sigma^F) \end{aligned} \tag{7.109}$$

and note

$$\widehat{\nabla}_3\alpha^F + \frac{1}{2}tr\underline{\chi}\alpha^F = -\widehat{\nabla}\rho^F + {}^*\widehat{\nabla}\sigma^F - 2\,{}^*\eta\sigma^F + 2\eta\rho^F + 2\underline{\omega}\alpha^F - \hat{\chi}\cdot\underline{\alpha}^F, \tag{7.110}$$

$$\widehat{\nabla}_4\rho^F = -\hat{\mathrm{div}}\alpha^F - \mathrm{tr}\chi\rho^F - (\eta - \underline{\eta})\cdot\alpha^F \tag{7.111}$$

$$\widehat{\nabla}_4\sigma^F = -\hat{\mathrm{curl}}\alpha^F - \mathrm{tr}\chi\sigma^F + (\eta - \underline{\eta})\cdot{}^*\alpha^F \tag{7.112}$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\mathrm{tr}\underline{\chi}\alpha^F\cdot\alpha^F| \lesssim \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)\left(\sup_{\mathcal{K}}(r^2|\mathrm{tr}\underline{\chi} + \frac{2}{r}|)(\int_{u_0}^{u} r^{-\frac{5}{2}}|u^{'}|^{-1}du^{'}) + (\int_{u_0}^{u} r^{-\frac{3}{2}}|u^{'}|^{-1}du^{'})\right)$$

$$\int_H (u^2r^2\rho^2 + r^2|\alpha^F|^2) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\alpha^F\cdot\widehat{\nabla}\rho^F| \lesssim \sup_{\mathcal{K}} r^{\frac{5}{2}}|\alpha^F|(\int_{u_0}^{u} r^{-\frac{1}{2}}|u^{'}|^{-2}du^{'}) \int_H (u^2r^2\rho^2 + u^2r^2|\widehat{\nabla}\rho^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\alpha^F\eta(\rho^F,\sigma^F)| \lesssim \sup_{\mathcal{K}}(r^2|\eta|) \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|(\rho^F,\sigma^F)|)(\int_{u_0}^{u} r^{-2}|u^{'}|^{-\frac{3}{2}}du^{'}) \int_H (u^2r^2\rho^2 + r^2|\alpha^F|^2)$$

$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\underline{\omega}\alpha^F\cdot\alpha^F| \lesssim \sup_{\mathcal{K}} r^{\frac{5}{2}}|\alpha^F| \sup_{\mathcal{K}} r|u||\underline{\omega}|(\int_{u_0}^{u} r^{-\frac{3}{2}}|u^{'}|^{-2}du^{'})\int_H (u^2r^2\rho^2+r^2|\alpha^F|^2)$$

$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\widehat{\chi}\alpha^F\cdot\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}} r^2|\widehat{\chi}|\sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)\int_H(u^2r^2\rho^2+r^2|\alpha^F|^2) \lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\rho^F\cdot\widehat{\nabla}\alpha^F| \lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^{u} r^{-1}|u^{'}|^{-\frac{3}{2}}du^{'})\int_H(u^2r^2\rho^2+r^4|\widehat{\nabla}\alpha^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\rho^F\cdot\rho^F\mathrm{tr}\chi| \lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)\left(\sup_{\mathcal{K}}(r^2|\mathrm{tr}\chi-\frac{2}{r}|)(\int_{u_0}^{u} r^{-1}|u^{'}|^{-\frac{5}{2}}du^{'})+(\int_{u_0}^{u}|u^{'}|^{-\frac{5}{2}}du^{'})\right)$$

$$\int_H(u^2r^2\rho^2+u^2|\alpha^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\rho^F\cdot\underline{\alpha}^F\underline{\eta}| \lesssim \sup_{\mathcal{K}} r^2|\underline{\eta}|\sup_{\mathcal{K}}(r|u|^{\frac{3}{2}}|\underline{\alpha}^F|)(\int_{u_0}^{u}|u^{'}|^{-\frac{7}{2}}du^{'})\int_H(u^2r^2\rho^2+u^2|\rho^F|^2)$$

$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^4\rho\rho^F\cdot\sigma^F\underline{\eta}| \lesssim \sup_{\mathcal{K}}(r^2|\underline{\eta}|)\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\sigma^F|)(\int_{u_0}^{u} r^{-1}|u^{'}|^{-\frac{5}{2}}du^{'})\int_H(u^2r^2\rho^2+u^2|\rho^F|^2)$$

$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0).$$

For σ, the self-gravitation is of the same form as ρ and therefore verifies a similar estimate.

$$|\int_{D_{u,\underline{u}}} r^4\sigma(D_\mu\mathfrak{T}_{4\nu}-D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34}| \tag{7.113}$$

Explicit computations yield

$$(D_\mu\mathfrak{T}_{4\nu}-D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34} \sim \widehat{\nabla}(\alpha^F\cdot\rho^F-\alpha^F\cdot\sigma^F)-\chi(\rho^F\cdot\rho^F+\alpha^F\cdot\underline{\alpha}^F+\sigma^F\cdot\sigma^F) \tag{7.114}$$
$$+(\eta-\underline{\eta})\alpha^F\cdot(\rho^F+\sigma^F).$$

We estimate each term separately

$$|\int_{D_{u,\underline{u}}} r^4\sigma\widehat{\nabla}\alpha^F\cdot\rho^F| \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1) \tag{7.115}$$

$$|\int_{D_{u,\underline{u}}} r^4\sigma\alpha^F\cdot\widehat{\nabla}\rho^F| \lesssim \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{u_0}^{u} r^{-\frac{1}{2}}|u^{'}|^{-2}du^{'})\int_H(u^2r^2\sigma^2+u^2r^2|\widehat{\nabla}\rho^F|^2) \tag{7.116}$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1)$$

$$|\int_{D_{u,\underline{u}}} r^4\sigma\mathrm{tr}\chi\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0), \tag{7.117}$$

$$|\int_{D_{u,\underline{u}}} r^4\sigma\widehat{\chi}\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0) \tag{7.118}$$

Collecting all the terms together yields

$$|\int_{D_{u,\underline{u}}} r^4\sigma(D_\mu\mathfrak{T}_{4\nu} - D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1) \tag{7.119}$$

and therefore

$$|\mathcal{E}_{\beta,\rho,\sigma}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{Y}_0 + \mathcal{Y}_1). \tag{7.120}$$

□

Lemma 7.4. *The following estimate holds in the exterior domain*

$$\int_H r^4|\beta|^2 + \int_{\underline{H}} r^4(\rho^2+\sigma^2) \lesssim \int_{\widehat{\Sigma}_0} r^4|\beta|^2 + \int_{\widehat{\Sigma}_0} r^4(\rho^2+\sigma^2) + \epsilon(\mathcal{W}+\mathcal{Y}). \tag{7.121}$$

Proof. A consequence of proposition (6.9) and $2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}} - \mathrm{tr}\underline{\chi} = -\frac{2}{r}) + \frac{C\epsilon}{r^2}$ as well as $(\mathrm{tr}\chi - \Omega^{-1}\overline{\Omega\mathrm{tr}\chi}) \lesssim \frac{\epsilon}{r^2}$. □

Now consider the last remaining pairs $||r^2u^2\underline{\beta}||_{\underline{H}}$ and $||r^2u^2(\rho,\sigma)||_H$ and write

$$\begin{aligned}\int_{D_{u,\underline{u}}} \nabla_4(r^2u^2|\underline{\beta}|^2) + \int_{D_{u,\underline{u}}} \nabla_3(r^2u^2(\rho^2+\sigma^2)) &= \int_{\underline{H}} r^2u^2|\underline{\beta}|^2 + \int_H r^2u^2(\rho^2+\sigma^2)\\ &- \int_{\widehat{\Sigma}_0} r^2u^2|\underline{\beta}|^2 - \int_{\widehat{\Sigma}_0} r^2u^2(\rho^2+\sigma^2) + \int_{D_{u,\underline{u}}} (2\omega - \mathrm{tr}\chi)r^2u^2|\underline{\beta}|^2\\ &+ \int_{D_{u,\underline{u}}} (2\underline{\omega} - \mathrm{tr}\underline{\chi})r^2u^2(\rho^2+\sigma^2).\end{aligned}$$

Explicit calculations yield

$$\begin{aligned}&\int_{D_{u,\underline{u}}} \nabla_4(r^2u^2|\underline{\beta}|^2) + \int_{D_{u,\underline{u}}} \nabla_3(r^2u^2(\rho^2+\sigma^2)) \qquad (7.122)\\ &= \int_{D_{u,\underline{u}}} \left(2r\Omega^{-1}\frac{dr}{d\underline{u}}u^2|\underline{\beta}|^2 + 2r^2u^2\langle\underline{\beta},\nabla_4\underline{\beta}\rangle\right) + \int_{D_{u,\underline{u}}} \left(2r\Omega^{-1}u^2\frac{dr}{du} + 2r^2u\Omega^{-1}\right)(\rho^2+\sigma^2)\\ &\quad + \int_{D_{u,\underline{u}}} 2r^2u^2(\langle\rho,\nabla_3\rho\rangle + \langle\sigma,\nabla_3\sigma\rangle)\\ &= \int_{D_{u,\underline{u}}} (r^2u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\underline{\beta}|^2 + \left(r^2u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}} + 2r^2u\Omega^{-1}\right)(\rho^2+\sigma^2))\\ &\quad + \int_{D_{u,\underline{u}}} 2r^2u^2\langle\underline{\beta}, -\mathrm{tr}\chi\underline{\beta}_a - \nabla_a\rho + {}^*\nabla_a\sigma + 2\omega\underline{\beta}_a + 2(\hat{\underline{\chi}}\cdot\beta)_a - 3(\underline{\eta}\rho - {}^*\underline{\eta}\sigma)_a\\ &\quad -\frac{1}{2}(D_a\mathfrak{T}_{43} - D_3\mathfrak{T}_{4a})\rangle + \int_{D_{u,\underline{u}}} 2r^2u^2\langle\rho, -\frac{3}{2}tr\underline{\chi}\rho - \mathrm{div}\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} + \zeta\cdot\underline{\beta} - 2\eta\cdot\underline{\beta}\\ &\quad +\frac{1}{4}(D_3\mathfrak{T}_{34} - D_4\mathfrak{T}_{33})\rangle\end{aligned}$$

Explicit calculations yield

$$\int_{\underline{H}} r^2u^2|\underline{\beta}|^2 + \int_H r^2u^2(\rho^2+\sigma^2) + \int_{D_{u,\underline{u}}} r^2u^2(\mathrm{tr}\chi - \Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}})|\underline{\beta}|^2 \tag{7.123}$$

$$+\int_{D_{u,\underline{u}}}(2r^2u^2\mathrm{tr}\underline{\chi}-2\Omega^{-1}r^2u-r^2u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}})(\rho^2+\sigma^2)$$
$$=\int_{\widehat{\Sigma}_0}r^2u^2|\underline{\beta}|^2+\int_{\widehat{\Sigma}_0}r^2u^2(\rho^2+\sigma^2)+\mathcal{E}_{\underline{\beta},\rho,\sigma}$$

where the error term $\mathcal{E}_{\underline{\beta},\rho,\sigma}$ reads

$$\mathcal{E}_{\underline{\beta},\rho,\sigma}\sim\int_{D_{u,\underline{u}}}2r^2u^2\langle\underline{\beta},2\omega\underline{\beta}_a+2(\underline{\hat{\chi}}\cdot\beta)_a-3(\underline{\eta}\rho-\ {}^*\underline{\eta}\sigma)_a$$
$$-\frac{1}{2}(D_a\mathfrak{T}_{43}-D_3\mathfrak{T}_{4a})\rangle+\int_{D_{u,\underline{u}}}2r^2u^2\langle\rho,-\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}+\zeta\cdot\underline{\beta}-2\eta\cdot\underline{\beta}$$
$$+\frac{1}{4}(D_3\mathfrak{T}_{34}-D_4\mathfrak{T}_{33})\rangle+\int_{D_{u,\underline{u}}}2r^2u^2\langle\sigma,\frac{1}{2}\hat{\chi}\cdot\ {}^*\underline{\alpha}-\zeta\cdot\ {}^*\underline{\beta}-2\eta\cdot\ {}^*\underline{\beta}$$
$$+\frac{1}{4}(D_\mu\mathfrak{T}_{3\nu}-D_\nu\mathfrak{T}_{3\mu})\epsilon^{\mu\nu}{}_{34}\rangle.$$

Proposition 7.10. *The error term $\mathcal{E}_{\underline{\beta},\rho,\sigma}$ verifies the following estimates*

$$|\mathcal{E}_{\underline{\beta},\rho,\sigma}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{Y}_0+\mathcal{Y}_1)\tag{7.124}$$

Proof.

$$|\int_{D_{u,\underline{u}}}u^2r^2\underline{\beta}\omega\underline{\beta}|\lesssim\sup_{\mathcal{K}}r^2|\omega|(\int_{\underline{u}_0}^{\underline{u}}r^{-2}d\underline{u}')\int_{\underline{H}}u^2r^2|\underline{\beta}|^2\lesssim\epsilon\mathcal{W}_0,\tag{7.125}$$
$$|\int_{D_{u,\underline{u}}}u^2r^2\underline{\beta}\underline{\widehat{\chi}}\beta|\lesssim\sup_{\mathcal{K}}(|u|r|\underline{\widehat{\chi}}|)(\int_{\underline{u}_0}^{\underline{u}}r^{-2}d\underline{u}')\int_{\underline{H}}(u^2r^2|\underline{\beta}|^2+r^4|\beta|^2)\lesssim\epsilon\mathcal{W}_0,\tag{7.126}$$
$$|\int_{D_{u,\underline{u}}}u^2r^2\underline{\beta}\underline{\eta}(\rho,\sigma)|\lesssim\sup_{\mathcal{K}}(r^2|\eta|)(\int_{u_0}^{u}|u'|^{-2}du')\int_{H}(u^4|\underline{\beta}|^2+u^4|\rho,\sigma|^2)\lesssim\epsilon\mathcal{W}_0.\tag{7.127}$$

Now we ought to estimate the coupling terms

$$|\int_{D_{u,\underline{u}}}u^2r^2\langle\underline{\beta},\nabla_A\mathfrak{T}_{43}-\nabla_3\mathfrak{T}_{4A}\rangle|.\tag{7.128}$$

Explicit calculations yield

$$D_A\mathfrak{T}_{43}-D_3\mathfrak{T}_{4A}\sim\langle\rho^F,\widehat{\nabla}\rho^F\rangle+\langle\sigma^F,\widehat{\nabla}\sigma^F\rangle-\underline{\chi}\alpha^F(\rho^F+\sigma^F)+\chi\underline{\alpha}^F(\rho^F+\sigma^F).\tag{7.129}$$

We estimate each term separately

$$|\int_{D_{u,\underline{u}}}u^2r^2\underline{\beta}\rho^F\cdot\widehat{\nabla}\rho^F|\lesssim\sup_{\mathcal{K}}r^2|u|^{\frac{1}{2}}|\rho^F|(\int_{u_0}^{u}r^{-1}|u|^{-\frac{3}{2}})\int_{H}(u^4|\underline{\beta}|^2+u^2r^2|\widehat{\nabla}\rho^F|^2)\tag{7.130}$$
$$\lesssim\epsilon(\mathcal{W}_0+\mathcal{Y}_1)$$

Similarly

$$|\int_{D_{u,\underline{u}}}u^2r^2\underline{\beta}\sigma^F\cdot\widehat{\nabla}\sigma^F|\lesssim\sup_{\mathcal{K}}r^2|u|^{\frac{1}{2}}|\sigma^F|(\int_{u_0}^{u}r^{-1}|u|^{-\frac{3}{2}})\int_{H}(u^4|\underline{\beta}|^2+u^2r^2|\widehat{\nabla}\sigma^F|^2)\tag{7.131}$$
$$\lesssim\epsilon(\mathcal{W}_0+\mathcal{Y}_1)$$

$$|\int_{D_{u,\underline{u}}} u^2r^2\underline{\beta}\mathrm{tr}\underline{\chi}\alpha^F\cdot(\rho^F,\sigma^F)| \lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F,\sigma^F|)\sup_{\mathcal{K}}(r^2|\mathrm{tr}\underline{\chi}+\frac{2}{r}|)(\int_{u_0}^{u} r^{-3}|u^{'}|^{-\frac{1}{2}}du^{'})$$
$$\int_H(u^4|\beta|^2+r^2|\alpha^F|^2)+\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F,\sigma^F|)(\int_{u_0}^{u} r^{-2}|u^{'}|^{-\frac{1}{2}}du^{'})\int_H(u^4|\beta|^2+r^2|\alpha^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} u^2r^2\underline{\beta\widehat{\chi}}\alpha^F\cdot(\rho^F,\sigma^F)| \tag{7.132}$$
$$\lesssim \sup_{\mathcal{K}} r^2|u|^{\frac{1}{2}}|\rho^F,\sigma^F|\sup_{\mathcal{K}}(r|u||\widehat{\chi}|)(\int_{u_0}^{u} r^{-2}|u^{'}|^{-\frac{3}{2}}du^{'})\int_H(u^4|\underline{\beta}|^2+r^2|\alpha^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} u^2r^2\underline{\beta}\mathrm{tr}\chi\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| \tag{7.133}$$
$$\lesssim \sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r|\underline{\alpha}^F|)\left(\sup_{\mathcal{K}} r^2|\mathrm{tr}\chi-\frac{2}{r}|(\int_{u_0}^{u} r^{-1}|u^{'}|^{-\frac{5}{2}}du^{'})+(\int_{u_0}^{u}|u^{'}|^{-\frac{5}{2}}du^{'})\right)$$
$$\int_H(u^4|\underline{\beta}|^2+u^2|\rho^F,\sigma^F|^2)\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0),$$
$$|\int_{D_{u,\underline{u}}} u^2r^2\underline{\beta}\widehat{\chi}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| \lesssim \sup_{\mathcal{K}}(r^2|\widehat{\chi}|)\sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r|\underline{\alpha}^F|)\int_H(u^4|\underline{\beta}|^2+u^2|\rho^F,\sigma^F|^2) \tag{7.134}$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0).$$

Collecting all the terms yields

$$|\int_{D_{u,\underline{u}}} u^2r^2\langle\underline{\beta},\nabla_A\mathfrak{T}_{43}-\nabla_3\mathfrak{T}_{4A}\rangle| \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0+\mathcal{Y}_1) \tag{7.135}$$

Now we need to estimate

$$|\int_{D_{u,\underline{u}}} u^2r^2\rho(D_3\mathfrak{T}_{43}-D_4\mathfrak{T}_{33})| \tag{7.136}$$

Explicit calculation yields

$$D_3\mathfrak{T}_{43}-D_4\mathfrak{T}_{33}\sim 2\langle\rho^F,-\hat{\mathrm{div}}\underline{\alpha}^F+tr\underline{\chi}\rho^F+(\eta-\underline{\eta})\underline{\alpha}^F\rangle+2\langle\sigma_F,-\hat{\mathrm{curl}}\underline{\alpha}^F-tr\underline{\chi}\sigma^F$$
$$+(\eta-\underline{\eta})\,{}^*\underline{\alpha}^F\rangle-2\eta(\rho^F\underline{\alpha}^F+\rho^F\sigma^F)-\langle\underline{\alpha}^F,-\frac{1}{2}\mathrm{tr}\chi\underline{\alpha}^F-\widehat{\nabla}\rho^F-\,{}^*\widehat{\nabla}\sigma^F-2\,{}^*\underline{\eta}\cdot\sigma^F$$
$$-2\underline{\eta}\cdot\rho^F+2\omega\underline{\alpha}^F-\underline{\hat{\chi}}\cdot\alpha^F\rangle+2\omega|\underline{\alpha}^F|^2-4\underline{\eta}_a(\underline{\alpha}^F\cdot\rho^F-\underline{\alpha}^F\cdot\sigma^F)$$

We estimate each term separately

$$|\int_{D_{u,\underline{u}}} u^2r^2\rho\rho^F\cdot\widehat{\nabla}\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\rho^F|)(\int_{\underline{u}_0}^{\underline{u}}|u|^{\frac{1}{2}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}(r^4\rho^2+u^2r^2|\widehat{\nabla}\rho^F|^2) \tag{7.137}$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1)$$

$$|\int_{D_{u,\underline{u}}} u^2r^2\rho\underline{\alpha}^F\cdot\widehat{\nabla}\rho^F| \lesssim \sup_{\mathcal{K}}|u|^{\frac{3}{2}}r|\underline{\alpha}^F|(\int_{u_0}^{u}|u^{'}|^{-\frac{3}{2}}r^{-1}du^{'})\int_H(u^2r^2\rho^2+u^2r^2|\widehat{\nabla}\rho^F|^2) \tag{7.138}$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1)$$

$$|\int_{D_{u,\underline{u}}} u^2r^2\rho\omega|\underline{\alpha}^F|^2| \lesssim \sup_{\mathcal{K}}(r^2|\omega|)\sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r|\underline{\alpha}^F|)(\int_{\underline{u}_0}^{\underline{u}} r^{-3}|u|^{-\frac{1}{2}}d\underline{u}^{'})\int_{\underline{H}}(r^4\rho^2+u^2|\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_0+\mathcal{Y}_0)$$

Similar estimates hold for σ^F and therefore we omit writing it here. The next important term to estimate is the following

$$|\int_{D_{u,\underline{u}}} u^2r^2\sigma(D_\mu\mathfrak{T}_{3\nu}-D_\nu\mathfrak{T}_{3\mu})\epsilon^{\mu\nu}{}_{34}| \tag{7.139}$$

Explicit calculations yield

$$(D_\mu\mathfrak{T}_{3\nu}-D_\nu\mathfrak{T}_{3\mu})\epsilon^{\mu\nu}{}_{34} \sim \widehat{\nabla}(\underline{\alpha}^F\rho^F-\underline{\alpha}^F\sigma^F)-\underline{\chi}(\rho^F\rho^F+\underline{\alpha}^F\alpha^F+\sigma^F\sigma^F)$$
$$+(\eta+\underline{\eta})\underline{\alpha}^F(\rho^F+\sigma^F) \tag{7.140}$$

These interaction terms are estimated in an exact similar way as the previous ones

$$|\int_{D_{u,\underline{u}}} u^2r^2\sigma\rho^F\cdot\widehat{\nabla}\underline{\alpha}^F| \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1), \tag{7.141}$$
$$|\int_{D_{u,\underline{u}}} u^2r^2\sigma\widehat{\nabla}\rho^F\cdot\underline{\alpha}^F| \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1) \tag{7.142}$$

and similarly

$$|\int_{D_{u,\underline{u}}} u^2r^2\sigma(\underline{\chi}(\rho^F\rho^F+\underline{\alpha}^F\alpha^F+\sigma^F\sigma^F)+(\eta+\underline{\eta})\underline{\alpha}^F(\rho^F+\sigma^F))| \lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0). \tag{7.143}$$

□

Lemma 7.5. *The following estimate holds in the exterior region*

$$\int_{\underline{H}} r^2u^2|\underline{\beta}|^2+\int_H r^2u^2(\rho^2+\sigma^2)+\int_{D_{u,\underline{u}}} r^2u^2(tr\chi-\Omega^{-1}\overline{\Omega tr\chi})|\underline{\beta}|^2 \tag{7.144}$$
$$+\int_{D_{u,\underline{u}}}(2r^2u^2tr\underline{\chi}-2\Omega^{-1}r^2u-r^2u^2\Omega^{-1}\overline{\Omega tr\underline{\chi}})(\rho^2+\sigma^2)$$
$$=\int_{\widehat{\Sigma}_0} r^2u^2|\underline{\beta}|^2+\int_{\widehat{\Sigma}_0} r^2u^2(\rho^2+\sigma^2)+\mathcal{E}_{\underline{\beta},\rho,\sigma}$$

Proof. A consequence of the proposition (6.10) in the exterior domain ($r(u,\underline{u})>|u|$). □

7.2.2. Lowest order Yang-Mills energy estimates

Now we focus on the Yang-Mills curvature estimation. We utilize the gauge invariant characteristic of the norm associated with the Yang-Mills fields as well as the symmetric hyperbolicity of the null Yang-Mills equations. First we prove a short lemma regarding the symmetric hyperbolicity of the null Yang-Mills equations.

Proposition 7.11. *Null Yang-Mills equations are manifestly hyperbolic.*

Remark 5. *Note that this is natural since Yang-Mills equations are derivable from a Lagrangian and as such it is equipped with a canonical stress-energy tensor.*

Proof. The integration lemma can be utilized to prove the manifestly hyperbolic characteristics of the Yang-Mills equations while expressed in the double null coordinates. First consider the null triple $(\underline{\alpha}^F, \rho^F, \sigma^F)$ and recall their gauge covariant evolution equations

$$\widehat{\nabla}_4\underline{\alpha}^F + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}^F = -\widehat{\nabla}\rho^F + {}^*\widehat{\nabla}\sigma^F - 2\,{}^*\underline{\eta}\sigma^F - 2\underline{\eta}\rho^F + 2\omega\underline{\alpha}^F - \underline{\hat{\chi}}\cdot\alpha^F, \tag{7.145}$$

$$\widehat{\nabla}_3\rho^F + \mathrm{tr}\underline{\chi}\rho^F = -\widehat{\mathrm{div}}\underline{\alpha}^F + \left(\eta - \underline{\eta}\right)\cdot\underline{\alpha}^F, \tag{7.146}$$

$$\widehat{\nabla}_3\sigma^F + \mathrm{tr}\underline{\chi}\sigma^F = -\widehat{\mathrm{curl}}\underline{\alpha}^F + \left(\eta - \underline{\eta}\right)\cdot{}^*\underline{\alpha}^F. \tag{7.147}$$

Now define $f_1 := |\underline{\alpha}^F|^2_{\gamma,\delta}$, $f_2 = |\rho^F|^2_{\gamma,\delta}$, and $f_3 := |\sigma^F|^2_{\gamma,\delta}$. With these definitions in mind, let us apply the integration lemma to f_1, f_2, and f_3 to yield

$$\begin{aligned}\int_{D_{u,\underline{u}}}\nabla_4 f_1 + \int_{D_{u,\underline{u}}}\nabla_3 f_2 + \int_{D_{u,\underline{u}}}\nabla_3 f_3 &= \int_{\underline{H}_{\underline{u}}} f_1 + \int_{H_u} f_2 + \int_{H_u} f_3 - \int_{\underline{H}_{\underline{u}_0}} f_1 \\ &- \int_{H_{u_0}} f_2 - \int_{H_{u_0}} f_3 + \int_{D_{u,\underline{u}}}(2\omega - \mathrm{tr}\chi)f_1 + \int_{D_{u,\underline{u}}}(2\underline{\omega} - \mathrm{tr}\underline{\chi})(f_2 + f_3).\end{aligned} \tag{7.148}$$

In order for these equations to exhibit a hyperbolic characteristic, the left hand side should simplify to terms that are algebraic in $\underline{\alpha}^F$, ρ^F, and σ^F upon using the null evolution equations. Now we note the most important point: f_1, f_2, and f_3 are gauge invariant object and therefore we have the following as a consequence of the compatibility of the gauge covariant connection $\hat{\nabla}$ with the metrics γ and δ of the fibres (γ is a Riemannian metric on the topological 2-sphere $S_{u,\underline{u}}$ and δ is a Riemannian metric on the fibres of the associated gauge bundle)

$$\nabla_4 f_1 = 2\langle\underline{\alpha}^F, \widehat{\nabla}_4\underline{\alpha}^F\rangle_{\gamma,\delta},\ \nabla_3 f_2 = 2\langle\rho^F, \widehat{\nabla}_3\rho^F\rangle_{\gamma,\delta},\ \nabla_3 f_3 = 2\langle\sigma^F, \widehat{\nabla}_3\sigma^F\rangle_{\gamma,\delta}. \tag{7.149}$$

Now we only focus on the principal terms for hyperbolicity argument

$$\begin{aligned}\langle\underline{\alpha}^F, \widehat{\nabla}_4\underline{\alpha}^F\rangle_{\gamma,\delta} &= \langle\underline{\alpha}^F, -\widehat{\nabla}\rho^F + {}^*\widehat{\nabla}\sigma^F + \cdots\rangle_{\gamma,\delta} \\ = -\mathrm{div}\langle\underline{\alpha}^F, \rho^F\rangle_{\gamma,\delta} + \langle\widehat{\mathrm{div}}\underline{\alpha}^F, \rho^F\rangle_{\gamma,\delta} &+ \mathrm{div}\ {}^*\langle\underline{\alpha}^F, \sigma^F\rangle_{\gamma,\delta} + \langle\widehat{\mathrm{curl}}\underline{\alpha}^F, \sigma^F\rangle + \mathrm{l.o.t}\end{aligned} \tag{7.150}$$

$$\langle\rho^F, \widehat{\nabla}_3\rho^F\rangle_{\gamma,\delta} = \langle\rho^F, -\widehat{\mathrm{div}}\underline{\alpha}^F + \cdots\rangle_{\gamma,\delta}$$

$$\langle\sigma^F, \widehat{\nabla}_3\sigma^F\rangle_{\gamma,\delta} = \langle\sigma^F, -\widehat{\mathrm{curl}}\underline{\alpha}^F + \cdots\rangle_{\gamma,\delta}. \tag{7.151}$$

Now upon addition, we have

$$\begin{aligned}&\langle\underline{\alpha}^F, \widehat{\nabla}_4\underline{\alpha}^F\rangle_{\gamma,\delta} + \langle\rho^F, \widehat{\nabla}_3\rho^F\rangle_{\gamma,\delta} + \langle\sigma^F, \widehat{\nabla}_3\sigma^F\rangle_{\gamma,\delta} \\ &= -\mathrm{div}\langle\underline{\alpha}^F, \rho^F\rangle_{\gamma,\delta} + \langle\widehat{\mathrm{div}}\underline{\alpha}^F, \rho^F\rangle_{\gamma,\delta} + \mathrm{div}\ {}^*\langle\underline{\alpha}^F, \sigma^F\rangle_{\gamma,\delta} + \langle\widehat{\mathrm{curl}}\underline{\alpha}^F, \sigma^F\rangle \\ &\quad -\langle\rho^F, \widehat{\mathrm{div}}\underline{\alpha}^F\rangle_{\gamma,\delta} - \langle\sigma^F, \widehat{\mathrm{curl}}\underline{\alpha}^F\rangle_{\gamma,\delta} + \mathrm{l.o.t} \\ &= -\mathrm{div}\langle\underline{\alpha}^F, \rho^F\rangle_{\gamma,\delta} + \mathrm{div}\ {}^*\langle\underline{\alpha}^F, \sigma^F\rangle_{\gamma,\delta} + \mathrm{l.o.t}\end{aligned} \tag{7.152}$$

which upon integration over the topological 2−sphere yields terms that are algebraic in $\underline{\alpha}^F, \rho^F$, and σ^F. Here $\langle\underline{\alpha}^F, \rho^F\rangle_{\gamma,\delta} = (\underline{\alpha}^F)^P{}_{Qab}(\rho^F)^Q{}_P{}^b$ and ${}^*\langle\underline{\alpha}^F, \sigma^F\rangle_{\gamma,\delta} = \epsilon^{ca}(\underline{\alpha}^F)^P{}_{Qab}\rho^Q{}_P{}^b$. The most vital property that is utilized here is the compatibility of the connection $\widehat{\nabla}$ with the inner product $\langle\ ,\ \rangle_{\gamma,\delta}$ induced by the fibre metrics γ and δ together with the Hodge structure present in the null Yang-Mills equations. Notice that nowhere in the procedure we need the information about the Yang-Mills connection 1−form $A^P{}_{Q\mu}dx^\mu$. The remaining Yang-Mills null evolution equations may be utilized in a manner similar to obtain energy identities associated with the triple $\alpha^F, \rho^F, \sigma^F$. This concludes the proof of the hyperbolic characteristics of the null Yang-Mills equations. ∎

