Math 230B

Semi-Riemannian Geometry

topology: study of "open" sets on abstract objects like sets

Def^n | A topological space is a set S with a topology $\mathcal{O}$ which satisfies the following axioms
↳ collection of open sets

1) $\emptyset \in \mathcal{O}$, $S \in \mathcal{O}$
2) $U_1 \in \mathcal{O}, U_2 \in \mathcal{O} \Rightarrow U_1 \cap U_2 \in \mathcal{O}$
3) $U_1 \cup U_2 \in \mathcal{O}$, $U_1 \in \mathcal{O}$, $U_2 \in \mathcal{O}$

Def^n | A map f between two top. spaces A, B is continuous if preimage of open sets of B is open under f

Def^n | A topological mfld is a top. space that is locally "homeomorphic" to $\mathbb{R}^n$

Def^n | $f: A \to B$ is a homeomorphism if 1-to-1, onto, continuous, continuous inverse

Charts: $\{U_i, \varphi_i\}$ open set + homeo to $\mathbb{R}^n$
Atlas: collection of charts s.t. $M = \bigcup_i U_i$

Differentiable mfld: transition maps $\varphi_u \circ \varphi_v^{-1} : \varphi_v(U \cap V) \to \varphi_u(U \cap V)$ are C^k

Ex) 1) $Gl(n, \mathbb{R})$
↳ $\det(A) \neq 0$ open condition b/c $\det(A) =$ polynomial (A) $\to$ continuous in a_{ij}

2) S^2
↳ topology induced by topology on $\mathbb{R}^3$
↳ charts from stereographic projection

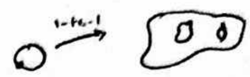
Always consider C^∞ mflds ↓

Hausdorff: for $a, b \in M$, $\exists$ $U \ni a$, $V \ni b$ opens s.t. $U \cap V = \emptyset$ $\forall a \neq b$

Claim: let M be Hausdorff mfld. Then every "sequence" has unique limit if it exists

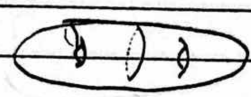
Pf sketch |
Limit/accumulation point: $\{u_i\}$ has a limit $\lim_{i\to\infty} u_i = A$ if $\exists$ N s.t. $\forall$ open sets A_i of A, $u_n \in A_i$ $\forall n > N$.
Assume limit is non-unique: two limits A_1, A_2. $\Rightarrow$ contradiction w/ Hausdorff

Def^n | A mfld M is sequentially compact if every sequence has a convergent subsequence.

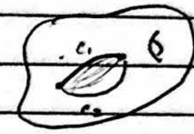


Def^n If all loops $(\ell : S^1 \to M)$ are contractible to a point, the mfld is simply connected.

Ex) $\mathbb{R}^n - \{0\}$, $n > 2$.

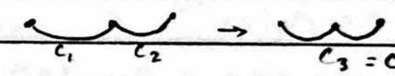
Σ_2  ← neither loop contractible

A smooth manifold M, a curve $c : I \subset \mathbb{R} \to M$
$t \mapsto c(t)$

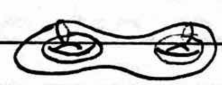


Two curves c_1 and c_2 are homotopic when you can smoothly deform one to another. Formally, there is a smooth map $h(t, x)$ s.t. $h(t, 0) = c_1(t)$
$h(t, 1) = c_2(t)$

Claim: paths form a group under "concatenation"



Def^n
Let $x \in M$ be a point. The space of loops up to homotopy based at x forms a group under concatenation called the Fundamental Group $\pi_1(M)$

Ex)  Fundamental group generated by 4 loops

$0 \neq \ell_1 \in H_1(\Sigma_2; \mathbb{Z})$
$0 = \ell_2 \in H_1(\Sigma; \mathbb{Z})$

Tangent Spaces, bundles, Vector fields

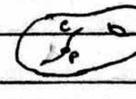
Differentiable: you can approximate maps by "linear" maps

Curve point of view: $c : I \subset \mathbb{R} \to M$ 

c_1 & c_2 are "equivalent" if $c_1'(p) = c_2'(p)$

"the space of curves"$/\sim$ gives all curves with different tangent vectors. If have dim n manifold, then n curves give n vectors to locally span a vector space.

Def^n The tangent space at $p \in M$ = {equivalence class of curves passing through p}
Vectors at $p \in T_pM \simeq \mathbb{R}^n$



Directional Derivative view:

$\mathcal{F} = \{C^\infty$ functions on $M\}$. Take $f : M \to \mathbb{R}$, $c : I \subset \mathbb{R} \to M$

Def^n Directional derivative of $f \in C^\infty(M)$ along c at p is defined as $\frac{d}{dt}(f \circ c)\big|_p$
$= \frac{\partial f}{\partial x^\mu} \cdot c'(t)^\mu \big|_p = V^\mu \frac{\partial f}{\partial x^\mu}$.

A vector field V acts on f as $V^\mu \frac{\partial f}{\partial x^\mu}$. Can represent V in local coords as $V := V^\mu \frac{\partial}{\partial x^\mu}$. The $\{\frac{\partial}{\partial x^\mu}\}_{\mu=1}^{n}$ forms basis of tangent space at p.

Claim: T_pM is the space of derivations at p.

Derivation D is map $D : \mathcal{F}(M) \to \mathcal{F}(M)$ s.t. $D(fg) = (Df)g + f(Dg)$

$$f \mapsto Df$$

Semi-Riemannian Manifolds

Def^{n}|

A Semi-Riemannian $n+1$ mfld of type (p,q) is a pair $(M, \hat{g})$ that satisfies the following

1) M is a smooth, Hausdorff $n+1$ mfld

2) $\hat{g}$ is a symmetric, bilinear, non-degenerate form on $\Gamma(TM)$ where $\Gamma(TM) := \{$vector space of vector fields on $M\}$. So $\hat{g}_x : T_xM \times T_xM \to \mathbb{R}$ (locally)

$(X, Y) \mapsto \hat{g}(X,Y)$

Moreover, at each x, $\hat{g}$ can be expressed as

up to a possible change of chart $\rightarrow$ $\hat{g} = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{\times p}, \underbrace{1, \ldots, 1}_{\times q})$ s.t. $p + q = n+1$. p is called index of $(M, \hat{g})$

Lorentzian mfld when $p = 1$.

Q^{n}| does $p = 0 \Rightarrow$ Riemannian? $\rightarrow$ yes

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Def^{n}| $\hat{g}$ is non-degenerate as a quadratic form if $\hat{g}_x(X,Y) = 0 \;\forall Y \in T_xM \Rightarrow X = 0$

Claim: if $\hat{g}$ is non-degenerate, then $\det(\hat{g}) \neq 0$

Pf| Prove contrapositive: assume $\det(\hat{g}) = 0$. Then columns in $\hat{g}$ linearly dependent so $\exists$ nontrivial coefficients s.t.

$X^0 \hat{g}_{0\nu} + \cdots + X^n \hat{g}_{n\nu} = 0 \Rightarrow X^\mu \hat{g}_{\mu\nu} = 0 \Rightarrow X^\mu \hat{g}(\partial_\mu, \partial_\nu) = 0 \Rightarrow \hat{g}(\underbrace{X^\mu \partial_\mu}_{\neq 0}, \underbrace{\partial_\nu}_{\neq 0}) = 0 \Rightarrow$ degenerate ∎

Volume Form

Non-degenerate $\hat{g}$ induces a volume form on M which is written in local chart as

$$\mu_{\hat{g}} := \underbrace{\sqrt{-\det \hat{g}}}_{\text{"density"}}\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n$$

Claim: This volume form is diffeo-invariant

Pf| $\hat{g}_{\mu\nu}$ in chart x, $\hat{g}'_{\mu\nu}$ in chart y:

$$\hat{g}'\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right) = \hat{g}'\left(\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial}{\partial x^\alpha}, \frac{\partial x^\beta}{\partial y^\nu}\frac{\partial}{\partial x^\beta}\right) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\hat{g}'\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\hat{g}_{\alpha\beta} = (d\Phi)^{-1^T} g\, (d\Phi)^{-1}$$

$$dx^\mu \mapsto \frac{\partial x^\mu}{\partial y^\nu} dy^\nu$$

$$\Rightarrow \mu_{\hat{g}'} = \sqrt{-\det \hat{g}'_{\mu\nu}}\, dy^0 \wedge \cdots \wedge dy^n = \sqrt{-\det\left(\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\hat{g}_{\alpha\beta}\right)}\, \frac{\partial y^0}{\partial x^{\alpha_0}} dx^{\alpha_0} \wedge \cdots \wedge \frac{\partial y^n}{\partial x^{\alpha_n}} dx^{\alpha_n}$$

$$= \det\left(\frac{\partial x^\alpha}{\partial y^\nu}\right)\det\left(\frac{\partial y^\delta}{\partial x^\gamma}\right)\sqrt{-\det \hat{g}}\, dx^0 \wedge \cdots \wedge dx^n = \det \Phi^{-1} \det \Phi \sqrt{\det \hat{g}}\, dx^0 \wedge \cdots \wedge dx^n$$

$$= \sqrt{\det \hat{g}}\, dx^0 \wedge \cdots \wedge dx^n$$ ∎

Connection on TM induced by $\hat{g}$

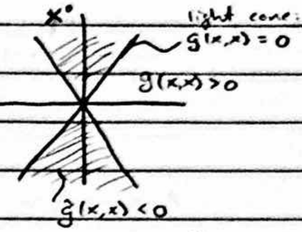
$\hat{g}_x : T_xM \times T_xM \to \mathbb{R} = (-\infty, 0) \cup \{0\} \cup (0, \infty)$

$\hookrightarrow$ can find $X \in T_xM$ s.t. $\hat{g}(X,X) < 0$

$\hat{g}(X,X) = 0$

$\hat{g}(X,X) > 0$

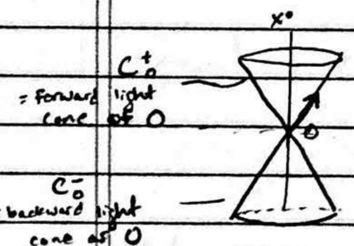
$\hat{g}_x(x,x) = -(x^0)^2 + (x^1)^2 + \cdots$



"Null geometry of tangent space"

Def^n

- $\{X \in T_xM \mid \hat{g}(X,X) < 0\}$ time-like vectors
- $\{X \in T_xM \mid \hat{g}(X,X) = 0\}$ null or light-like vectors
- $\{X \in T_xM \mid \hat{g}(X,X) > 0\}$ space-like vectors



$X \in T_0 C_0^+ \to \hat{g}(X,X) = 0$ ← tangent / normal coincide

Connection on TM

Def^n A connection on TM is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ that sends $(X,Y) \mapsto \nabla_X Y$ and satisfies

1) $\nabla_{fX} Y = f \nabla_X Y, \quad f \in C^\infty(M)$ ($C^\infty(M)$ linearity)

2) $\nabla_X (fY) = X(f)\, Y + f \nabla_X Y, \quad X(f) := X^\mu \partial_\mu f$

Def^n A connection ∇ is called Torsion-free if $\nabla_X Y - \nabla_Y X - [X,Y] = 0$

with $[X,Y] := (X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu)\partial_\mu$

Def^n A metric-compatible connection ∇ that is torsion free is defined as follows: $X, Y, Z \in \Gamma(TM)$, $X(\hat{g}(Y,Z)) = \hat{g}(\nabla_X Y, Z) + \hat{g}(Y, \nabla_X Z)$

$\Rightarrow \nabla_X \hat{g} = 0 \quad \forall X \in \Gamma(TM)$

Claim: This metric-compatible connection is unique!