Utilizing this manifestly hyperbolic character, we may now start the energy estimates for the Yang-Mills curvatures. Consider the pair $\{\alpha^F, (\rho^F, \sigma^F)\}$

$$\int_{D_{u,\underline{u}}} \nabla_3(r^2|\alpha^F|^2) + \int_{D_{u,\underline{u}}} \nabla_4(r^2(|\rho^F|^2 + |\sigma^F|^2)) = \int_H r^2|\alpha^F|^2 + \int_{\underline{H}} r^2(|\rho^F|^2 + |\sigma^F|^2)$$
$$- \int_{\widehat{\Sigma}_0} r^2|\alpha^F|^2 - \int_{\widehat{\Sigma}_0} r^2(|\rho^F|^2 + |\sigma^F|^2) + \int_{D_{u,\underline{u}}} (2\underline{\omega} - \mathrm{tr}\underline{\chi})r^2|\alpha^F|^2$$
$$+ \int_{D_{u,\underline{u}}} (2\omega - \mathrm{tr}\chi)r^2(|\rho^F|^2 + |\sigma^F|^2).$$

Explicit calculations yield

$$\int_{D_{u,\underline{u}}} \nabla_3(r^2|\alpha^F|^2) + \int_{D_{u,\underline{u}}} \nabla_4(r^2(|\rho^F|^2 + |\sigma^F|^2)) \tag{7.153}$$
$$= \int_{D_{u,\underline{u}}} \Big(2r\Omega^{-1}\frac{dr}{du}|\alpha^F|^2 + 2r^2\langle\alpha^F, \widehat{\nabla}_3\alpha^F\rangle + 2r\Omega^{-1}\frac{dr}{d\underline{u}}(|\rho^F|^2 + |\sigma^F|^2)$$
$$+2r^2(\langle\rho^F, \widehat{\nabla}_4\rho^F\rangle + \langle\sigma^F, \widehat{\nabla}_4\sigma^F\rangle)\Big)$$
$$= \int_{D_{u,\underline{u}}} (r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\alpha^F|^2 + 2r^2\langle\alpha^F, -\frac{1}{2}tr\underline{\chi}\alpha^F - \widehat{\nabla}\rho^F + {}^*\widehat{\nabla}\sigma^F - 2\,{}^*\eta\sigma^F + 2\eta\rho^F$$
$$+2\underline{\omega}\alpha^F - \hat{\chi}\cdot\underline{\alpha}^F\rangle + r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}(|\rho^F|^2 + |\sigma^F|^2) + 2r^2(\langle\rho^F, -\mathrm{tr}\chi\rho^F - \hat{\mathrm{div}}\alpha^F$$
$$-(\eta - \underline{\eta})\cdot\alpha^F\rangle + 2r^2(\langle\sigma^F, -\mathrm{tr}\chi\sigma^F - \hat{\mathrm{curl}}\alpha^F + (\eta - \underline{\eta})\cdot\,{}^*\alpha^F\rangle$$

Putting everything together, the energy integral reads

$$\int_H r^2|\alpha^F|^2 + \int_{\underline{H}} r^2(|\rho^F|^2 + |\sigma^F|^2) - \int_{D_{u,\underline{u}}} r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\alpha^F|^2 + \int_{D_{u,\underline{u}}} r^2(\mathrm{tr}\chi - \Omega^{-1}\overline{\Omega\mathrm{tr}\chi})$$
$$(|\rho^F|^2 + |\sigma^F|^2) = \int_{\widehat{\Sigma}_0} r^2|\alpha^F|^2 + \int_{\widehat{\Sigma}_0} r^2(|\rho^F|^2 + |\sigma^F|^2) + \mathcal{E}_{\alpha^F\rho^F\sigma^F},$$

where the error term $\mathcal{E}_{\alpha^F\rho^F\sigma^F}$ verifies the following schematic expression

$$\mathcal{E}_{\alpha^F\rho^F\sigma^F} \sim \int_{D_{u,\underline{u}}} 2r^2\langle\alpha^F, -2\,{}^*\eta\sigma^F + 2\eta\rho^F + 2\underline{\omega}\alpha^F - \hat{\chi}\cdot\underline{\alpha}^F\rangle$$
$$+2r^2(\langle\rho^F, -(\eta - \underline{\eta})\cdot\alpha^F\rangle + 2r^2(\langle\sigma^F, (\eta - \underline{\eta})\cdot\,{}^*\alpha^F\rangle$$

Proposition 7.12. *The error term $\mathcal{E}_{\alpha^F\rho^F\sigma^F}$ verifies the following estimate*

$$|\mathcal{E}_{\alpha^F\rho^F\sigma^F}| \lesssim \epsilon\mathcal{Y}_0 \tag{7.154}$$

Proof. Notice the special structure present in the null Yang-Mills equations (slowly decaying connection coefficient $\underline{\omega}$ appears with α^F instead of $\underline{\alpha}^F$ which would have caused serious problem)

$$|\int_{D_{u,\underline{u}}} r^2\alpha^F\cdot\alpha^F\underline{\omega}| \lesssim (\int_{u_0}^{u} r^{-1}|u'|^{-1}du')\int_H r^2|\alpha^F|^2 \lesssim \epsilon\mathcal{Y}_0 \tag{7.155}$$

The remaining terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^2\alpha^F\cdot\sigma^F\eta| \lesssim \sup_{\mathcal{K}}(r^2|\eta|)(\int_{u_0}^{u} r^{-1}|u'|^{-1}du')\int_H (r^2|\alpha^F|^2 + u^2|\sigma^F|^2) \tag{7.156}$$

$$\lesssim \epsilon\mathcal{Y}_0,$$

$$|\int_{D_{u,\underline{u}}} r^2\alpha^F\cdot\underline{\alpha}^F\hat{\chi}| \lesssim \sup_{\mathcal{K}}(r^2|\hat{\chi}|)\int_{D_{u,\underline{u}}}\left(\frac{r^2|\alpha^F|^2}{u^2}+\frac{u^2|\alpha^F|^2}{r^2}\right)$$

$$\lesssim \epsilon\left((\int_{u_0}^{u}|u^{'}|^{-2}du^{'})\int_H r^2|\alpha^F|^2+(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}^{'})\int_{\underline{H}} u^2|\underline{\alpha}^F|^2\right)\lesssim \epsilon\mathcal{Y}_0$$

$$|\int_{D_{u,\underline{u}}} r^2\rho^F\cdot\alpha^F(\eta-\underline{\eta})| \lesssim \sup_{\mathcal{K}}(r^2|\eta,\underline{\eta}|)(\int_{u_0}^{u} r^{-1}|u^{'}|^{-1}du^{'})\int_H(r^2|\alpha^F|^2+u^2|\rho^F|^2) \tag{7.157}$$

$$\lesssim \epsilon\mathcal{Y}_0$$

Lemma 7.6. *The following estimates hold true in the exterior region*

$$\int_H r^2|\alpha^F|^2+\int_{\underline{H}} r^2(|\rho^F|^2+|\sigma^F|^2)\lesssim \int_{\widehat{\Sigma}_0} r^2|\alpha^F|^2+\int_{\widehat{\Sigma}_0} r^2(|\rho^F|^2+|\sigma^F|^2)+\epsilon\mathbf{Y}.$$

Proof. Consequence of the previous proposition (7.12). □

Lemma 7.7. *The triple $\underline{\alpha}^F,\rho^F$, and σ^F verify the following estimates in the exterior region*

$$\int_{\underline{H}} u^2|\underline{\alpha}^F|^2+\int_H u^2(|\rho^F|^2+|\sigma^F|^2)\lesssim \int_{\widehat{\Sigma}_0} u^2|\underline{\alpha}^F|^2+\int_{\widehat{\Sigma}_0} u^2(|\rho^F|^2+|\sigma^F|^2)+\epsilon\mathbf{Y}. \tag{7.158}$$

Proof. Consider the last Yang-Mills pair $\underline{\alpha}^F$ and ρ^F,σ^F and use the integration lemma (6.1)

$$\int_{D_{u,\underline{u}}}\nabla_4(u^2|\underline{\alpha}^F|^2)+\int_{D_{u,\underline{u}}}\nabla_3(u^2(|\rho^F|^2+|\sigma^F|^2)) \tag{7.159}$$

$$=\int_{\underline{H}} u^2|\underline{\alpha}^F|^2+\int_H u^2(|\rho^F|^2+|\sigma^F|^2)-\int_{\widehat{\Sigma}_0} u^2|\underline{\alpha}^F|^2-\int_{\widehat{\Sigma}_0} u^2(|\rho^F|^2+|\sigma^F|^2)$$
$$+\int_{D_{u,\underline{u}}}(2\omega-\mathrm{tr}\chi)u^2|\underline{\alpha}^F|^2+\int_{D_{u,\underline{u}}}(2\underline{\omega}-\mathrm{tr}\underline{\chi})u^2(|\rho^F|^2+|\sigma^F|^2)$$

Explicit calculations yield

$$\int_{D_{u,\underline{u}}}\nabla_4(u^2|\underline{\alpha}^F|^2)+\int_{D_{u,\underline{u}}}\nabla_3(u^2(|\rho^F|^2+|\sigma^F|^2)) \tag{7.160}$$

$$=\int_{D_{u,\underline{u}}} 2u^2\langle\underline{\alpha}^F,-\frac{1}{2}\mathrm{tr}\chi\underline{\alpha}^F-\widehat{\nabla}\rho^F-\ {}^*\widehat{\nabla}\sigma^F-2\ {}^*\underline{\eta}\sigma^F-2\underline{\eta}\rho^F+2\omega\underline{\alpha}^F-\hat{\underline{\chi}}\cdot\alpha^F\rangle$$
$$+\int_{D_{u,\underline{u}}} 2\Omega^{-1}u(|\rho^F|^2+|\sigma^F|^2)+\int_{D_{u,\underline{u}}} 2u^2\Big(\langle\rho^F,-tr\underline{\chi}\rho^F-\hat{\mathrm{div}}\underline{\alpha}^F+(\eta-\underline{\eta})\cdot\underline{\alpha}^F\rangle$$
$$+\langle\sigma^F,-tr\underline{\chi}\sigma^F-\hat{\mathrm{curl}}\underline{\alpha}^F+(\eta-\underline{\eta})\cdot\ {}^*\underline{\alpha}^F\rangle\Big)$$

Collecting all the terms

$$\int_{\underline{H}} u^2|\underline{\alpha}^F|^2+\int_H u^2(|\rho^F|^2+|\sigma^F|^2)-\int_{D_{u,\underline{u}}}(2\Omega^{-1}u-\mathrm{tr}\underline{\chi}u^2)(|\rho^F|^2+|\sigma^F|^2)$$
$$=\int_{\widehat{\Sigma}_0} u^2|\underline{\alpha}^F|^2+\int_{\widehat{\Sigma}_0} u^2(|\rho^F|^2+|\sigma^F|^2)+\int_{D_{u,\underline{u}}} 2u^2\langle\underline{\alpha}^F,-2\ {}^*\underline{\eta}\sigma^F-2\underline{\eta}\rho^F+2\omega\underline{\alpha}^F-\hat{\underline{\chi}}\cdot\alpha^F\rangle$$

$$+|\sigma^F|^2) + \int_{D_{u,\underline{u}}} 2u^2 \left(\langle \rho^F, (\eta - \underline{\eta}) \cdot \underline{\alpha}^F \rangle + \langle \sigma^F, (\eta - \underline{\eta}) \cdot {}^*\underline{\alpha}^F \rangle \right).$$

We estimate the main terms as follows

$$|\int_{D_{u,\underline{u}}} u^2 \underline{\alpha}^F \cdot \underline{\alpha}^F \omega| \lesssim \sup_{\mathcal{K}} r^2 |\omega| (\int_{\underline{u}_0}^{\underline{u}} r^{-2} d\underline{u}) \int_{\underline{H}} u^2 |\underline{\alpha}^F|^2 \lesssim \epsilon \mathcal{Y}_0,$$

$$|\int_{D_{u,\underline{u}}} u^2 \underline{\alpha}^F \cdot \alpha^F \hat{\underline{\chi}}| \lesssim \sup_{\mathcal{K}} (r|u||\hat{\underline{\chi}}|) \left((\int_{u_0}^{u} r^{-2} du') \int_H r^2 |\alpha^F|^2 + (\int_{\underline{u}_0}^{\underline{u}} r^{-2} d\underline{u}') \int_{\underline{H}} u^2 |\underline{\alpha}^F|^2 \right)$$
$$\lesssim \epsilon \mathcal{Y}_0,$$

$$|\int_{D_{u,\underline{u}}} u^2 \rho^F \cdot \underline{\alpha}^F (\eta - \underline{\eta})| \lesssim \sup_{\mathcal{K}} (r^2 |\eta, \underline{\eta}|) (\int_{\underline{u}_0}^{\underline{u}} |u| r^{-3} d\underline{u}') \int_{\underline{H}} (u^2 |\underline{\alpha}^F|^2 + r^2 |\rho^F|^2) \lesssim \epsilon \mathcal{Y}_0.$$

Collection of all the estimates together with the fact that $u < 0$ in the exterior domain $(r(u, \underline{u}) > |u|)$ of consideration yields the result. $\square$

7.2.3. Derivative estimates

Once we have controlled the L^2 norms of the curvature components, we move onto estimating the derivatives. First we estimate the first derivatives of the Weyl curvature. Since the Yang-Mills source terms (in the Bianchi equations) appear in a differentiated manner, the first derivative norms of the Weyl curvature will contain second derivatives of the Yang-Mills fields.

$$\int_{D_{u,\underline{u}}} \nabla_3 (r^6 |\nabla \alpha|^2) + \int_{D_{u,\underline{u}}} \nabla_4 (r^6 |\nabla \beta|^2) = \int_H r^6 |\nabla \alpha|^2 + \int_{\underline{H}} r^6 |\nabla \beta|^2 \tag{7.161}$$
$$- \int_{\widehat{\Sigma}_0} r^6 |\nabla \alpha|^2 - \int_{\widehat{\Sigma}_0} r^6 |\nabla \beta|^2 + \int_{D_{u,\underline{u}}} (2\underline{\omega} - \mathrm{tr}\underline{\chi}) r^6 |\nabla \alpha|^2 + \int_{D_{u,\underline{u}}} (2\omega - \mathrm{tr}\chi) r^6 |\nabla \beta|^2$$

$$\int_{D_{u,\underline{u}}} \nabla_3 (r^6 |\nabla \alpha|^2) + \int_{D_{u,\underline{u}}} \nabla_4 (r^6 |\nabla \beta|^2) \tag{7.162}$$
$$= \int_{D_{u,\underline{u}}} \Big(6r^5 \Omega^{-1} \frac{dr}{du} |\nabla \alpha|^2 + 2r^6 \langle \nabla \alpha, \nabla \nabla_3 \alpha \rangle + 2r^6 \langle \nabla \alpha, [\nabla_3, \nabla] \alpha \rangle + 6r^5 \Omega^{-1} \frac{dr}{d\underline{u}} |\nabla \beta|^2$$
$$+ 2r^6 \langle \nabla \beta, \nabla \nabla_4 \beta \rangle + 2r^6 \langle \nabla \beta, [\nabla_4, \nabla] \beta \rangle \Big)$$

Now we make use of the following two explicit commutation relations (lemma 5.1)

$$[\nabla_4, \nabla_a] P_{bc} = -\chi_{ad} \nabla_d P_{bc} + \mathcal{E}_1, [\nabla_3, \nabla_a] P_{bc} = -\underline{\chi}_{ad} \nabla_d P_{bc} + \mathcal{E}_2, \tag{7.163}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy the following schematic expressions

$$\mathcal{E}_1 \sim (\eta + \underline{\eta}) \nabla_4 P + \underline{\eta} \chi P + \beta P, \; \mathcal{E}_2 \sim (\eta + \underline{\eta}) \nabla_3 P + \eta \underline{\chi} P + \underline{\beta} P. \tag{7.164}$$

The energy expression yields

$$\int_{D_{u,\underline{u}}} \nabla_3 (r^6 |\nabla \alpha|^2) + \int_{D_{u,\underline{u}}} \nabla_4 (r^6 |\nabla \beta|^2) \tag{7.165}$$
$$= \int_{D_{u,\underline{u}}} [3r^6 \Omega^{-1} \overline{\Omega \mathrm{tr}\underline{\chi}} |\nabla \alpha|^2 + 2r^6 \langle \nabla \alpha, \nabla \left(-\frac{1}{2} \mathrm{tr}\underline{\chi} \alpha + \nabla \widehat{\otimes} \beta + 4\underline{\omega} \alpha - 3(\widehat{\chi} \rho + {}^* \widehat{\chi} \sigma) \right.$$

$$+(\zeta+4\eta)\hat{\otimes}\beta+\frac{1}{2}(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})\gamma_{ab}-D_4\mathfrak{T}_{ab}+\frac{1}{2}\left(D_b\mathfrak{T}_{4a}+D_a\mathfrak{T}_{4b}\right)\Big)\rangle$$

$$+2r^6\langle\nabla\alpha,-\frac{\mathrm{tr}\underline{\chi}}{2}\nabla\alpha-\widehat{\underline{\chi}}\cdot\nabla\alpha+\mathcal{E}_2\rangle+3r^6\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\nabla\beta|^2$$

$$+2r^6\langle\nabla\beta,\nabla(-2\mathrm{tr}\chi\beta_a+(\mathrm{div}\alpha)_a-2\omega\beta_a+(\eta\cdot\alpha)_a-\frac{1}{2}(D_a\mathfrak{T}_{44}-D_4\mathfrak{T}_{4a})\rangle$$

$$+2r^6\langle\nabla\beta,-\frac{\mathrm{tr}\chi}{2}\nabla\beta-\widehat{\chi}\cdot\nabla\beta+\mathcal{E}_1\rangle]$$

Collecting all the terms yields

$$\int_H r^6|\nabla\alpha|^2+\int_{\underline{H}}r^6|\nabla\beta|^2-\int_{D_{u,\underline{u}}}r^6\left(3\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-\mathrm{tr}\underline{\chi}\right)|\nabla\alpha|^2 \tag{7.166}$$
$$+\int_{D_{u,\underline{u}}}r^6(4\mathrm{tr}\chi-3\Omega^{-1}\overline{\Omega\mathrm{tr}\chi})|\nabla\beta|^2=\int_{\widehat{\Sigma}_0}r^6|\nabla\alpha|^2+\int_{\widehat{\Sigma}_0}r^6|\nabla\beta|^2+\mathcal{E}_{\nabla\alpha,\nabla\beta},$$

where $\mathcal{E}_{\nabla\alpha,\nabla\beta}$ is the error term to be controlled

$$\mathcal{E}_{\nabla\alpha,\nabla\beta}\sim\int_{D_{u,\underline{u}}}r^6\langle\nabla\alpha,\nabla\mathrm{tr}\underline{\chi}\alpha\rangle+\int_{D_{u,\underline{u}}}[2r^6\langle\nabla\alpha,\nabla\left(4\underline{\omega}\alpha-3(\widehat{\chi}\rho+{}^*\widehat{\chi}\sigma)\right.$$
$$+(\zeta+4\eta)\hat{\otimes}\beta+\frac{1}{2}(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})\gamma_{ab}-D_4\mathfrak{T}_{ab}+\frac{1}{2}\left(D_b\mathfrak{T}_{4a}+D_a\mathfrak{T}_{4b}\right)\Big)\rangle$$
$$+2r^6\langle\nabla\alpha,-\widehat{\underline{\chi}}\cdot\nabla\alpha+\mathcal{E}_2\rangle+2r^6\langle\nabla\beta,-2\nabla(\mathrm{tr}\chi)\beta_a-2\nabla(\omega\beta_a)+\nabla(\eta\cdot\alpha)_a-\frac{1}{2}\nabla(D_a\mathfrak{T}_{44}-D_4\mathfrak{T}_{4a})\rangle$$
$$+2r^6\langle\nabla\beta,-\widehat{\chi}\cdot\nabla\beta+\mathcal{E}_1\rangle+2r^6\langle\nabla\alpha,[\nabla_3,\nabla]\alpha\rangle+2r^6\langle\nabla\beta,\nabla\nabla_4\beta\rangle+2r^6\nabla\beta K\beta].$$

Proposition 7.13. *The error term $\mathcal{E}_{\nabla\alpha,\nabla\beta}$ verifies the following estimate*

$$|\mathcal{E}_{\nabla\alpha,\nabla\beta}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2) \tag{7.167}$$

Proof.

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha(\nabla\mathrm{tr}\underline{\chi})\alpha|\lesssim\int_{u,\underline{u}}||r^3\nabla\alpha||_{L^2(S)}||r^{3-\frac{1}{2}}\nabla\mathrm{tr}\underline{\chi}||_{L^4(S)}||r^2\alpha||_{L^4(S)}r^{-\frac{3}{2}} \tag{7.168}$$
$$\lesssim\sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\mathrm{tr}\underline{\chi}||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u}r^{-\frac{3}{2}})\int_H(r^6|\nabla\alpha|^2+r^4|\alpha|^2)\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1)$$

First note the special structure: slowly decaying connection coefficient $\underline{\omega}$ appears with α (instead of $\underline{\alpha}$ which would have caused serious problem)

$$|\int_{D_{u,\underline{u}}}r^4|\nabla\alpha|^2\underline{\omega}|\lesssim\sup_{\mathcal{K}}(r|u||\underline{\omega}|)(\int_{u_0}^{u}r^{-1}|u'|^{-1}du')\int_H r^6|\nabla\alpha|^2\lesssim\epsilon\mathcal{W}_1. \tag{7.169}$$

Now we want to estimate

$$|\int_{D_{u,\underline{u}}}r^6\langle\nabla\alpha,\nabla D_4\mathfrak{T}_{AB}\rangle| \tag{7.170}$$

Explicit calculations yield

$$\nabla D_4\mathfrak{T}_{AB}\sim\widehat{\nabla}\rho^F\cdot\widehat{\nabla}\alpha^F+\rho^F\cdot\widehat{\nabla}^2\alpha^F+(\nabla\mathrm{tr}\chi)\rho^F\cdot\rho^F+\mathrm{tr}\chi\widehat{\nabla}\rho^F\cdot\rho^F+\nabla(\eta,\underline{\eta})\alpha^F\cdot\rho^F$$
$$+(\eta+\underline{\eta})\widehat{\nabla}\alpha^F\cdot\rho^F+\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}_4\alpha^F+\alpha^F\cdot\widehat{\nabla}^2\rho^F+\mathrm{tr}\chi\nabla\alpha^F\cdot\underline{\alpha}^F+\mathrm{tr}\chi\widehat{\nabla}\underline{\alpha}^F\cdot\alpha^F+\nabla\mathrm{tr}\chi\alpha^F\cdot\underline{\alpha}^F$$

$$+\underline{\alpha}^F\cdot\widehat{\nabla}\widehat{\nabla}_4\alpha^F$$

We estimate the top order terms since remaining terms scale accordingly. We will frequently use the global Sobolev inequality (or simply the trace inequality) on the null cones (lemma 4.2)

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\widehat{\nabla}\rho^F\cdot\widehat{\nabla}\alpha^F|\lesssim\sup_{u,\underline{u}}||r^3\widehat{\nabla}\rho^F||_{L^4(S_{u,\underline{u}})}\left(\int_H r^6|\nabla\alpha|^2\right)^{\frac{1}{2}}\sup_{u,\underline{u}}||ur^2\widehat{\nabla}\alpha^F||_{L^4(S_{u,\underline{u}})}$$
$$(\int_{u_0}^{u}\frac{1}{|u^{'}|}(\int_{\underline{u}_0}^{\underline{u}}r^{-4}d\underline{u}^{'})^{\frac{1}{2}}du^{'})\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2),$$
$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\rho^F\cdot\widehat{\nabla}^2\alpha^F|\lesssim\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^{u}r^{-2}|u^{'}|^{-\frac{1}{2}}du^{'})\int_H(r^6|\nabla\alpha|^2+r^6|\widehat{\nabla}^2\alpha^F|^2)$$
$$\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2),$$
$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\nabla\mathrm{tr}\chi\rho^F\cdot\rho^F|\lesssim\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)\sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u}|u^{'}|^{-\frac{3}{2}}r^{-2}du^{'})$$
$$\int_H(r^6|\nabla\alpha|^2+u^2|\rho^F|^2+u^2r^2|\widehat{\nabla}\rho^F|^2)\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1),$$
$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\alpha^F\cdot\widehat{\nabla}^2\rho^F|\lesssim\sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{u_0}^{u}|u^{'}|^{-1}r^{-\frac{3}{2}}du^{'})\int_H(r^6|\nabla\alpha|^2+u^2r^4|\widehat{\nabla}^2\rho^F|^2)$$
$$\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2),$$
$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\widehat{\nabla}\underline{\alpha}^F\cdot\alpha^F\mathrm{tr}\chi|\lesssim\epsilon\left((\int_{u_0}^{u}|u^{'}|^{-2}du^{'})\int_H r^6|\nabla\alpha|^2+(\int_{\underline{u}_0}^{\underline{u}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2\right)$$
$$\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1),$$
$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}_4\alpha^F|\lesssim\sup_{u,\underline{u}}(||r^{\frac{3}{2}}|u|^{\frac{3}{2}}\widehat{\nabla}\underline{\alpha}^F||)(\int_{u_0}^{u}|u^{'}|^{-\frac{3}{2}}r^{-1}du^{'})\int_H(r^6|\nabla\alpha|^2+r^4|\widehat{\nabla}_4\alpha^F|^2$$
$$+r^6|\widehat{\nabla}\widehat{\nabla}_4\alpha^F|^2)\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2)$$

The remaining terms are straightforward to estimate

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\nabla\mathrm{tr}\chi\alpha^F\cdot\underline{\alpha}^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0),\tag{7.171}$$
$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\underline{\alpha}^F\cdot\widehat{\nabla}\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2),\tag{7.172}$$
$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha(\eta+\underline{\eta})\widehat{\nabla}\alpha^F\cdot\rho^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_1),\tag{7.173}$$
$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\nabla(\eta,\underline{\eta})\alpha^F\cdot\rho^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0).$$

Now we need to estimate

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha_{AB}\nabla(D_B\mathfrak{T}_{4A}+D_A\mathfrak{T}_{4B}|.\tag{7.174}$$

An explicit computation yields

$$\nabla\nabla\mathfrak{T}_{4A}\sim\widehat{\nabla}^2\alpha^F\cdot(\rho^F,\sigma^F)+\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\alpha^F\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)+\nabla(\eta,\underline{\eta})(\rho^F,\sigma^F)\cdot\alpha^F$$
$$+(\eta,\underline{\eta})\widehat{\nabla}(\rho^F,\sigma^F)\cdot\alpha^F+(\eta,\underline{\eta})(\rho^F,\sigma^F)\cdot\widehat{\nabla}\alpha^F+\nabla(\mathrm{tr}\chi,\hat{\chi})(|\rho^F,\sigma^F|^2+\alpha^F\cdot\underline{\alpha}^F)$$

$$+(\mathrm{tr}\chi,\hat{\chi})(\rho^F\cdot\widehat{\nabla}\rho^F+\sigma^F\cdot\widehat{\nabla}\sigma^F+\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F+\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F).$$

We have already estimated most of the terms in the expression of $\nabla\nabla\mathfrak{T}_{4A}$ previously. The remaining terms are controlled in a similar way

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\nabla\mathrm{tr}\chi\rho^F\cdot\rho^F|\lesssim\int_{D_{u,\underline{u}}}||r^3\nabla\alpha||_{L^2(S_{u,\underline{u}})}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}||r^{\frac{1}{2}}|u|\rho^F||_{L^4(S_{u,\underline{u}})}r^{-2}|u|^{-\frac{3}{2}}$$

$$\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)$$

$$\lesssim\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)\sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u}r^{-2}|u^{'}|^{-\frac{3}{2}}du^{'})\int_H(r^6|\nabla\alpha|^2+u^2|\rho^F|^2+u^2r^2|\widehat{\nabla}\rho^F|^2)$$

$$\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1)$$

Similarly

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\nabla\hat{\chi}\rho^F\cdot\rho^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1),\tag{7.175}$$

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\mathrm{tr}\chi\rho^F\cdot\widehat{\nabla}\rho^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1),\tag{7.176}$$

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\nabla(\mathrm{tr}\chi,\hat{\chi})|\rho^F,\sigma^F|^2|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0),\tag{7.177}$$

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\nabla(\mathrm{tr}\chi,\hat{\chi})\alpha^F\cdot\underline{\alpha}^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0),\tag{7.178}$$

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\ \mathrm{tr}\chi\alpha^F\cdot\widehat{\nabla}\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1)\tag{7.179}$$

Collecting all the terms together, we obtain

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha_{AB}\nabla(D_B\mathfrak{T}_{4A}+D_A\mathfrak{T}_{4B})|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2)\tag{7.180}$$

First note the special structure: slowly decaying connection coefficient $\hat{\underline{\chi}}$ appears with α (instead of $\underline{\alpha}$ which would have caused serious problem)

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\hat{\underline{\chi}}\nabla\alpha|\lesssim\sup_{\mathcal{K}}(|u|r|\hat{\underline{\chi}}|)(\int_{u_0}^{u}|u^{'}|^{-1}r^{-1}du^{'})\int_H r^6|\nabla\alpha|^2\lesssim\epsilon\mathcal{W}_1.\tag{7.181}$$

Now we need to estimate the error term that is generated via commutation of ∇ and ∇_3 while acted on α. The error term reads

$$\mathcal{E}_2^{\alpha}\sim(\eta+\underline{\eta})\nabla_3\alpha+\eta\underline{\chi}\alpha+\underline{\beta}\alpha\tag{7.182}$$

which after using the equation of motion for α yields

$$\mathcal{E}_2\sim(\eta+\underline{\eta})(\mathrm{tr}\underline{\chi}\alpha+\nabla\beta+\underline{\omega}\alpha+\chi(\rho,\sigma)+(\eta,\underline{\eta})\beta+D_4\mathfrak{T}_{AB}+D_A\mathfrak{T}_{4B})+\eta\underline{\chi}\alpha+\underline{\beta}\alpha.$$

We estimate these terms as follows

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha(\eta+\underline{\eta})\mathrm{tr}\underline{\chi}\alpha|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1),\tag{7.183}$$

$$|\int_{D_{u,\underline{u}}}r^6\nabla\alpha\eta\underline{\chi}\alpha|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1),$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\alpha\underline{\beta}\alpha| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1),$$
$$|\int_{D_{u,\underline{u}}} r^6\nabla\alpha(\eta+\underline{\eta})\nabla\beta| \lesssim \epsilon\mathcal{W}_1,$$
$$|\int_{D_{u,\underline{u}}} r^6\nabla\alpha\underline{\omega}\alpha| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1),$$
$$|\int_{D_{u,\underline{u}}} r^6\nabla\alpha\chi(\rho,\sigma)(\eta,\underline{\eta})| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1),$$
$$|\int_{D_{u,\underline{u}}} r^6\nabla\alpha(\eta,\underline{\eta})\beta| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1),$$
$$|\int_{D_{u,\underline{u}}} r^6\nabla\alpha_{AB}D_4\mathfrak{T}_{AB}| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1),$$
$$|\int_{D_{u,\underline{u}}} r^6\nabla\alpha_{AB}D_A\mathfrak{T}_{4B}| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1).$$

Collecting all the terms together, we obtain

$$|\int_{D_{u,\underline{u}}} r^6\langle\nabla\alpha,\mathcal{E}_2\rangle| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1) \tag{7.184}$$

Here notice that $\int_{D_{u,\underline{u}}} r^6\langle\nabla\alpha,(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})\gamma\rangle=0$ due to the compatibility of γ with ∇ as well as the trace-free property of α.

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^6\nabla\beta\nabla(\mathrm{tr}\chi)\beta| &\lesssim \sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}||r^3\nabla\beta||_{L^2(Su,\underline{u})}||r^{\frac{5}{2}}\beta||_{L^4(S_{u,\underline{u}})}r^{-2} \\ &\lesssim \sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u} r^{-2}du')\int_H(r^6|\nabla\beta|^2+r^4|\beta|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1)\end{aligned} \tag{7.185}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^6\nabla\beta(\nabla\omega)\beta| &\lesssim \sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\omega||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}||r^3\nabla\beta||_{L^2(Su,\underline{u})}||r^{\frac{5}{2}}\beta||_{L^4(S_{u,\underline{u}})}r^{-2} \\ &\lesssim \sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\omega||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u} r^{-2}du')\int_H(r^6|\nabla\beta|^2+r^4|\beta|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1)\end{aligned} \tag{7.186}$$

$$|\int_{D_{u,\underline{u}}} r^6\omega|\nabla\beta|^2| \lesssim \epsilon\mathcal{W}_1,\ |\int_{D_{u,\underline{u}}} r^6\nabla\beta\nabla\eta\alpha| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1), \tag{7.187}$$
$$|\int_{D_{u,\underline{u}}} r^6\nabla\beta\eta\nabla\alpha| \lesssim \epsilon\mathcal{W}_1 \tag{7.188}$$

Now we need to estimate the error term that arises due to the commutation of ∇_4 and ∇ i.e.,

$$|\int_{D_{u,\underline{u}}} r^6\langle\nabla\beta,\mathcal{E}_1^{\beta}\rangle|. \tag{7.189}$$

An explicit calculation yields

$$\mathcal{E}_1 \sim \underline{\eta}\chi\beta+\beta\beta+(\eta,\underline{\eta})\Big(\mathrm{tr}\chi\beta+\nabla\alpha+\omega\beta+\eta\alpha+\alpha^F\cdot\widehat{\nabla}\alpha^F-\chi\alpha^F\cdot(\rho^F,\sigma^F)\tag{7.190}$$

$$+\eta|\alpha^F|^2+\omega\alpha^F\cdot(\rho^F,\sigma^F)-\widehat{\nabla}_4\alpha^F\cdot(\rho^F,\sigma^F)-\alpha^F\cdot(\widehat{\nabla}\alpha^F+\mathrm{tr}\chi\rho^F+(\eta-\underline{\eta})\alpha^F)\Big)$$

$$|\int_{D_{u,\underline{u}}}\nabla\beta\underline{\eta}\chi\beta|\lesssim(\mathcal{W}_0+\mathcal{W}_1),\ |\int_{D_{u,\underline{u}}}\nabla\beta\beta\beta|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1), \quad (7.191)$$
$$|\int_{D_{u,\underline{u}}}\nabla\beta\nabla\alpha(\eta,\underline{\eta})|\lesssim\epsilon\mathcal{W}_1,|\int_{D_{u,\underline{u}}}\nabla\beta\omega\beta|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1), \quad (7.192)$$
$$|\int_{D_{u,\underline{u}}}\nabla\beta\alpha^F\cdot\widehat{\nabla}\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1),|\int_{D_{u,\underline{u}}}\nabla\beta\chi\alpha^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_0), \quad (7.193)$$
$$|\int_{D_{u,\underline{u}}}\nabla\beta(\rho^F,\sigma^F)\cdot\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1),\ |\int_{D_{u,\underline{u}}}\nabla\beta(\eta,\underline{\eta})\alpha^F\cdot\alpha^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0).$$

Collecting all the terms, we obtain

$$|\int_{D_{u,\underline{u}}}r^6\langle\nabla\beta,\mathcal{E}_1^\beta\rangle|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1) \quad (7.194)$$

The last term to estimate is the coupling term

$$|\int_{D_{u,\underline{u}}}r^6\nabla\beta_A\nabla(D_A\mathfrak{T}_{44}-D_4\mathfrak{T}_{4A})|. \quad (7.195)$$

An explicit calculation yields

$$\nabla(D_A\mathfrak{T}_{44}-D_4\mathfrak{T}_{4A})\sim\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}\alpha^F+\alpha^F\cdot\widehat{\nabla}^2\alpha^F+\nabla\chi\alpha^F\cdot(\rho^F,\sigma^F)+\chi\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)$$
$$+\chi\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\nabla(\eta,\underline{\eta})\alpha^F\cdot\alpha^F+(\eta,\underline{\eta})\widehat{\nabla}\alpha^F\cdot\alpha^F+\nabla\omega\alpha^F\cdot(\rho^F,\sigma^F)+\omega\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)$$
$$+\omega\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\widehat{\nabla}\widehat{\nabla}_4\alpha^F\cdot(\rho^F,\sigma^F)+\widehat{\nabla}_4\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F).$$

Now we estimate the most non-trivial terms

$$|\int_{D_{u,\underline{u}}}r^6\nabla\beta\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}\alpha^F|\lesssim\sup_{\mathcal{K}}(r^{\frac{7}{2}}|\widehat{\nabla}\alpha^F|)(\int_{u_0}^{u}r^{-\frac{5}{2}}du')\int_H(r^6|\nabla\beta|^2+r^4|\widehat{\nabla}\alpha^F|^2) \quad (7.196)$$
$$\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1)$$

$$|\int_{D_{u,\underline{u}}}r^6\nabla\beta\alpha^F\cdot\widehat{\nabla}^2\alpha^F|\lesssim\sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{u_0}^{u}r^{-\frac{5}{2}}du')\int_H(r^6|\nabla\beta|^2+r^6|\widehat{\nabla}^2\alpha^F|^2) \quad (7.197)$$
$$\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2).$$

The last term involving sectional curvature K of the topological $2-$sphere $S_{u,\underline{u}}$

$$|\int_{D_{u,\underline{u}}}2r^6\nabla\beta K\beta|\lesssim\int_{\underline{u}_0}^{\underline{u}}r^{-2}(\int_{\underline{H}_{\underline{u}'}}r^6|\nabla\beta|^2)d\underline{u}+\epsilon(\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2), \quad (7.198)$$

which is controlled by mean of Grönwall. Collection of all terms yields the result[7]. $\square$

[7]The section curvature is estimated by means of the null Hamiltonian constraint 2.45

As a consequence of the previous proposition (7.13), and the connection estimates section 7.1), we obtain the following lemma

Lemma 7.8. *The following estimates hold true in the exterior region*

$$\int_{H} r^6|\nabla\alpha|^2 + \int_{\underline{H}} r^6|\nabla\beta|^2 \leq \int_{\widehat{\Sigma}_0} r^6|\nabla\alpha|^2 + \int_{\widehat{\Sigma}_0} r^6|\nabla\beta|^2 + \epsilon(\mathbf{W}+\mathbf{Y}). \tag{7.199}$$

Now we consider the pair $\underline{\alpha}$ and $\underline{\beta}$ and write

$$\int_{D_{u,\underline{u}}} \nabla_4(u^4r^2|\nabla\underline{\alpha}|^2) + \int_{D_{u,\underline{u}}} \nabla_3(u^4r^2|\nabla\underline{\beta}|^2) = \int_{H_u} u^4r^2|\nabla\underline{\beta}|^2 + \int_{\underline{H}_{\underline{u}}} u^4r^2|\nabla\underline{\alpha}|^2$$
$$- \int_{\widehat{\Sigma}_0} u^4r^2|\nabla\underline{\alpha}|^2 - \int_{\widehat{\Sigma}_0} u^4r^2|\nabla\underline{\beta}|^2 + \int_{D_{u,\underline{u}}} (2\underline{\omega}-\mathrm{tr}\underline{\chi})u^4r^2|\nabla\underline{\beta}|^2 + \int_{D_{u,\underline{u}}} (2\omega-\mathrm{tr}\chi)u^4r^2|\nabla\underline{\alpha}|^2.$$

Now calculate

$$\int_{D_{u,\underline{u}}} \nabla_4(u^4r^2|\nabla\underline{\alpha}|^2) + \int_{D_{u,\underline{u}}} \nabla_3(u^4r^2|\nabla\underline{\beta}|^2) \tag{7.200}$$
$$= \int_{D_{u,\underline{u}}} u^4r\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\nabla\underline{\alpha}|^2 + 2u^4r^2\langle\nabla\underline{\alpha}, \nabla\nabla_4\underline{\alpha} + [\nabla_4,\nabla]\underline{\alpha}\rangle + 4\Omega^{-1}u^3r^2|\nabla\underline{\beta}|^2$$
$$+u^4r\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\nabla\underline{\beta}|^2 + 2u^4r^2\langle\nabla\underline{\beta}, \nabla\nabla_3\underline{\beta} + [\nabla_3,\nabla]\underline{\beta}\rangle$$
$$= \int_{D_{u,\underline{u}}} u^4r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\nabla\underline{\alpha}|^2 + 2u^4r^2\langle\nabla\underline{\alpha}, \nabla(-\frac{1}{2}\mathrm{tr}\chi\underline{\alpha} - (\nabla\hat{\otimes}\underline{\beta})_{ab} + 4\omega\underline{\alpha}_{ab}$$
$$-3(\underline{\hat{\chi}}_{ab}\rho - {}^*\underline{\hat{\chi}}_{ab}\sigma) + ((\zeta-4\underline{\eta})\hat{\otimes}\underline{\beta})_{ab} + \frac{1}{2}(D_4\mathfrak{T}_{33} - D_3\mathfrak{T}_{34})\gamma_{ab} - D_3\mathfrak{T}_{ab}$$
$$+\frac{1}{2}\left(D_a\mathfrak{T}_{3b} + D_b\mathfrak{T}_{3a}\right)\rangle + 2u^4r^2\langle\nabla\underline{\alpha}, -\frac{\mathrm{tr}\chi}{2}\nabla\underline{\alpha} - \widehat{\chi}\cdot\nabla\underline{\alpha} + \mathcal{E}_1^{[\nabla_4,\nabla]\underline{\alpha}}\rangle$$
$$+4\Omega^{-1}u^3r^2|\nabla\underline{\beta}|^2 + u^4r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\nabla\underline{\beta}|^2 + 2u^4r^2\langle\nabla\underline{\beta}, \nabla(-2tr\underline{\chi}\underline{\beta}_a - (\mathrm{div}\underline{\alpha})_a - 2\underline{\omega}\underline{\beta}_a$$
$$+(\underline{\eta}\cdot\underline{\alpha})_a + \frac{1}{2}(D_a\mathfrak{T}_{33} - D_3\mathfrak{T}_{3a})\rangle + 2u^4r^2\langle\nabla\underline{\beta}, -\frac{\mathrm{tr}\underline{\chi}}{2}\nabla\underline{\beta} - \widehat{\underline{\chi}}\cdot\nabla\underline{\beta} + \mathcal{E}_2^{[\nabla_3,\nabla]\underline{\beta}}\rangle.$$

Collecting all the terms yields

$$\int_{H_u} u^4r^2|\nabla\underline{\beta}|^2 + \int_{\underline{H}_{\underline{u}}} u^4r^2|\nabla\underline{\alpha}|^2 - \int_{D_{u,\underline{u}}} u^4r^2(\Omega^{-1}\overline{\Omega\mathrm{tr}\chi} - \mathrm{tr}\chi)|\nabla\underline{\alpha}|^2 \tag{7.201}$$
$$- \int_{D_{u,\underline{u}}} (4\Omega^{-1}u^3r^2 - 3u^4r^2\mathrm{tr}\underline{\chi})|\nabla\underline{\beta}|^2 + \int_{D_{u,\underline{u}}} u^4r^2(\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}} - \mathrm{tr}\underline{\chi})|\nabla\underline{\beta}|^2$$
$$= \int_{\widehat{\Sigma}_0} u^4r^2|\nabla\underline{\alpha}|^2 + \int_{\widehat{\Sigma}_0} u^4r^2|\nabla\underline{\beta}|^2 + \mathcal{E}^{\nabla\underline{\alpha},\nabla\underline{\beta}},$$

where the error term $\mathcal{E}^{\nabla\underline{\alpha},\nabla\underline{\beta}}$ is given as follows

$$\mathcal{E}^{\nabla\underline{\alpha},\nabla\underline{\beta}} \sim \int_{D_{u,\underline{u}}} 2u^4r^2\langle\nabla\underline{\alpha}, -\frac{1}{2}\nabla(\mathrm{tr}\chi)\underline{\alpha} + \nabla\left(4\omega\underline{\alpha}_{ab} - 3(\underline{\hat{\chi}}_{ab}\rho - {}^*\underline{\hat{\chi}}_{ab}\sigma) + ((\zeta-4\underline{\eta})\hat{\otimes}\underline{\beta})_{ab}\right.$$
$$\left.+\frac{1}{2}(D_4\mathfrak{T}_{33} - D_3\mathfrak{T}_{34})\gamma_{ab} - D_3\mathfrak{T}_{ab} + \frac{1}{2}\left(D_a\mathfrak{T}_{3b} + D_b\mathfrak{T}_{3a}\right)\right)\rangle + 2u^4r^2\langle\nabla\underline{\alpha}, -\widehat{\chi}\cdot\nabla\underline{\alpha} + \mathcal{E}_1^{[\nabla_4,\nabla]\underline{\alpha}}\rangle$$
$$+2u^4r^2\langle\nabla\underline{\beta}, -2\nabla(\mathrm{tr}\underline{\chi})\underline{\beta}_a - \nabla\left(2\underline{\omega}\underline{\beta}_a + (\underline{\eta}\cdot\underline{\alpha})_a + \frac{1}{2}(D_a\mathfrak{T}_{33} - D_3\mathfrak{T}_{3a})\right)\rangle$$
$$+2u^4r^2\langle\nabla\underline{\beta}, -\widehat{\underline{\chi}}\cdot\nabla\underline{\beta} + \mathcal{E}_2^{[\nabla_3,\nabla]\underline{\beta}}\rangle$$

Proposition 7.14. *The error term $\mathcal{E}_{\nabla\alpha,\nabla\beta}$ verifies the following estimate*

$$|\mathcal{E}_{\nabla\underline{\alpha},\nabla\underline{\beta}}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{W}_1 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2) \tag{7.202}$$

Proof.

$$\begin{aligned}|\int_{D_{u,\underline{u}}} u^4 r^2 \nabla\underline{\alpha}\nabla(\mathrm{tr}\chi)\underline{\alpha}| &\lesssim \int_{u,\underline{u}} ||u^2 r\nabla\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}||r^{\frac{1}{2}}|u|^2\underline{\alpha}||_{L^4(S_{u,\underline{u}})}r^{-2}\\ &\lesssim \sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + u^4r^2|\nabla\underline{\alpha}|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1)\end{aligned}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2|\nabla\underline{\alpha}|^2\omega| \lesssim \sup_{\mathcal{K}}(r^2|\omega|)(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}^{'})\int_{\underline{H}} u^4r^2|\nabla\underline{\alpha}|^2 \lesssim \epsilon\mathcal{W}_1 \tag{7.203}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}(\nabla\omega)\underline{\alpha}| &\lesssim \int_{u,\underline{u}} ||u^2r\nabla\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{5}{2}}\nabla\omega||_{L^4(S_{u,\underline{u}})}||r^{\frac{1}{2}}|u|^2\underline{\alpha}||_{L^4(S_{u,\underline{u}})}r^{-2}\\ &\lesssim \sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\omega||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2+u^4r^2|\nabla\underline{\alpha}|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1)\end{aligned}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\nabla\hat{\underline{\chi}}\rho| &\lesssim \int_{u,\underline{u}} ||u^2r\nabla\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}|u|\nabla\hat{\underline{\chi}}||_{L^4(S_{u,\underline{u}})}||r^{\frac{5}{2}}\rho||_{L^4(S_{u,\underline{u}})}|u|r^{-3}\\ &\lesssim \sup_{u,\underline{u}}||r^{\frac{3}{2}}|u|\nabla\hat{\underline{\chi}}||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} |u|r^{-3}d\underline{u}^{'})\int_{\underline{H}}(r^4\rho^2+r^6|\nabla\rho|^2+u^4r^2|\nabla\underline{\alpha}|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1)\end{aligned}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}(\nabla(\eta,\underline{\eta}))\underline{\beta}| &\lesssim \int_{u,\underline{u}} ||u^2r\nabla\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{5}{2}}\nabla(\eta,\underline{\eta})||_{L^4(S_{u,\underline{u}})}||r^{\frac{3}{2}}|u|\underline{\beta}||_{L^4(S_{u,\underline{u}})}|u|r^{-3}\\ &\lesssim \sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla(\eta,\underline{\eta})||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} |u|r^{-3}d\underline{u}^{'})\int_{\underline{H}}(u^4|\underline{\alpha}|^2+u^4r^2|\nabla\underline{\alpha}|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1).\end{aligned}$$

The remaining pure gravity terms are of lower order. Now we estimate the Yang-Mills coupling terms

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}_{AB}\nabla D_3\mathfrak{T}_{AB}|. \tag{7.204}$$

An explicit calculation yields

$$\begin{aligned}\nabla D_3\mathfrak{T}_{AB} \sim\ & \widehat{\nabla}\rho^F\cdot\widehat{\nabla}\underline{\alpha}^F + \rho^F\cdot\widehat{\nabla}^2\underline{\alpha}^F + (\nabla\mathrm{tr}\underline{\chi})\rho^F\cdot\rho^F + \mathrm{tr}\underline{\chi}\widehat{\nabla}\rho^F\cdot\rho^F + \nabla(\eta,\underline{\eta})\underline{\alpha}^F\cdot\rho^F\\ &+(\eta+\underline{\eta})\widehat{\nabla}\underline{\alpha}^F\cdot\rho^F + \widehat{\nabla}\alpha^F\cdot\widehat{\nabla}_3\underline{\alpha}^F + \underline{\alpha}^F\cdot\widehat{\nabla}^2\rho^F + \mathrm{tr}\underline{\chi}\nabla\underline{\alpha}^F\cdot\alpha^F + \mathrm{tr}\underline{\chi}\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F + \nabla\mathrm{tr}\underline{\chi}\alpha^F\cdot\underline{\alpha}^F\end{aligned}$$

We estimate each term separately

$$\begin{aligned}|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\widehat{\nabla}\rho^F\cdot\widehat{\nabla}\underline{\alpha}^F| &\lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^3|\widehat{\nabla}\rho^F|)(\int_{u_0}^{\underline{u}}|u|r^{-3}d\underline{u}^{'})\int_{\underline{H}}(u^4r^2|\nabla\underline{\alpha}|^2+u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2)\\ &\lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_1)\end{aligned}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\rho^F\cdot\widehat{\nabla}^2\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}} r^2|u|^{\frac{1}{2}}|\rho^F|(\int_{\underline{u}_0}^{\underline{u}}|u|r^{-3}d\underline{u}')\int_{\underline{H}}(u^4r^2|\nabla\underline{\alpha}|^2+r^4|u|^2|\widehat{\nabla}^2\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_2),$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\nabla\mathrm{tr}\underline{\chi}\rho^F\cdot\rho^F| \lesssim \sup_{u,\underline{u}}(||r^{\frac{5}{2}}\nabla\mathrm{tr}\underline{\chi}||_{L^4(S_{u,\underline{u}})})\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)$$
$$\int_{u,\underline{u}}||u^2r\nabla\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}\rho^F||_{L^4(S_{u,\underline{u}})}|u|^{\frac{1}{2}}r^{-5} \lesssim \epsilon^2\int_{\underline{H}}(u^4r^2|\nabla\underline{\alpha}|^2+r^2|\rho^F|^2+r^4|\widehat{\nabla}\rho^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}_3\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}} r^{\frac{7}{2}}|\widehat{\nabla}\alpha^F|(\int_{\underline{u}_0}^{u}r^{-\frac{5}{2}}d\underline{u}')\int_{\underline{H}}(u^4r^2|\nabla\underline{\alpha}|^2+u^4|\widehat{\nabla}_3\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_1)$$

Similarly

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\mathrm{tr}\underline{\chi}\widehat{\nabla}\rho^F\cdot\rho^F| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.205}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\nabla(\eta,\underline{\eta})\underline{\alpha}^F\cdot\rho^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.206}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}(\eta,\underline{\eta})\widehat{\nabla}\underline{\alpha}^F\cdot\rho^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.207}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\underline{\alpha}^F\cdot\widehat{\nabla}^2\rho^F| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_2), \tag{7.208}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\mathrm{tr}\underline{\chi}\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.209}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\nabla\mathrm{tr}\underline{\chi}\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.210}$$

Collecting all the terms, we obtain

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}_{AB}\nabla D_3\mathfrak{T}_{AB}| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.211}$$

Now we need to estimate the coupling term

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}_{AB}\nabla\nabla_A\mathfrak{T}_{3B}| \tag{7.212}$$

An explicit calculation yields

$$\nabla\nabla\mathfrak{T}_{3A} \sim \widehat{\nabla}^2\underline{\alpha}^F\cdot(\rho^F,\sigma^F)+\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\underline{\alpha}^F\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)+\nabla(\eta,\underline{\eta})(\rho^F,\sigma^F)\cdot\underline{\alpha}^F$$
$$+(\eta,\underline{\eta})\widehat{\nabla}(\rho^F,\sigma^F)\cdot\underline{\alpha}^F+(\eta,\underline{\eta})(\rho^F,\sigma^F)\cdot\widehat{\nabla}\underline{\alpha}^F+\nabla(\mathrm{tr}\chi,\hat{\chi})(|\rho^F,\sigma^F|^2+\alpha^F\cdot\underline{\alpha}^F)$$
$$+(\mathrm{tr}\underline{\chi},\underline{\hat{\chi}})(\rho^F\cdot\widehat{\nabla}\rho^F+\sigma^F\cdot\widehat{\nabla}\sigma^F+\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F+\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F)$$

We estimate

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\alpha}\widehat{\nabla}^2\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| \lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|(\rho^F,\sigma^F)|(\int_{\underline{u}_0}^{\underline{u}}|u|^{\frac{1}{2}}r^{-3}d\underline{u}') \tag{7.213}$$

$$\int_{\underline{H}}(u^4r^2|\nabla\alpha|^2+u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2)\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2)$$

$$|\int_{D_{u,\underline{u}}}u^4r^2\nabla\underline{\alpha}\widehat{\nabla}^2(\rho^F,\sigma^F)\cdot\underline{\alpha}^F|\lesssim\sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r|\underline{\alpha}^F|)|(\int_{\underline{u}_0}^{\underline{u}}|u|^{\frac{1}{2}}r^{-3}d\underline{u}') \tag{7.214}$$
$$\int_{\underline{H}}(u^4r^2|\nabla\alpha|^2+r^6|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2)\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2)$$

Similarly we obtain

$$|\int_{D_{u,\underline{u}}}u^4r^2\nabla\underline{\alpha}\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}\underline{\alpha}^F|\lesssim\sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r^2|\widehat{\nabla}\underline{\alpha}^F|)|(\int_{\underline{u}_0}^{\underline{u}}|u|^{\frac{1}{2}}r^{-3}d\underline{u}') \tag{7.215}$$
$$\int_{\underline{H}}(u^4r^2|\nabla\alpha|^2+r^4|\widehat{\nabla}(\rho^F,\sigma^F)|^2)\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1)$$

Remaining terms are of lower order and can be estimated by means of the boot-strap assumption. Collecting all the terms one obtains

$$|\int_{D_{u,\underline{u}}}u^4r^2\nabla\underline{\alpha}_{AB}\nabla\nabla_A\mathfrak{T}_{3B}|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.216}$$

Notice that all three terms are equivalent in terms of the scaling. Now consider

$$|\int_{D_{u,\underline{u}}}u^4r^2\nabla_A\underline{\beta}_B\nabla_A(D_B\mathfrak{T}_{33}-D_3\mathfrak{T}_{B3})|. \tag{7.217}$$

An explicit calculations yield

$$\begin{aligned}\nabla(D_B\mathfrak{T}_{33}-D_3\mathfrak{T}_{B3})&\sim\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\underline{\alpha}^F+\underline{\alpha}^F\cdot\widehat{\nabla}^2\underline{\alpha}^F+\nabla\underline{\chi}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)+\underline{\chi}\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)\\&+\underline{\chi}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\nabla(\eta,\underline{\eta})\underline{\alpha}^F\cdot\underline{\alpha}^F+(\eta,\underline{\eta})\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F+\nabla\underline{\omega}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)+\underline{\omega}\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)\\&+\underline{\omega}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\widehat{\nabla}\widehat{\nabla}_3\underline{\alpha}^F\cdot(\rho^F,\sigma^F)+\widehat{\nabla}_3\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F).\end{aligned}$$

We estimate the top order terms

$$\begin{aligned}|\int_{D_{u,\underline{u}}}u^4r^2\nabla\underline{\beta}\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\underline{\alpha}^F|&\lesssim\sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r^2|\widehat{\nabla}\underline{\alpha}^F|)(\int_{\underline{u}_0}^{\underline{u}}|u|^{-\frac{1}{2}}r^{-2}d\underline{u}')\int_{\underline{H}}(u^4r^2|\nabla\underline{\beta}|^2+u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2)\\&\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1)\end{aligned}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}}u^4r^2\nabla\underline{\beta}\underline{\alpha}^F\cdot\widehat{\nabla}^2\underline{\alpha}^F|&\lesssim\sup_{\mathcal{K}}|u|^{\frac{3}{2}}r|\underline{\alpha}^F|(\int_{\underline{u}_0}^{\underline{u}}|u|^{-\frac{1}{2}}r^{-2}d\underline{u}')\int_{\underline{H}}(u^4r^2|\nabla\underline{\beta}|^2+u^2r^4|\widehat{\nabla}^2\alpha^F|^2)\\&\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1)\end{aligned}$$

$$|\int_{D_{u,\underline{u}}}u^4r^2\nabla\underline{\beta}\widehat{\nabla}\widehat{\nabla}_3\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| \tag{7.218}$$
$$\begin{aligned}&\lesssim\sup_{\mathcal{K}}|u|^{\frac{1}{2}}r^2|\rho^F,\sigma^F|(\int_{\underline{u}_0}^{\underline{u}}|u|^{-\frac{1}{2}}r^{-2}d\underline{u}')\int_{\underline{H}}(u^4r^2|\nabla\underline{\beta}|^2+u^4r^2|\widehat{\nabla}\widehat{\nabla}_3\underline{\alpha}^F|^2)\\&\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2).\end{aligned}$$

The remaining terms are lower orders and are controlled as follows

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta}\nabla\mathrm{tr}\chi\underline{\alpha}^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.219}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta\chi}\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.220}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta\chi\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.221}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta}\nabla(\eta,\underline{\eta})\underline{\alpha}^F\cdot\underline{\alpha}^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.222}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta}(\eta,\underline{\eta})\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.223}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta}\nabla\underline{\omega\alpha}^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.224}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta\omega}\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.225}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta\omega\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.226}$$

$$|\int_{D_{u,\underline{u}}} u^4r^2\nabla\underline{\beta}\widehat{\nabla}_3\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1). \tag{7.227}$$

□

As a consequence, we obtain the following lemma in the exterior domain

Lemma 7.9.

$$\int_{H_u} u^4r^2|\nabla\underline{\beta}|^2+\int_{\underline{H}_{\underline{u}}} u^4r^2|\nabla\underline{\alpha}|^2\le\int_{\widehat{\Sigma}_0} u^4r^2|\nabla\underline{\alpha}|^2+\int_{\widehat{\Sigma}_0} u^4r^2|\nabla\underline{\beta}|^2+\epsilon(\mathbf{W}+\mathbf{Y}) \tag{7.228}$$

Remark 6. *The Sectional curvature K is estimated by means of the null Hamiltonian constraint (2.45).*

Now consider β and (ρ,σ) and write

$$\begin{aligned}\int_{D_{u,\underline{u}}}\nabla_3(r^6|\nabla\beta|^2)+\int_{D_{u,\underline{u}}}\nabla_4(r^6(|\nabla\rho|^2+|\nabla\sigma|^2))&=\int_H r^6|\nabla\beta|^2+\int_{\underline{H}} r^6(|\nabla\rho|^2+|\nabla\sigma|^2)\\&-\int_{\widehat{\Sigma}_0} r^6|\nabla\beta|^2-\int_{\widehat{\Sigma}_0} r^6(|\nabla\rho|^2+|\nabla\sigma|^2)+\int_{D_{u,\underline{u}}}(2\omega-\mathrm{tr}\chi)r^6(|\nabla\rho|^2+|\nabla\sigma|^2)\\&+\int_{D_{u,\underline{u}}}(2\underline{\omega}-\mathrm{tr}\underline{\chi})r^6|\nabla\beta|^2.\end{aligned}$$

Explicit calculations yield

$$\begin{aligned}&\int_{D_{u,\underline{u}}}\nabla_3(r^6|\nabla\beta|^2)+\int_{D_{u,\underline{u}}}\nabla_4(r^6(|\nabla\rho|^2+|\nabla\sigma|^2)) \\&=\int_{D_{u,\underline{u}}}3r^6\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}(|\nabla\rho|^2+|\nabla\sigma|^2)+\int_{D_{u,\underline{u}}}2r^6\left(\langle\nabla\rho,\nabla\nabla_4\rho+[\nabla_4,\nabla]\rho\rangle\right.\end{aligned} \tag{7.229}$$

$$+\langle\nabla\sigma,\nabla\nabla_4\sigma+[\nabla_4,\nabla]\sigma\rangle\big)+3\int_{D_{u,\underline{u}}}r^6\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\nabla\beta|^2$$
$$+\int_{D_{u,\underline{u}}}2r^6\langle\nabla\beta,\nabla\nabla_3\beta+[\nabla_3,\nabla]\beta\rangle$$
$$=\int_{D_{u,\underline{u}}}3r^6\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}(|\nabla\rho|^2+|\nabla\sigma|^2)+3\int_{D_{u,\underline{u}}}r^6\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\nabla\beta|^2$$
$$+\int_{D_{u,\underline{u}}}2r^6\langle\nabla\rho,\nabla(-\frac{3}{2}\mathrm{tr}\chi\rho+\mathrm{div}\beta-\frac{1}{2}\hat{\underline{\chi}}\cdot\alpha+\zeta\cdot\beta+2\underline{\eta}\cdot\beta-\frac{1}{4}(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{34})$$
$$-\frac{\mathrm{tr}\chi}{2}\nabla\rho-\widehat{\chi}\cdot\nabla\rho+\mathcal{E}^{[\nabla_4,\nabla]\rho}\rangle+\int_{D_{u,\underline{u}}}2r^6\langle\nabla\sigma,\nabla(-\frac{3}{2}\mathrm{tr}\chi\sigma-\mathrm{div}\ {}^*\beta+\frac{1}{2}\hat{\underline{\chi}}\cdot\ {}^*\alpha$$
$$-\zeta\cdot\ {}^*\beta-2\underline{\eta}\cdot\ {}^*\beta-\frac{1}{4}(D_\mu\mathfrak{T}_{4\nu}-D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34})-\frac{\mathrm{tr}\chi}{2}\nabla\sigma-\widehat{\chi}\cdot\nabla\sigma+\mathcal{E}^{[\nabla_4,\nabla]\sigma}\rangle$$
$$+\int_{D_{u,\underline{u}}}3r^6\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\nabla\beta|^2+2r^6(\langle\nabla\beta,\nabla(-tr\underline{\chi}\beta_a+\nabla_a\rho+\ {}^*\nabla_a\sigma+2\underline{\omega}\beta_a+2(\hat{\chi}\cdot\underline{\beta})_a$$
$$+3(\eta\rho+\ {}^*\eta\sigma)_a+\frac{1}{2}(D_a\mathfrak{T}_{34}-D_4\mathfrak{T}_{3a}))-\frac{\mathrm{tr}\underline{\chi}}{2}\nabla\beta-\widehat{\chi}\cdot\nabla\beta+\mathcal{E}^{[\nabla_3,\nabla]\beta}\rangle.$$