Pf $X, Y, Z \in \Gamma(TM)$, need (1) torsion-free, (2) metric compatibility conditions

① $X(\hat{g}(Y,Z)) = \hat{g}(\nabla_X Y, Z) + \hat{g}(Y, \nabla_X Z)$

② $Y(\hat{g}(Z,X)) = \hat{g}(\nabla_Y Z, X) + \hat{g}(Z, \nabla_Y X)$

③ $Z(\hat{g}(X,Y)) = \hat{g}(\nabla_Z X, Y) + \hat{g}(X, \nabla_Z Y)$

want eq. $\hat{g}(\nabla_X Y, Z) =$ not dependent on ∇

①+②−③ $\Rightarrow$ want $\hat{g}(Y, \nabla_X Z)$ and $\hat{g}(\nabla_Z X, Y)$ to cancel

$\hat{g}(\nabla_Y Z, X)$ and $\hat{g}(X, \nabla_Z Y)$ to cancel

Pf cont. Use torsion-free conditions: $\nabla_Z X = \nabla_X Z + [Z, X]$

$$\nabla_Z Y = \nabla_Y Z + [Z, Y]$$

Gives $\hat{g}(\nabla_X Z, Y) + \hat{g}([Z,X], Y) = \hat{g}(\nabla_Z X, Y)$

$\hat{g}(X, \nabla_Y Z) + \hat{g}(X, [Z,Y]) = \hat{g}(X, \nabla_Z Y)$

↳ repeat for other terms to cancel

Combine:
$$\boxed{2\hat{g}(\nabla_X Y, Z) = X(\hat{g}(Y,Z)) + Y(\hat{g}(Z,X)) - Z(\hat{g}(X,Y)) - \hat{g}([Z,X],Y) - \hat{g}(X,[Z,Y]) + \hat{g}(Z,[Y,X])}$$

Now suppose $\exists\, \hat{\nabla}$ which is metric compatible & torsion free. Then

$2\hat{g}(\nabla_X Y - \hat{\nabla}_X Y, Z) = 0$ (b/c RHS indep of ∇) $\forall Z \in \Gamma(TM)$.

So non-degeneracy $\Rightarrow \nabla_X Y = \hat{\nabla}_X Y \Rightarrow$ uniqueness ☒

Lemma $X, Y \in \Gamma(TM)$ and in local coord. $X = X^\mu \partial_\mu$, $Y = Y^\mu \partial_\mu$. Then,

$$\nabla_X Y = X^\lambda(\partial_\lambda Y^\mu + \Gamma^\mu_{\lambda\nu} Y^\nu)\partial_\mu \quad \text{where}$$

$g_{\mu\alpha}\Gamma^\alpha_{\lambda\nu} := \hat{g}(\nabla_{\partial_\lambda}\partial_\nu, \partial_\mu)$ and in local coord $\Gamma^\alpha_{\lambda\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_\lambda g_{\beta\nu} + \partial_\nu g_{\beta\lambda} - \partial_\beta g_{\lambda\nu})$

NB| $[\partial_\mu, \partial_\nu] = 0$

Pf $\nabla_X Y = \nabla_{X^\mu\partial_\mu}(Y^\nu\partial_\nu) = X^\mu \nabla_{\partial_\mu}(Y^\nu\partial_\nu) = X^\mu((\partial_\mu Y^\nu)\partial_\nu + \nabla_{\partial_\mu}\partial_\nu Y^\nu)$

$= X^\mu((\partial_\mu Y^\nu)\partial_\nu + Y^\nu \Gamma^\lambda_{\mu\nu}\partial_\lambda) = X^\mu(\partial_\mu Y^\nu + \Gamma^\nu_{\mu\lambda} Y^\lambda)\partial_\nu$ $\quad \because \Gamma^\lambda_{\mu\nu}\partial_\lambda$

$\nabla_X Y$ a tensor, but $\partial_\lambda Y^\mu$, $\Gamma^\mu_{\lambda\nu} Y^\nu$ not tensors

Now $2\hat{g}(\nabla_{\partial_\mu}\partial_\nu, \partial_\lambda) = \partial_\mu(\hat{g}(\partial_\nu, \partial_\lambda)) + \partial_\nu(\hat{g}(\partial_\lambda, \partial_\mu)) - \partial_\lambda(\hat{g}(\partial_\mu, \partial_\nu))$

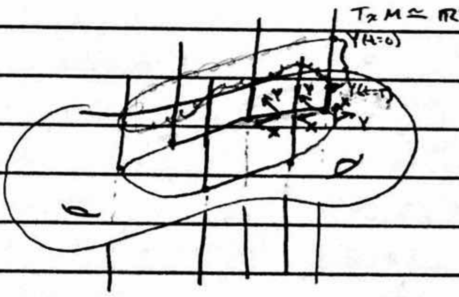
$= \partial_\mu \hat{g}_{\nu\lambda} + \partial_\nu \hat{g}_{\lambda\mu} - \partial_\lambda \hat{g}_{\mu\nu}$

$\Rightarrow 2\hat{g}(\Gamma^\alpha_{\mu\nu}\partial_\alpha, \partial_\lambda) = \partial_\mu \hat{g}_{\nu\lambda} + \partial_\nu \hat{g}_{\lambda\mu} - \partial_\lambda \hat{g}_{\mu\nu}$

$\Rightarrow 2\hat{g}_{\alpha\lambda}\Gamma^\alpha_{\mu\nu} = \partial_\mu \hat{g}_{\nu\lambda} + \partial_\nu \hat{g}_{\lambda\mu} - \partial_\lambda \hat{g}_{\mu\nu}$ ✓

Particle Motion and Parallel Transport

Defn A vector field along a curve $c(t)$ is a curve on the tangent bundle s.t. $X_{c(t)} \in T_{c(t)}M$.



X tangent to curve

Y vector field to be transported

Defn A vector field Y is said to be parallely transported along a curve $c(t)$ if $\nabla_X Y = 0$ along the curve where X is the tangent vector field to $c(t)$.

Take curve $t \mapsto c(t) = (c^0(t), \ldots, c^n(t))$ with $t \in [0,T]$, $c(0) = c(T)$.

Want to solve $\nabla_X Y = 0$. If $Y(c(0)) = Y_1$, define map $P: Y_1 \mapsto Y(c(T))$

$T_{c(0)}M \to T_{c(T)}M$.

In local coords, $X = c'^\mu(t)\partial_\mu$.

WTS: $\nabla_{\frac{dc^\mu}{dt}\partial_\mu} Y = \frac{dc^\mu}{dt}\nabla_{\partial_\mu} Y = 0$.

Claim: Parallel transport map $P: T_{c(0)}M \to T_{c(t)}M$ $\forall t \in [0,T]$ is an isometry. (wrt $\hat{g}$)

$\frac{d}{dt} = \frac{dc^\mu}{dt}\frac{\partial}{\partial c^\mu} = \dot{c}(t)\partial_\mu$

Covariant deriv. along curve

Pf Fix X tangent vector field to $t \mapsto c(t)$ curve. Let Y, Z be vector fields along $c(t)$ which are parallely transported. To show isometry, need to show $\hat{g}(Y,Z) = \hat{g}(PY, PZ)$.

$$\frac{d}{dt}\hat{g}(Y,Z) = \hat{g}(\nabla_X Y, Z) + \hat{g}(Y, \nabla_X Z) \quad \text{from metric compatibility}$$
$$= 0 + 0 \quad \text{b/c def. of parallel transport}$$

Have ODE: $\frac{d}{dt}\hat{g}(Y,Z) = 0$ along the curve. With $Y(t=0) = Y_1$, $Z(t=0) = Z_1$

$\Rightarrow \hat{g}(Y,Z) = \text{const.}$ along the curve $\Rightarrow \hat{g}(Y,Z) = \hat{g}(PY,PZ)$. ∎

Can find map $End(T_pM) = End(\mathbb{R}^{n+1}) = Gl(n+1, \mathbb{R})$. to move endpoint of Y to startpoint

$P: T_pM \to T_pM$, $Y_p \mapsto P_X Y$ given by solution to $\frac{d}{d\lambda} Y \Big|_{\text{along curve}} = 0 \Rightarrow \nabla_X Y = 0$

$$\Rightarrow \underbrace{X^\mu \partial_\mu Y^\nu + \Gamma_{\mu\alpha}^{\ \ \nu} X^\mu Y^\alpha = 0}$$

given $Y(\lambda=0)$, does eqn. have solution?

$$\Rightarrow \frac{dY^\nu}{d\lambda} + \Gamma_{\mu\alpha}^{\ \ \nu} X^\mu Y^\alpha = 0, \quad Y(\lambda=0) = Y_p. \quad (*)$$

Claim: $(*)$ has a unique sol^n:

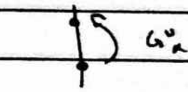
$$Y^\nu(\lambda) - Y^\nu(0) = -\int_0^\lambda \Gamma_{\mu\alpha}^{\ \ \nu} X^\mu Y^\alpha \, d\lambda$$
$$\Rightarrow Y^\nu(\lambda) = Y^\nu(0) - \int_0^\lambda \Gamma_{\mu\alpha}^{\ \ \nu} X^\mu \left(Y^\alpha(0) - \int_0^\lambda \Gamma_{\beta_1\beta_2}^{\ \ \alpha} X^{\beta_1} Y^{\beta_2} \, d\lambda' \right) d\lambda = \cdots$$
$$Y^\nu(\lambda) := Y^\alpha \boxed{\mathcal{P} e^{\int_0^\lambda \Gamma_{\mu\alpha}^{\ \ \nu} X^\mu d\lambda}} = G^\nu_\alpha$$

$e^{\oint_0^1 \Gamma \cdot X \, d\lambda}$ → 'wilson loop', 'holonomy'

└ partial exponential ← doesn't give matrix exponential as would require matrices to commute

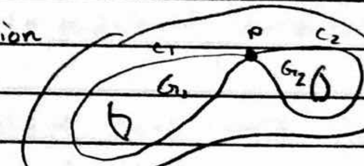
So $Y^\nu(1) = G^\nu_\alpha(1) \cdot Y^\alpha(0)$

$[G] \in Isom(T_pM) \subset Gl(n+1, \mathbb{R})$ ('matrix that moves vectors along fibre')



Def^n A holonomy group at p is collection of elements of the type G which 'fills the gap' created by parallel transport.

↳ $Hol_p(M) := \{ G \in Isom(T_pM) \subset Gl(n+1, \mathbb{R}) : \underbrace{P}_{\text{parallel transport map}} \cdot Y(0) = G \cdot Y(0) = Y(1) \ \forall Y \in T_pM \}$



$C_1, C_2 \in \pi_1(M)$

$C_1 * C_2 \Rightarrow G_2 \cdot G_1$

Fact: Hol_p is a Lie group

Curvature

$$R: \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$(X, Y, Z) \mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X,Y)Z$$

If $X = \partial_\mu$, $Y = \partial_\nu$: $(\partial_\mu, \partial_\nu, Z^\alpha \partial_\alpha) \mapsto \nabla_{\partial_\mu} \nabla_{\partial_\nu} Z^\alpha - \nabla_{\partial_\nu} \nabla_{\partial_\mu} Z^\alpha = R^\alpha_{\ \beta\mu\nu} Z^\beta$

$$\boxed{R^\alpha_{\ \beta\mu\nu} Z^\beta = \nabla_\mu \nabla_\nu Z^\alpha - \nabla_\nu \nabla_\mu Z^\alpha} \quad \text{Ricci identity}$$

Define $R(X, Y) : \Gamma(TM) \to \Gamma(TM)$ ← $R(X,Y)$ at p is endomorphism of T_pM

(X, Y given)

$$Z \mapsto R(X, Y, Z) \quad , \quad Z^\alpha \mapsto R^\alpha{}_{\beta\mu\nu} Z^\beta X^\mu Y^\nu$$

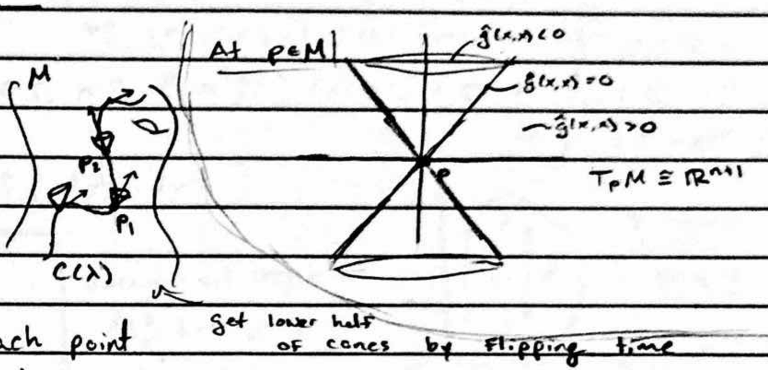
$\boxed{\text{Th}^{\text{m}} \text{ (Ambrose-Singer thm)}}$

The Lie algebra of $\text{Hol}_p(M)$ is precisely generated by $R_p(X, Y)$

$\text{Hol}_p(M) \ni G \sim \exp(R_p(X,Y))$

Causal Geometry

Consider $\lambda \mapsto c(\lambda)$



Def$^{\text{n}}$ A curve $\lambda \mapsto c(\lambda)$ is called timelike (resp. spacelike, null) if its tangent vector at each point is timelike (spacelike, null)

Def$^{\text{n}}$ Time is a function $t : M \to \mathbb{R}$ such that $\underbrace{\nabla t}_{\text{time vector field}}$ is timelike everywhere on M. (i.e. $\hat{g}(\nabla t, \nabla t) < 0$)

$\nabla t := \nabla^\mu t \, \partial_\mu$

in local coord chart $(x^0, x^1, \ldots x^n)$, identify $x^0 \equiv t$

Fact: Using ∇t, can define flow $\frac{dx^\mu}{d\lambda} = (\nabla t)^\mu$ to get a timelike curve

- Integral curve of $\nabla t \Rightarrow$ "time line"

Can expand: $\hat{g}_{\mu\nu} dx^\mu \otimes dx^\nu = \hat{g}_{00} dx^0 \otimes dx^0 + \hat{g}_{0i} dx^0 \otimes dx^i + \hat{g}_{i0} dx^i \otimes dx^0 + \hat{g}_{ij} dx^i \otimes dx^j$

Th$^{\text{m}}$ There exists a sufficiently small coord. chart $(x^{0'}, \ldots, x^{n'})$ around each point in which $\hat{g} := \hat{g}_{00} dx^{0'} \otimes dx^{0'} + \hat{g}_{ij} dx^{i'} \otimes dx^{j'}$

Pf Choose a change of chart s.t. $x^{0'} = x^0$, $x^{i'} = \overbrace{x^{i'}(x^0, \ldots, x^n)}^{\text{fn. of old coords.}}$ with $\frac{\partial x^{i'}}{\partial x^j}$ is non-singular (to have a well-defined diffeo).

$\hookrightarrow J = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial x^{i'}}{\partial x^j} \end{pmatrix}$

Now $\hat{g}'_{\mu\nu} = \frac{\partial x^{\alpha'}}{\partial x^\mu} \frac{\partial x^{\beta'}}{\partial x^\nu} \hat{g}_{\alpha\beta}$. To kill cross-terms, demand $\hat{g}_{0i} = 0$, so

$$\frac{\partial x^{\alpha'}}{\partial x^0} \frac{\partial x^{\beta'}}{\partial x^i} \hat{g}_{\alpha\beta} = 0 \Rightarrow \frac{\partial x'^0}{\partial x^0} \cancelto{0}{\frac{\partial x'^0}{\partial x^i}} \hat{g}_{00} + \frac{\partial x'^0}{\partial x^0} \frac{\partial x'^j}{\partial x^i} \hat{g}_{0j} + \frac{\partial x'^j}{\partial x^0} \cancelto{0}{\frac{\partial x'^0}{\partial x^i}} \hat{g}_{j0} + \frac{\partial x'^j}{\partial x^0} \frac{\partial x'^k}{\partial x^i} \hat{g}_{jk} = 0$$

$$\Rightarrow \frac{\partial x'^j}{\partial x^i} \hat{g}_{0j} + \frac{\partial x'^j}{\partial x^0} \frac{\partial x'^k}{\partial x^i} \hat{g}_{jk} = 0 \Rightarrow \frac{\partial x'^j}{\partial x^i} \left(\hat{g}_{0i} + \frac{\partial x'^k}{\partial x^0} \hat{g}_{ik} \right) = 0$$

sym. of $\hat{g}_{jk}$ to swap partial indices

since $\frac{\partial x'^j}{\partial x^i}$ is non-singular, mult. by inverse $\Rightarrow \frac{\partial x'^k}{\partial x^0} \hat{g}_{ik} + \hat{g}_{0i} = 0$

$\hookrightarrow$ system of linear PDE $\to$ from PDE theory, $\exists$ solution! ∎

From now on, will be able to choose this chart to work in.

T (p,q) tensor: ⤳ use $\hat{g}$ to raise/lower

$$T := T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \frac{\partial}{\partial x^{\mu_p}} \otimes dx^{\nu_1} \ldots \otimes dx^{\nu_q}$$

(0,2) tensor

$\frac{\partial}{\partial x^\mu}(dx^\nu) = \delta^\nu_\mu$

$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$

Have metric $\hat{g} := \hat{g}_{00}\, dx^0 \otimes dx^0 + \hat{g}_{ij}\, dx^i \otimes dx^j$. Question: What is inverse?

$\mapsto \hat{g}^{-1} := \frac{1}{\hat{g}_{00}} \frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x^0} + (g^{-1})^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \equiv \frac{1}{\hat{g}_{00}} \frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x^0} + \hat{g}^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$

(2,0) tensor

As we took $x^0 \leftrightarrow t$, where $\hat{g}(\nabla t, \nabla t) < 0 \Rightarrow \hat{g}(dt, dt) < 0$

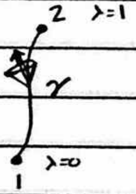
$\hat{g}(\nabla t, \nabla t) = \hat{g}_{\mu\nu} \nabla^\mu t \nabla^\nu t$, $\quad \hat{g}(dt,dt) = \hat{g}^{\mu\nu} \nabla_\mu t \nabla_\nu t$

$= \hat{g}_{\mu\nu} \delta^\mu_\alpha \nabla^\alpha t \nabla^\nu t = \hat{g}_{\mu\nu} \hat{g}^{\mu\beta} \hat{g}_{\beta\alpha} \nabla^\alpha t \nabla^\nu t = \hat{g}^{\mu\beta} \nabla_\beta t \nabla_\mu t$

With this condition, $\left(\frac{1}{\hat{g}_{00}} \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} + \hat{g}_{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}\right)(dt, dt) < 0 \Rightarrow \hat{g}_{00} < 0$

$\Rightarrow \hat{g}_{00} := -N^2$

So write $\hat{g} = -N^2\, dt \otimes dt + \hat{g}_{ij}\, dx^i \otimes dx^j$



Curve $\lambda \mapsto (x^0(\lambda), x^1(\lambda), \ldots) = x^\mu(\lambda)$

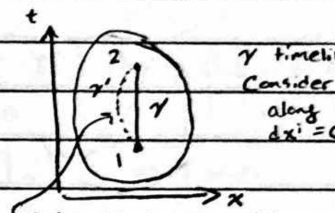
Length of Timelike Curve

$$l_{12} := \int_0^1 \sqrt{-g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda}}\, d\lambda$$

Infentesimal length

$ds^2_\gamma = |-N^2 dt^2|$

$ds^2_{\gamma'} < ds^2_\gamma$



γ timelike. Consider along γ $dx^i = 0$

γ' has shorter length b/c time and space parts of metric have opposite sign

Claim: l_{12} is $\underbrace{\text{parametrization}}$ invariant

reparametrization $\equiv$ diffeo of an open interval

$\boxed{\text{Pf}}$ $\lambda' = \lambda'(\lambda)$, so $d\lambda' = \frac{d\lambda'}{d\lambda} d\lambda$ $\left(d\lambda = \frac{d\lambda}{d\lambda'} d\lambda'\right)$

Then $l_{12} = \int_0^1 \sqrt{-g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda'} \frac{\partial x^\nu}{\partial \lambda'} \left(\frac{\partial \lambda'}{\partial \lambda}\right)^2}\, d\lambda' \cdot \frac{d\lambda}{d\lambda'} = \int_0^1 \sqrt{-g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda'} \frac{\partial x^\nu}{\partial \lambda'}}\, d\lambda'$ ☒

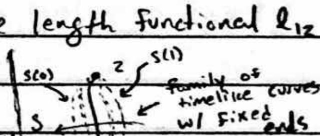
Explicitly, $l_{12} = \int_0^1 \sqrt{-N^2 \left(\frac{dt}{d\lambda}\right)^2 + \hat{g}_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}\, d\lambda$.