Collecting all the terms

$$\int_H r^6|\nabla\beta|^2+\int_{\underline{H}}r^6(|\nabla\rho|^2+|\nabla\sigma|^2)+\int_{D_{u,\underline{u}}}3r^6(\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-3\mathrm{tr}\chi)(|\nabla\rho|^2+|\nabla\sigma|^2)$$
$$+\int_{D_{u,\underline{u}}}r^6(3\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-2\mathrm{tr}\underline{\chi})|\nabla\beta|^2=\int_{\widehat{\Sigma}_0}r^6|\nabla\beta|^2+\int_{\widehat{\Sigma}_0}r^6(|\nabla\rho|^2+|\nabla\sigma|^2)+\mathcal{E}^{\nabla\beta,\nabla(\rho,\sigma)},$$

where the error term $\mathcal{E}^{\nabla\beta,\nabla(\rho,\sigma)}$ can be written explicitly as follows

$$\mathcal{E}^{\nabla\beta,\nabla(\rho,\sigma)}=\int_{D_{u,\underline{u}}}2r^6\langle\nabla\rho,-\frac{3}{2}\nabla(\mathrm{tr}\chi)\rho-\nabla\left(\frac{1}{2}\hat{\underline{\chi}}\cdot\alpha+\zeta\cdot\beta+2\underline{\eta}\cdot\beta-\frac{1}{4}(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{34})\right)$$
$$-\widehat{\chi}\cdot\nabla\rho+\mathcal{E}^{[\nabla_4,\nabla]\rho}\rangle+\int_{D_{u,\underline{u}}}2r^6\langle\nabla\sigma,-\frac{3}{2}\nabla(\mathrm{tr}\chi)\sigma+\frac{1}{2}\hat{\underline{\chi}}\cdot\ {}^*\alpha$$
$$-\zeta\cdot\ {}^*\beta-2\underline{\eta}\cdot\ {}^*\beta-\frac{1}{4}(D_\mu\mathfrak{T}_{4\nu}-D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34})-\widehat{\chi}\cdot\nabla\sigma+\mathcal{E}^{[\nabla_4,\nabla]\sigma}\rangle$$
$$+\int_{D_{u,\underline{u}}}2r^6(\langle\nabla\beta,-\nabla(tr\underline{\chi})\beta_a+\nabla\Big(2\underline{\omega}\beta_a+2(\hat{\chi}\cdot\underline{\beta})_a$$
$$+3(\eta\rho+\ {}^*\eta\sigma)_a+\frac{1}{2}(D_a\mathfrak{T}_{34}-D_4\mathfrak{T}_{3a})\Big)-\widehat{\chi}\cdot\nabla\beta+\mathcal{E}^{[\nabla_3,\nabla]\beta}\rangle.$$

Proposition 7.15. *The error term $\mathcal{E}^{\nabla\beta,\nabla(\rho,\sigma)}$ verifies the following estimate*

$$|\mathcal{E}^{\nabla\beta,\nabla(\rho,\sigma)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2)\tag{7.230}$$

Proof. We estimate each term separately.

$$|\int_{D_{u,\underline{u}}}r^6\nabla\rho\nabla\mathrm{tr}\chi\rho|\lesssim\int_{u,\underline{u}}||r^3\nabla\rho||_{L^2(S_{u,\underline{u}})}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}||r^{\frac{5}{2}}\rho||_{L^4(S_{u,\underline{u}})}r^{-2}\tag{7.231}$$
$$\lesssim\sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}}r^{-2})\int_{\underline{H}}(r^6|\nabla\rho|^2+r^4|\rho|^2)\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1).$$

The remaining gravity self-coupling terms are of lower order. We focus on the Yang-Mills coupling terms. σ enjoys same property as ρ and therefore estimating only one these coupling is sufficient. We want to estimate

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\nabla(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})|. \tag{7.232}$$

An explicit calculation yields

$$\begin{aligned}\nabla(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43}) &\sim \mathrm{tr}\underline{\chi}\alpha^F\cdot\widehat{\nabla}\alpha^F+\nabla\mathrm{tr}\underline{\chi}\alpha^F\cdot\alpha^F+\alpha^F\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)+\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)\\ &+(\rho^F,\sigma^F)\cdot\widehat{\nabla}^2\alpha^F+\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\nabla\eta\alpha^F\cdot(\rho^F,\sigma^F)+\eta\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\underline{\omega}\widehat{\nabla}\alpha^F\cdot\alpha^F\\ +\nabla\underline{\omega}\alpha^F\cdot\alpha^F&+\nabla\hat{\chi}\alpha^F\cdot\underline{\alpha}^F+\hat{\chi}\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F+\hat{\chi}\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F+\underline{\eta}\widehat{\nabla}\rho^F\cdot\underline{\alpha}^F+\underline{\eta}\rho^F\cdot\widehat{\nabla}\underline{\alpha}^F+\widehat{\nabla}\underline{\eta}\rho^F\cdot\underline{\alpha}^F\end{aligned}$$

The top order terms are estimated as follows

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^6\nabla\rho\alpha^F\cdot\widehat{\nabla}^2(\sigma^F,\rho^F)| &\lesssim \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{\underline{u}_0}^{\underline{u}} r^{-\frac{5}{2}})\int_{\underline{H}}(r^6|\nabla\rho|^2+r^6|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2)\\ &\lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_2)\end{aligned}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^6\nabla\rho\rho^F\cdot\widehat{\nabla}^2\alpha^F| &\lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\rho^F|)(\int_{u_0}^{u}|u'|^{-\frac{3}{2}}r^{-1}du')\int_{H}(u^2r^4|\nabla\rho|^2+r^6|\widehat{\nabla}^2\alpha^F|^2)\\ &\lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_2).\end{aligned}$$

The remaining terms satisfy the following estimates

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.233}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\nabla\eta\alpha^F\cdot(\rho^F,\sigma^F)| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.234}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\eta\alpha^F\widehat{\nabla}(\rho^F,\sigma^F)| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.235}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\underline{\omega}\widehat{\nabla}\alpha^F\cdot\alpha^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.236}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\nabla\underline{\omega}\alpha^F\cdot\alpha^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.237}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\nabla\hat{\chi}\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.238}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\hat{\chi}\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.239}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\hat{\chi}\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.240}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\underline{\eta}\widehat{\nabla}\rho^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.241}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\underline{\eta}\rho^F\cdot\widehat{\nabla}\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.242}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\nabla\underline{\eta}\rho^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.243}$$

Collecting all the terms yields

$$|\int_{D_{u,\underline{u}}} r^6\nabla\rho\nabla(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.244}$$

Now we estimate

$$|\int_{D_{u,\underline{u}}} r^6\nabla_B\beta_A\nabla_B(D_A\mathfrak{T}_{34}-D_4\mathfrak{T}_{3A})|. \tag{7.245}$$

An explicit calculation yields

$$\begin{aligned}\nabla(D_A\mathfrak{T}_{34}-D_4\mathfrak{T}_{3A})\sim\ &\widehat{\nabla}^2(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)+\nabla\underline{\chi}\alpha^F\cdot(\rho^F,\sigma^F)+\underline{\chi}\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)+\underline{\chi}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)\\ &+\nabla\mathrm{tr}\chi\underline{\alpha}^F\cdot(\rho^F,\sigma^F)+\mathrm{tr}\chi\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)+\mathrm{tr}\chi\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\nabla\underline{\eta}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)\\ &+\underline{\eta}\widehat{\nabla}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)+\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\alpha^F+\underline{\alpha}^F\cdot\widehat{\nabla}^2\alpha^F+\nabla(\eta,\underline{\eta})\underline{\alpha}^F\cdot\alpha^F+(\eta,\underline{\eta})\underline{\alpha}^F\cdot\widehat{\nabla}\alpha^F\\ &+(\eta,\underline{\eta})\widehat{\nabla}\underline{\alpha}^F\cdot\alpha^F\end{aligned}$$

The top order terms are estimated as follows

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^6\nabla\beta\widehat{\nabla}^2(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)|&\lesssim\sup_{\mathcal{K}}(r^2|u|^{\frac12}|\rho^F|)(\int_{u_0}^{u}|u^{'}|^{-\frac32}r^{-1}du^{'})\int_H(r^6|\nabla\beta|^2+u^2r^4|\widehat{\nabla}^2\rho^F|^2)\\ &\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2).\end{aligned}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^6\nabla\beta\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\alpha^F|&\lesssim\int_{u,\underline{u}}||r^3\nabla\beta||_{L^2(S_{u,\underline{u}})}||r^{\frac32}|u|^{\frac32}|\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}||r^{\frac52}\widehat{\nabla}\alpha^F||_{L^4(S_{u,\underline{u}})}r^{-1}|u|^{-\frac32}\\ &\lesssim\sup_{u,\underline{u}}||r^{\frac32}|u|^{\frac32}|\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u}|u^{'}|^{-\frac32}r^{-1}du^{'})\int_H(r^6|\nabla\beta|^2+r^4|\widehat{\nabla}\alpha^F|^2+r^6|\widehat{\nabla}^2\alpha^F|^2)\\ &\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2),\end{aligned}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} r^6\nabla\beta\underline{\alpha}^F\cdot\widehat{\nabla}^2\alpha^F|&\lesssim\sup_{\mathcal{K}}(|u|^{\frac32}r^{-1}|\underline{\alpha}^F|)(\int_{u_0}^{u}|u^{'}|^{-\frac32}r^{-1}du^{'})\int_H(r^6|\nabla\beta|^2+r^6|\widehat{\nabla}^2\alpha^F|^2)\\ &\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2).\end{aligned}$$

The remaining terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^6\nabla\beta\nabla\underline{\chi}\alpha^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.246}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\beta\underline{\chi}\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.247}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\beta\underline{\chi}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.248}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\beta\nabla\mathrm{tr}\chi\underline{\alpha}^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.249}$$

$$|\int_{D_{u,\underline{u}}} r^6\nabla\beta\mathrm{tr}\chi\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.250}$$

$$|\int_{D_{u,\underline{u}}} r^6 \nabla\beta \mathrm{tr}\chi \underline{\alpha}^F \cdot \widehat{\nabla}(\rho^F,\sigma^F)| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.251}$$

$$|\int_{D_{u,\underline{u}}} r^6 \nabla\beta\nabla\underline{\eta}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F) \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.252}$$

$$|\int_{D_{u,\underline{u}}} r^6 \nabla\beta\underline{\eta}\widehat{\nabla}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F) \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.253}$$

$$|\int_{D_{u,\underline{u}}} r^6 \nabla\beta\nabla(\eta,\underline{\eta})\underline{\alpha}^F\cdot\alpha^F \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.254}$$

$$|\int_{D_{u,\underline{u}}} r^6 \nabla\beta(\eta,\underline{\eta})\widehat{\nabla}\underline{\alpha}^F\cdot\alpha^F \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.255}$$

$$|\int_{D_{u,\underline{u}}} r^6 \nabla\beta(\eta,\underline{\eta})\underline{\alpha}^F\cdot\widehat{\nabla}\alpha^F \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1) \tag{7.256}$$

Collection of all the terms concludes the proof. □

As a consequence of the previous proposition, we obtain the following lemma in the exterior domain

Lemma 7.10. *The following estimates hold in the exterior domain*

$$\int_H r^6|\nabla\beta|^2+\int_{\underline{H}} r^6(|\nabla\rho|^2+|\nabla\sigma|^2) \leq \int_{\widehat{\Sigma}_0} r^6|\nabla\beta|^2+\int_{\widehat{\Sigma}_0} r^6(|\nabla\rho|^2+|\nabla\sigma|^2)+\epsilon(\mathbf{W}+\mathbf{Y}),$$

The last remaining pair that need to be estimated is $\{\underline{\beta},(\rho,\sigma)\}$. An application of the integration lemma (7.1) yields

$$\begin{aligned}
&\int_{D_{u,\underline{u}}} \nabla_4(r^4u^2|\nabla\underline{\beta}|^2)+\int_{D_{u,\underline{u}}}\nabla_3(r^4u^2(|\nabla\rho|^2+\nabla\sigma|^2)) \\
&= \int_{\underline{H}} r^2u^2|\nabla\underline{\beta}|^2+\int_H r^2u^2(|\nabla\rho|^2+|\nabla\sigma|^2) \\
&-\int_{\widehat{\Sigma}_0} r^4u^2|\nabla\underline{\beta}|^2-\int_{\widehat{\Sigma}_0} r^4u^2(|\nabla\rho|^2+|\nabla\sigma|^2)+\int_{D_{u,\underline{u}}}(2\omega-\mathrm{tr}\chi)r^2u^2|\nabla\underline{\beta}|^2 \\
&+\int_{D_{u,\underline{u}}}(2\underline{\omega}-\mathrm{tr}\underline{\chi})r^2u^2(|\nabla\rho|^2+|\nabla\sigma|^2).
\end{aligned} \tag{7.257}$$

Now explicitly evaluate

$$\begin{aligned}
&\int_{D_{u,\underline{u}}} \nabla_4(r^4u^2|\nabla\underline{\beta}|^2)+\int_{D_{u,\underline{u}}}\nabla_3(r^4u^2(|\nabla\rho|^2+\nabla\sigma|^2)) \\
&= \int_{D_{u,\underline{u}}} 2r^4u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\nabla\underline{\beta}|^2+\int_{D_{u,\underline{u}}}2r^4u^2\langle\nabla\underline{\beta},\nabla\nabla_4\underline{\beta}+[\nabla_4,\nabla]\beta\rangle \\
&+\int_{D_{u,\underline{u}}}(2r^4u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}+2r^4u\Omega^{-1})|\nabla(\rho,\sigma)|^2 \\
&+\int_{D_{u,\underline{u}}}2r^4u^2\left(\langle\nabla(\rho,\sigma),\nabla\nabla_3(\rho,\sigma)+[\nabla_3,\nabla](\rho,\sigma)\rangle\right) \\
&\sim \int_{D_{u,\underline{u}}} 2r^4u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\nabla\underline{\beta}|^2+\int_{D_{u,\underline{u}}}2r^4u^2\langle\nabla\underline{\beta},\nabla(-\mathrm{tr}\chi\underline{\beta}_a-\nabla_a\rho+{}^*\nabla_a\sigma+2\omega\underline{\beta}_a \\
&+2(\underline{\hat{\chi}}\cdot\beta)_a-3(\underline{\eta}\rho-{}^*\underline{\eta}\sigma)_a-\frac{1}{2}(D_a\mathfrak{T}_{43}-D_3\mathfrak{T}_{4a}))-\frac{\mathrm{tr}\chi}{2}\nabla\underline{\beta}-\widehat{\chi}\cdot\nabla\underline{\beta}+\mathcal{E}^{[\nabla_4,\nabla]\underline{\beta}}\rangle
\end{aligned} \tag{7.258}$$

$$
\begin{aligned}
&+\int_{D_{u,\underline{u}}}(2r^4u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}+2r^4u\Omega^{-1})|\nabla(\rho,\sigma)|^2\\
&+\int_{D_{u,\underline{u}}}2r^4u^2\Bigg(\langle\nabla\rho,\nabla(-\frac{3}{2}\mathrm{tr}\underline{\chi}\rho-\mathrm{div}\underline{\beta}-\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}+\zeta\cdot\underline{\beta}-2\eta\cdot\underline{\beta}+\frac{1}{4}(D_3\mathfrak{T}_{34}-D_4\mathfrak{T}_{33}))\\
&-\frac{\mathrm{tr}\underline{\chi}}{2}\nabla\rho-\widehat{\chi}\cdot\nabla\rho+\mathcal{E}^{[\nabla_3,\nabla]\rho}\rangle\Bigg)+\int_{D_{u,\underline{u}}}2r^4u^2\langle\nabla\rho,\nabla(-\frac{3}{2}tr\underline{\chi}\sigma-\mathrm{div}\ {}^*\underline{\beta}+\frac{1}{2}\hat{\chi}\cdot\ {}^*\underline{\alpha}\\
&-\zeta\cdot\ {}^*\underline{\beta}-2\eta\cdot\ {}^*\underline{\beta}+\frac{1}{4}(D_\mu\mathfrak{T}_{3\nu}-D_\nu\mathfrak{T}_{3\mu})\epsilon^{\mu\nu}{}_{34}-\frac{\mathrm{tr}\underline{\chi}}{2}\nabla\sigma-\widehat{\chi}\cdot\nabla\sigma+\mathcal{E}^{[\nabla_3,\nabla]\sigma}\rangle.
\end{aligned}
$$

Collecting all the terms

$$
\begin{aligned}
&\int_{\underline{H}}r^4u^2|\nabla\underline{\beta}|^2+\int_H r^4u^2(|\nabla\rho|^2+|\nabla\sigma|^2)-\int_{D_{u,\underline{u}}}2r^4u^2(\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-\mathrm{tr}\chi)|\nabla\underline{\beta}|^2\\
&-\int_{D_{u,\underline{u}}}(2r^4u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}+2r^4u\Omega^{-1}-3r^4u^2\mathrm{tr}\underline{\chi})|\nabla(\rho,\sigma)|^2\\
&=\int_{\widehat{\Sigma}_0}r^4u^2|\nabla\underline{\beta}|^2+\int_{\widehat{\Sigma}_0}r^4u^2(|\nabla\rho|^2+|\nabla\sigma|^2)+\mathcal{E}^{\nabla\underline{\beta},\nabla(\rho,\sigma)},
\end{aligned}
$$

where the error term is explicitly computed as follows

$$
\begin{aligned}
\mathcal{E}^{\nabla\underline{\beta},\nabla(\rho,\sigma)}&\sim\int_{D_{u,\underline{u}}}2r^4u^2\langle\nabla\underline{\beta},-\nabla(\mathrm{tr}\chi)\underline{\beta}_a+2\nabla\Big(\omega\underline{\beta}_a\\
&+2(\underline{\hat{\chi}}\cdot\beta)_a-3(\underline{\eta}\rho-\ {}^*\underline{\eta}\sigma)_a-\frac{1}{2}(D_a\mathfrak{T}_{43}-D_3\mathfrak{T}_{4a})\Big)-\widehat{\chi}\cdot\nabla\underline{\beta}+\mathcal{E}^{[\nabla_4,\nabla]\underline{\beta}}\rangle\\
&+\int_{D_{u,\underline{u}}}2r^4u^2\Bigg(\langle\nabla\rho,-\nabla(\frac{3}{2}\mathrm{tr}\underline{\chi})\rho-\nabla\Big(\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}+\zeta\cdot\underline{\beta}-2\eta\cdot\underline{\beta}+\frac{1}{4}(D_3\mathfrak{T}_{34}-D_4\mathfrak{T}_{33})\Big)\\
&-\widehat{\chi}\cdot\nabla\rho+\mathcal{E}^{[\nabla_3,\nabla]\rho}\rangle\Bigg)+\int_{D_{u,\underline{u}}}2r^4u^2\langle\nabla\sigma,-\nabla(\frac{3}{2}tr\underline{\chi})\sigma+\nabla\Big(\frac{1}{2}\hat{\chi}\cdot\ {}^*\underline{\alpha}\\
&-\zeta\cdot\ {}^*\underline{\beta}-2\eta\cdot\ {}^*\underline{\beta}+\frac{1}{4}(D_\mu\mathfrak{T}_{3\nu}-D_\nu\mathfrak{T}_{3\mu})\epsilon^{\mu\nu}{}_{34}\Big)-\widehat{\chi}\cdot\nabla\sigma+\mathcal{E}^{[\nabla_3,\nabla]\sigma}\rangle
\end{aligned}
$$

Proposition 7.16. *The error term $\mathcal{E}_{\nabla\underline{\beta},\nabla(\rho,\sigma)}$ verifies the following estimates*

$$|\mathcal{E}_{\nabla\underline{\beta},\nabla(\rho,\sigma)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.259}$$

Proof. Once again, we only focus on the potentially dangerous coupling terms since gravitational self-coupling terms are easily controlled under the boot-strap assumption. In particular we ought to estimate

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla_B\underline{\beta}_A\nabla_B(D_A\mathfrak{T}_{43}-D_3\mathfrak{T}_{4A})|. \tag{7.260}$$

An explicit calculation yields

$$
\begin{aligned}
\nabla(D_A\mathfrak{T}_{43}-D_3\mathfrak{T}_{4A})&\sim\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}(\rho^F,\sigma^F)+(\rho^F,\sigma^F)\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)\\
&+\nabla\underline{\chi}\alpha^F\cdot(\rho^F,\sigma^F)+\underline{\chi}\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)+\underline{\chi}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\nabla\chi\underline{\alpha}^F\cdot(\rho^F,\sigma^F)\\
&+\chi\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)+\chi\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F).
\end{aligned}
\tag{7.261}
$$

$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\underline{\beta}(\rho^F,\sigma^F)\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)| \tag{7.262}$$
$$\lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\rho^F,\sigma^F|)(\int_{\underline{u}_0}^{\underline{u}}|u|^{\frac{1}{2}}r^{-3}d\underline{u}')\int_{\underline{H}}(u^2r^4|\nabla\underline{\beta}|^2+r^6|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2)$$
$$\lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_2),$$

$$\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\underline{\beta}\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}(\rho^F,\sigma^F)| \tag{7.263}$$
$$\lesssim \sup_{u,\underline{u}}(|||u|^{\frac{3}{2}}r^{\frac{3}{2}}\widehat{\nabla}(\rho^F,\sigma^F)||_{L^4(S_{u,\underline{u}})})(\int_{\underline{u}_0}^{\underline{u}}|u|^{-\frac{1}{2}}r^{-2}d\underline{u}')$$
$$\int_{\underline{H}}(u^2r^4|\nabla\underline{\beta}|^2+r^4|\widehat{\nabla}(\rho^F,\sigma^F)|^2+r^6|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2)\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2).$$

Similarly,

$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\underline{\beta}\nabla\underline{\chi}\alpha^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.264}$$
$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\underline{\beta}\underline{\chi}\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.265}$$
$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\underline{\beta}\underline{\chi}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.266}$$
$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\underline{\beta}\nabla\chi\underline{\alpha}^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.267}$$
$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\underline{\beta}\chi\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.268}$$
$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\underline{\beta}\chi\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_1). \tag{7.269}$$

Now we need to estimate the following term

$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\rho\nabla(D_3\mathfrak{T}_{34}-D_4\mathfrak{T}_{33})|. \tag{7.270}$$

An explicit calculation yields

$$\nabla(D_3\mathfrak{T}_{34}-D_4\mathfrak{T}_{33})\sim \mathrm{tr}\chi\underline{\alpha}^F\cdot\widehat{\nabla}\underline{\alpha}^F+\nabla\mathrm{tr}\chi\underline{\alpha}^F\cdot\underline{\alpha}^F+\underline{\alpha}^F\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)+\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)$$
$$+(\rho^F,\sigma^F)\cdot\widehat{\nabla}^2\underline{\alpha}^F+\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\nabla\underline{\eta}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)+\underline{\eta}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)+\omega\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F$$
$$+\nabla\omega\underline{\alpha}^F\cdot\underline{\alpha}^F+\nabla\underline{\hat{\chi}}\alpha^F\cdot\underline{\alpha}^F+\underline{\hat{\chi}}\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F+\underline{\hat{\chi}}\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F+\underline{\eta}\widehat{\nabla}\rho^F\cdot\underline{\alpha}^F+\eta\rho^F\cdot\widehat{\nabla}\alpha^F+\nabla\eta\rho^F\cdot\alpha^F.$$

$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\rho\underline{\alpha}^F\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)|\lesssim\sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r|\underline{\alpha}^F|)(\int_{u_0}^{u}|u|^{-\frac{3}{2}}r^{-1})\int_{H}(u^2r^4|\nabla\rho|^2+u^2r^4|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2$$
$$\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2),$$

$$|\int_{D_{u,\underline{u}}} u^2 r^4 \nabla\rho\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)|$$

$$\lesssim \sup_{u,\underline{u}}(|||u|^{\frac{3}{2}}r^{\frac{3}{2}}\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})})(\int_{u_0}^{u}|u|^{-\frac{3}{2}}r^{-1}du^{'})\int_H(u^2r^4|\nabla\rho|^2+u^2r^2|\widehat{\nabla}(\rho^F,\sigma^F)|^2+u^2r^6|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2$$

$$\lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2),$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\widehat{\nabla}^2\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| \lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\rho^F,\sigma^F|)(\int_{\underline{u}_0}^{\underline{u}}|u|^{\frac{1}{2}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}(r^6|\nabla\rho|^2+u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_2).$$

Similarly the other terms are estimated as follows

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\mathrm{tr}\chi\underline{\alpha}^F\cdot\widehat{\nabla}\underline{\alpha}^F| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.271}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\nabla\mathrm{tr}\chi\underline{\alpha}^F\cdot\underline{\alpha}^F| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.272}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\nabla\underline{\eta}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.273}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\underline{\eta}\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.274}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\underline{\eta}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.275}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\omega\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.276}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\nabla\omega\underline{\alpha}^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.277}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\nabla\hat{\chi}\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.278}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\hat{\chi}\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.279}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\hat{\chi}\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.280}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\eta\rho^F\cdot\widehat{\nabla}\alpha^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.281}$$

$$|\int_{D_{u,\underline{u}}}u^2r^4\nabla\rho\nabla\eta\rho^F\cdot\alpha^F| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.282}$$

Collecting of the all the terms, one obtains

$$|\mathcal{E}_{\nabla\underline{\beta},\nabla(\rho,\sigma)}| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.283}$$

$\square$

Now consider the Yang-Mills pairs (first place where non-linearities show up)

$$\int_{D_{u,\underline{u}}}\nabla_3(r^4|\widehat{\nabla}\alpha^F|^2)+\int_{D_{u,\underline{u}}}\nabla_4(r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)) = \int_H r^4|\widehat{\nabla}\alpha^F|^2 \tag{7.284}$$
$$+\int_{\underline{H}}r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)-\int_{\widehat{\Sigma}_0}r^4|\widehat{\nabla}\alpha^F|^2-\int_{\widehat{\Sigma}_0}r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)$$

$$+\int_{D_{u,\underline{u}}}(2\underline{\omega}-\mathrm{tr}\underline{\chi})r^4|\widehat{\nabla}\alpha^F|^2+\int_{D_{u,\underline{u}}}(2\omega-\mathrm{tr}\chi)r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)$$

Explicitly compute

$$\int_{D_{u,\underline{u}}}\nabla_3(r^4|\widehat{\nabla}\alpha^F|^2)+\int_{D_{u,\underline{u}}}\nabla_4(r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2))$$

$$=\int_{D_{u,\underline{u}}}2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2))+\int_{D_{u,\underline{u}}}2r^4\langle\widehat{\nabla}\rho^F,\widehat{\nabla}\widehat{\nabla}_4(\rho^F,\sigma^F)$$

$$+[\widehat{\nabla}_4,\widehat{\nabla}](\rho^F,\sigma^F))\rangle+\int_{D_{u,\underline{u}}}2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\widehat{\nabla}\alpha^F|^2+\int_{D_{u,\underline{u}}}2r^4\langle\widehat{\nabla}\alpha^F,\widehat{\nabla}\widehat{\nabla}_3\alpha^F+[\widehat{\nabla}_3,\widehat{\nabla}]\alpha^F\rangle$$

$$=\int_{D_{u,\underline{u}}}2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2))+\int_{D_{u,\underline{u}}}2r^4\langle\widehat{\nabla}\rho^F,\widehat{\nabla}(-\hat{\mathrm{div}}\alpha^F-\mathrm{tr}\chi\rho^F-(\eta-\underline{\eta})\cdot\alpha^F)$$

$$-\frac{\mathrm{tr}\chi}{2}\widehat{\nabla}\rho^F-\widehat{\chi}\cdot\widehat{\nabla}\rho^F+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\rho^F}\rangle+\int_{D_{u,\underline{u}}}2r^4\langle\widehat{\nabla}\sigma^F,\widehat{\nabla}(-\hat{\mathrm{curl}}\alpha^F-\mathrm{tr}\chi\sigma^F+(\eta-\underline{\eta})\cdot{}^*\alpha^F)$$

$$-\frac{\mathrm{tr}\chi}{2}\widehat{\nabla}\sigma^F-\widehat{\chi}\cdot\widehat{\nabla}\sigma+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\sigma^F}\rangle+\int_{D_{u,\underline{u}}}2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}|\widehat{\nabla}\alpha^F|^2+\int_{D_{u,\underline{u}}}2r^4\langle\widehat{\nabla}\alpha^F,\widehat{\nabla}(-\frac{1}{2}tr\underline{\chi}\alpha^F$$

$$-\widehat{\nabla}\rho^F+{}^*\widehat{\nabla}\sigma^F-2\,{}^*\eta\sigma^F+2\eta\rho^F+2\underline{\omega}\alpha^F-\hat{\chi}\cdot\underline{\alpha}^F)-\frac{\mathrm{tr}\underline{\chi}}{2}\widehat{\nabla}\alpha^F-\widehat{\chi}\cdot\widehat{\nabla}\alpha^F+\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}]\alpha^F}\rangle.$$

Collecting all the terms, we obtain

$$\int_H r^4|\widehat{\nabla}\alpha^F|^2+\int_{\underline{H}}r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)-\int_{D_{u,\underline{u}}}(2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-2r^4\mathrm{tr}\chi)|\widehat{\nabla}(\rho^F,\sigma^F)|^2$$

$$-\int_{D_{u,\underline{u}}}r^4(2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-\mathrm{tr}\underline{\chi})|\widehat{\nabla}\alpha^F|^2=\int_{\widehat{\Sigma}_0}r^4|\widehat{\nabla}\alpha^F|^2+\int_{\widehat{\Sigma}_0}r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)+\mathcal{E}^{\nabla\alpha^F,\nabla(\rho^F,\sigma^F)},$$

where $\mathcal{E}^{\nabla\alpha^F,\nabla(\rho^F,\sigma^F)}$ verifies

$$\mathcal{E}^{\nabla\alpha^F,\nabla(\rho^F,\sigma^F)}\sim\int_{D_{u,\underline{u}}}2r^4\langle\widehat{\nabla}\rho^F,-\nabla(\mathrm{tr}\chi)\rho^F-\widehat{\nabla}((\eta-\underline{\eta})\cdot\alpha^F))$$

$$-\widehat{\chi}\cdot\widehat{\nabla}\rho^F+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\rho^F}\rangle+\int_{D_{u,\underline{u}}}2r^4\langle\widehat{\nabla}\sigma^F,-\widehat{\nabla}(\mathrm{tr}\chi)\sigma^F+\widehat{\nabla}((\eta-\underline{\eta})\cdot{}^*\alpha^F)$$

$$-\widehat{\chi}\cdot\widehat{\nabla}\sigma+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\sigma^F}\rangle+\int_{D_{u,\underline{u}}}2r^4\langle\widehat{\nabla}\alpha^F,-\frac{1}{2}\widehat{\nabla}(\mathrm{tr}\underline{\chi})\alpha^F$$

$$-2\,{}^*\widehat{\nabla}(\eta\sigma^F)+2\widehat{\nabla}(\eta\rho^F)+2\widehat{\nabla}(\underline{\omega}\alpha^F)-\widehat{\nabla}(\hat{\chi}\cdot\underline{\alpha}^F)-\widehat{\chi}\cdot\widehat{\nabla}\alpha^F+\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}]\alpha^F}\rangle.$$

Proposition 7.17. *The error term $\mathcal{E}^{\nabla\alpha^F,\nabla(\rho^F,\sigma^F)}$ verifies the following estimate*

$$|\mathcal{E}^{\nabla\alpha^F,\nabla(\rho^F,\sigma^F)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{Y}_0+\mathcal{Y}_1).\tag{7.285}$$

Proof. The proof relies on noting the presence of null-structure and appropriate scaling. The slowly decaying term $\underline{\omega}$ appears with α^F i.e.,

$$|\int_{D_{u,\underline{u}}}r^4\underline{\omega}\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}\alpha^F|\lesssim\sup_{\mathcal{K}}(r|u||\underline{\omega}|)(\int_{u_0}^u|u'|^{-1}r^{-1}du')\int_H r^4|\widehat{\nabla}\alpha^F|^2\lesssim\epsilon\mathcal{Y}_1\tag{7.286}$$

$$|\int_{D_{u,\underline{u}}}r^4\widehat{\nabla}\alpha^F\cdot\alpha^F\nabla\omega|\lesssim\sup_{u,\underline{u}}||r^{\frac{3}{2}}|u|\nabla\omega||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}(||r^2\widehat{\nabla}\alpha^F||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}\alpha^F||_{L^4(S_{u,\underline{u}})}|u|^{-1}r^{-1})$$

$$\lesssim \epsilon\left(\int_{u_0}^{u}|u^{'}|^{-1}r^{-1}\right)\int_{H}(r^2|\alpha^F|^2+r^4|\widehat{\nabla}\alpha^F|^2)$$
$$\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1),$$

where we have made use of the weighted co-dimension 1 trace inequality (lemma 4.3).