Recall Y is parallely transported along integral curve of X if $\nabla_X Y|_{\text{curve}} = 0$

$\boxed{\text{Def}^{\text{n}}}$ Geodesics are auto-parallel curves (i.e. $\nabla_X X = 0$)

doesn't make sense for null curves since $g(X,X) = 0 \Rightarrow l_{12} = 0$ ⤳

$\boxed{\text{Thm}}$ Timelike (spacelike) geodesics are 'critical points' of the length functional l_{12}



$\boxed{\text{Pf}}$ Take a family of timelike curves w/ fixed endpoints at $\lambda = 0$, $\lambda = 1$. A critical point of the length functional in the space of curves is defined as a zero of the Frechet derivative of l_{12}:

Frechet derivative of l at γ in direction of ξ

$$D l(\gamma) \cdot \xi = \frac{d}{ds} l_{12}(y^\mu + s\xi^\mu)\Big|_{s=0} = 0.$$

'transversal to $\frac{dy^\mu}{d\lambda}$ on γ'

Expand: $D l(\gamma) \cdot \xi = \frac{d}{ds} \int_0^1 \sqrt{-\hat{g}_{\mu\nu}(y + s\xi) \frac{d}{d\lambda}(y^\mu + s\xi^\mu) \frac{d}{d\lambda}(y^\nu + s\xi^\nu)}\, d\lambda \Big|_{s=0}$

$= \frac{1}{2} \int_0^1 \frac{1}{\sqrt{\cdots}} \frac{d}{ds}\left(-\hat{g}_{\mu\nu}(y + s\xi) \frac{d}{d\lambda}(y^\mu + s\xi^\mu) \frac{d}{d\lambda}(y^\nu + s\xi^\nu)\right) d\lambda \Big|_{s=0}$

From chain rule, $\frac{d}{ds} g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \xi^\alpha} \frac{d\xi^\alpha}{ds}$. So,

$$D l(\gamma) \cdot \xi = -\frac{1}{2} \int_0^1 \frac{1}{\sqrt{-g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda}}} \cdot \left(\frac{\partial g_{\mu\nu}}{\partial y^\alpha} \xi^\alpha \frac{dy^\mu}{d\lambda} \frac{dy^\nu}{d\lambda} + 2\hat{g}_{\mu\nu}(y) \frac{d\xi^\mu}{d\lambda} \frac{dy^\nu}{d\lambda}\right) d\lambda$$

Pf cont ⟶

Lemma Along geodesic, $\hat{g}_{\mu\nu} \frac{dy^\mu}{d\lambda} \frac{dy^\nu}{d\lambda}$ is conserved

Pf $\hat{g}_{\mu\nu} \frac{dy^\mu}{d\lambda} \frac{dy^\nu}{d\lambda} = \hat{g}(X, X) \Rightarrow$ WTS: $\frac{d}{d\lambda} \hat{g}(X,X) = 0$

Recall $\frac{d}{d\lambda} = \frac{\partial y^\mu}{\partial \lambda} \frac{\partial}{\partial y^\mu} = X^\mu \frac{\partial}{\partial y^\mu} \Rightarrow \frac{d}{d\lambda} \hat{g}(X,X) = \nabla_X \hat{g}(X,X)$

$= \hat{g}(\nabla_X X, X) + \hat{g}(X, \nabla_X X) = 0$ $\blacksquare$

Thm Pf cont. Since $\frac{d}{d\lambda} \hat{g}(X,X) = 0 \Rightarrow \hat{g}(X,X) = \text{const.}$ along $\gamma \Rightarrow$ can set $= -1$ by normalizi

$$\text{So } -2 \cdot D\ell(\gamma) \cdot \xi = \int_0^1 \Big(\underbrace{\frac{\partial \hat{g}_{\mu\nu}}{\partial y^\alpha} \frac{dy^\mu}{d\lambda} \frac{dy^\nu}{d\lambda} \xi^\alpha} + 2 \hat{g}_{\mu\nu}(y) \frac{d\xi^\mu}{d\lambda} \frac{dy^\nu}{d\lambda} \Big) d\lambda$$

$$= \int_0^1 \; = \; + 2 \int_0^1 \frac{d}{d\lambda} \Big(\hat{g}_{\mu\nu}(y) \frac{dy^\nu}{d\lambda} \xi^\mu \Big) d\lambda - 2 \int_0^1 \frac{d}{d\lambda} \Big(\hat{g}_{\mu\nu} \frac{dy^\nu}{d\lambda} \Big) \xi^\mu d\lambda$$

$$= \int_0^1 [=] - 2 \int_0^1 \frac{d}{d\lambda} \Big(\hat{g}_{\mu\nu} \frac{dy^\nu}{d\lambda} \Big) \xi^\mu + \Big[\cancelto{0}{\hat{g}_{\mu\nu} \frac{dy^\nu}{d\lambda} \xi^\mu} \Big]_0^1 \leftarrow \text{endpoints fixed} \Rightarrow \xi = 0$$

$$= \int_0^1 \partial_\alpha \hat{g}_{\mu\nu} \frac{dy^\mu}{d\lambda} \frac{dy^\nu}{d\lambda} \xi^\alpha - 2 \frac{d}{d\lambda} \Big(\hat{g}_{\alpha\nu} \frac{dy^\nu}{d\lambda} \Big) \xi^\alpha \, d\lambda \leftarrow \xi \text{ arbitrary, so show rest vanishes}$$

$$= 2 \int_0^1 \Big(\frac{1}{2} \partial_\alpha \hat{g}_{\mu\nu} X^\mu X^\nu - \frac{dX_\alpha}{d\lambda} \Big) \xi^\alpha \, d\lambda$$

$$= 2 \int_0^1 \Big(\underbrace{\frac{1}{2} \partial_\alpha \hat{g}_{\mu\nu} X^\mu X^\nu - X^\mu \partial_\mu X_\alpha}_{\text{should vanish on geodesic}} \Big) \xi^\alpha \, d\lambda$$

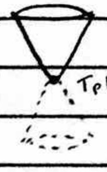
$\nabla \hat{g} = 0$

$\nabla_\alpha \hat{g}_{\mu\nu} = (\nabla_{\partial_\alpha} \hat{g})(\partial_\mu, \partial_\nu) = 0 \Rightarrow$ $\nabla_\alpha \hat{g}_{\mu\nu} = \partial_\alpha \hat{g}_{\mu\nu} - \Gamma_{\alpha\mu}^{\lambda} \hat{g}_{\lambda\nu} - \Gamma_{\nu\alpha}^{\lambda} \hat{g}_{\mu\lambda} = 0$

$$\Rightarrow \frac{1}{2} \partial_\alpha \hat{g}_{\mu\nu} X^\mu X^\nu - X^\mu \partial_\mu X_\alpha = \frac{1}{2} \big(\Gamma_{\alpha\mu}^{\lambda} \hat{g}_{\lambda\nu} + \Gamma_{\nu\alpha}^{\lambda} \hat{g}_{\mu\lambda} \big) X^\mu X^\nu - X^\mu \partial_\mu X_\alpha$$

$$= \frac{1}{2} \big(\Gamma_{\alpha\mu}^{\lambda} X^\mu X_\lambda + \Gamma_{\nu\alpha}^{\lambda} X_\lambda X^\nu \big) - X^\mu \partial_\mu X_\alpha$$

$$= \Gamma_{\alpha\mu}^{\lambda} X^\mu X_\lambda - X^\mu \partial_\mu X_\alpha = X^\mu \nabla_\mu X_\alpha = 0 \text{ b/c geodesic}$$ $\blacksquare$



T_pM, $g = \text{diag}(-1,1,1,1)$ $\Rightarrow d\lambda^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = 0$ for null vectors

For null vectors, $\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2 = 1 \Rightarrow |v|^2 = 1 \Rightarrow c := 1$

When $-dt^2 + dr^2 < 0$, $\left(\frac{dr}{dt}\right)^2 < 1 \Rightarrow v_{\text{timelike}} < c$

Postulates of Relativity

① At each point $p \in M$, speed of light is 1 and therefore universal

② Any physical (time like) observer / particle moves at a speed less than 1

③ ℓ_{12} is the actual physical 'time' experienced by any particle = "proper time"

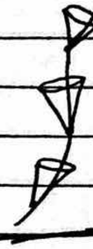
(coordinate / chart time t is manifestly relative)

b/c can change chart

b/c ℓ_{12} invariant under change of chart

Suppose moving very slow $\Rightarrow d\tau^2 \ll dt^2$

$\Rightarrow d\lambda^2 \approx -dt^2 \Rightarrow$ coord. time and proper time start to coincide



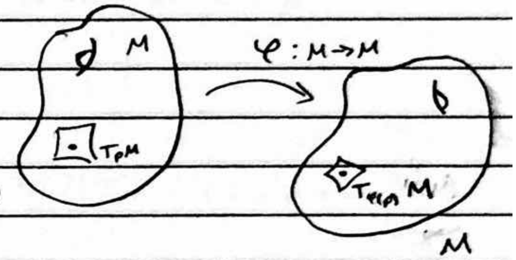
Def$^{\underline{n}}$ | A relativistic point particle is a curve $\lambda \mapsto (x^0(\lambda), x^1(\lambda), \cdots, x^n(\lambda))$ in the spacetime M that has speed at each point comparable to the speed of light

Special Relativity: when $\hat{g} = \text{diag}(-1,1,1,1) = \eta \quad \forall p \in M \quad (\mathbb{R}^{1+n})$

Isometry, Killing Vector Fields, Killing Tensors

Pushforward: $\underset{T_pM}{X} \mapsto \underset{T_{\varphi(p)}M}{\varphi_* X}$

can't pull $\hat{g}$ back, since $\hat{g} \in \text{section}(T^*M \otimes T^*M)$



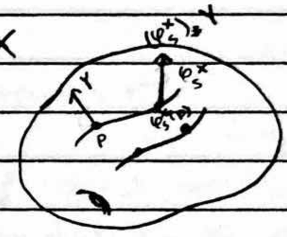
Pullback: pulling $T^*_{\varphi(p)}M$ back to T_p^*M along φ

$\hookrightarrow (\varphi^* \hat{g})_p (X,Y) := \hat{g}_{\varphi(p)} (\varphi_* X, \varphi_* Y)$.

Consider 1-parameter diffeo (from M to M) φ_s^X, with $\left.\frac{d\varphi_s}{ds}\right|_{s=0} = X$

Define $\boxed{L_X \hat{g} := \left.\frac{d}{ds}\left((\varphi_s^X)^* \hat{g}\right)\right|_{s=0}} \in T^*M \otimes T^*M$ (not necessarily a metric)

$\hookrightarrow$ measure how metric changes along flow φ_s^X



Th$^{\underline{m}}$ | Let X be a vector field on $(M, \hat{g})$ with flow φ_s^X. The Lie derivative of $\hat{g}$ wrt X reads in local coords. as:

$$(L_X \hat{g})_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu$$

Def$^{\underline{n}}$ | φ_s^X is an isometry if $(\varphi_s^X)^* \hat{g} = \hat{g} \Rightarrow L_X \hat{g} = 0$.