The other connections are estimated in a similar way i.e.,

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F)\nabla\eta|\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.287}$$

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F)\eta|\lesssim \epsilon\mathcal{Y}_1, \tag{7.288}$$

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F\hat{\chi}|\lesssim \sup_{\mathcal{K}}(r^2|\hat{\chi}|)\int_{D_{u,\underline{u}}}\left(\frac{r^4|\widehat{\nabla}\alpha^F|^2}{u^2}+\frac{u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2}{r^2}\right) \tag{7.289}$$
$$\lesssim \epsilon\left(\left(\int_{u_0}^{u}|u^{'}|^{-2}\right)\int_{H}r^4|\widehat{\nabla}\alpha^F|^2+\left(\int_{\underline{u}_0}^{\underline{u}}|r|^{-2}d\underline{u}^{'}\right)\int_{\underline{H}}u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2\right)\lesssim \epsilon\mathcal{Y}_1.$$

The most vital term that we want to focus is the error term $\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}]\alpha^F}$ since this is where the *true nonlinearity* of the Yang-Mills equations shows up. Recall the commutation formulae and write

$$\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}]\alpha^F}\sim(\underline{\beta}+\underline{\alpha}^F\cdot(\rho^F,\sigma^F))\alpha^F+\underbrace{\underline{\alpha}^F\alpha^F}_{I}+(\eta,\underline{\eta})\widehat{\nabla}_3\alpha^F, \tag{7.290}$$

where the term of type I is is obtained as a result of the *non-linear* character of the Yang-Mills equations. This is a vital benefit of working in a purely gauge-invariant framework since one does not need to deal with the non-linearities at the level of connection which would require an additional choice of Yang-Mills gauge rendering the problem substantially complicated. Through this commutator term, one observes that gravity is coupled to Yang-Mills by $\underline{\beta}$. We estimate this error term as follows

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F\alpha^F|\lesssim \sup_{u,\underline{u}}|||u|^{\frac{3}{2}}r^{\frac{1}{2}}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}||r^2\widehat{\nabla}\alpha^F||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}\alpha^F||_{L^4(S_{u,\underline{u}})}|u|^{-\frac{3}{2}}$$
$$\lesssim \epsilon\left(\int_{u_0}^{u}|u|^{-\frac{3}{2}}\right)\int_{H}(r^2|\alpha^F|^2+r^4|\widehat{\nabla}\alpha^F|^2)\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1),$$

where we have utilized the L^4-H^1 Sobolev inequality on the topological 2−sphere **??**. The Yang-Mills-gravity coupling term satisfies

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\alpha^F\cdot\alpha^F\underline{\beta}|\lesssim \sup_{\mathcal{K}}r^{\frac{5}{2}}|\alpha^F\left(\int_{u_0}^{u}|u|^{-1}r^{-\frac{3}{2}}du^{'}\right)\int_{H}(r^4|\widehat{\nabla}\alpha^F|^2+u^2r^2|\underline{\beta}|^2) \tag{7.291}$$
$$\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1).$$

Similarly we obtain

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}_3\alpha^F(\eta,\underline{\eta})|\lesssim \epsilon^2(\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.292}$$

Once again, we only estimate the top order terms involving ρ^F since σ^F satisfies similar estimates. Explicit calculations yields

$$\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\rho^F}\sim(\beta+\alpha^F\cdot(\rho^F,\sigma^F))\rho^F+\underbrace{\alpha^F\rho^F}_{II}+(\eta,\underline{\eta})\widehat{\nabla}_4\rho^F. \tag{7.293}$$

Once again note that the term II is the manifestation of Yang-Mills non-linearity.

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\rho^F\cdot\alpha^F\rho^F|\lesssim\sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{\underline{u}_0}^{\underline{u}}r^{-\frac{3}{2}}d\underline{u}^{'})\int_{\underline{H}}(r^4|\widehat{\nabla}\rho^F|^2+r^2|\rho^F|^2)\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.294}$$

Similarly, the remaining terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\rho^F\cdot\rho^F\beta|\lesssim\epsilon(\mathcal{W}_0+\mathcal{Y}_1), \tag{7.295}$$

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\rho^F\cdot\widehat{\nabla}_4\rho^F(\eta,\underline{\eta})|\lesssim\epsilon^2(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.296}$$

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\rho^F\cdot\rho^F\nabla\mathrm{tr}\chi|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.297}$$

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\rho^F\cdot\alpha^F\nabla(\eta,\underline{\eta})|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.298}$$

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}\rho^F\cdot\widehat{\nabla}\rho^F\hat{\chi}|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.299}$$

Collecting all the terms yields

$$|\mathcal{E}^{\nabla\alpha^F,\nabla(\rho^F,\sigma^F)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.300}$$

This concludes the proof. □

Similar to the previous cases, we obtain the following exterior estimates as a consequence of the proposition (7.17)

Lemma 7.11. *The following estimates hold in the exterior domain*

$$\int_H r^4|\widehat{\nabla}\alpha^F|^2+\int_{\underline{H}} r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)\leq\int_{\widehat{\Sigma}_0} r^4|\widehat{\nabla}\alpha^F|^2+\int_{\widehat{\Sigma}_0} r^4(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)+\epsilon(\mathbf{W}+\mathbf{Y}).$$

Now consider the last two Yang-Mills pairs $\underline{\alpha}^F$ and (ρ^F,σ^F)

$$\begin{aligned}\int_{D_{u,\underline{u}}}\nabla_4(u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2)+\int_{D_{u,\underline{u}}}\nabla_3(u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2))\\ =\int_{\underline{H}} u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2+\int_H u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)-\int_{\widehat{\Sigma}_0} u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2\\ -\int_{\widehat{\Sigma}_0} u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)+\int_{D_{u,\underline{u}}}(2\omega-\mathrm{tr}\chi)u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2\\ +\int_{D_{u,\underline{u}}}(2\underline{\omega}-\mathrm{tr}\underline{\chi})u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2).\end{aligned} \tag{7.301}$$

Explicit calculations yield

$$\begin{aligned}\int_{D_{u,\underline{u}}}\nabla_4(u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2)+\int_{D_{u,\underline{u}}}\nabla_3(u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2))\\ =\int_{D_{u,\underline{u}}} u^2r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\widehat{\nabla}\underline{\alpha}^F|^2+\int_{D_{u,\underline{u}}} 2u^2r^2\langle\widehat{\nabla}\underline{\alpha}^F,\widehat{\nabla}\widehat{\nabla}_4\underline{\alpha}^F+[\widehat{\nabla}_4,\widehat{\nabla}]\underline{\alpha}^F\rangle\end{aligned} \tag{7.302}$$

$$+\int_{D_{u,\underline{u}}}(2\Omega^{-1}ur^2+u^2r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}})(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)$$

$$+\int_{D_{u,\underline{u}}}2u^2r^2\Big(\langle\widehat{\nabla}\rho^F,\widehat{\nabla}\widehat{\nabla}_3\rho^F+[\widehat{\nabla}_3,\widehat{\nabla}]\rho^F\rangle+\langle\widehat{\nabla}\sigma^F,\widehat{\nabla}\widehat{\nabla}_3\sigma^F+[\widehat{\nabla}_3,\widehat{\nabla}]\sigma^F\rangle\Big)$$

$$=\int_{D_{u,\underline{u}}}u^2r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\widehat{\nabla}\underline{\alpha}^F|^2+\int_{D_{u,\underline{u}}}2u^2r^2\langle\widehat{\nabla}\underline{\alpha}^F,\widehat{\nabla}(-\frac{1}{2}\mathrm{tr}\chi\underline{\alpha}^F-\widehat{\nabla}\rho^F-\ {}^*\widehat{\nabla}\sigma^F$$

$$-2\ {}^*\underline{\eta}\sigma^F-2\underline{\eta}\rho^F+2\omega\underline{\alpha}^F-\hat{\underline{\chi}}\cdot\alpha^F)-\frac{\mathrm{tr}\chi}{2}\widehat{\nabla}\underline{\alpha}^F-\widehat{\chi}\cdot\widehat{\nabla}\alpha^F+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\underline{\alpha}^F}\rangle$$

$$+\int_{D_{u,\underline{u}}}(2\Omega^{-1}ur^2+u^2r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}})(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)$$

$$+\int_{D_{u,\underline{u}}}2u^2r^2\Big(\langle\widehat{\nabla}\rho^F,\widehat{\nabla}(-\widehat{\mathrm{div}}\underline{\alpha}^F-\mathrm{tr}\underline{\chi}\rho^F+(\eta-\underline{\eta})\cdot\underline{\alpha}^F)-\frac{\mathrm{tr}\underline{\chi}}{2}\widehat{\nabla}\rho^F-\widehat{\chi}\cdot\widehat{\nabla}\rho^F\rangle$$

$$+\langle\widehat{\nabla}\sigma^F,\widehat{\nabla}(-\hat{\mathrm{curl}}\underline{\alpha}^F-tr\underline{\chi}\sigma^F+(\eta-\underline{\eta})\cdot\ {}^*\underline{\alpha}^F)-\frac{\mathrm{tr}\underline{\chi}}{2}\widehat{\nabla}\sigma^F-\widehat{\chi}\cdot\widehat{\nabla}\sigma^F+\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}]\sigma^F}\rangle\Big).$$

Collecting all the terms yield

$$\int_{\underline{H}}u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2+\int_H u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)-\int_{D_{u,\underline{u}}}u^2r^2(\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-\mathrm{tr}\chi)|\widehat{\nabla}\underline{\alpha}^F|^2 \quad (7.303)$$

$$-\int_{D_{u,\underline{u}}}(2\Omega^{-1}ur^2+u^2r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-2\mathrm{tr}\underline{\chi})|\widehat{\nabla}(\rho^F,\sigma^F)|^2$$

$$=\int_{\widehat{\Sigma}_0}u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2+\int_{\widehat{\Sigma}_0}u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)+\mathcal{E}^{\widehat{\nabla}\underline{\alpha}^F,\widehat{\nabla}(\rho^F,\sigma^F)},$$

where the error term is defined as follows

$$\mathcal{E}^{\widehat{\nabla}\underline{\alpha}^F,\widehat{\nabla}(\rho^F,\sigma^F)}=\int_{D_{u,\underline{u}}}u^2r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}|\widehat{\nabla}\underline{\alpha}^F|^2+\int_{D_{u,\underline{u}}}2u^2r^2\langle\widehat{\nabla}\underline{\alpha}^F,\widehat{\nabla}(-\frac{1}{2}\mathrm{tr}\chi)\underline{\alpha}^F \quad (7.304)$$

$$-\widehat{\nabla}(2\ {}^*\underline{\eta}\sigma^F-2\underline{\eta}\rho^F+2\omega\underline{\alpha}^F-\hat{\underline{\chi}}\cdot\alpha^F))-\widehat{\chi}\cdot\widehat{\nabla}\alpha^F+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\underline{\alpha}^F}\rangle$$

$$+\int_{D_{u,\underline{u}}}(2\Omega^{-1}ur^2+u^2r^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}})(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)$$

$$+\int_{D_{u,\underline{u}}}2u^2r^2\Big(\langle\widehat{\nabla}\rho^F,-\nabla(\mathrm{tr}\underline{\chi})\rho^F+\widehat{\nabla}((\eta-\underline{\eta})\cdot\underline{\alpha}^F)-\widehat{\chi}\cdot\widehat{\nabla}\rho^F\rangle$$

$$+\langle\widehat{\nabla}\sigma^F,-\widehat{\nabla}(\mathrm{tr}\underline{\chi})\sigma^F+\widehat{\nabla}((\eta-\underline{\eta})\cdot\ {}^*\underline{\alpha}^F)-\widehat{\chi}\cdot\widehat{\nabla}\sigma^F+\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}](\sigma^F,\rho^F)}\rangle\Big).$$

Proposition 7.18. *The error term $\mathcal{E}^{\nabla\underline{\alpha}^F,\nabla(\rho^F,\sigma^F)}$ verifies the following estimate*

$$|\mathcal{E}^{\nabla\underline{\alpha}^F,\nabla(\rho^F,\sigma^F)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{Y}_0+\mathcal{Y}_1). \quad (7.305)$$

Proof. Once again, we note that the nonlinearity of the Yang-Mills fields manifests itself as the commutator term $\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\underline{\alpha}^F}$. An explicit computation yields

$$\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}]\underline{\alpha}^F}\sim(\beta+\alpha^F\cdot(\rho^F+\sigma^F))\underline{\alpha}^F+\underbrace{\alpha^F\underline{\alpha}^F}_{III}+(\eta+\bar{\eta})\widehat{\nabla}_4\underline{\alpha}^F, \quad (7.306)$$

where III is the non-linear characteristics of the Yang-Mills equations. This term is estimated as follows

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}\underline{\alpha}^F\cdot\alpha^F\underline{\alpha}^F|\lesssim\sup_{u,\underline{u}}||r^2\alpha^F||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}(||ur\widehat{\nabla}\underline{\alpha}^F||_{L^2(S_{u,\underline{u}})}||u|r^{\frac{1}{2}}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}r^{-\frac{3}{2}})$$

$$\lesssim \epsilon(\int_{\underline{H}}(u^2|\underline{\alpha}^F|^2+u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2)\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1).$$

This is essentially the most dangerous term at the level of first derivative estimates of the Yang-Mills fields. Another important feature is the presence of the null structure. Notice that $\hat{\chi}$ appears with $|\widehat{\nabla}\underline{\alpha}^F|^2$ (recall $\underline{\hat{\chi}}$ appeared multiplied with $|\widehat{\nabla}\alpha^F|^2$ which verifies a weaker decay along the outgoing direction but such a property did not make a difference since α^F is only controlled on H) and satisfies a stronger decay along the outgoing null direction. More specifically, we have

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\underline{\alpha}^F\hat{\chi}|\lesssim \sup_{\mathcal{K}}(r^2|\hat{\chi}|)(\int_{\underline{u}_0}^{\underline{u}}r^{-2}d\underline{u}^{'})\int_{\underline{H}}u^2r^4|\widehat{\nabla}\underline{\alpha}^F|^2\lesssim \epsilon\mathcal{Y}_1. \tag{7.307}$$

The remaining terms satisfy the following estimates (we use the available null evolution equations)

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F\beta|\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_1), \tag{7.308}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F\alpha^F\cdot(\rho^F,\sigma^F)|\lesssim \epsilon^2(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.309}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}_4\underline{\alpha}^F(\eta,\underline{\eta})|\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.310}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F\nabla\mathrm{tr}\chi|\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.311}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}(\sigma^F,\rho^F)\underline{\eta}|\lesssim \epsilon\mathcal{Y}_1, \tag{7.312}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\alpha^F\underline{\hat{\chi}}|\lesssim \epsilon\mathcal{Y}_1, \tag{7.313}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)\nabla\mathrm{tr}\underline{\chi}|\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.314}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)\nabla(\eta,\underline{\eta})|\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.315}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}(\rho^F,\sigma^F)(\eta,\underline{\eta})|\lesssim \epsilon\mathcal{Y}_1, \tag{7.316}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}(\rho^F,\sigma^F)\hat{\chi}|\lesssim \epsilon\mathcal{Y}_1, \tag{7.317}$$

$$|\int_{D_{u,\underline{u}}}u^2r^2\widehat{\nabla}(\rho^F,\sigma^F)\cdot\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}](\sigma^F,\rho^F)}|\lesssim \epsilon(\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.318}$$

Collecting all the terms yields the desired estimate

$$|\mathcal{E}^{\nabla\underline{\alpha}^F,\nabla(\rho^F,\sigma^F)}|\lesssim \epsilon(\mathcal{W}_0+\mathcal{Y}_0+\mathcal{Y}_1). \tag{7.319}$$

□

Lemma 7.12. *The following estimates hold in the exterior domain*

$$\int_{\underline{H}}u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2+\int_H u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)\leq \int_{\widehat{\Sigma}_0}u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2+\int_{\widehat{\Sigma}_0}u^2r^2(|\widehat{\nabla}\rho^F|^2+|\widehat{\nabla}\sigma^F|^2)$$
$$+\epsilon(\mathbf{W}+\mathbf{Y}).$$

7.2.4. Second derivative estimates for the Weyl Curvature

$$\int_H r^8|\nabla^2\alpha|^2+\int_{\underline{H}} r^8|\nabla^2\beta|^2-\int_{D_{u,\underline{u}}} r^8\left(4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-2\mathrm{tr}\underline{\chi}\right)|\nabla^2\alpha|^2 \tag{7.320}$$
$$+\int_{D_{u,\underline{u}}} r^8(5\mathrm{tr}\chi-4\Omega^{-1}\overline{\Omega\mathrm{tr}\chi})|\nabla^2\beta|^2=\int_{\widehat{\Sigma}_0} r^8|\nabla^2\alpha|^2+\int_{\widehat{\Sigma}_0} r^8|\nabla^2\beta|^2+\mathcal{E}_{\nabla^2\alpha,\nabla^2\beta},$$

where $\mathcal{E}_{\nabla^2\alpha,\nabla^2\beta}$ is the error term to be controlled

$$\mathcal{E}_{\nabla^2\alpha,\nabla^2\beta}=\int_{D_{u,\underline{u}}} r^8\langle\nabla^2\alpha,\nabla^2\mathrm{tr}\underline{\chi}\alpha+\nabla\mathrm{tr}\underline{\chi}\nabla\alpha\rangle+\int_{D_{u,\underline{u}}}[2r^8\langle\nabla^2\alpha,\nabla^2\left(4\underline{\omega}\alpha-3(\widehat{\chi}\rho+{}^*\widehat{\chi}\sigma)\right.$$
$$\left.\left.+(\zeta+4\eta)\hat{\otimes}\beta+\frac{1}{2}(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})\gamma_{ab}-D_4\mathfrak{T}_{ab}+\frac{1}{2}\left(D_b\mathfrak{T}_{4a}+D_a\mathfrak{T}_{4b}\right)\right)\right\rangle$$
$$+2r^8\langle\nabla^2\alpha,-\underline{\widehat{\chi}}\cdot\nabla^2\alpha-\nabla\underline{\widehat{\chi}}\cdot\nabla\alpha+\mathcal{E}_2\rangle+2r^8\langle\nabla^2\beta,-2\nabla^2(\mathrm{tr}\chi)\beta_a-2\nabla(\mathrm{tr}\chi)\nabla\beta_a-2\nabla^2(\omega\beta_a)$$
$$+\nabla^2(\eta\cdot\alpha)_a-\frac{1}{2}\nabla^2(D_a\mathfrak{T}_{44}-D_4\mathfrak{T}_{4a})\rangle+2r^8\langle\nabla^2\beta,-\widehat{\chi}\cdot\nabla^2\beta-\nabla\widehat{\chi}\cdot\nabla\beta+\mathcal{E}_1\rangle].$$

Proposition 7.19. *The error term $\mathcal{E}_{\nabla^2\alpha,\nabla^2\beta}$ verifies the following estimate*

$$|\mathcal{E}_{\nabla^2\alpha,\nabla^2\beta}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3) \tag{7.321}$$

Proof.

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha(\nabla\mathrm{tr}\underline{\chi})\nabla\alpha|\lesssim\int_{u,\underline{u}}||r^4\nabla^2\alpha||_{L^2(S)}||r^{3-\frac{1}{2}}\nabla\mathrm{tr}\underline{\chi}||_{L^4(S)}||r^{\frac{7}{2}}\nabla\alpha||_{L^4(S)}r^{-2} \tag{7.322}$$
$$\lesssim\sup_{u,\underline{u}}||r^{\frac{5}{2}}\nabla\mathrm{tr}\underline{\chi}||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u} r^{-2})\int_H(r^8|\nabla^2\alpha|^2+r^4|\alpha|^2)\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_2).$$

Notice that the slowly decaying connection coefficient $\underline{\omega}$ appears with α

$$|\int_{D_{u,\underline{u}}} r^8|\nabla^2\alpha|^2\underline{\omega}|\lesssim\sup_{\mathcal{K}}(r|u||\underline{\omega}|)(\int_{u_0}^{u} r^{-1}|u^{'}|^{-1}du^{'})\int_H r^8|\nabla^2\alpha|^2\lesssim\epsilon\mathcal{W}_2. \tag{7.323}$$

Now we want to estimate

$$|\int_{D_{u,\underline{u}}} r^8\langle\nabla^2\alpha,\nabla^2 D_4\mathfrak{T}_{AB}\rangle| \tag{7.324}$$

We estimate the top order terms since remaining terms scale accordingly. We will frequently use the global Sobolev inequality (or simply the trace inequality) on the null cones (lemma 4.3)

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\widehat{\nabla}^2\rho^F\cdot\widehat{\nabla}\alpha^F|$$
$$\lesssim\sup_{u,\underline{u}}|||u|r^{\frac{5}{2}}\widehat{\nabla}^2\rho^F||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u}|u^{'}|^{-1}r^{-1}du^{'})\int_H(r^8|\nabla^2\alpha|^2+r^4|\widehat{\nabla}\alpha^F|^2+r^6|\widehat{\nabla}^2\alpha^F|^2)$$
$$\lesssim\epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2),$$
$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\rho^F\cdot\widehat{\nabla}^3\alpha^F|\lesssim\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^{u} r^{-2}|u^{'}|^{-\frac{1}{2}}du^{'})\int_H(r^8|\nabla^2\alpha|^2+r^8|\widehat{\nabla}^3\alpha^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_3),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla^2\mathrm{tr}\chi\rho^F\cdot\rho^F| \lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)\sup_{u,\underline{u}}||r^{\frac{7}{2}}\nabla\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u}|u^{'}|^{-\frac{1}{2}}r^{-\frac{3}{2}}du^{'})$$

$$\int_H (r^8|\nabla^2\alpha|^2+u^2|\rho^F|^2+u^2r^2|\widehat{\nabla}\rho^F|^2)\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}\rho^F|$$

$$\lesssim \sup_{u,\underline{u}}|||u|r^{\frac{3}{2}}\widehat{\nabla}\rho^F||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u}|u^{'}|^{-1}r^{-1}du^{'})\int_H(r^8|\nabla^2\alpha|^2+r^6|\widehat{\nabla}^2\alpha^F|^2+r^8|\widehat{\nabla}^3\alpha^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\alpha^F\mathrm{tr}\chi|\lesssim \epsilon\left((\int_{u_0}^{u}|u^{'}|^{-2}du^{'})\int_H r^8|\nabla^2\alpha|^2+(\int_{\underline{u}_0}^{\underline{u}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2\right)$$

$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\widehat{\nabla}_4\alpha^F|$$

$$\lesssim \sup_{u,\underline{u}}(||r^{\frac{3}{2}}|u|^{\frac{3}{2}}\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})})(\int_{u_0}^{u}|u^{'}|^{-\frac{3}{2}}r^{-1}du^{'})\int_H(r^8|\nabla^2\alpha|^2+r^6|\nabla\widehat{\nabla}_4\alpha^F|^2+r^8|\widehat{\nabla}^2\widehat{\nabla}_4\alpha^F|^2)$$

$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_2+\mathcal{Y}_3).$$

The remaining terms are straightforward to estimate

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla^2\mathrm{tr}\chi\alpha^F\cdot\underline{\alpha}^F|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.325}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}\widehat{\nabla}_4\alpha^F|\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_2+\mathcal{Y}_3), \tag{7.326}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha(\eta+\underline{\eta})\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}\rho^F|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.327}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla^2(\eta,\underline{\eta})\alpha^F\cdot\rho^F|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1).$$

Now we need to estimate

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha_{AB}\nabla^2(D_B\mathfrak{T}_{4A}+D_A\mathfrak{T}_{4B}|. \tag{7.328}$$

We have already estimated most of the terms in the expression of $\nabla^2\nabla\mathfrak{T}_{4A}$ previously. The remaining terms are controlled in a similar way

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla^2\mathrm{tr}\chi\rho^F\cdot\rho^F|\lesssim \int_{D_{u,\underline{u}}}||r^4\nabla^2\alpha||_{L^2(S_{u,\underline{u}})}||r^{\frac{7}{2}}\nabla^2\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}||r^{\frac{1}{2}}|u|\rho^F||_{L^4(S_{u,\underline{u}})}r^{-2}|u|^{-\frac{3}{2}}$$

$$\sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)$$

$$\lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)\sup_{u,\underline{u}}||r^{\frac{7}{2}}\nabla^2\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u}r^{-2}|u^{'}|^{-\frac{3}{2}}du^{'})\int_H(r^8|\nabla^2\alpha|^2+u^2|\rho^F|^2+u^2r^2|\widehat{\nabla}\rho^F|^2)$$

$$\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1)$$

Similarly

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla^2\hat{\chi}\rho^F\cdot\rho^F|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.329}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla\hat{\chi}\widehat{\nabla}\rho^F\cdot\rho^F| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.330}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\mathrm{tr}\chi\rho^F\cdot\widehat{\nabla}^2\rho^F| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_2), \tag{7.331}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla(\mathrm{tr}\chi,\hat{\chi})\widehat{\nabla}(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F) \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.332}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla(\mathrm{tr}\chi,\hat{\chi})\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.333}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla(\mathrm{tr}\chi,\hat{\chi})\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.334}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\ \mathrm{tr}\chi\alpha^F\cdot\widehat{\nabla}^2\alpha^F| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2) \tag{7.335}$$

Collecting all the terms together, we obtain

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha_{AB}\nabla^2(D_B\mathfrak{T}_{4A}+D_A\mathfrak{T}_{4B})| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3). \tag{7.336}$$

Notice the following term with special structure (slowly decaying connection $\hat{\chi}$ appears with α)

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\hat{\underline{\chi}}\nabla^2\alpha| \lesssim \sup_{\mathcal{K}}(|u|r|\hat{\underline{\chi}}|)(\int_{u_0}^{u}|u^{'}|^{-1}r^{-1}du^{'})\int_H r^8|\nabla^2\alpha|^2 \lesssim \epsilon\mathcal{W}_2. \tag{7.337}$$

Now we need to estimate the error term that is generated via commutation of ∇^2 and ∇_3 while acted on α. We estimate these terms as follows

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla((\eta+\underline{\eta})\mathrm{tr}\underline{\chi}\alpha)| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2), \tag{7.338}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla(\eta\underline{\chi}\alpha)| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla(\underline{\beta}\alpha)| \lesssim \epsilon(\mathcal{W}_1+\mathcal{W}_2),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla((\eta+\underline{\eta})\nabla\beta)| \lesssim \epsilon(\mathcal{W}_1+\mathcal{W}_2),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla(\underline{\omega}\alpha)| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla(\chi(\rho,\sigma)(\eta,\underline{\eta}))| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha\nabla((\eta,\underline{\eta})\beta)| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha_{AB}\nabla(D_4\mathfrak{T}_{AB})| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\alpha_{AB}\nabla D_A\mathfrak{T}_{4B}| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2).$$

Here notice that $\int_{D_{u,\underline{u}}} r^8\langle\nabla\alpha,\nabla^2(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})\gamma\rangle = 0$ due to the compatibility of γ with ∇ as well as the trace-free property of α.

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla^2(\text{tr}\chi)\beta| \lesssim \sup_{u,\underline{u}} ||r^{\frac{7}{2}}\nabla^2\text{tr}\chi||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}} ||r^4\nabla^2\beta||_{L^2(Su,\underline{u})}||r^{\frac{5}{2}}\beta||_{L^4(S_{u,\underline{u}})}r^{-2} \tag{7.339}$$

$$\lesssim \sup_{u,\underline{u}} ||r^{\frac{7}{2}}\nabla^2\text{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u} r^{-2}du^{'})\int_H(r^8|\nabla^2\beta|^2+r^4|\beta|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_2)$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta(\nabla^2\omega)\beta| \lesssim \sup_{u,\underline{u}} ||r^{\frac{7}{2}}\nabla^2\omega||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}} ||r^4\nabla^2\beta||_{L^2(Su,\underline{u})}||r^{\frac{5}{2}}\beta||_{L^4(S_{u,\underline{u}})}r^{-2} \tag{7.340}$$

$$\lesssim \sup_{u,\underline{u}} ||r^{\frac{7}{2}}\nabla^2\omega||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u} r^{-2}du^{'})\int_H(r^8|\nabla^2\beta|^2+r^4|\beta|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_2)$$

$$|\int_{D_{u,\underline{u}}} r^8\omega|\nabla^2\beta|^2| \lesssim \epsilon\mathcal{W}_2,\ |\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\nabla\eta\alpha)| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2), \tag{7.341}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\eta\nabla\alpha)| \lesssim \epsilon(\mathcal{W}_1+\mathcal{W}_2). \tag{7.342}$$

Now we need to estimate the error term that arises due to the commutation of ∇_4 and ∇^2 i.e.,

$$|\int_{D_{u,\underline{u}}} r^8\nabla\beta\nabla(\underline{\eta}\chi\beta)| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2),\ |\int_{D_{u,\underline{u}}} r^8\nabla\beta\nabla\beta\beta| \lesssim \epsilon\mathcal{W}_1,$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla\beta\nabla(\alpha(\eta,\underline{\eta}))| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2), |\int_{D_{u,\underline{u}}} r^8\nabla\beta\nabla(\omega\beta)| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla\beta\alpha^F\cdot\widehat{\nabla}^2\alpha^F| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_2), |\int_{D_{u,\underline{u}}} r^8\nabla\beta\nabla(\chi\alpha^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla\beta(\rho^F,\sigma^F)\cdot\widehat{\nabla}\widehat{\nabla}_4\alpha^F| \lesssim \epsilon(\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla\beta\nabla((\eta,\underline{\eta})\alpha^F\cdot\alpha^F)| \lesssim \epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1).$$

The last term to estimate is the coupling term

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta_A\nabla^2(D_A\mathfrak{T}_{44}-D_4\mathfrak{T}_{4A})|. \tag{7.343}$$

Now we estimate the most non-trivial terms

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}^2\alpha^F| \lesssim \sup_{\mathcal{K}}(r^{\frac{7}{2}}|\widehat{\nabla}\alpha^F|)(\int_{u_0}^{u} r^{-\frac{5}{2}}du^{'})\int_H(r^8|\nabla^2\beta|^2+r^6|\widehat{\nabla}^2\alpha^F|^2) \tag{7.344}$$

$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_2)$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\alpha^F\cdot\widehat{\nabla}^3\alpha^F| \lesssim \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{u_0}^{u} r^{-\frac{5}{2}}du^{'})\int_H(r^8|\nabla^2\beta|^2+r^8|\widehat{\nabla}^3\alpha^F|^2) \tag{7.345}$$

$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_3).$$

Collection of all the terms yields the desired result

$$|\mathcal{E}_{\nabla^2\alpha,\nabla^2\beta}| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3). \tag{7.346}$$

□

Lemma 7.13. *The following estimate holds in the exterior domain*

$$\int_{H} r^8|\nabla^2\alpha|^2 + \int_{\underline{H}} r^8|\nabla^2\beta|^2 \leq \int_{\widehat{\Sigma}_0} r^8|\nabla^2\alpha|^2 + \int_{\widehat{\Sigma}_0} r^8|\nabla^2\beta|^2 + \epsilon(\mathbf{W}+\mathbf{Y}), \tag{7.347}$$

Proof. A consequence of the previous proposition (7.19) and the connection estimates (section 7.1) $\square$

An application of the integration lemma (lemma 7.1) on the Bianchi pair $(\underline{\alpha}, \underline{\beta})$ yields

$$\begin{aligned} &\int_{H_u} u^4r^4|\nabla^2\underline{\beta}|^2 + \int_{\underline{H}_{\underline{u}}} u^4r^4|\nabla^2\underline{\alpha}|^2 - 2\int_{D_{u,\underline{u}}} u^4r^4(\Omega^{-1}\overline{\Omega\mathrm{tr}\chi} - \mathrm{tr}\chi)|\nabla^2\underline{\alpha}|^2 \\ &- \int_{D_{u,\underline{u}}} (4\Omega^{-1}u^3r^4 - 3u^4r^4\mathrm{tr}\underline{\chi})|\nabla^2\underline{\beta}|^2 + 2\int_{D_{u,\underline{u}}} u^4r^4(\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}} - \mathrm{tr}\underline{\chi})|\nabla^2\underline{\beta}|^2 \\ &= \int_{\widehat{\Sigma}_0} u^4r^4|\nabla^2\underline{\alpha}|^2 + \int_{\widehat{\Sigma}_0} u^4r^4|\nabla^2\underline{\beta}|^2 + \mathcal{E}^{\nabla^2\underline{\alpha},\nabla^2\underline{\beta}}, \end{aligned} \tag{7.348}$$

where the error term $\mathcal{E}^{\nabla\underline{\alpha},\nabla\underline{\beta}}$ is given as follows

$$\begin{aligned} \mathcal{E}^{\nabla^2\underline{\alpha},\nabla^2\underline{\beta}} \sim \int_{D_{u,\underline{u}}} & 2u^4r^4\langle\nabla^2\underline{\alpha}, -\frac{1}{2}\nabla(\nabla(\mathrm{tr}\chi)\underline{\alpha}) + \nabla^2\Big(4\omega\underline{\alpha}_{ab} - 3(\hat{\underline{\chi}}_{ab}\rho - {}^*\hat{\underline{\chi}}_{ab}\sigma) + ((\zeta - 4\underline{\eta})\hat{\otimes}\underline{\beta})_{ab} \\ &+\frac{1}{2}(D_4\mathfrak{T}_{33} - D_3\mathfrak{T}_{34})\gamma_{ab} - D_3\mathfrak{T}_{ab} + \frac{1}{2}(D_a\mathfrak{T}_{3b} + D_b\mathfrak{T}_{3a})\Big)\rangle + 2u^4r^4\langle\nabla^2\underline{\alpha}, -\nabla(\widehat{\chi}\cdot\nabla\underline{\alpha}) + \mathcal{E}_1^{[\nabla_4,\nabla^2]\underline{\alpha}}\rangle \\ &+2u^4r^4\langle\nabla^2\underline{\beta}, -2\nabla(\nabla(\mathrm{tr}\underline{\chi})\underline{\beta}) - \nabla^2\Big(2\underline{\omega}\underline{\beta}_a + (\underline{\eta}\cdot\underline{\alpha})_a + \frac{1}{2}(D_a\mathfrak{T}_{33} - D_3\mathfrak{T}_{3a})\Big)\rangle \\ &+2u^4r^4\langle\nabla^2\underline{\beta}, -\nabla(\widehat{\underline{\chi}}\cdot\nabla\underline{\beta}) + \mathcal{E}_2^{[\nabla_3,\nabla^2]\underline{\beta}}\rangle. \end{aligned}$$

Proposition 7.20. *The error term $\mathcal{E}_{\nabla\alpha,\nabla\beta}$ verifies the following estimate*

$$|\mathcal{E}_{\nabla^2\underline{\alpha},\nabla^2\underline{\beta}}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3). \tag{7.349}$$

Proof. Once again we start each types of term separately.

$$\begin{aligned} &|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}\nabla^2(\mathrm{tr}\chi)\underline{\alpha}| \lesssim \int_{u,\underline{u}} ||u^2r^2\nabla^2\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{7}{2}}\nabla^2\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}||r^{\frac{1}{2}}|u|^2\underline{\alpha}||_{L^4(S_{u,\underline{u}})}r^{-2} \\ &\lesssim \sup_{u,\underline{u}}||r^{\frac{7}{2}}\nabla^2\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}')\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + u^4r^2|\nabla\underline{\alpha}|^2 + u^4r^4|\nabla^2\underline{\alpha}|^2) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2) \end{aligned}$$

Notice the following term with special structure

$$|\int_{D_{u,\underline{u}}} u^4r^4|\nabla^2\underline{\alpha}|^2\omega| \lesssim \sup_{\mathcal{K}}(r^2|\omega|)(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}')\int_{\underline{H}} u^4r^4|\nabla^2\underline{\alpha}|^2 \lesssim \epsilon\mathcal{W}_2. \tag{7.350}$$

$$\begin{aligned} &|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}(\nabla^2\omega)\underline{\alpha}| \lesssim \int_{u,\underline{u}} ||u^2r^2\nabla^2\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{7}{2}}\nabla^2\omega||_{L^4(S_{u,\underline{u}})}||r^{\frac{1}{2}}|u|^2\underline{\alpha}||_{L^4(S_{u,\underline{u}})}r^{-2} \\ &\lesssim \sup_{u,\underline{u}}||r^{\frac{7}{2}}\nabla^2\omega||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}')\int_{\underline{H}}(u^4|\underline{\alpha}|^2 + u^4r^2|\nabla\underline{\alpha}|^2 + u^4r^4|\nabla^2\underline{\alpha}|^2) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2) \end{aligned}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}\nabla^2\hat{\underline{\chi}}\rho| \lesssim \int_{u,\underline{u}} ||u^2r^2\nabla^2\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{5}{2}}|u|\nabla^2\hat{\underline{\chi}}||_{L^4(S_{u,\underline{u}})}||r^{\frac{5}{2}}\rho||_{L^4(S_{u,\underline{u}})}|u|r^{-3}$$

$$\lesssim \sup_{u,\underline{u}} ||r^{\frac{5}{2}}|u|\nabla^2\hat{\underline{\chi}}||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} |u|r^{-3}d\underline{u}^{'}) \int_{\underline{H}}(r^4\rho^2 + r^6|\nabla\rho|^2 + u^4r^4|\nabla^2\underline{\alpha}|^2) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2)$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}(\nabla^2(\eta,\underline{\eta}))\underline{\beta}| \lesssim \int_{u,\underline{u}} ||u^2r^2\nabla^2\underline{\alpha}||_{L^2(S_{u,\underline{u}})}||r^{\frac{7}{2}}\nabla^2(\eta,\underline{\eta})||_{L^4(S_{u,\underline{u}})}||r^{\frac{3}{2}}|u|\underline{\beta}||_{L^4(S_{u,\underline{u}})}|u|r^{-3}$$

$$\lesssim \sup_{u,\underline{u}} ||r^{\frac{7}{2}}\nabla^2(\eta,\underline{\eta})||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} |u|r^{-3}d\underline{u}^{'}) \int_{\underline{H}}(u^4r^4|\nabla^2\underline{\alpha}|^2 + u^2r^2|\underline{\beta}|^2 + u^2r^4|\nabla\underline{\beta}|^2) \lesssim \epsilon(\mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2).$$

The remaining pure gravity terms are of lower order. Now we estimate the Yang-Mills coupling terms

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}_{AB}\nabla^2 D_3\mathfrak{T}_{AB}|. \tag{7.351}$$

We estimate each term separately

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}\widehat{\nabla}\rho^F\cdot\widehat{\nabla}^2\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^3|\widehat{\nabla}\rho^F|)(\int_{u_0}^{\underline{u}} |u|^{-\frac{1}{2}}r^{-3}d\underline{u}^{'}) \int_{\underline{H}}(u^4r^4|\nabla^2\underline{\alpha}|^2 + u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_2)$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}\rho^F\cdot\widehat{\nabla}^2\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}} r^2|u|^{\frac{1}{2}}|\rho^F|(\int_{\underline{u}_0}^{\underline{u}} |u|r^{-3}d\underline{u}^{'}) \int_{\underline{H}}(u^4r^4|\nabla^2\underline{\alpha}|^2 + r^4|u|^2|\widehat{\nabla}^2\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_2),$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}\nabla^2\mathrm{tr}\underline{\chi}\rho^F\cdot\rho^F| \lesssim \sup_{u,\underline{u}}(||r^{\frac{7}{2}}\nabla^2\mathrm{tr}\underline{\chi}||_{L^4(S_{u,\underline{u}})}) \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)$$
$$\int_{u,\underline{u}} ||u^2r^2\nabla^2\underline{\alpha}||_{L^2(S_{u,\underline{u}}}||r^{\frac{3}{2}}\rho^F||_{L^4(S_{u,\underline{u}})}|u|^{\frac{1}{2}}r^{-5} \lesssim \epsilon^2\int_{\underline{H}}(u^4r^4|\nabla^2\underline{\alpha}|^2 + r^2|\rho^F|^2 + r^4|\widehat{\nabla}\rho^F|^2)$$
$$\lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}\widehat{\nabla}_3\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}} r^{\frac{7}{2}}|\widehat{\nabla}\alpha^F|(\int_{\underline{u}_0}^{\underline{u}} r^{-\frac{5}{2}}d\underline{u}^{'}) \int_{\underline{H}}(u^4r^4|\nabla^2\underline{\alpha}|^2 + u^4r^2|\widehat{\nabla}\widehat{\nabla}_3\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_1 + \mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\alpha}\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}_3\underline{\alpha}^F| \tag{7.352}$$
$$\lesssim \sup_{u,\underline{u}} ||r^3\widehat{\nabla}^2\alpha^F||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} r^{-\frac{3}{2}}d\underline{u}^{'}) \int_{\underline{H}}(u^4r^4|\nabla^2\underline{\alpha}|^2 + u^4|\widehat{\nabla}_3\underline{\alpha}^F|^2 + u^4r^2|\widehat{\nabla}\widehat{\nabla}_3\underline{\alpha}^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_1 + \mathcal{Y}_1)$$

Similarly

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \nabla(\mathrm{tr}\underline{\chi}\widehat{\nabla}\rho^F \cdot \rho^F)| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_1 + \mathcal{Y}_2), \tag{7.353}$$

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \nabla(\nabla(\eta, \underline{\eta})\underline{\alpha}^F \cdot \rho^F)| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2), \tag{7.354}$$

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \nabla((\eta, \underline{\eta})\widehat{\nabla}\underline{\alpha}^F \cdot \rho^F)| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_1 + \mathcal{Y}_2), \tag{7.355}$$

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \nabla(\underline{\alpha}^F \cdot \widehat{\nabla}^2\rho^F)| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_2 + \mathcal{Y}_3), \tag{7.356}$$

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \nabla(\mathrm{tr}\underline{\chi}\widehat{\nabla}\alpha^F \cdot \underline{\alpha}^F)| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_1 + \mathcal{Y}_2), \tag{7.357}$$

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \nabla(\nabla\mathrm{tr}\underline{\chi}\alpha^F \cdot \underline{\alpha}^F)| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2). \tag{7.358}$$

Collecting all the terms, we obtain

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha}_{AB} \nabla^2 D_3 \mathfrak{T}_{AB}| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3). \tag{7.359}$$

Now we need to estimate the coupling term

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha}_{AB} \nabla\nabla_A \mathfrak{T}_{3B}|. \tag{7.360}$$

We estimate

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \widehat{\nabla}^3 \underline{\alpha}^F \cdot (\rho^F, \sigma^F)| \lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|(\rho^F, \sigma^F)|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}} r^{-3} d\underline{u}') \tag{7.361}$$
$$\int_{\underline{H}} (u^4 r^4 |\nabla\alpha|^2 + u^2 r^6 |\widehat{\nabla}^3 \underline{\alpha}^F|^2) \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_3)$$

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \widehat{\nabla}^3 (\rho^F, \sigma^F) \cdot \underline{\alpha}^F| \lesssim \sup_{\mathcal{K}}(|u|^{\frac{3}{2}} r |\underline{\alpha}^F|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}} r^{-3} d\underline{u}') \tag{7.362}$$
$$\int_{\underline{H}} (u^4 r^4 |\nabla\alpha|^2 + r^8 |\widehat{\nabla}^3 (\rho^F, \sigma^F)|^2) \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_3)$$

Similarly we obtain

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha} \widehat{\nabla}^2 (\rho^F, \sigma^F) \cdot \widehat{\nabla}\underline{\alpha}^F| \lesssim \sup_{\mathcal{K}}(|u|^{\frac{3}{2}} r^2 |\widehat{\nabla}\underline{\alpha}^F|)(\int_{\underline{u}_0}^{\underline{u}} |u|^{\frac{1}{2}} r^{-3} d\underline{u}') \tag{7.363}$$
$$\int_{\underline{H}} (u^4 r^4 |\nabla^2\underline{\alpha}|^2 + r^6 |\widehat{\nabla}^2 (\rho^F, \sigma^F)|^2) \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_2)$$

Remaining terms are of lower order and can be estimated by means of the boot-strap assumption in an straightforward way. Collecting all the terms one obtains

$$|\int_{D_{u,\underline{u}}} u^4 r^4 \nabla^2 \underline{\alpha}_{AB} \nabla^2 \nabla_A \mathfrak{T}_{3B}| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3). \tag{7.364}$$

Notice that all three terms are equivalent in terms of the scaling. Now consider

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla\nabla_A\underline{\beta}_B\nabla\nabla_A(D_B\mathfrak{T}_{33}-D_3\mathfrak{T}_{B3})|. \tag{7.365}$$

We estimate the top order terms

$$\begin{aligned}|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\widehat{\nabla}^2\underline{\alpha}^F\cdot\widehat{\nabla}\underline{\alpha}^F| &\lesssim \sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r^2|\widehat{\nabla}\underline{\alpha}^F|)(\int_{\underline{u}_0}^{\underline{u}}|u|^{-\frac{1}{2}}r^{-2}d\underline{u}^{'})\int_{\underline{H}}(u^4r^4|\nabla\underline{\beta}|^2+u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2)\\ &\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_2)\end{aligned}$$

$$\begin{aligned}|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\underline{\alpha}^F\cdot\widehat{\nabla}^3\underline{\alpha}^F| &\lesssim \sup_{\mathcal{K}}|u|^{\frac{3}{2}}r|\underline{\alpha}^F|(\int_{\underline{u}_0}^{\underline{u}}|u|^{-\frac{1}{2}}r^{-2}d\underline{u}^{'})\int_{\underline{H}}(u^4r^4|\nabla\underline{\beta}|^2+u^2r^6|\widehat{\nabla}^3\underline{\alpha}^F|^2)\\ &\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_3)\end{aligned}$$

$$\begin{aligned}&|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\widehat{\nabla}^2\widehat{\nabla}_3\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| \qquad (7.366)\\ &\lesssim \sup_{\mathcal{K}}|u|^{\frac{1}{2}}r^2|\rho^F,\sigma^F|(\int_{\underline{u}_0}^{\underline{u}}|u|^{-\frac{1}{2}}r^{-2}d\underline{u}^{'})\int_{\underline{H}}(u^4r^4|\nabla\underline{\beta}|^2+u^4r^4|\widehat{\nabla}^2\widehat{\nabla}_3\underline{\alpha}^F|^2)\\ &\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_3).\end{aligned}$$

The remaining terms are lower orders and are controlled as follows

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla(\nabla\mathrm{tr}\chi\underline{\alpha}^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.367}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla(\underline{\chi}\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.368}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla(\underline{\chi}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.369}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla(\nabla(\eta,\underline{\eta})\underline{\alpha}^F\cdot\underline{\alpha}^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.370}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla((\eta,\underline{\eta})\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.371}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla(\nabla\underline{\omega}\underline{\alpha}^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.372}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla(\underline{\omega}\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.373}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla(\underline{\omega}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.374}$$

$$|\int_{D_{u,\underline{u}}} u^4r^4\nabla^2\underline{\beta}\nabla(\widehat{\nabla}_3\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.375}$$

Collection of all the terms yields the desired estimate

$$|\mathcal{E}_{\nabla^2\underline{\alpha},\nabla^2\underline{\beta}}| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3). \tag{7.376}$$

$\square$

As a consequence of the previous proposition (7.20), we obtain the following lemma

Lemma 7.14. *Following estimate holds in the exterior domain*

$$\int_{H_u} u^4r^4|\nabla^2\underline{\beta}|^2 + \int_{\underline{H}_{\underline{u}}} u^4r^4|\nabla^2\underline{\alpha}|^2 \leq \int_{\widehat{\Sigma}_0} u^4r^4|\nabla^2\underline{\alpha}|^2 + \int_{\widehat{\Sigma}_0} u^4r^4|\nabla^2\underline{\beta}|^2 + \epsilon(\mathbf{W}+\mathbf{Y}).$$

Now consider the Bianchi pair $\{\beta,(\rho,\sigma)\}$ and apply the integration lemma (7.1) to yield

$$\int_H r^8|\nabla^2\beta|^2 + \int_{\underline{H}} r^8(|\nabla^2\rho|^2+|\nabla^2\sigma|^2) + \int_{D_{u,\underline{u}}} r^8(4\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-3\mathrm{tr}\chi)(|\nabla^2\rho|^2+|\nabla^2\sigma|^2)$$
$$+\int_{D_{u,\underline{u}}} r^8(4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-3\mathrm{tr}\underline{\chi})|\nabla^2\beta|^2 = \int_{\widehat{\Sigma}_0} r^8|\nabla^2\beta|^2 + \int_{\widehat{\Sigma}_0} r^8(|\nabla^2\rho|^2+|\nabla^2\sigma|^2) + \mathcal{E}^{\nabla^2\beta,\nabla^2(\rho,\sigma)},$$

where the error term $\mathcal{E}^{\nabla^2\beta,\nabla(\rho,\sigma)}$ can be written explicitly as follows

$$\mathcal{E}^{\nabla^2\beta,\nabla^2(\rho,\sigma)} = \int_{D_{u,\underline{u}}} 2r^8\langle\nabla^2\rho, -\frac{3}{2}\nabla(\nabla(\mathrm{tr}\chi)\rho) - \nabla^2\left(\frac{1}{2}\underline{\hat{\chi}}\cdot\alpha+\zeta\cdot\beta+2\underline{\eta}\cdot\beta-\frac{1}{4}(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{34})\right)$$
$$-\widehat{\chi}\cdot\nabla^2\rho+\mathcal{E}^{[\nabla_4,\nabla^2]\rho}\rangle + \int_{D_{u,\underline{u}}} 2r^8\langle\nabla^2\sigma, -\frac{3}{2}\nabla(\nabla(\mathrm{tr}\chi)\sigma)+\nabla\left(\frac{1}{2}\underline{\hat{\chi}}\cdot{}^*\alpha\right.$$
$$\left.-\zeta\cdot{}^*\beta-2\underline{\eta}\cdot{}^*\beta-\frac{1}{4}(D_\mu\mathfrak{T}_{4\nu}-D_\nu\mathfrak{T}_{4\mu})\epsilon^{\mu\nu}{}_{34})\right)-\widehat{\chi}\cdot\nabla^2\sigma+\mathcal{E}^{[\nabla_4,\nabla^2]\sigma}\rangle$$
$$+\int_{D_{u,\underline{u}}} 2r^8(\langle\nabla^2\beta,-\nabla(\nabla(tr\underline{\chi})\beta)+\nabla^2\left(2\underline{\omega}\beta_a+2(\hat{\chi}\cdot\underline{\beta})_a\right.$$
$$\left.+3(\eta\rho+{}^*\eta\sigma)_a+\frac{1}{2}(D_a\mathfrak{T}_{34}-D_4\mathfrak{T}_{3a})\right)-\widehat{\chi}\cdot\nabla^2\beta+\mathcal{E}^{[\nabla_3,\nabla^2]\beta}\rangle.$$

Proposition 7.21. *The error term $\mathcal{E}^{\nabla^2\beta,\nabla^2(\rho,\sigma)}$ verifies the following estimate*

$$|\mathcal{E}^{\nabla^2\beta,\nabla^2(\rho,\sigma)}| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3) \tag{7.377}$$

Proof. We estimate each term separately.

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla^2\mathrm{tr}\chi\rho| \lesssim \int_{u,\underline{u}} ||r^4\nabla^2\rho||_{L^2(S_{u,\underline{u}})}||r^{\frac{7}{2}}\nabla^2\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}||r^{\frac{5}{2}}\rho||_{L^4(S_{u,\underline{u}})}r^{-2} \tag{7.378}$$
$$\lesssim \sup_{u,\underline{u}}||r^{\frac{7}{2}}\nabla^2\mathrm{tr}\chi||_{L^4(S_{u,\underline{u}})}(\int_{\underline{u}_0}^{\underline{u}} r^{-2})\int_{\underline{H}}(r^8|\nabla^2\rho|^2+r^4|\rho|^2) \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_2).$$

The remaining gravity self-coupling terms are of lower order. We focus on the Yang-Mills coupling terms. σ enjoys same property as ρ and therefore estimating only one these coupling is sufficient. We want to estimate

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla^2(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})|. \tag{7.379}$$

The top order terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\alpha^F\cdot\widehat{\nabla}^3(\sigma^F,\rho^F)| \lesssim \sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{\underline{u}_0}^{\underline{u}} r^{-\frac{5}{2}})\int_{\underline{H}}(r^8|\nabla^2\rho|^2+r^8|\widehat{\nabla}^3(\rho^F,\sigma^F)|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_3)$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\rho^F\cdot\widehat{\nabla}^3\alpha^F|\lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\rho^F|)(\int_{u_0}^{u}|u^{'}|^{-\frac{3}{2}}r^{-1}du^{'})\int_H(u^2r^6|\nabla^2\rho|^2+r^8|\widehat{\nabla}^2\alpha^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_3).$$

The remaining terms satisfy the following estimates

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F))|\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3), \tag{7.380}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\nabla\eta\alpha^F\cdot(\rho^F,\sigma^F))|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.381}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\eta\alpha^F\widehat{\nabla}(\rho^F,\sigma^F))|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.382}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\underline{\omega}\widehat{\nabla}\alpha^F\cdot\alpha^F)|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.383}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\nabla\underline{\omega}\alpha^F\cdot\alpha^F)|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.384}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\nabla\hat{\chi}\alpha^F\cdot\underline{\alpha}^F)|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.385}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\hat{\chi}\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F)|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.386}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\hat{\chi}\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F)|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.387}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\underline{\eta}\widehat{\nabla}\rho^F\cdot\underline{\alpha}^F)|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.388}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\underline{\eta}\rho^F\cdot\widehat{\nabla}\underline{\alpha}^F)|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.389}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla(\nabla\underline{\eta}\rho^F\cdot\underline{\alpha}^F)|\lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.390}$$

Collecting all the terms yields

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\rho\nabla^2(D_3\mathfrak{T}_{44}-D_4\mathfrak{T}_{43})|\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3). \tag{7.391}$$

Now we estimate

$$|\int_{D_{u,\underline{u}}} r^8\nabla\nabla_B\beta_A\nabla\nabla_B(D_A\mathfrak{T}_{34}-D_4\mathfrak{T}_{3A})|. \tag{7.392}$$

The top order terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\widehat{\nabla}^3(\rho^F,\sigma^F)\cdot(\rho^F,\sigma^F)|\lesssim \sup_{\mathcal{K}}(r^2|u|^{\frac{1}{2}}|\rho^F|)(\int_{u_0}^{u}|u^{'}|^{-\frac{3}{2}}r^{-1}du^{'})\int_H(r^8|\nabla^2\beta|^2+u^2r^6|\widehat{\nabla}^3\rho^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_2+\mathcal{Y}_3).$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}^2\alpha^F|\lesssim \int_{u,\underline{u}}||r^4\nabla^2\beta||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}|u|^{\frac{3}{2}}|\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}||r^{\frac{7}{2}}\widehat{\nabla}^2\alpha^F||_{L^4(S_{u,\underline{u}})}r^{-1}|u|^{-\frac{3}{2}}$$

$$\lesssim \sup_{u,\underline{u}} ||r^{\frac{3}{2}}|u|^{\frac{3}{2}}|\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}(\int_{u_0}^{u} |u^{'}|^{-\frac{3}{2}}r^{-1}du^{'}) \int_H (r^8|\nabla^2\beta|^2 + r^6|\widehat{\nabla}^2\alpha^F|^2 + r^8|\widehat{\nabla}^3\alpha^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_2 + \mathcal{Y}_3),$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\underline{\alpha}^F \cdot \widehat{\nabla}^3\alpha^F| \lesssim \sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r^{-1}|\underline{\alpha}^F|)(\int_{u_0}^{u} |u^{'}|^{-\frac{3}{2}}r^{-1}du^{'}) \int_H (r^8|\nabla^2\beta|^2 + r^8|\widehat{\nabla}^3\alpha^F|^2)$$
$$\lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_3).$$

The remaining terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\nabla\underline{\chi}\alpha^F \cdot (\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1), \tag{7.393}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\underline{\chi}\widehat{\nabla}\alpha^F \cdot (\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_1), \tag{7.394}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\underline{\chi}\alpha^F \cdot \widehat{\nabla}(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_1), \tag{7.395}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\nabla\mathrm{tr}\chi\underline{\alpha}^F \cdot (\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1), \tag{7.396}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\mathrm{tr}\chi\widehat{\nabla}\underline{\alpha}^F \cdot (\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_1), \tag{7.397}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\mathrm{tr}\chi\underline{\alpha}^F \cdot \widehat{\nabla}(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2 + \mathcal{Y}_1), \tag{7.398}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\nabla\underline{\eta}(\rho^F,\sigma^F) \cdot (\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1), \tag{7.399}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\underline{\eta}\widehat{\nabla}(\rho^F,\sigma^F) \cdot (\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1), \tag{7.400}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla(\nabla(\eta,\underline{\eta})\underline{\alpha}^F \cdot \alpha^F)| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1), \tag{7.401}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla((\eta,\underline{\eta})\widehat{\nabla}\underline{\alpha}^F \cdot \alpha^F)| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_1), \tag{7.402}$$

$$|\int_{D_{u,\underline{u}}} r^8\nabla^2\beta\nabla((\eta,\underline{\eta})\underline{\alpha}^F \cdot \widehat{\nabla}\alpha^F)| \lesssim \epsilon^2(\mathcal{W}_2 + \mathcal{Y}_1) \tag{7.403}$$

Collection of all the terms concludes the proof. □

Lemma 7.15.

$$\int_H r^8|\nabla^2\beta|^2 + \int_{\underline{H}} r^8(|\nabla^2\rho|^2 + |\nabla^2\sigma|^2) \leq \int_{\widehat{\Sigma}_0} r^8|\nabla^2\beta|^2 + \int_{\widehat{\Sigma}_0} r^8(|\nabla^2\rho|^2 + |\nabla^2\sigma|^2) + \epsilon(\mathbf{W} + \mathbf{Y}).$$

Proof. A consequence of the proposition (7.21) in the exterior domain. □

An application of the integration lemma on the last Bianchi pair $\{\underline{\beta},(\rho,\sigma)\}$ yields

$$\int_{\underline{H}} r^6u^2|\nabla^2\underline{\beta}|^2+\int_{H} r^6u^2(|\nabla^2\rho|^2+|\nabla^2\sigma|^2)-\int_{D_{u,\underline{u}}} 3r^6u^2(\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-\mathrm{tr}\chi)|\nabla^2\underline{\beta}|^2$$
$$-\int_{D_{u,\underline{u}}}(3r^6u^2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}+2r^4u\Omega^{-1}-4r^6u^2\mathrm{tr}\underline{\chi})|\nabla^2(\rho,\sigma)|^2$$
$$=\int_{\widehat{\Sigma}_0} r^6u^2|\nabla^2\underline{\beta}|^2+\int_{\widehat{\Sigma}_0} r^6u^2(|\nabla^2\rho|^2+|\nabla^2\sigma|^2)+\mathcal{E}^{\nabla^2\underline{\beta},\nabla^2(\rho,\sigma)},$$

where the error term is explicitly computed as follows

$$\mathcal{E}^{\nabla^2\underline{\beta},\nabla^2(\rho,\sigma)}=\int_{D_{u,\underline{u}}} 2r^6u^2\langle\nabla^2\underline{\beta},-\nabla(\nabla(\mathrm{tr}\chi)\underline{\beta})+2\nabla^2\Big(\omega\underline{\beta}_a$$
$$+2(\underline{\hat{\chi}}\cdot\beta)_a-3(\underline{\eta}\rho-{}^*\underline{\eta}\sigma)_a-\frac{1}{2}(D_a\mathfrak{T}_{43}-D_3\mathfrak{T}_{4a})\Big)-\widehat{\chi}\cdot\nabla^2\underline{\beta}+\mathcal{E}^{[\nabla_4,\nabla^2]\underline{\beta}}\rangle$$
$$+\int_{D_{u,\underline{u}}} 2r^6u^2\Bigg(\langle\nabla^2\rho,-\nabla(\nabla(\frac{3}{2}\mathrm{tr}\underline{\chi})\rho)-\nabla^2\left(\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}+\zeta\cdot\underline{\beta}-2\eta\cdot\underline{\beta}+\frac{1}{4}(D_3\mathfrak{T}_{34}-D_4\mathfrak{T}_{33})\right)$$
$$-\widehat{\chi}\cdot\nabla^2\rho+\mathcal{E}^{[\nabla_3,\nabla^2]\rho}\rangle\Bigg)+\int_{D_{u,\underline{u}}} 2r^6u^2\langle\nabla^2\sigma,-\nabla(\nabla(\frac{3}{2}tr\underline{\chi})\sigma)+\nabla^2\Big(\frac{1}{2}\hat{\chi}\cdot{}^*\underline{\alpha}$$
$$-\zeta\cdot{}^*\underline{\beta}-2\eta\cdot{}^*\underline{\beta}+\frac{1}{4}(D_\mu\mathfrak{T}_{3\nu}-D_\nu\mathfrak{T}_{3\mu})\epsilon^{\mu\nu}{}_{34}\Big)-\widehat{\chi}\cdot\nabla^2\sigma+\mathcal{E}^{[\nabla_3,\nabla^2]\sigma}\rangle$$