Pf of Th$^{\underline{m}}$ | $(\varphi_s^* \hat{g})(X,Y) = \hat{g}((\varphi_s)_* X, (\varphi_s)_* Y)$

$$\Rightarrow (\varphi_s^* \hat{g})_{\mu\nu} = \frac{\partial \varphi_s^\alpha}{\partial x^\mu} \frac{\partial \varphi^\beta}{\partial x^\nu} \hat{g}_{\alpha\beta}$$

So have
$$(L_X \hat{g})_{\mu\nu} = \frac{d}{ds}\left((\varphi_s^*)\hat{g}_{\mu\nu}\right)\Big|_{s=0} = \frac{d}{ds}\left(\frac{\partial \varphi_s^\alpha}{\partial x^\mu} \frac{\partial \varphi_s^\beta}{\partial x^\nu} \hat{g}_{\alpha\beta}\right)\Big|_{s=0}$$
$$= \frac{\partial}{\partial x^\mu}\frac{d\varphi_s^\alpha}{ds} \frac{\partial \varphi_s^\beta}{\partial x^\nu} \hat{g}_{\alpha\beta} + \frac{\partial \varphi^\alpha}{\partial x^\mu} \frac{\partial}{\partial x^\nu}\frac{d\varphi^\beta}{ds} \hat{g}_{\alpha\beta} + \frac{\partial \varphi_s^\alpha}{\partial x^\mu}\frac{\partial \varphi_s^\beta}{\partial x^\nu}\frac{d}{ds}\hat{g}_{\alpha\beta}\Big|_{s=0}$$
$$= \frac{\partial X^\alpha}{\partial x^\mu} \delta^\beta{}_\nu \hat{g}_{\alpha\beta} + \delta^\alpha{}_\mu \frac{\partial X^\beta}{\partial x^\nu} \hat{g}_{\alpha\beta} + \delta^\alpha_\mu \delta^\beta{}_\nu X^\lambda \partial_\lambda \hat{g}_{\mu\nu}$$
$$= \hat{g}_{\nu\alpha} \frac{\partial X^\alpha}{\partial x^\mu} + \hat{g}_{\mu\alpha}\frac{\partial X^\alpha}{\partial x^\nu} + X^\lambda \partial_\lambda \hat{g}_{\mu\nu}$$

NB | All ∇s are induced by metric $\hat{g}$ and metric compatible. $\nabla \hat{g} = 0$

Recall $\nabla_\mu X^\lambda = \partial_\mu X^\lambda + \Gamma_{\mu\alpha}^\lambda X^\alpha$

$$= X^\lambda \partial_\lambda \hat{g}_{\mu\nu} + \hat{g}_{\mu\lambda}(\nabla_\nu X^\lambda - \Gamma_{\nu\alpha}^\lambda X^\alpha) + \hat{g}_{\nu\lambda}(\nabla_\mu X^\lambda - \Gamma_{\mu\alpha}^\lambda X^\alpha)$$
$$= \underline{\nabla_\nu X_\mu + \nabla_\mu X_\nu} + X^\lambda \partial_\lambda \hat{g}_{\mu\nu} - \hat{g}_{\mu\lambda}\Gamma_{\nu\alpha}^\lambda X^\alpha - \hat{g}_{\lambda\nu}\Gamma_{\mu\alpha}^\lambda X^\alpha$$
$$= + X^\lambda(\partial_\lambda \hat{g}_{\mu\nu} - \Gamma_{\nu\lambda}^\alpha \hat{g}_{\mu\alpha} - \Gamma_{\mu\lambda}^\alpha \hat{g}_{\alpha\nu}) = X^\lambda \underbrace{\nabla_\lambda \hat{g}_{\mu\nu}}_{0}$$

Def$^{\underline{n}}$ φ_s^X is an isometry if $\frac{d}{ds}\left((\varphi_s^X)^* \hat{g}_{\mu\nu}\right) = (L_X g)_{\mu\nu} = 0$

Question: is this symmetry continuous or discrete?
↳ Continuous (in parameter s)

Symmetry groups = Isometry group = $ISO_M = \{ \varphi_s \in \text{Diff}(M) \mid \varphi_s^* \hat{g} = \hat{g} \}$

one-param group of — $\text{Diff}(M) = \{\text{group of diffeos of } M\}$

↳ generators of $ISO_M \Rightarrow X = \frac{d}{ds} \varphi_s^X \big|_{s=0}$

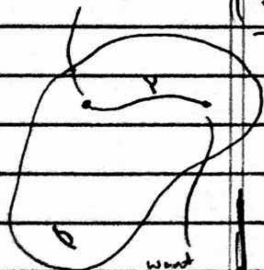
Fact: ISO_M is a Lie group
↳ $\mathfrak{iso}_M = \{\text{space spanned by } X\}$ = Lie algebra of ISO_M

X called a Killing field

$\mathfrak{iso}_M = \{ X \in \text{sections}(TM) \mid L_X \hat{g} = 0 \}$

Claim: the dimension of $\mathfrak{iso}_M$ can be at most $\frac{(n+1)(n+2)}{2}$ $(\dim M = n+1)$

$\frac{(n+1)(n+2)}{2}$ components



want # components needed to be conserved

Pf sketch: $(L_X \hat{g})_{\mu\nu} = 0 \Rightarrow \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$.

\# of components of $\nabla_\mu X_\nu = (n+1)^2$

With $\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$ constraint, # independent components: $(n+1)^2 - \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}$

- Now take vector field $Y = \frac{d}{d\lambda} = Y^\mu \frac{\partial}{\partial x^\mu}$ and study $\frac{d}{d\lambda}(\nabla_\mu X_\nu)$ and show $\frac{(n+1)(n+2)}{2}$ independent components is enough to describe $(X, \nabla X)$ along the flow

Lemma Let Y be a vector field $\frac{d}{d\lambda} = Y^\mu \frac{\partial}{\partial x^\mu}$ and X be a generator of isometry / Killing field. Then the following is satisfied along flow of Y:

$$\underbrace{\frac{d}{d\lambda} \nabla_\mu X^\nu}_{= Y^\alpha \nabla_\alpha \nabla_\mu X^\nu} = R^\nu{}_{\mu\alpha\delta} X^\delta Y^\alpha$$

Pf

$\nabla_\mu X_\nu + \nabla_\nu X_\mu = \pi_{\mu\nu}$ — deformation tensor (measures how much metric changes → how much deforms)

↓

$\nabla_\lambda \nabla_\mu X_\nu + \nabla_\lambda \nabla_\nu X_\mu = \nabla_\lambda \pi_{\mu\nu}$ ①

$\lambda \to \mu$, $\mu \to \nu$, ~~$\nu \to \lambda$~~ ⇒ $\nabla_\mu \nabla_\nu X_\lambda + \nabla_\mu \nabla_\lambda X_\nu = \nabla_\mu \pi_{\nu\lambda}$ ②

$\mu \to \nu$, $\nu \to \lambda$, $\lambda \to \mu$ ⇒ $\nabla_\nu \nabla_\lambda X_\mu + \nabla_\nu \nabla_\mu X_\nu = \nabla_\nu \pi_{\lambda\mu}$ ③

Use $\nabla_\mu \nabla_\nu X_\lambda - \nabla_\nu \nabla_\mu X_\lambda = -R^\alpha{}_{\lambda\mu\nu} X_\alpha$

↳ $① + ② - ③ = [\nabla_\lambda \nabla_\mu X_\nu + \nabla_\mu \nabla_\lambda X_\nu] + [\nabla_\lambda \nabla_\nu X_\mu - \nabla_\nu \nabla_\lambda X_\mu]$
$+ [\nabla_\mu \nabla_\nu X_\lambda - \nabla_\nu \nabla_\mu X_\lambda]$
$= \nabla_\lambda \pi_{\mu\nu} + \nabla_\mu \pi_{\nu\lambda} - \nabla_\nu \pi_{\lambda\mu}$

Pf cont. | But from Ricci identity,

$$①+②-③ = [2\nabla_\lambda \nabla_\mu X_\nu + R^\alpha{}_{\nu\lambda\mu} X_\alpha] + [-R^\alpha{}_{\mu\lambda\nu} X_\alpha] + [-R^\alpha{}_{\lambda\mu\nu} X_\alpha]$$

$$\Rightarrow 2\nabla_\lambda \nabla_\mu X_\nu + \cancel{R^\alpha{}_{\nu\lambda\mu} X_\alpha} \underbrace{- R^\alpha{}_{\mu\lambda\nu} X_\alpha - R^\alpha{}_{\lambda\mu\nu} X_\alpha} = \Pi_{\mu\nu\lambda} \quad \leftarrow \text{from other page}$$

$$(R^\alpha{}_{\mu\lambda\nu} + R^\alpha{}_{\lambda\mu\nu}) X_\alpha = (2R^\alpha{}_{\mu\lambda\nu} + \cancel{R^\alpha{}_{\nu\mu\lambda}}) X_\alpha$$

$$\Rightarrow 2\nabla_\lambda \nabla_\mu X_\nu - 2R^\alpha{}_{\mu\lambda\nu} X_\alpha = \Pi_{\mu\nu\lambda} \quad \leftarrow \text{from algebraic Bianchi identity}$$

$$\Rightarrow Y^\lambda \nabla_\lambda \nabla_\mu X_\nu = Y^\lambda R^\alpha{}_{\mu\lambda\nu} X_\alpha + \underbrace{\tfrac{1}{2} Y^\lambda \Pi_{\mu\nu\lambda}}_{=0 \text{ for killing field}}$$

∎

Back to Claim pf: X killing $\Rightarrow Y^\alpha \nabla_\alpha \nabla_\beta X^\mu = Y^\alpha R^\mu{}_{\beta\alpha\delta} X^\delta$

$$\frac{d}{d\lambda}(\nabla_\beta X^\mu) = R^\mu{}_{\beta\alpha\delta} X^\delta Y^\alpha \rightarrow \text{algebraic in } X$$

$$\Rightarrow \nabla_\beta X^\mu(\lambda) = \underbrace{\nabla_\beta X^\mu(\lambda=0)}_{(n+1)(n+2)/2} + \int_{\text{along flow of } Y} R^\mu{}_{\beta\alpha\delta} X^\delta Y^\alpha \, d\lambda$$

since algebraic in X, cannot contribute more components

$\Rightarrow$ Completely described by $(n+1)(n+2)/2$ indep. components.

∎

$\varphi_s^X \rightarrow$ 1-parameter family of ISO

ISOm $\Rightarrow X$ = Killing field $= \frac{d}{ds} \varphi_s^X \big|_{s=0}$

$(\mathcal{L}_X \hat{g})_{\mu\nu} = 0 \Rightarrow \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$

---

* Finite dim. version of Noether's thm.

Conservation Thm |

Let $\lambda \mapsto x(\lambda)$ be a time-like geodesic with tangent vector field $X = X^\mu \frac{d}{dx^\mu} \equiv \frac{d}{d\lambda}$ and let $Y = Y^\alpha \partial_\alpha$ be a killing field. Then, $\hat{g}(X,Y)$ is conserved along $\lambda \mapsto x(\lambda)$.

Pf | WTS: $\frac{d}{d\lambda} \hat{g}(X,Y) = 0$

$$\rightarrow \frac{d}{d\lambda} \hat{g}(X,Y) = \nabla_X \hat{g}(X,Y) = \hat{g}(\overset{0 \text{ b/c geodesic}}{\cancel{\nabla_X X}}, Y) + \hat{g}(X, \nabla_X Y)$$

$$= \hat{g}(X^\mu \partial_\mu, X^\lambda \nabla_\lambda Y_\alpha dx^\alpha) = X^\mu X^\lambda \nabla_\lambda Y_\alpha \underbrace{\hat{g}(\partial_\mu, dx^\alpha)}_{=\delta^\alpha_\mu}$$

$$= \nabla_\lambda Y_\mu X^\mu X^\lambda$$

$$\Rightarrow \frac{d}{d\lambda} \hat{g}(X,Y) = \nabla_\mu Y_\lambda X^\mu X^\lambda = \frac{1}{2}(\nabla_\mu Y_\lambda + \nabla_\lambda Y_\mu) X^\mu X^\lambda = 0 \quad \text{b/c } Y \text{ killing field}$$

∎

From here work w/ 3+1 spaces

Special Relativity: $(M, \hat{g})$ when $g = \eta = \mathrm{diag}(-1, +1, +1, +1)$ everywhere

$\eta = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz$

↳ only one chart needed to cover space

Derive the isometries of η

$\nabla_\mu Y_\nu := (\nabla_\mu Y)\left(\frac{\partial}{\partial x^\nu}\right)$

Recall Killing eqn: $\nabla_\mu Y_\nu + \nabla_\nu Y_\mu = 0$

- $\Gamma[\eta] = \partial \eta = 0 \Rightarrow R = 0$ everywhere

↳ Killing eqn simplifies: $\partial_\mu Y_\nu + \partial_\nu Y_\mu = 0$.

Also have $\nabla_\alpha \nabla_\beta Y_\mu = \cancel{R^\lambda{}_{\beta\alpha\mu}}^{\,0}\, Y_\lambda \Rightarrow \partial_\alpha \partial_\beta Y_\mu = 0$.

↳ general solution: $Y_\mu = a_{\mu\nu} x^\nu + b_\mu$, where $a_{\mu\nu}, b_\mu$ constants.

Substitute into Killing eqn: $\partial_\mu x^\nu = \delta_\mu^\nu \Rightarrow$ get $a_{\mu\nu} + a_{\nu\mu} = 0$.

↳ $Y = Y^\mu \partial_\mu = \eta^{\mu\nu} Y_\nu \partial_\mu = \eta^{\mu\nu}(a_{\nu\beta} x^\beta + b_\nu)\partial_\mu$

$= \eta^{\mu\nu} a_{\nu\beta} x^\beta \partial_\mu + b^\mu \partial_\mu$

$= \sum_{\beta=0}^3 \sum_{\nu=0}^3 \eta^{\mu\nu} a_{\nu\beta} x^\beta \partial_\mu + \sum_{\mu=0}^3 b^\mu \partial_\mu$

$a_{\nu\nu} = 0$ by anti-sym

$= \sum_{\beta=0}^3 \left(\sum_{\nu>\beta} + \sum_{\nu<\beta}\right) a_{\nu\beta} x^\beta \eta^{\mu\nu} \partial_\mu + b^\mu \partial_\mu$

$= \sum_{\beta=0}^3 \sum_{\nu>\beta} a_{\nu\beta} x^\beta \eta^{\mu\nu} \partial_\mu + \sum_{\beta=0}^3 \sum_{\nu<\beta} a_{\nu\beta} x^\beta \eta^{\mu\nu} \partial_\mu + b^\mu \partial_\mu$

switch sum $\Big\{$ $= \sum_{\nu=0}^3 \sum_{\beta<\nu} a_{\beta\nu} x^\nu \eta^{\mu\beta} \partial_\mu +$ " "

relabel dummy $\Big\{$ $= \sum_{\beta=0}^3 \sum_{\nu<\beta} a_{\beta\nu} x^\nu \eta^{\mu\beta} \partial_\mu + \sum_{\beta=0}^3 \sum_{\nu<\beta} a_{\nu\beta} x^\beta \eta^{\mu\nu} \partial_\mu + b^\mu \partial_\mu$

$= \sum_{\beta=0}^3 \sum_{\beta>\nu} \left(a_{\beta\nu} x^\nu \eta^{\mu\beta} - a_{\beta\nu} x^\beta \eta^{\mu\nu}\right)\partial_\mu + b^\mu \partial_\mu$

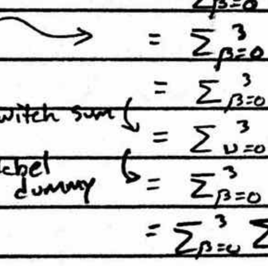
$= a_{\beta\nu}\left(x^\nu \eta^{\mu\beta} - x^\beta \eta^{\mu\nu}\right) + b^\mu \partial_\mu$

Th$^{\underline{m}}$ Every generator (in Lie algebra) of the isometry group of $(\mathbb{R}^{1+3}, \eta)$ can be written as linear combinations of the following vector field:

(1) $(x^\nu \eta^{\mu\beta} - x^\beta \eta^{\mu\nu})\partial_\mu = (\Omega^{\nu\beta})^\mu \partial_\mu$

(2) ∂_μ

$(\Omega^{\nu\beta})^\mu \partial_\mu = (x^\nu \eta^{\mu\beta} - x^\beta \eta^{\mu\nu})\partial_\mu$ in 3+1 dim gives $\frac{16-4}{2} = 6$ generators



(I) $\nu = 0, \beta = 1$: $(x^0 \eta^{\mu 1} - x^1 \eta^{\mu 0})\partial_\mu = x^0 \eta^{11} \partial_1 - x^1 \eta^{00} \partial_0 = t\partial_x + x\partial_t$

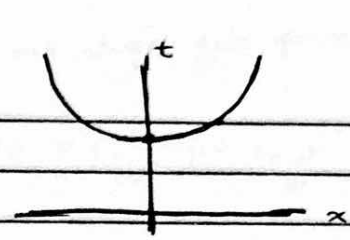
Def^{n} of flow: $\frac{dx^\mu}{d\lambda} = I^\mu(t, x, \dots) \Rightarrow \frac{dt}{d\lambda} = x, \ \frac{dx}{d\lambda} = t$

$\Rightarrow \frac{d^2 t}{d\lambda^2} = t \Rightarrow t(\lambda) = ae^\lambda + be^{-\lambda}$

$x(\lambda) = ae^\lambda - be^{-\lambda}$

Suppose $t(\lambda=0) = t_0$, $x(\lambda=0) = x_0$ $\Rightarrow$ $t_0 = a + b$, $x_0 = a - b$ $\Rightarrow$ $a = \frac{t_0 + x_0}{2}$, $b = \frac{t_0 - x_0}{2}$ $\Rightarrow$

$$\boxed{\begin{aligned} t(\lambda) &= \frac{t_0 + x_0}{2} e^\lambda + \frac{t_0 - x_0}{2} e^{-\lambda} \\ x(\lambda) &= \frac{t_0 + x_0}{2} e^\lambda - \frac{t_0 - x_0}{2} e^{-\lambda} \end{aligned}}$$



Then $t^2 - x^2 = t_0^2 - x_0^2$,

So $I = t\partial_x + x\partial_t$ preserves "$t^2 - x^2 =$ constant" curves

$\Rightarrow$ Boost / Lorentz boost

Similar for y, z planes:

$$\boxed{\begin{aligned} I &= t\partial_x + x\partial_t \\ II &= t\partial_y + y\partial_t \\ III &= t\partial_z + z\partial_t \end{aligned}}$$

generators of boosts

Also have generators of rotations

$$\boxed{\begin{aligned} R_1 &= x\partial_y - y\partial_x \\ R_2 &= x\partial_z - z\partial_x \\ R_3 &= y\partial_z - z\partial_y \end{aligned}}$$

Lorentz group $SO(1,3)$

Add in translations ($T = \partial_\mu$) to get Poincaré Group

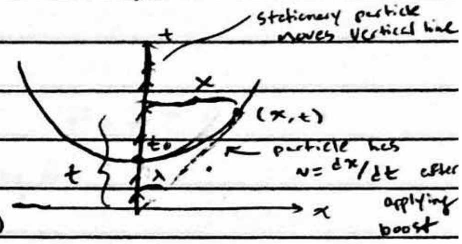
(resp. y, z)

Lemma: Lorentz boost I (resp. II, III) preserves the hyperbola $t^2 - x^2 = \text{const}$

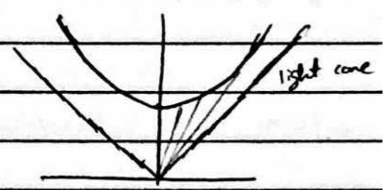
Set $t_0 = t_0$, $x = 0$ $\rightarrow$ $t(\lambda) = t_0 \cosh(\lambda)$

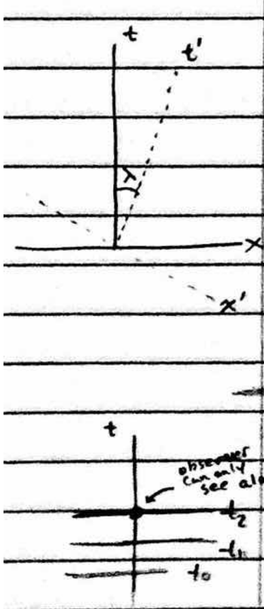
$x(\lambda) = t_0 \sinh(\lambda)$

Boost gives particle velocity: Stationary particle starts moving w/ velocity $v = \frac{x}{t} = \tanh(\lambda)$



For finite λ: $\begin{pmatrix} t(\lambda) \\ x(\lambda) \\ y(\lambda) \\ z(\lambda) \end{pmatrix} = \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix}$

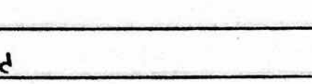


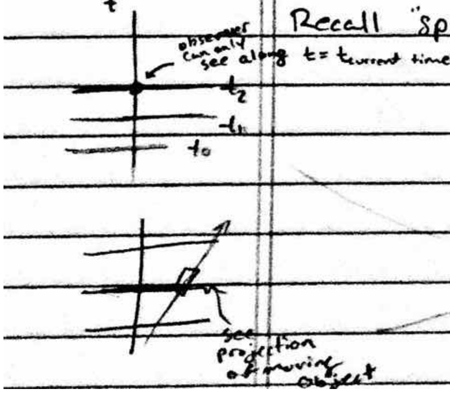


Recall the infinetesimal / algebra limit: $x\partial_t + t\partial_x$

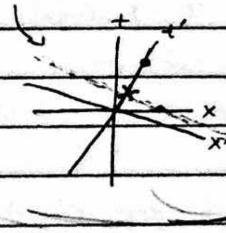
$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{``\Lambda_1} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \longrightarrow \exp(\Lambda_1 \lambda) = \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & \\ \sinh\lambda & \cosh\lambda & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}$$

As $\lambda \to \infty$, $v \to 1$ $\Rightarrow$ postulates of SR not violated





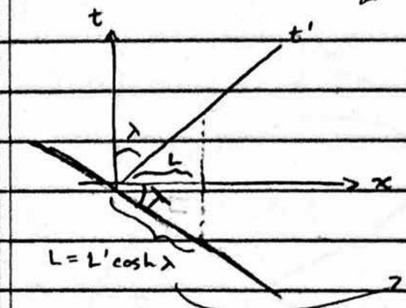
Recall "space of simultaneity" (where $t = $ const.)



Th^m (Length contraction th^m) Let A be a 1-dimensional object lying on the x axis and is stationary. Let a particle move with speed v along x-axis. Then the length L' observed by particle of A is given by $L' = L\sqrt{1 - v^2}$ $< L$.

Sketch of Pf

times are relative,
but invariant interval always gets preserved



$\tanh\lambda = v$

"Lorentz-Fitzgerald" length contraction

$L = L'\cosh\lambda$

$$\frac{\sinh\lambda}{\cosh\lambda} = v \Rightarrow \frac{\sqrt{\cosh^2\lambda - 1}}{\cosh\lambda} = v \Rightarrow \cosh\lambda = \frac{1}{\sqrt{1-v^2}}$$

$$\Rightarrow L = L'\cosh\lambda \Rightarrow L' = L\sqrt{1-v^2}$$

Basis Killing fields

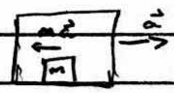
$$\left\{ \begin{array}{l|l|l} t\partial_x + x\partial_t & x\partial_y - y\partial_x & \\ t\partial_y + y\partial_t & x\partial_z - z\partial_x & \partial_t, \partial_x, \partial_y, \partial_z \\ t\partial_z + z\partial_z & y\partial_z - z\partial_y & \end{array} \right\}$$

If $X = x^\mu \partial_\mu$ a geodesic, $\eta(X, \partial_t) = \text{const}$
$\eta(X, \partial_x) = \text{const}$ ... $\Rightarrow$ 10 conserved quantities
Using conservation th$^{\underline{m}}$.