Proposition 7.22. *The error term $\mathcal{E}_{\nabla\underline{\beta},\nabla(\rho,\sigma)}$ verifies the following estimates*

$$|\mathcal{E}_{\nabla^2\underline{\beta},\nabla^2(\rho,\sigma)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3). \tag{7.404}$$

Proof. Once again, we only focus on the potentially dangerous coupling terms since gravitational self-coupling terms are easily controlled under the boot-strap assumption. In particular we ought to estimate

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla\nabla_B\underline{\beta}_A\nabla\nabla_B(D_A\mathfrak{T}_{43}-D_3\mathfrak{T}_{4A})|. \tag{7.405}$$

We estimate each term separately

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\underline{\beta}(\rho^F,\sigma^F)\cdot\widehat{\nabla}^3(\rho^F,\sigma^F)| \tag{7.406}$$
$$\lesssim\sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\rho^F,\sigma^F|)(\int_{\underline{u}_0}^{\underline{u}}|u|^{\frac{1}{2}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}(u^2r^6|\nabla^2\underline{\beta}|^2+r^8|\widehat{\nabla}^3(\rho^F,\sigma^F)|^2)$$
$$\lesssim\epsilon(\mathcal{W}_2+\mathcal{Y}_3),$$
$$\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\underline{\beta}\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)| \tag{7.407}$$
$$\lesssim\sup_{u,\underline{u}}(|||u|^{\frac{3}{2}}r^{\frac{3}{2}}\widehat{\nabla}(\rho^F,\sigma^F)||_{L^4(S_{u,\underline{u}})})(\int_{\underline{u}_0}^{\underline{u}}|u|^{-\frac{1}{2}}r^{-2}d\underline{u}^{'})$$
$$\int_{\underline{H}}(u^2r^6|\nabla^2\underline{\beta}|^2+r^6|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2+r^8|\widehat{\nabla}^3(\rho^F,\sigma^F)|^2)\lesssim\epsilon(\mathcal{W}_2+\mathcal{Y}_2+\mathcal{Y}_3).$$

Similarly,

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\underline{\beta}\nabla(\nabla\underline{\chi}\alpha^F\cdot(\rho^F,\sigma^F))|\lesssim\epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.408}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\underline{\beta}\nabla(\underline{\chi}\widehat{\nabla}\alpha^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.409}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\underline{\beta}\nabla(\underline{\chi}\alpha^F\cdot\widehat{\nabla}(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.410}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\underline{\beta}\nabla(\nabla\chi\underline{\alpha}^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.411}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\underline{\beta}\nabla(\chi\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.412}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\underline{\beta}\nabla(\chi\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.413}$$

Now we need to estimate the following term

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(D_3\mathfrak{T}_{34}-D_4\mathfrak{T}_{33})|. \tag{7.414}$$

$$\begin{aligned}
|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\underline{\alpha}^F\cdot\widehat{\nabla}^3(\rho^F,\sigma^F)| &\lesssim \sup_{\mathcal{K}}(|u|^{\frac{3}{2}}r|\underline{\alpha}^F|)(\int_{u_0}^{u}|u|^{-\frac{3}{2}}r^{-1})\int_H(u^2r^6|\nabla^2\rho|^2+u^2r^6|\widehat{\nabla}^3(\rho^F,\sigma^F)|^2 \\
&\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_3), \\
&|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\widehat{\nabla}\underline{\alpha}^F\cdot\widehat{\nabla}^2(\rho^F,\sigma^F)| \\
\lesssim \sup_{u,\underline{u}}(|||u|^{\frac{3}{2}}r^{\frac{3}{2}}\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})})(\int_{u_0}^{u}|u|^{-\frac{3}{2}}r^{-1}du^{'})&\int_H(u^2r^6|\nabla^2\rho|^2+u^2r^4|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2+u^2r^6|\widehat{\nabla}^3(\rho^F,\sigma^F)|^2 \\
&\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_2+\mathcal{Y}_3), \\
|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\widehat{\nabla}^3\underline{\alpha}^F\cdot(\rho^F,\sigma^F)| &\lesssim \sup_{\mathcal{K}}(|u|^{\frac{1}{2}}r^2|\rho^F,\sigma^F|)(\int_{\underline{u}_0}^{\underline{u}}|u|^{\frac{1}{2}}r^{-3}d\underline{u}^{'})\int_{\underline{H}}(u^2r^6|\nabla^2\rho|^2+u^2r^6|\widehat{\nabla}^3\underline{\alpha}^F|^2) \\
&\lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_2).
\end{aligned}$$

Similarly the other terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\mathrm{tr}\chi\underline{\alpha}^F\cdot\widehat{\nabla}\underline{\alpha}^F)| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.415}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\nabla\mathrm{tr}\chi\underline{\alpha}^F\cdot\underline{\alpha}^F)| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.416}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\nabla\underline{\eta}\underline{\alpha}^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.417}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\underline{\eta}\widehat{\nabla}\underline{\alpha}^F\cdot(\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.418}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\underline{\eta}\underline{\alpha}^F\cdot\widehat{\nabla}(\rho^F,\sigma^F))| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.419}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\omega\widehat{\nabla}\underline{\alpha}^F\cdot\underline{\alpha}^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.420}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\nabla\omega\underline{\alpha}^F\cdot\underline{\alpha}^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.421}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\nabla\hat{\chi}\alpha^F\cdot\underline{\alpha}^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.422}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\hat{\chi}\widehat{\nabla}\alpha^F\cdot\underline{\alpha}^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.423}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\hat{\chi}\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.424}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\eta\rho^F\cdot\widehat{\nabla}\alpha^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.425}$$

$$|\int_{D_{u,\underline{u}}} u^2r^6\nabla^2\rho\nabla(\nabla\eta\rho^F\cdot\alpha^F)| \lesssim \epsilon^2(\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.426}$$

Collecting of the all the terms, one obtains

$$|\mathcal{E}_{\nabla^2\underline{\beta},\nabla^2(\rho,\sigma)}| \lesssim \epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3). \tag{7.427}$$

□

Lemma 7.16. *The following estimates hold for the second derivatives*

$$\int_{\underline{H}} r^6u^2|\nabla^2\underline{\beta}|^2+\int_H r^6u^2(|\nabla^2\rho|^2+|\nabla^2\sigma|^2) \tag{7.428}$$
$$\lesssim \int_{\widehat{\Sigma}_0} r^6u^2|\nabla^2\underline{\beta}|^2+\int_{\widehat{\Sigma}_0} r^6u^2(|\nabla^2\rho|^2+|\nabla^2\sigma|^2)+\epsilon(\mathbf{W}+\mathbf{Y}),$$

Proof. The proof is a consequence of the previous proposition (7.22) as well as the fact that $|(\Omega^{-1}\overline{\Omega\text{tr}\chi}-\text{tr}\chi)| \lesssim \frac{\epsilon}{r^2}$ and $(3r^6u^2\Omega^{-1}\overline{\Omega\text{tr}\underline{\chi}}+2r^4u\Omega^{-1}-4r^6u^2\text{tr}\underline{\chi})<0$ modulo $\frac{\epsilon}{r^2}$ terms in the exterior domain (i.e, $r(u,\underline{u})>|u|$). □

Now we need to estimate the mixed derivatives. The procedure is exactly similar as before. Therefore, we only estimate the non-trivial gravity-Yang-Mills coupling terms only. Define the following

$$\mathfrak{D}_4\alpha := \mathfrak{D}_4\alpha+\frac{5}{2}\text{tr}\chi\alpha,\ \mathfrak{D}_4\beta := \mathfrak{D}_4\beta+2\text{tr}\chi\beta,\ \mathfrak{D}_3\underline{\alpha}=\mathfrak{D}_3\underline{\alpha}+\frac{5}{2}\text{tr}\underline{\chi}\underline{\alpha}, \tag{7.429}$$
$$\mathfrak{D}_3\underline{\beta}=\mathfrak{D}_3\underline{\beta}+2\text{tr}\underline{\chi}\underline{\beta}.$$

We have the following lemma regarding the mixed norm of the Weyl curvature

Lemma 7.17.

$$\int_H r^6|\mathfrak{D}_4\alpha|^2+\int_{\underline{H}} r^6|\mathfrak{D}_4\beta|^2 \lesssim \int_{\widehat{\Sigma}_0} r^6|\mathfrak{D}_4\alpha|^2+\int_{\widehat{\Sigma}_0} r^6|\mathfrak{D}_4\beta|^2+\epsilon(\mathbf{W}+\mathbf{Y}), \tag{7.430}$$

$$\int_{H_u} u^6|\mathfrak{D}_3\underline{\beta}|^2+\int_{\underline{H}_{\underline{u}}} u^6|\mathfrak{D}_3\underline{\alpha}|^2 \lesssim \int_{\widehat{\Sigma}_0} u^4r^2|\mathfrak{D}_3\underline{\alpha}|^2+\int_{\widehat{\Sigma}_0} u^4r^2|\mathfrak{D}_3\underline{\beta}|^2+\epsilon(\mathbf{W}+\mathbf{Y}), \tag{7.431}$$

$$\int_H r^8|\nabla\mathfrak{D}_4\alpha|^2+\int_{\underline{H}} r^8|\nabla\mathfrak{D}_4\beta|^2 \lesssim \int_{\widehat{\Sigma}_0} r^6|\nabla\mathfrak{D}_4\alpha|^2+\int_{\widehat{\Sigma}_0} r^6|\nabla\mathfrak{D}_4\beta|^2+\epsilon(\mathbf{W}+\mathbf{Y}), \tag{7.432}$$

$$\int_{H_u} u^6r^2|\nabla\mathfrak{D}_3\underline{\beta}|^2+\int_{\underline{H}_{\underline{u}}} u^6r^2|\nabla\mathfrak{D}_3\underline{\alpha}|^2 \lesssim \int_{\widehat{\Sigma}_0} u^6r^2|\nabla\mathfrak{D}_3\underline{\alpha}|^2+\int_{\widehat{\Sigma}_0} u^6r^2|\nabla\mathfrak{D}_3\underline{\beta}|^2$$
$$+\epsilon(\mathbf{W}+\mathbf{Y}),$$

Proof. The procedure is the same as the previous cases. But here we will have terms of the type $\widehat{\nabla}_4\widehat{\nabla}_4\alpha^F$ (and $\widehat{\nabla}_3\widehat{\nabla}_3\underline{\alpha}^F$ resp.) appearing from the term $D_4D_4\mathfrak{T}_{ab}$ ($D_3D_3\mathfrak{T}_{ab}$ resp.). This is precisely why we needed to include the term $\int_H r^6|\widehat{\nabla}_4\widehat{\nabla}_4\alpha^F_4|^2$ ($\int_{\underline{H}}|\widehat{\nabla}^2_3\underline{\alpha}^F$ resp.) in the Yang-Mills energy. Also note that ∇_4 and ∇ are similar in terms of scaling property. Main estimates read

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\widehat{\nabla}\rho^F\cdot\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2),$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\rho^F\cdot\widehat{\nabla}^2_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2), \tag{7.433}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\nabla_4\mathrm{tr}\chi\rho^F\cdot\rho^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1), \tag{7.434}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\alpha^F\cdot\widehat{\nabla}^2_4\rho^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2), \tag{7.435}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\widehat{\nabla}_4\underline{\alpha}^F\cdot\alpha^F\mathrm{tr}\chi|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.436}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\widehat{\nabla}_4\underline{\alpha}^F\cdot\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.437}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\nabla_4\mathrm{tr}\chi\alpha^F\cdot\underline{\alpha}^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0), \tag{7.438}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\underline{\alpha}^F\cdot\widehat{\nabla}_4\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2), \tag{7.439}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha(\eta+\underline{\eta})\widehat{\nabla}_4\alpha^F\cdot\rho^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.440}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\alpha\nabla_4\eta\alpha^F\cdot\rho^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0),$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\beta\nabla_4(\mathrm{tr}\chi)\beta|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1),|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\beta(\nabla_4\omega)\beta|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1), \tag{7.441}$$

$$|\int_{D_{u,\underline{u}}} r^6\omega|\mathfrak{D}_4\beta|^2|\lesssim\epsilon\mathcal{W}_1,\ |\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\beta\nabla_4\eta\alpha|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1), \tag{7.442}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\beta\eta\mathfrak{D}_4\alpha|\lesssim\epsilon\mathcal{W}_1, \tag{7.443}$$

$$|\int_{D_{u,\underline{u}}} \mathfrak{D}_4\beta\mathfrak{D}_4\alpha(\eta,\underline{\eta})|\lesssim\epsilon\mathcal{W}_1,|\int_{D_{u,\underline{u}}} \mathfrak{D}_4\beta\alpha^F\cdot\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.444}$$

$$|\int_{D_{u,\underline{u}}} \mathfrak{D}_4\beta(\rho^F,\sigma^F)\cdot\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1),$$

$$|\int_{D_{u,\underline{u}}} \mathfrak{D}_4\beta(\eta,\underline{\eta})\alpha^F\cdot\alpha^F|\lesssim\epsilon^2(\mathcal{W}_1+\mathcal{Y}_0), \tag{7.445}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\beta\widehat{\nabla}\alpha^F\cdot\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_1), \tag{7.446}$$

$$|\int_{D_{u,\underline{u}}} r^6\mathfrak{D}_4\beta\alpha^F\cdot\widehat{\nabla}\widehat{\nabla}_4\alpha^F|\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2) \tag{7.447}$$

Here we estimate $\nabla_4\omega$ by means of the estimates from proposition (6.4)-(6.5). Collection of all the terms yields the proof. ☐

Remark 7. *Notice an important fact that we can not simply replace $\mathfrak{D}_4\beta$ ($\mathfrak{D}_3\underline{\beta}$ resp.) by its evolution equation since that would imply controlling $\nabla\alpha$ ($\nabla\underline{\alpha}$) on $\underline{H}$ (H) which is impossible. Therefore, we need to include $\mathfrak{D}_4\beta$ ($\mathfrak{D}_3\underline{\beta}$) in the initial data.*

□

7.2.5. Second and third Derivative Estimates for the Yang-Mills curvature
We estimate the second and third derivatives of the Yang-Mills curvature in this section. Since the calculations are quite similar to the lower derivative estimates, we only include the most non-trivial terms such as the one arising purely due to Yang-Mills interaction

$$\int_H r^6|\widehat{\nabla}^2\alpha^F|^2+\int_{\underline{H}} r^6(|\widehat{\nabla}^2\rho^F|^2+|\widehat{\nabla}^2\sigma^F|^2)-\int_{D_{u,\underline{u}}}(3r^6\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-3r^6\mathrm{tr}\chi)|\widehat{\nabla}(\rho^F,\sigma^F)|^2$$
$$-\int_{D_{u,\underline{u}}} r^6(3\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-2\mathrm{tr}\underline{\chi})|\widehat{\nabla}^2\alpha^F|^2=\int_{\widehat{\Sigma}_0} r^6|\widehat{\nabla}^2\alpha^F|^2+\int_{\widehat{\Sigma}_0} r^6(|\widehat{\nabla}^2\rho^F|^2+|\widehat{\nabla}^2\sigma^F|^2)$$
$$+\mathcal{E}^{\nabla^2\alpha^F,\nabla^2(\rho^F,\sigma^F)},$$

where $\mathcal{E}^{\widehat{\nabla}^2\alpha^F,\widehat{\nabla}^2(\rho^F,\sigma^F)}$ verifies

$$\mathcal{E}^{\widehat{\nabla}^2\alpha^F,\widehat{\nabla}^2(\rho^F,\sigma^F)}=\int_{D_{u,\underline{u}}} 2r^6\langle\widehat{\nabla}^2\rho^F,-\widehat{\nabla}(\nabla(\mathrm{tr}\chi)\rho^F)-\widehat{\nabla}^2((\eta-\underline{\eta})\cdot\alpha^F))$$
$$-\widehat{\chi}\cdot\widehat{\nabla}^2\rho^F+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}^2]\rho^F}+\mathcal{E}^{[\widehat{\nabla},\widehat{\nabla}^2](\rho^F,\sigma^F)}\rangle+\int_{D_{u,\underline{u}}} 2r^6\langle\widehat{\nabla}^2\sigma^F,-\widehat{\nabla}(\widehat{\nabla}(\mathrm{tr}\chi)\sigma^F)+\widehat{\nabla}^2((\eta-\underline{\eta})\cdot\ {}^*\alpha^F)$$
$$-\widehat{\nabla}(\widehat{\chi}\cdot\widehat{\nabla}\sigma)+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}^2]\sigma^F}\rangle+\int_{D_{u,\underline{u}}} 2r^6\langle\widehat{\nabla}^2\alpha^F,-\frac{1}{2}\widehat{\nabla}(\widehat{\nabla}(\mathrm{tr}\underline{\chi})\alpha^F)$$
$$-2\widehat{\nabla}\ {}^*\widehat{\nabla}(\eta\sigma^F)+2\widehat{\nabla}^2(\eta\rho^F)+2\widehat{\nabla}^2(\underline{\omega}\alpha^F)-\widehat{\nabla}^2(\hat{\chi}\cdot\underline{\alpha}^F)+\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}^2]\alpha^F}\rangle.$$

Proposition 7.23. *The error term $\mathcal{E}^{\widehat{\nabla}^2\alpha^F,\widehat{\nabla}^2(\rho^F,\sigma^F)}$ verifies the following estimate*

$$|\mathcal{E}^{\widehat{\nabla}^2\alpha^F,\widehat{\nabla}^2(\rho^F,\sigma^F)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.448}$$

Proof. The proof relies on noting the presence of null-structure and appropriate scaling. The slowly decaying term $\underline{\omega}$ appears with α^F i.e.,

$$|\int_{D_{u,\underline{u}}} r^6\underline{\omega}\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}^2\alpha^F|\lesssim\sup_{\mathcal{K}}(r|u||\underline{\omega}|)(\int_{u_0}^{u}|u^{'}|^{-1}r^{-1}du^{'})\int_H r^6|\widehat{\nabla}^2\alpha^F|^2\lesssim\epsilon\mathcal{Y}_2 \tag{7.449}$$

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}\alpha^F\nabla\omega| \tag{7.450}$$
$$\lesssim\sup_{u,\underline{u}}||r^{\frac{3}{2}}|u|\nabla\omega||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}(||r^3\widehat{\nabla}^2\alpha^F||_{L^2(S_{u,\underline{u}})}||r^{\frac{5}{2}}\widehat{\nabla}\alpha^F||_{L^4(S_{u,\underline{u}})}|u|^{-1}r^{-1})$$
$$\lesssim\epsilon(\int_{u_0}^{u}|u^{'}|^{-1}r^{-1})\int_H(r^4|\widehat{\nabla}\alpha^F|^2+r^6|\widehat{\nabla}^2\alpha^F|^2)\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2),$$

where we have made use of the weighted co-dimension 1 trace inequality (**??**). The other connections are estimated in a similar way i.e.,

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}((\rho^F,\sigma^F)\nabla\eta)|\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.451}$$

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}(\widehat{\nabla}(\rho^F,\sigma^F)\eta)|\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.452}$$

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}^2\underline{\alpha}^F\hat{\chi}|\lesssim\sup_{\mathcal{K}}(r^2|\hat{\chi}|)\int_{D_{u,\underline{u}}}(\frac{r^6|\widehat{\nabla}^2\alpha^F|^2}{u^2}+\frac{u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2}{r^2}) \tag{7.453}$$

$$\lesssim\epsilon\left((\int_{u_0}^{u}|u'|^{-2})\int_H r^6|\widehat{\nabla}^2\alpha^F|^2+(\int_{\underline{u}_0}^{\underline{u}}|r|^{-2}d\underline{u}')\int_{\underline{H}}u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2\right)\lesssim\epsilon\mathcal{Y}_2.$$

The most vital term that we want to focus is the error term $\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}^2]\alpha^F}$ since this is where the *true nonlinearity* of the Yang-Mills equations shows up. We estimate this error term as follows

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}\underline{\alpha}^F\alpha^F|\lesssim\sup_{u,\underline{u}}|||u|^{\frac{3}{2}}r^{\frac{3}{2}}\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}||r^3\widehat{\nabla}^2\alpha^F||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}\alpha^F||_{L^4(S_{u,\underline{u}})}|u|^{-\frac{3}{2}}$$

$$\lesssim\epsilon(\int_{u_0}^{u}|u|^{-\frac{3}{2}})\int_H(r^2|\alpha^F|^2+r^6|\widehat{\nabla}\alpha^F|^2)\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_2),$$

where we have utilized the L^4-H^1 Sobolev inequality on the topological $2-$sphere **??**. The Yang-Mills-gravity coupling term satisfies

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\alpha^F\cdot\alpha^F\nabla\underline{\beta}|\lesssim\sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{u_0}^{u}|u|^{-1}r^{-\frac{3}{2}}du')\int_H(r^6|\widehat{\nabla}^2\alpha^F|^2+u^2r^4|\nabla\underline{\beta}|^2) \tag{7.454}$$

$$\lesssim\epsilon(\mathcal{W}_1+\mathcal{Y}_2).$$

Similarly we obtain

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\alpha^F\cdot\widehat{\nabla}\widehat{\nabla}_3\alpha^F(\eta,\underline{\eta})|\lesssim\epsilon^2(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.455}$$

Once again, we only estimate the top order terms involving ρ^F since σ^F satisfies similar estimates. Once again note that the commutator terms are the manifestation of Yang-Mills non-linearity. We estimate the error terms as follows

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\rho^F\cdot\alpha^F\widehat{\nabla}\rho^F|\lesssim\sup_{\mathcal{K}}(r^{\frac{5}{2}}|\alpha^F|)(\int_{\underline{u}_0}^{\underline{u}}r^{-\frac{3}{2}}d\underline{u}')\int_{\underline{H}}(r^6|\widehat{\nabla}^2\rho^F|^2+r^4|\widehat{\nabla}\rho^F|^2) \tag{7.456}$$

$$\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2).$$

Similarly, the remaining terms are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\rho^F\cdot\widehat{\nabla}(\rho^F\beta)|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.457}$$

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\rho^F\cdot\widehat{\nabla}(\widehat{\nabla}_4\rho^F(\eta,\underline{\eta}))|\lesssim\epsilon^2(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.458}$$

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\rho^F\cdot\widehat{\nabla}(\rho^F\nabla\mathrm{tr}\chi)|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.459}$$

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\rho^F\cdot\widehat{\nabla}(\alpha^F\nabla(\eta,\underline{\eta}))|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.460}$$

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}^2\rho^F\cdot\widehat{\nabla}(\widehat{\nabla}\rho^F\hat{\chi})|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.461}$$

Collecting all the terms yields

$$|\mathcal{E}^{\widehat{\nabla}^2\alpha^F,\widehat{\nabla}^2(\rho^F,\sigma^F)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.462}$$

$\square$

Lemma 7.18. *The following estimate holds*

$$\int_H r^6|\widehat{\nabla}^2\alpha^F|^2+\int_{\underline{H}} r^6(|\widehat{\nabla}^2\rho^F|^2+|\widehat{\nabla}^2\sigma^F|^2)\lesssim\int_{\widehat{\Sigma}_0} r^6|\widehat{\nabla}^2\alpha^F|^2+\int_{\widehat{\Sigma}_0} r^6(|\widehat{\nabla}^2\rho^F|^2+|\widehat{\nabla}^2\sigma^F|^2)$$
$$+\epsilon(\mathbf{W}+\mathbf{Y}),$$

Proof. The proof is a consequence of the previous proposition (7.23) and the fact that $(3\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-3\mathrm{tr}\chi)\lesssim\frac{\epsilon}{r^2}$, $(3\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-2\mathrm{tr}\underline{\chi})<0$ modulo $\frac{\epsilon}{r^2}$ terms. $\square$

Now we consider the Null Yang-Mills pair $\underline{\alpha}^F$ and (ρ^F,σ^F) as usual, use the integration lemma (6.1) and write down the following identity

$$\int_{\underline{H}} u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2+\int_H u^2r^4(|\widehat{\nabla}^2\rho^F|^2+|\widehat{\nabla}^2\sigma^F|^2)-2\int_{D_{u,\underline{u}}} u^2r^4(\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-\mathrm{tr}\chi)|\widehat{\nabla}^2\underline{\alpha}^F|^2$$
$$-\int_{D_{u,\underline{u}}}(2\Omega^{-1}ur^2+2u^2r^4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-3u^2r^4\mathrm{tr}\underline{\chi})|\widehat{\nabla}^2(\rho^F,\sigma^F)|^2$$
$$=\int_{\widehat{\Sigma}_0} u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2+\int_{\widehat{\Sigma}_0} u^2r^4(|\widehat{\nabla}^2\rho^F|^2+|\widehat{\nabla}^2\sigma^F|^2)+\mathcal{E}^{\widehat{\nabla}^2\underline{\alpha}^F,\widehat{\nabla}^2(\rho^F,\sigma^F)},$$

where the error term is defined as follows

$$\mathcal{E}^{\widehat{\nabla}^2\underline{\alpha}^F,\widehat{\nabla}^2(\rho^F,\sigma^F)}\sim\int_{D_{u,\underline{u}}} 2u^2r^4\langle\widehat{\nabla}^2\underline{\alpha}^F,\widehat{\nabla}(\widehat{\nabla}(-\frac{1}{2}\mathrm{tr}\chi)\underline{\alpha}^F)\tag{7.463}$$
$$-\widehat{\nabla}^2(2\,{}^*\underline{\eta}\sigma^F-2\underline{\eta}\rho^F+2\omega\underline{\alpha}^F-\underline{\hat{\chi}}\cdot\alpha^F))-\widehat{\nabla}(\widehat{\chi}\cdot\widehat{\nabla}\alpha^F)+\mathcal{E}^{[\widehat{\nabla}_4,\widehat{\nabla}^2]\underline{\alpha}^F}+\mathcal{E}^{[\widehat{\nabla},\widehat{\nabla}^2](\rho^F,\sigma^F)}\rangle$$
$$+\int_{D_{u,\underline{u}}} 2u^2r^4\Big(\langle\widehat{\nabla}^2\rho^F,-\widehat{\nabla}(\nabla(\mathrm{tr}\underline{\chi})\rho^F)+\widehat{\nabla}^2((\eta-\underline{\eta})\cdot\underline{\alpha}^F)-\widehat{\nabla}(\widehat{\chi}\cdot\widehat{\nabla}\rho^F)\rangle$$
$$+\langle\widehat{\nabla}^2\sigma^F,-\widehat{\nabla}(\widehat{\nabla}(\mathrm{tr}\underline{\chi})\sigma^F)+\widehat{\nabla}^2((\eta-\underline{\eta})\cdot\,{}^*\underline{\alpha}^F)-\widehat{\nabla}(\widehat{\chi}\cdot\widehat{\nabla}\sigma^F)+\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}^2](\sigma^F,\rho^F)}\rangle\Big)$$

Proposition 7.24. *The error term $\mathcal{E}^{\widehat{\nabla}^2\underline{\alpha}^F,\widehat{\nabla}^2(\rho^F,\sigma^F)}$ verifies the following estimate*

$$|\mathcal{E}^{\nabla^2\underline{\alpha}^F,\nabla^2(\rho^F,\sigma^F)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2).\tag{7.464}$$

Proof.

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}^2\underline{\alpha}^F\cdot\alpha^F\widehat{\nabla}\underline{\alpha}^F|\lesssim\sup_{u,\underline{u}}||r^2\alpha^F||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}(||ur^2\widehat{\nabla}^2\underline{\alpha}^F||_{L^2(S_{u,\underline{u}})}||u|r^{\frac{3}{2}}\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}r^{-\frac{3}{2}})$$
$$\lesssim\epsilon(\int_{\underline{H}}(u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2+u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2)\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2).$$

This is essentially the most dangerous term at the level of second derivative estimates of the Yang-Mills fields. Notice that all the derivative estimates are essentially the same since the appearance of a derivative is compensated by the scale factor r. Another important feature is the presence of the null structure. Notice that $\hat{\chi}$ appears with $|\widehat{\nabla}^2\underline{\alpha}^F|^2$ (recall $\underline{\hat{\chi}}$ appeared multiplied with $|\widehat{\nabla}\alpha^F|^2$ which verifies a weaker decay along the outgoing direction but such a property did not make a difference since α^F is only controlled on H) and satisfies a stronger decay along the outgoing null direction. More specifically, we have

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}^2\underline{\alpha}^F\cdot\widehat{\nabla}^2\underline{\alpha}^F\hat{\chi}|\lesssim\sup_{\mathcal{K}}(r^2|\hat{\chi}|)(\int_{\underline{u}_0}^{\underline{u}} r^{-2}d\underline{u}')\int_{\underline{H}} u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2\lesssim\epsilon\mathcal{Y}_2.\tag{7.465}$$

The remaining terms satisfy the following estimates (we use the available null evolution equations)

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}^2\underline{\alpha}^F\cdot\widehat{\nabla}(\underline{\alpha}^F\beta)|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_1+\mathcal{Y}_2)), \tag{7.466}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}^2\underline{\alpha}^F\cdot\widehat{\nabla}(\underline{\alpha}^F\alpha^F\cdot(\rho^F,\sigma^F))|\lesssim\epsilon^2(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2)), \tag{7.467}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}^2\underline{\alpha}^F\cdot\widehat{\nabla}(\widehat{\nabla}_4\underline{\alpha}^F(\eta,\underline{\eta}))|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.468}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}^2\underline{\alpha}^F\cdot\widehat{\nabla}(\underline{\alpha}^F\nabla\mathrm{tr}\chi)|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.469}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}^2\underline{\alpha}^F\cdot\widehat{\nabla}(\widehat{\nabla}(\sigma^F,\rho^F)\underline{\eta})|\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2) \tag{7.470}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}^2\underline{\alpha}^F\cdot\widehat{\nabla}(\widehat{\nabla}\alpha^F\underline{\hat{\chi}})|\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.471}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}((\rho^F,\sigma^F)\nabla\mathrm{tr}\underline{\chi})|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.472}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}((\rho^F,\sigma^F)\nabla(\eta,\underline{\eta}))|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.473}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}(\widehat{\nabla}(\rho^F,\sigma^F)(\eta,\underline{\eta}))|\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.474}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}(\rho^F,\sigma^F)\cdot\widehat{\nabla}(\widehat{\nabla}(\rho^F,\sigma^F)\hat{\chi})|\lesssim\epsilon(\mathcal{Y}_1+\mathcal{Y}_2), \tag{7.475}$$

$$|\int_{D_{u,\underline{u}}} u^2r^4\widehat{\nabla}(\rho^F,\sigma^F)\cdot\mathcal{E}^{[\widehat{\nabla}_3,\widehat{\nabla}^2](\sigma^F,\rho^F)}|\lesssim\epsilon(\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2)). \tag{7.476}$$

Collecting all the terms yields the desired estimate

$$|\mathcal{E}^{\nabla^2\underline{\alpha}^F,\nabla^2(\rho^F,\sigma^F)}|\lesssim\epsilon(\mathcal{W}_0+\mathcal{W}_1+\mathcal{Y}_0+\mathcal{Y}_1+\mathcal{Y}_2). \tag{7.477}$$

□

Lemma 7.19. *The following estimate holds*

$$\begin{aligned}&\int_{\underline{H}} u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2+\int_H u^2r^4(|\widehat{\nabla}^2\rho^F|^2+|\widehat{\nabla}^2\sigma^F|^2) \\ \lesssim&\int_{\widehat{\Sigma}_0} u^2r^4|\widehat{\nabla}^2\underline{\alpha}^F|^2+\int_{\widehat{\Sigma}_0} u^2r^4(|\widehat{\nabla}^2\rho^F|^2+|\widehat{\nabla}^2\sigma^F|^2)+\epsilon(\mathbf{W}+\mathbf{Y})\end{aligned} \tag{7.478}$$

Proof. The proof is a consequence of the lemma (7.24) and the fact that $|(\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-\mathrm{tr}\chi)|\lesssim\frac{\epsilon}{r^2}$ and $2\Omega^{-1}u^{-1}+2\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}}-3\mathrm{tr}\underline{\chi}<0$ modulo $\frac{\epsilon}{r^2}$ terms. □

Now we proceed to finish the third derivative estimate for the Yang-Mills curvature components. An application of the integration by parts formula (lemma 6.1) yields

$$\int_H r^8|\widehat{\nabla}^3\alpha^F|^2+\int_{\underline{H}} r^8(|\widehat{\nabla}^3\rho^F|^2+|\widehat{\nabla}^3\sigma^F|^2)-\int_{D_{u,\underline{u}}}(4r^8\Omega^{-1}\overline{\Omega\mathrm{tr}\chi}-4r^8\mathrm{tr}\chi)|\widehat{\nabla}(\rho^F,\sigma^F)|^2$$

$$-\int_{D_{u,\underline{u}}} r^4(4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}} - 3\mathrm{tr}\underline{\chi})|\widehat{\nabla}^3\alpha^F|^2 = \int_{\widehat{\Sigma}_0} r^8|\widehat{\nabla}^3\alpha^F|^2 + \int_{\widehat{\Sigma}_0} r^8(|\widehat{\nabla}^3\rho^F|^2 + |\widehat{\nabla}^3\sigma^F|^2) + \mathcal{E}^{\widehat{\nabla}^3\alpha^F,\widehat{\nabla}^3(\rho^F,\sigma^F)}.$$

Proposition 7.25. *The error term $\mathcal{E}^{\widehat{\nabla}^3\alpha^F,\widehat{\nabla}^3(\rho^F,\sigma^F)}$ verifies the following estimate*

$$|\mathcal{E}^{\widehat{\nabla}^3\alpha^F,\widehat{\nabla}^3(\rho^F,\sigma^F)}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3). \tag{7.479}$$

Proof. The proof is exactly similar to the second derivative estimates. Notice that introducing one higher derivative is compensated by an additional factor of the scale. The most dangerous term, once is the following

$$|\int_{D_{u,\underline{u}}} r^8\widehat{\nabla}^3\alpha^F \cdot \widehat{\nabla}^2\underline{\alpha}^F\alpha^F| \tag{7.480}$$

which can be estimated as follows

$$\begin{aligned}
|\int_{D_{u,\underline{u}}} r^8\widehat{\nabla}^3\alpha^F \cdot \widehat{\nabla}^2\underline{\alpha}^F\alpha^F| &\lesssim \sup_{u,\underline{u}} |||u|^{\frac{3}{2}}r^{\frac{5}{2}}\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})} \int_{u,\underline{u}} ||r^4\widehat{\nabla}^3\alpha^F||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}\alpha^F||_{L^4(S_{u,\underline{u}})}|u|^{-\frac{3}{2}} \\
&\lesssim \epsilon(\int_{u_0}^{u} |u|^{-\frac{3}{2}}) \int_H (r^2|\alpha^F|^2 + r^4|\widehat{\nabla}\alpha^F|^2 + r^8|\widehat{\nabla}\alpha^F|^3) \lesssim \epsilon(\mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_3),
\end{aligned}$$

where we have estimated the the term $\sup_{u,\underline{u}} |||u|^{\frac{3}{2}}r^{\frac{5}{2}}\widehat{\nabla}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}$ by means of co-dimension 1 gauge-invariant trace inequality (lemma 4.3) □

Lemma 7.20. *The third derivatives of the Yang-Mills curvature components α^F, ρ^F, and σ^F verify the following estimate*

$$\int_H r^8|\widehat{\nabla}^3\alpha^F|^2 + \int_{\underline{H}} r^8(|\widehat{\nabla}^3\rho^F|^2 + |\widehat{\nabla}^3\sigma^F|^2) \lesssim \int_{\widehat{\Sigma}_0} r^8|\widehat{\nabla}^3\alpha^F|^2 + \int_{\widehat{\Sigma}_0} r^8(|\widehat{\nabla}^3\rho^F|^2 + |\widehat{\nabla}^3\sigma^F|^2) + \epsilon(\mathbf{W} + \mathbf{Y})$$

Proof. Notice that the terms the terms $(4r^8\Omega^{-1}\overline{\Omega\mathrm{tr}\chi} - 4r^8\mathrm{tr}\chi)$ and $(4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}} - 3\mathrm{tr}\underline{\chi})$ verify

$$4r^8\Omega^{-1}\overline{\Omega\mathrm{tr}\chi} - 4r^8\mathrm{tr}\chi \lesssim \epsilon\frac{1}{r^2} \tag{7.481}$$

$$(4\Omega^{-1}\overline{\Omega\mathrm{tr}\underline{\chi}} - 3\mathrm{tr}\underline{\chi}) = -\frac{2}{r} + \epsilon\frac{C}{r^2} \tag{7.482}$$

in the light of the connection estimates (proposition 7.1-7.4). Therefore we obtain

$$\begin{aligned}
\int_H r^8|\widehat{\nabla}^3\alpha^F|^2 + \int_{\underline{H}} r^8(|\widehat{\nabla}^3\rho^F|^2 + |\widehat{\nabla}^3\sigma^F|^2) &\lesssim \int_{\widehat{\Sigma}_0} r^8|\widehat{\nabla}^3\alpha^F|^2 + \int_{\widehat{\Sigma}_0} r^8(|\widehat{\nabla}^3\rho^F|^2 + |\widehat{\nabla}^3\sigma^F|^2) \\
&\quad + \epsilon(\mathbf{W} + \mathbf{Y}) + \epsilon|\mathcal{E}^{\widehat{\nabla}^3\alpha^F,\widehat{\nabla}^3(\rho^F,\sigma^F)}| \\
&\lesssim \int_{\widehat{\Sigma}_0} r^8|\widehat{\nabla}^3\alpha^F|^2 + \int_{\widehat{\Sigma}_0} r^8(|\widehat{\nabla}^3\rho^F|^2 + |\widehat{\nabla}^3\sigma^F|^2) + \epsilon(\mathbf{W} + \mathbf{Y}).
\end{aligned}$$

□

The last remaining topmost derivative is that of the Yang-Mills curvatures $\underline{\alpha}^F$. Utilizing the integration by parts formula (lemma 7.1), we obtain

$$\int_{\underline{H}} u^2 r^6 |\widehat{\nabla}^3 \underline{\alpha}^F|^2 + \int_H u^2 r^6 (|\widehat{\nabla}^3 \rho^F|^2 + |\widehat{\nabla}^3 \sigma^F|^2) - \int_{D_{u,\underline{u}}} 3u^2 r^6 (\mathrm{tr}\chi - \Omega^{-1}\overline{\Omega \mathrm{tr}\chi}) |\widehat{\nabla}^3 \underline{\alpha}^F|^2$$
$$- \int_{D_{u,\underline{u}}} (2\Omega^{-1}u - \mathrm{tr}\underline{\chi} u^2)(|\widehat{\nabla}^3 \rho^F|^2 + |\widehat{\nabla}^3 \sigma^F|^2) - \int_{D_{u,\underline{u}}} 3(\mathrm{tr}\underline{\chi} - \Omega^{-1}\overline{\Omega \mathrm{tr}\underline{\chi}})(|\widehat{\nabla}^3 \rho^F|^2 + |\widehat{\nabla}^3 \sigma^F|^2)$$
$$= \int_{\widehat{\Sigma}_0} u^2 r^6 |\widehat{\nabla}^3 \underline{\alpha}^F|^2 + \int_{\widehat{\Sigma}_0} u^2 r^6 (|\widehat{\nabla}^3 \rho^F|^2 + |\widehat{\nabla}^3 \sigma^F|^2) + \mathcal{E}^{\widehat{\nabla}^3 \underline{\alpha}^F, \widehat{\nabla}^3 (\rho^F, \sigma^F)}$$

Proposition 7.26. *The error term $\mathcal{E}^{\widehat{\nabla}^3 \underline{\alpha}^F, \widehat{\nabla}^3 (\rho^F, \sigma^F)}$ verifies the following estimate*

$$|\mathcal{E}^{\nabla^3 \underline{\alpha}^F, \nabla^3 (\rho^F, \sigma^F)}| \lesssim \epsilon(\mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3). \tag{7.483}$$

Proof. Once again, we only estimate the most dangerous term that differentiates the Yang-Mills coupling from that of Maxwell coupling (i.e., the true non-linear term of the Yang-Mills theory)

$$|\int_{D_{u,\underline{u}}} u^2 r^6 \widehat{\nabla}^3 \underline{\alpha}^F \cdot \widehat{\nabla} \alpha^F \widehat{\nabla} \underline{\alpha}^F| \lesssim \sup_{u,\underline{u}} ||r^3 \widehat{\nabla} \alpha^F||_{L^4(S_{u,\underline{u}})} \int_{u,\underline{u}} (||u r^3 \widehat{\nabla}^3 \underline{\alpha}^F||_{L^2(S_{u,\underline{u}})} ||u| r^{\frac{3}{2}} \widehat{\nabla} \underline{\alpha}^F||_{L^4(S_{u,\underline{u}})} r^{-\frac{3}{2}})$$
$$\lesssim \epsilon(\int_{\underline{H}} (u^2 r^2 |\widehat{\nabla} \underline{\alpha}^F|^2 + u^2 r^4 |\widehat{\nabla}^2 \underline{\alpha}^F|^2 + u^2 r^6 |\widehat{\nabla}^3 \underline{\alpha}^F|^2) \lesssim \epsilon(\mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3).$$

In addition, notice that the null structure persists at the level of all derivatives i.e., $\hat{\chi}$ appears with $\underline{\alpha}^F$ instead of $\hat{\underline{\chi}}$. The remaining terms are estimated in a similar way as that of the lower derivative terms and carefully keeping track of the weights. ☐

Lemma 7.21. *The third derivatives of the Yang-Mills curvature components $\underline{\alpha}^F, \rho^F$, and σ^F verify the following estimate*

$$\int_{\underline{H}} u^2 r^6 |\widehat{\nabla}^3 \underline{\alpha}^F|^2 + \int_H u^2 r^6 (|\widehat{\nabla}^3 \rho^F|^2 + |\widehat{\nabla}^3 \sigma^F|^2) \lesssim \int_{\widehat{\Sigma}_0} u^2 r^6 |\widehat{\nabla}^3 \underline{\alpha}^F|^2 + \int_{\widehat{\Sigma}_0} u^2 r^6 (|\widehat{\nabla}^3 \rho^F|^2 + |\widehat{\nabla}^3 \sigma^F|^2)$$
$$+ \epsilon(\mathbf{W} + \mathbf{Y})$$

Proof. Notice that in the exterior domain ($r(u, \underline{u}) > |u|$), we have $(\mathrm{tr}\chi - \Omega^{-1}\overline{\Omega \mathrm{tr}\chi}) \lesssim \epsilon \frac{1}{r^2}$ and $2\Omega^{-1}u^{-1} - \mathrm{tr}\underline{\chi} < 0$ modulo $\frac{\epsilon}{r^2}$ terms yielding

$$\int_{\underline{H}} u^2 r^6 |\widehat{\nabla}^3 \underline{\alpha}^F|^2 + \int_H u^2 r^6 (|\widehat{\nabla}^3 \rho^F|^2 + |\widehat{\nabla}^3 \sigma^F|^2) \lesssim \int_{\widehat{\Sigma}_0} u^2 r^6 |\widehat{\nabla}^3 \underline{\alpha}^F|^2 + \int_{\widehat{\Sigma}_0} u^2 r^6 (|\widehat{\nabla}^3 \rho^F|^2 + |\widehat{\nabla}^3 \sigma^F|^2)$$
$$+ \epsilon(\mathbf{W} + \mathbf{Y}),$$

which upon utilizing proposition (7.26) yields the desired result. ☐

Remark 8. *We need the third derivative estimates for the connections (lemma) in estimating the top-most derivatives of the Yang-Mills curvature.*

In an exact similar way but commuting with a combination of $\widehat{\nabla}_4$ and $\widehat{\nabla}$, we obtain the following lemma.

Lemma 7.22. *The following estimates hold for the mixed derivatives of the Yang-Mills curvature components*

$$\int_H r^4|\widehat{\nabla}_4\alpha^F|^2 + \int_{\underline{H}} r^4(|\widehat{\nabla}_4\rho^F|^2 + |\widehat{\nabla}_4\sigma^F|^2) \lesssim \int_{\widehat{\Sigma}_0} r^4|\widehat{\nabla}_4\alpha^F|^2 + \int_{\widehat{\Sigma}_0} r^4(|\widehat{\nabla}_4\rho^F|^2 + |\widehat{\nabla}_4\sigma^F|^2) + \int_{\widehat{\Sigma}_0} r^4|\widehat{\nabla}\alpha^F|^2 + \epsilon(\mathbf{W}+\mathbf{Y}),$$

$$\int_{\underline{H}} u^4|\widehat{\nabla}_3\underline{\alpha}^F|^2 + \int_H u^4(|\widehat{\nabla}_3\rho^F|^2 + |\widehat{\nabla}_3\sigma^F|^2) \lesssim \int_{\widehat{\Sigma}_0} u^4|\widehat{\nabla}_3\underline{\alpha}^F|^2 + \int_{\widehat{\Sigma}_0} u^4(|\widehat{\nabla}_3\rho^F|^2 + |\widehat{\nabla}_3\sigma^F|^2) + \epsilon(\mathbf{W}+\mathbf{Y}),$$

$$\int_H r^6|\widehat{\nabla}\widehat{\nabla}_4\alpha^F|^2 + \int_{\underline{H}} r^6(|\widehat{\nabla}\widehat{\nabla}_4\rho^F|^2 + |\widehat{\nabla}\widehat{\nabla}_4\sigma^F|^2)$$

$$\lesssim \int_{\widehat{\Sigma}_0} r^6|\widehat{\nabla}\widehat{\nabla}_4\alpha^F|^2 + \int_{\widehat{\Sigma}_0} r^6(|\widehat{\nabla}\widehat{\nabla}_4\rho^F|^2 + |\widehat{\nabla}\widehat{\nabla}_4\sigma^F|^2) + \int_{\widehat{\Sigma}_0} r^6|\widehat{\nabla}^2\alpha^F|^2 + \epsilon(\mathbf{W}+\mathbf{Y}),$$

$$\int_{\underline{H}} u^4r^2|\widehat{\nabla}\widehat{\nabla}_3\underline{\alpha}^F|^2 + \int_H u^4r^2(|\widehat{\nabla}\widehat{\nabla}_3\rho^F|^2 + |\widehat{\nabla}\widehat{\nabla}_3\sigma^F|^2)$$

$$\lesssim \int_{\widehat{\Sigma}_0} u^4r^2|\widehat{\nabla}\widehat{\nabla}_3\underline{\alpha}^F|^2 + \int_{\widehat{\Sigma}_0} u^4r^2(|\widehat{\nabla}\widehat{\nabla}_3\rho^F|^2 + |\widehat{\nabla}\widehat{\nabla}_3\sigma^F|^2) + \epsilon(\mathbf{W}+\mathbf{Y}),$$

$$\int_H r^6|\widehat{\nabla}_4\widehat{\nabla}_4\alpha^F|^2 + \int_{\underline{H}} r^6(|\widehat{\nabla}_4\widehat{\nabla}_4\rho^F|^2 + |\widehat{\nabla}_4\widehat{\nabla}_4\sigma^F|^2)$$

$$\lesssim \int_{\widehat{\Sigma}_0} r^6|\widehat{\nabla}_4\widehat{\nabla}_4\alpha^F|^2 + \int_{\widehat{\Sigma}_0} r^6(|\widehat{\nabla}_4\widehat{\nabla}_4\rho^F|^2 + |\widehat{\nabla}_4\widehat{\nabla}_4\sigma^F|^2) + \int_{\widehat{\Sigma}_0} r^6|\widehat{\nabla}\widehat{\nabla}\alpha^F|^2 + \epsilon(\mathbf{W}+\mathbf{Y}),$$

$$\int_{\underline{H}} u^6|\widehat{\nabla}_3\widehat{\nabla}_3\underline{\alpha}^F|^2 + \int_H u^6(|\widehat{\nabla}_3\widehat{\nabla}_3\rho^F|^2 + |\widehat{\nabla}_3\widehat{\nabla}_3\sigma^F|^2)$$

$$\lesssim \int_{\widehat{\Sigma}_0} u^4r^2|\widehat{\nabla}_3\widehat{\nabla}_3\underline{\alpha}^F|^2 + \int_{\widehat{\Sigma}_0} u^4r^2(|\widehat{\nabla}_3\widehat{\nabla}_3\rho^F|^2 + |\widehat{\nabla}_3\widehat{\nabla}_3\sigma^F|^2) + \epsilon(\mathbf{W}+\mathbf{Y}).$$

Proof. The proof is exactly similar as the Einstein-Maxwell case except for that the fact that one needs to keep track of the non-linear terms that appear as a result of the commutation of the gauge-covariant derivatives. These true non-linear terms (feature of the Yang-Mills theory) are estimated as follows

$$|\int_{D_{u,\underline{u}}} r^4\widehat{\nabla}_4\alpha^F\cdot\rho^F\alpha^F| \lesssim \sup_{u,\underline{u}}|||u|r\rho^F||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}||r^2\widehat{\nabla}_4\alpha^F||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}\alpha^F||_{L^4(S_{u,\underline{u}})}|u|^{-1}r^{-\frac{1}{2}}$$

$$\lesssim \epsilon(\int_{u_0}^{u}|u|^{-1}r^{-\frac{1}{2}})\int_H (r^2|\alpha^F|^2 + r^4|\widehat{\nabla}\alpha^F|^2 + r^4|\widehat{\nabla}_4\alpha^F|^2) \lesssim \epsilon(\mathbf{W}+\mathbf{Y}),$$

$$|\int_{D_{u,\underline{u}}} u^4\widehat{\nabla}_3\underline{\alpha}^F\cdot\rho^F\underline{\alpha}^F| \lesssim \sup_{u,\underline{u}}||r^2\rho^F||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}(||u^2\widehat{\nabla}_3\underline{\alpha}^F||_{L^2(S_{u,\underline{u}})}|||u|r^{\frac{1}{2}}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}|u|r^{-\frac{5}{2}})$$

$$\lesssim \epsilon(\int_{\underline{H}}(u^2|\underline{\alpha}^F|^2 + u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2 + u^4|\widehat{\nabla}\underline{\alpha}^F|^2) \lesssim \epsilon(\mathbf{W}+\mathbf{Y}),$$

$$|\int_{D_{u,\underline{u}}} r^6\widehat{\nabla}_4^2\alpha^F\cdot\widehat{\nabla}_4\rho^F\alpha^F| \lesssim \sup_{u,\underline{u}}|||u|r^2\widehat{\nabla}_4\rho^F||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}||r^3\widehat{\nabla}_4^2\alpha^F||_{L^2(S_{u,\underline{u}})}||r^{\frac{3}{2}}\alpha^F||_{L^4(S_{u,\underline{u}})}|u|^{-1}r^{-\frac{1}{2}}$$

$$\lesssim \epsilon(\int_{u_0}^{u}|u|^{-1}r^{-\frac{1}{2}})\int_H(r^2|\alpha^F|^2+r^4|\widehat{\nabla}\alpha^F|^2+r^6|\widehat{\nabla}_4^2\alpha^F|^2)\lesssim \epsilon(\mathbf{W}+\mathbf{Y}),$$

$$|\int_{D_{u,\underline{u}}}u^6\widehat{\nabla}_3^2\underline{\alpha}^F\cdot\widehat{\nabla}_3\rho^F\underline{\alpha}^F|\lesssim \sup_{u,\underline{u}}||r^2|u|\widehat{\nabla}_3\rho^F||_{L^4(S_{u,\underline{u}})}\int_{u,\underline{u}}(||u^3\widehat{\nabla}_3^2\underline{\alpha}^F||_{L^2(S_{u,\underline{u}})}|||u|r^{\frac{1}{2}}\underline{\alpha}^F||_{L^4(S_{u,\underline{u}})}|u|r^{-\frac{5}{2}})$$

$$\lesssim \epsilon(\int_{\underline{H}}(u^2|\underline{\alpha}^F|^2+u^2r^2|\widehat{\nabla}\underline{\alpha}^F|^2+u^6|\widehat{\nabla}_3^2\underline{\alpha}^F|^2)\lesssim \epsilon(\mathbf{W}+\mathbf{Y})$$

and similar other terms. Collecting all these terms yields the desired estimates. □

Remark 9. *In estimating these terms, we need the estimates for* $\nabla_4\omega$, $\nabla\nabla_4\omega$, $\nabla^2\nabla_4\omega$, $\nabla_3\underline{\omega}$, $\nabla\nabla_3\omega$, *and* $\nabla^2\nabla_3\omega$.

Lemma 7.23. *The total energy verifies the following uniform estimate uniform in* u *and* $\underline{u}$

$$\mathbf{W}+\mathbf{Y}\lesssim \mathbf{W}_0+\mathbf{Y}_0. \tag{7.484}$$

Proof. Collecting all the estimates proved in lemma (6.2)-lemma (6.14) yields

$$\mathbf{W}+\mathbf{Y}\leq \mathbf{W}_0+\mathbf{Y}_0+C\epsilon(\mathbf{W}+\mathbf{Y}), \tag{7.485}$$

for a constant C independent of $u,\underline{u}$ and ϵ. Therefore for sufficiently small ϵ, we obtain

$$\mathbf{W}+\mathbf{Y}\lesssim \mathbf{W}_0+\mathbf{Y}_0. \tag{7.486}$$

Now we can choose the initial data $\mathbf{W}_0+\mathbf{Y}_0$ to be sufficiently small to make sure $\mathbf{W}+\mathbf{Y}\leq\frac{\epsilon}{2}$ thereby closing the bootstrap assumption. □

Remark 10. *Since the connection norm* $\mathcal{O}$ *verifies* $\mathcal{O}\lesssim \mathbf{W}+\mathbf{Y}$, *this closes the bootstrap assumption on the connections too.*

Using these gauge-invariant estimates and standard arguments for the Cauchy problem up to a suitably defined 'time' $\tau=u+\underline{u}$, one can prove the existence of a solution to the Einstein-Yang-Mills equations through a continuity argument in space-time harmonic generalized Coloumb/temporal gauge. We omit such detail since they are standard (see section 6 of [19] for example for vacuum case, Yang-Mills equations in temporal gauge are symmetric hyperbolic and so follows in a similar way). For the moment, if we consider the metric in the ADM form (suitable for Cauchy problem) ${}^{1+3}g=-N^2dt\otimes dt+g_{ij}(dx^i+X^idt)\otimes(dx^j+X^jdt)$, where $\{x^i\}_{i=0}^3=(t,x^1,x^2,x^3)$ is local chart, N is the lapse function, and X is the shift vector field. Also, let $\mathbf{n}:=\frac{1}{N}(\partial_t-X)$ be the t =constant hypersurface orthogonal future directed unit normal field. In the frame-work of a Cauchy problem, the gauge covariant Yang-Mills equations read [8]

$$\widehat{\mathcal{L}}_{\partial_t}\mathcal{E}_i=\widehat{\mathcal{L}}_X\mathcal{E}_i-N\epsilon_i{}^{jk}\widehat{\nabla}_jH_k-2NK_i{}^j\mathcal{E}_j+Ntr_gK\mathcal{E}_i-\epsilon_i{}^{jk}\nabla_jNH_k, \tag{7.487}$$

$$\widehat{\mathcal{L}}_{\partial_t}H_i=\widehat{\mathcal{L}}_XH_i+N\epsilon_i{}^{jk}\widehat{\nabla}_j\mathcal{E}_k-2NK_i{}^jH_j+Ntr_gKH_i+\epsilon_i{}^{jk}\nabla_jN\mathcal{E}_k, \tag{7.488}$$

where $\mathcal{E}_i:=F(\mathbf{n},\partial_i),H_i={}^*F(\mathbf{n},e_i)$ are the chromo-electric and chromo-magnetic fields, respectively. Here $\widehat{\mathcal{L}}$ denotes the gauge-covariant Lie derivative operator. More explicitly,

[8] These coupled equations are essentially the first order formulation of the wave equation for the Yang-Mills curvature $\widehat{D}^\alpha\widehat{D}_\alpha F^{\hat{a}}{}_{\hat{b}\mu\nu}=2F^{\hat{a}}{}_{\hat{c}\mu\beta}F^{\hat{c}}{}_{\hat{b}\nu}{}^{\beta}-2F^{\hat{a}}{}_{\hat{c}\nu\beta}F^{\hat{c}}{}_{\hat{b}\mu}{}^{\beta}-R^\gamma{}_{\beta\mu\nu}F^{\hat{a}}{}_{\hat{b}\gamma}{}^{\beta}-R^\gamma{}_\nu F^{\hat{a}}{}_{\hat{b}\gamma\mu}-R^\gamma{}_\mu F^{\hat{a}}{}_{\hat{b}\nu\gamma}$

$\widehat{\mathcal{L}}_{\partial_t} := \partial_t + [A_0, \cdot]$ and $\widehat{\mathcal{L}}_X \mathcal{E}_i := \mathcal{L}_X \mathcal{E}_i + X^j [A_j, \mathcal{E}_i]$. These system of equations are symmetric hyperbolic and one may obtain a local well-posedness result in the temporal gauge ($A_0 = 0$). Since we obtained all the estimates for the Yang-Mills curvature $\mathcal{E}$ and $\mathcal{H}$, it is straightforward to obtain a global well-posedness through a continuity argument.

This concludes the proof of the global existence of the Minkowski space in the exterior region.

8. Concluding Remarks

We established the exterior global stability of the the trivial flat solutions of the Einstein-Yang-Mills system. The main challenge is to obtain suitable gauge invariant estimates for the physical degrees of freedom. In the coupled Einstein-Yang-Mills system, the total *energy* is contributed by the Weyl curvature (pure gravity) and the Yang-Mills curvature. The idea is to perturb the trivial solution the Minkowski space (and flat Yang-Mills solutions) and study whether the energy associated with these perturbations escape to null infinity or can the non-linearities associate with the coupled equations concentrate the energy to yield the incompleteness of the spacetime. This incompleteness can be understood in terms of a trapped surface formation (which by Penrose's theorem would imply a null geodesic incompleteness [25]) or a naked singularity. To this end, it is natural to work in a double null framework where one may explicitly study the energy dispersion along the outgoing cone and energy concentration along the incoming cone. Since both the Einstein and Yang-Mills equations possess the same characteristics, it is relatively easy to study the radiation problem in the double-null framework. Our result confirms that the geometric decay is just sufficient to yield a global existence result for the coupled theory. The point-wise decay of the Yang-Mills curvature is weaker than that of the Weyl curvature components. On the other hand Yang-Mills non-linearities are weaker than that of gravity (while the former is semi-linear the latter is quasi-linear). Therefore, one is able to close the argument with a less decay. In summary, the main idea of the proof essentially relies on the geometric scaling of different curvature components. Once the curvature components (Weyl and Yang-Mills) are estimated uniformly, the total degrees of freedom is exhausted and as a consequence the global well-posedness of the Cauchy problem follows.

Estimation of the gravitational radiation at the null infinity is of fundamental importance in general relativity for a detailed understanding of the gravitational waves. A double null foliation seems to be a natural choice for such studies as one can explicitly study the energy dispersion along the outgoing null cone. Our results suggest that if one disturbs an exterior domain of the Minkowski space by means of a coupled Einstein-Yang-Mills perturbations, the perturbations escape to null infinity and as such the resulting spacetime is close the Minkowski space in a global sense. It is known that the passing of gravitational radiation causes permanent displacement of the test particles (Chirstodoulou's non-linear memory effect [23]). Therefore, one would like to understand the impact of the coupled gravitational-Yang-Mills radiation on the memory effect. [24] proved that the electromagnetic field contributes at highest order to the nonlinear memory effect of gravitational waves, enlarging the permanent displacement of test masses. One would expect that a similar feature is observed in the Einstein-Yang-Mills case. In particular, we want to understand the limits of several masses (e.g., Wang-Yau quasi-local mass, Hawking mass etc.) associated with the topological $2-$ spheres foliating the future null infinity $\mathcal{I}^+$. Can we discern the information about the gauge group by studying the Yang-Mills contribution to the memory effect? We would like to investigate such issue in near future.

Lastly, we want to mention the fact that our estimates associated with the Yang-Mills curvature components are completely gauge-invariant. In particular, we obtain estimates for the fully gauge covariant angular derivatives. This essentially hints at an apparent similarity with the Maxwell and Yang-Mills theory despite the fact that the latter is a fully non-linear theory because the gauge covariant derivative *hides* the information of the connection and the non-linear coupling shows up only as the commutator of the fully gauge covariant derivatives (this non-linear term is the most dangerous term among all the other non-linear terms present in the Einstein-Yang-Mills theory and we note that this term causes serious problem in case of the stability of Milne universe under coupled gravity-Yang-Mills perturbations). This does not cause a problem in the context of obtaining estimates since all the associated inequalities are formulated in terms of the gauge covariant derivatives and the linear decay is just sufficient to control the nonlinear terms (and the estimates required for a local existence theory for coupled Yang-Mills equations can be obtained in a gauge-invariant way; note that at the end one ought to choose a gauge and work with the equation for connection). However our proffered choice of gauge is temporal gauge/Coulomb gauge where the Yang-Mills equations takes the form of a symmetric hyperbolic/elliptic-hyperbolic system and the spatial connection can be determined in terms of the gauge invariant norms of the Yang-Mills curvature). There is of course a physical motivation behind this. Since the double null framework in some sense encodes the information about the *physical* nature of the Yang-Mills fields, the choice of the gauge should not matter and one should expect to obtain gauge-invariant estimates as is done in the current context. In this framework of the gauge-invariant estimates, therefore, the exterior stability of the Minkowski space under coupled gravity-Yang-Mills perturbations is proven to hold as a consequence of the main theorem.

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