Road to General Relativity / Gravity

Ordinary Newtonian Mechanics: time t is absolute; $\vec{F} = m\vec{a}$

↳ "pseudoforce"

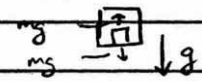


absolute (t, x, y, z) wrt big box

↳ $\downarrow \vec{g}$ = acceleration due to gravity, M

net force on small box: $\vec{F} = m\vec{g} - m\vec{g} = 0$

↳



↳ nothing special
about gravity - can mimic by means of accelerated coord. system.

Special Relativity: geodesics are straight lines
$\Rightarrow$ geodesics are force-free motion

Prop| The motion of a particle on $(\mathbb{R}^{1+3}, \eta)$ under a force is described by $\underbrace{\nabla_X X = F_{\text{external}}}$ (vector field)
with $X = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$ tangent to trajectory

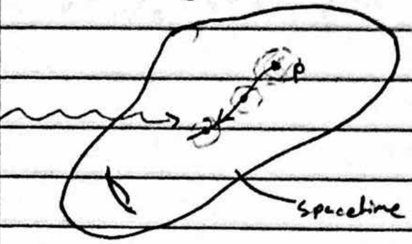
Def$^{\underline{n}}$| An inertial coord. system is one in which $F = 0$.

Def$^{\underline{n}}$| A mfld is inertial if all geodesics are straight lines
↳ $(\mathbb{R}^{1+3}, \eta)$ is globally inertial if $F_{ext} = 0$.
↳ there cannot be gravity in Minkowski space.

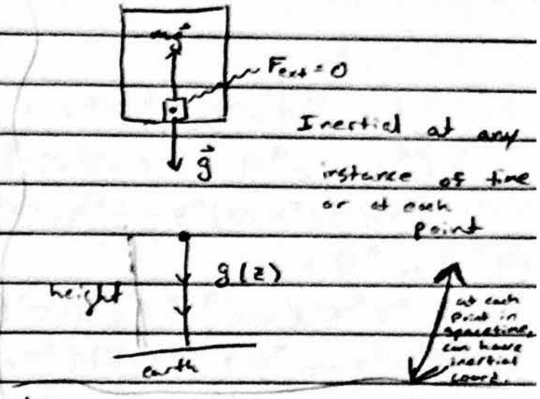
Since $(\mathbb{R}^{1+3}, \eta)$ doesn't have gravity $\Rightarrow$ SR not enough to describe physics of gravity
$\Rightarrow$ there must be some geometric interpretation.

Take a general Semi-Riemannian mfld

'falling' from point to point is path on spacetime



At each point,
$\hat{g}_p = \mathrm{diag}(-1, 1, 1, 1)$



Inertial at any instance of time or at each point

Lemma| An exponential map

$\exp_p : U \subset T_pM \to V \subset M$ is a local diffeomorphism and at p, $\hat{g}_p \equiv \eta$ and $\Gamma_p \equiv 0$

"geodesic normal coord. system"

$\Rightarrow$ Gravity must be a subtle geometric phenomenon

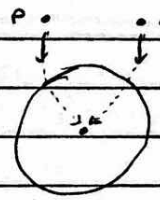
gravitational and inertial mass are equivalent

$\Rightarrow$ Equivalence Principle : "existence of an exponential map at each point"
"can get inertial frame at each point"

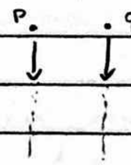
Look at some gravitational phenomena

no external forces; encoding gravity into spacetime

w/ gravity:



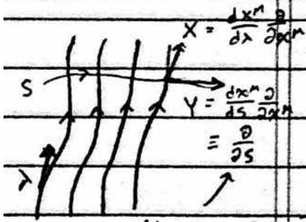
w/o gravity:



$\Rightarrow$ gravity causing acceleration, motion described by geodesic

$\Rightarrow$ since geodesics intersect w/ gravity, curvature must be responsible for gravity

$\Rightarrow$ look at geodesic deviations and compare to curvature



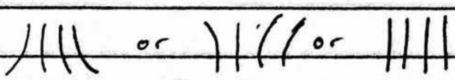
Take family of timelike geodesics on a general Semi-Riemannian mfld $(M, \hat{g})$

NB| $\left[\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial s}\right] = [X, Y] = 0.$

$\hookrightarrow X^\alpha \partial_\alpha Y^\mu - Y^\alpha \partial_\alpha X^\mu = 0$

$(\nabla_\alpha Y^\mu) = (\nabla_\alpha Y)(dx^\mu)$

$\Rightarrow X^\alpha \nabla_\alpha Y^\mu - Y^\alpha \nabla_\alpha X^\mu = 0$

Want to see if 

Acceleration: $\frac{d^2 Y^\mu}{d\lambda^2} = \frac{d}{d\lambda}\frac{dY^\mu}{d\lambda} \equiv \frac{d}{d\lambda}\left(X^\alpha \nabla_\alpha Y^\mu\right) = X^\beta \nabla_\beta \left(X^\alpha \nabla_\alpha Y^\mu\right)$

Also have since X is geodesic vec. field, $\nabla_X X = 0 \Rightarrow X^\alpha \nabla_\alpha X^\mu = 0.$

Lie deriv. defn $\longrightarrow$ $$X^\alpha \nabla_\alpha Y^\mu - Y^\alpha \nabla_\alpha X^\mu = 0$$

$$\Rightarrow X^\beta \nabla_\beta (X^\alpha \nabla_\alpha Y^\mu) = X^\beta \nabla_\beta (Y^\alpha \nabla_\alpha X^\mu)$$

$$= (X^\beta \nabla_\beta Y^\alpha)(\nabla_\alpha X^\mu) + X^\beta Y^\alpha \nabla_\beta \nabla_\alpha X^\mu$$

Ricci identity $\longrightarrow$ $$= \quad '' \quad + X^\beta Y^\alpha (\nabla_\alpha \nabla_\beta X^\mu + R^\mu{}_{\lambda\beta\alpha} X^\lambda)$$

$$= \quad '' \quad + Y^\alpha \nabla_\alpha (\underbrace{\cancel{X^\beta \nabla_\beta X^\mu}}_{0}) - Y^\alpha (\nabla_\alpha X^\beta)(\nabla_\beta X^\mu)$$

X geodesic vec. field $\longrightarrow$ $$+ R^\mu{}_{\lambda\beta\alpha} X^\lambda X^\beta Y^\alpha$$

$$= (X^\beta \nabla_\beta Y^\alpha)(\nabla_\alpha X^\mu) - Y^\alpha (\nabla_\alpha X^\beta)(\nabla_\beta X^\mu) + R^\mu{}_{\lambda\beta\alpha} X^\lambda X^\beta Y^\alpha$$

Lie deriv. $\longrightarrow$ $$= (Y^\beta \nabla_\beta X^\alpha)\cancel{(\nabla_\alpha X^\mu) - (Y^\alpha \nabla_\alpha X^\beta)}(\nabla_\beta X^\mu) + R^\mu{}_{\lambda\beta\alpha} X^\lambda X^\beta Y^\alpha$$

$$\Rightarrow \frac{d^2 Y^\mu}{d\lambda^2} = R^\mu{}_{\lambda\beta\alpha} X^\lambda X^\beta Y^\alpha$$

"Gravity is the physical manifestation of spacetime curvature"

Th$^{\underline{m}}$ (Geodesic Deviation Th$^{\underline{m}}$)

Let $(\lambda, s) \mapsto x^\mu(\lambda, s)$ be a one-parameter (s) family of timelike geodesics with affine parameter λ and velocity $X = x^\mu \partial_\mu$ and transversal vector field describing the separation be $Y = y^\mu \partial_\mu$. Then the relative acceleration between geodesics in the family (given by s) is controlled by the space-time curvature. More precisely, the acceleration is given as follows:

$$\frac{d^2 y^\mu}{d\lambda^2} = R^\mu{}_{\alpha\beta\nu} Y^\alpha X^\beta X^\gamma.$$

A relativistic puzzle

"In order to stand still, I must move faster than light. What am I?"

Lagrangian Field theory

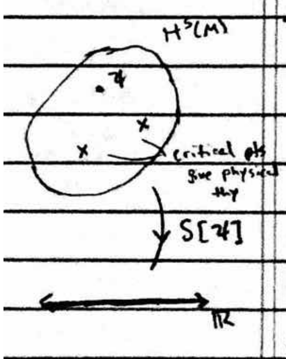
Let $(M, \hat{g})$ be a Lorentzian manifold and let $\mathcal{F}(M)$ be the space of functions on M (i.e. $\{f \mid f : M \to \mathbb{R}\}$)

Then $L^2(M) := \{ f \in \mathcal{F}(M) \mid \int_M |f|^2 \mu_{\hat{g}} < \infty \} / \sim$ $\longleftarrow$ Banach space/ Hilbert space

where $f \sim g$ if f and g agree on every measurable set

Define $\overset{\text{Sobolev space of order } s}{H^s(M)} := \{ f \in \mathcal{F}(M) \mid \overset{\text{derivatives up to } s}{\nabla^s} f \in L^2(M) \}$, $s \in \mathbb{Z} > 0$ $\quad$ $L^2(M) = H^0(M)$

Postulates of Classical Field Theory



- A physical theory is described by the critical points of an "Action" functional
  - ↳ $S[\psi] := \int_M L[\psi] \mu_{\hat{g}}$, $\mu_{\hat{g}} = \sqrt{-\det \hat{g}}\, dx^0 \wedge \dots \wedge dx^n$.
  - ↳ ψ is the physical field describing the theory
    - ↳ Note: $\psi \in \mathcal{F}(M)$, or more generally, $\psi \in H^s(M)$ for some $s \geq 0$ and $S : H^s(M) \to \mathbb{R}$ and $L : H^s(M) \to \mathcal{F}(M)$

Ex Massive scalar field $\varphi : M \to \mathbb{R}$ w/ mass m on $(M^{1+3}, \hat{g})$

$$S[\varphi] := -\frac{1}{2} \int (\hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m^2 \varphi^2) \underbrace{\sqrt{-\det g}\, dx^0 \wedge \dots \wedge dx^3}_{\text{diffeo invariant}}$$

whole action is invariant under diffeo (physics indep. of coord. system)

↳ $\partial\varphi := (g^{\alpha\beta} \partial_\beta \varphi) \partial_\alpha \to g(\partial\varphi, \partial\varphi)$ invariant under diffeo

↳ For integral to be finite, need $\varphi \in H^1(M)$

$H^s(M) = \{ f \in \mathcal{F}(M) \mid \nabla^s f \in L^2,\ f \in L^2 \}$

$\|f\|_{H^s} := \left(\int |\nabla^s f|^2 + f^2 \, \mu_g \right)^{1/2}$

$\|f\|_{\dot{H}^s} = \left(\int |\nabla^s f|^2 \mu_g \right)^{1/2}$

$\Rightarrow S[\varphi] \lesssim \int |\partial\varphi|^2 + \varphi^2 < \infty \Rightarrow$ bounded ✓

Goal: look for critical points of $S[\varphi]$ in $H^1(M)$.

Recall

$$S[\varphi] = -\frac{1}{2} \int (\hat{g}^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi + m^2 \varphi^2) \sqrt{-\det \hat{g}} \; dx^0 \wedge \cdots \wedge dx^n$$

Def$^{\underline{n}}$| φ_0 is a critical point of $S[\varphi]$ in $H^1(M)$ if the Frechet derivative of $S[\varphi]$ vanishes at $\varphi = \varphi_0$

φ could in general be sections of any vector bundle

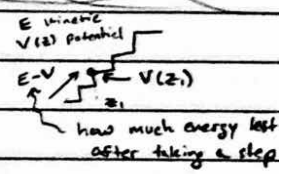
Def$^{\underline{n}}$ (Frechet derivative)| Frechet deriv. of S at φ_0 in direction $h \in \mathcal{F}(M)$ is:

$$DS[\varphi_0] \cdot h := \frac{d}{ds} S[\varphi_0 + sh] \Big|_{s=0}$$

Physical Motivation

$$S_{\text{Minkowski}}[\varphi] := -\frac{1}{2} \int -(\partial_t \varphi)^2 + (\partial_i \varphi)^2 + m^2 \varphi^2 \; \mu_{\hat{g}}$$

$$= \int \underbrace{\frac{1}{2} (\partial_t \varphi)^2}_{\text{Kinetic energy}} \underbrace{- \frac{1}{2} (\partial_i \varphi)^2 - \frac{1}{2} m^2 \varphi^2}_{-\ \text{Potential energy}} \; \mu_{\hat{g}} \equiv \text{free energy}$$

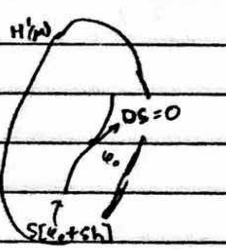


Postulates of Thermodynamics

- Free energy (kinetic − potential) is minimized for a physical state (or at best is a critical point)

$\Rightarrow$ Lagrangian generalizes free energy, action generalizes postulate

Back to critical point calculation:



$$DS[\varphi_0] \cdot h \equiv \frac{d}{ds} S[\varphi_0 + sh] \Big|_{s=0} = -\frac{1}{2} \frac{d}{ds} \int_M \left[g^{\mu\nu} \partial_\mu (\varphi_0 + sh) \partial_\nu (\varphi_0 + sh) + m^2 (\varphi_0 + sh)^2 \right] \mu_{\hat{g}} \Big|_{s=0}$$

$$= -\frac{1}{2} \int_M 2 g^{\mu\nu} \partial_\mu \varphi_0 \partial_\nu h + m^2 \frac{d}{ds} (\varphi_0^2 + 2s \varphi_0 h + s^2 h^2) \Big|_{s=0} \; \mu_{\hat{g}}$$

$$= -\frac{1}{2} \int_M 2 g^{\mu\nu} \partial_\mu \varphi_0 \partial_\nu h + 2m^2 \varphi_0 h \; \mu_{\hat{g}}$$

$$= -\int_M \left[g^{\mu\nu} \nabla_\mu \varphi_0 \nabla_\nu h + m^2 \varphi_0 h \right] \mu_{\hat{g}}$$

$$= -\int_M \left[\nabla_\nu (g^{\mu\nu} \nabla_\mu \varphi_0) h) - \nabla_\nu (g^{\mu\nu} \nabla_\mu \varphi_0) h + m^2 \varphi_0 h \right] \mu_{\hat{g}}$$

$\hookrightarrow$ by Stokes thm $\int_M \nabla_\mu J^\mu \mu_{\hat{g}} = \int_{\partial M} \hat{g}(J, n) \, \mu_{\partial M}$

$$= \underbrace{-\int_{\partial M} g^{\mu\nu} \nabla_\mu \varphi_0 \, h \, n_\nu \, \mu_{\partial M}}_{} + \int_M \left[\nabla_\nu (g^{\mu\nu} \nabla_\mu \varphi_0) - m^2 \varphi_0 \right] h \, \mu_{\hat{g}}$$

$\hookrightarrow$ Show boundary term $\to 0$ in suitable sense

$L: H^1(M) \to \mathcal{F}(M)$ is infinitly differentiable wrt φ (b/c polynomial in φ).

"molification"

Lemma| Compactly supported smooth functions (Schwartz functions) are dense in $H^s(M)$ — denote $SC(M)$

$\Rightarrow \varphi \in H^s(M), \ \exists \{\phi_k\}_{k=0}^{\infty} \subset SC(M)$ s.t. $\lim_{k \to \infty} \|\phi_k - \varphi\|_{H^s} = 0$.

$\phi \subset SC(M)$ then $|\nabla^\alpha \phi| \leq \frac{C}{(1+|x|^2)^{\beta/2}}$ $\forall \alpha, \beta \in \mathbb{Z} > 0$

Take $\xi = g^{\mu\nu} \nabla_\mu \varphi_0 h n_\nu$

$$\Rightarrow \left| -\int_{\partial M} \xi \, \mu_{\partial M} \right| = \left| -\int_{\partial M} \xi - \phi_k \, \mu_{\partial M} + \int_{\partial M} \phi_k \, \mu_{\partial M} \right| \leq \underbrace{\int_{\partial M} |\xi - \phi_k| \, \mu_{\partial M}}_{\to 0 \text{ by lemma}} + \underbrace{\int_{\partial M} |\phi_k| \, \mu_{\partial M}}_{\to 0 \text{ by}}$$

$\Rightarrow$ Boundary terms do not contribute to equations of motion but will play role in energy in GR.

Now have $DS[\varphi_0]\cdot h = \int_M \left[g^{\mu\nu}\nabla_\nu\nabla_\mu\varphi_0 - m^2\varphi_0\right] h\, \mu_{\hat g} = 0$ for all h

$\Rightarrow \boxed{g^{\mu\nu}\nabla_\nu\nabla_\mu\varphi_0 - m^2\varphi_0 = 0}$

$\Psi \in H^1(M)$

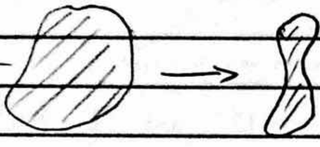
$\hat g$ smooth metric

With Ψ = scalar field, section of vector bundle,

given $S[\hat g, \Psi] = \int_M L[\hat g, \Psi]\, \mu_{\hat g}$ and the physics: $DS[\Psi]\cdot h = 0$ $\forall h \in$ sections of suitable vec. bundle

Want to understand energy, momentum, etc.

viscous fluid/ chunk of metal



to deform, need to do work $\longrightarrow$ amount of work needed encoded in "stress tensor"

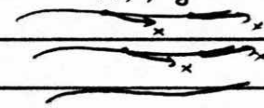
Analogy to spacetime: Change the metric and see if it gives a stress?

↳ how does action change when changing metric?

Consider a one parameter group of diffeo φ_s^X with flow X : $\frac{d}{ds}\varphi_s^X\big|_{s=0} = X$

and $\hat g \mapsto \hat g + \epsilon \underbrace{h}_{h = L_X \hat g}$ and $\Psi \mapsto \Psi + s\underbrace{\delta\varphi}_{\delta\varphi = L_X\varphi}$

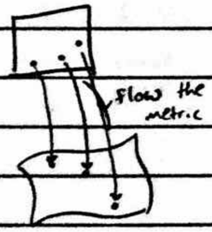
$L_X \hat g = \lim_{s\to 0} \frac{\varphi_s^* \hat g - \hat g}{s}$



$(\varphi_s^X)^* g$

infinitesimal

Change of $S[\hat g, \Psi]$ due to flow of φ_s^X:



flow the metric

$$DS(g_0, \Psi_0)\cdot(h, \delta\varphi) = \frac{d}{ds} S[\hat g_0 + sh, \Psi_0 + s\,\delta\varphi]\big|_{s=0}$$
$$= D_{\hat g} S(\hat g_0, \Psi_0)\cdot h + D_\Psi S(\hat g_0, \Psi_0)\cdot\delta\varphi$$

Recall for physical system, $D_\Psi S(\hat g_0, \Psi_0)\cdot\delta\varphi = 0$. So have left

$$DS(g_0, \Psi_0)\cdot(h, \delta\varphi) = \frac{d}{ds} S[\hat g_0 + sh, \Psi_0]\big|_{s=0}$$
$$= \frac{d}{ds}\int_M L[\hat g_0 + sh, \Psi_0]\, \mu_{\hat g_0 + sh}\big|_{s=0}$$
$$= \int_M \left\{ \frac{d}{ds} L[\hat g_0 + sh, \Psi_0]\big|_{s=0}\, \mu_{\hat g_0} + L[\hat g_0, \Psi_0]\frac{d}{ds}\mu_{\hat g_0 + sh}\big|_{s=0} \right\}$$
$$= \int_M \left\{ D_{\hat g} L[\hat g_0, \Psi_0]\cdot h\, \mu_{\hat g_0} + L[\hat g_0, \Psi_0]\frac{d}{ds}\mu_{\hat g_0 + sh}\big|_{s=0} \right\}$$

Lemma (1st Variation) $\frac{d}{ds}\mu_{\hat g_0 + sh} = \frac{1}{2}\mu_{\hat g_0}\, \mathrm{tr}(\hat g_0^{-1} h)$

$\mu_{\hat g_0 + sh} = \sqrt{-\det(\hat g_0 + sh)}\, dx^0 \wedge \dots \wedge dx^n$

Pf $\frac{d}{ds}\mu_{\hat g_0 + sh}\big|_{s=0} = \frac{d}{ds}\sqrt{-\det(\hat g_0 + sh)}\,\big|_{s=0} = \frac{1}{2\sqrt{-\det(\hat g_0 + sh)}}\frac{d}{ds}\left(-\det(\hat g_0 + sh)\right)\big|_{s=0}$

Now $-\det(\hat g_0 + sh) = -\det(\hat g_0(\mathbb{1} + s\hat g_0^{-1}h)) = -\det(\hat g_0)\underbrace{\det(\mathbb{1} + s\hat g_0^{-1}h)}$

$$\det(\mathbb{1} + s\hat g_0^{-1}h) = (1 + s(\hat g_0^{-1}h)_{00})(1 + s(\hat g_0^{-1}h)_{11})\cdots(1 + s(\hat g_0^{-1}h)_{nn}) + \mathcal{O}(s^2)$$
$$= 1 + s\underbrace{\left((\hat g_0^{-1}h)_{00} + \cdots + (\hat g_0^{-1}h)_{nn}\right)}_{\mathrm{tr}(\hat g_0^{-1}h)} + \mathcal{O}(s^2)$$

$\det\left[\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} + s\begin{pmatrix} (\hat g_0^{-1}h)_{00} & & \\ & (\hat g_0^{-1}h)_{11} & \\ & & \ddots \end{pmatrix}\right]$

$= \det\left[\begin{pmatrix} 1 + s(\hat g_0^{-1}h)_{00} & \cdots \\ (\hat g_0^{-1}h)_{10} & \ddots \\ \vdots & \end{pmatrix}\right]$

So now:

$$\frac{d}{ds}\mu_{\hat g_0 + sh}\big|_{s=0} = \frac{-1}{2\sqrt{-\det(\hat g_0)}}\cdot\det(\hat g_0)\cdot\frac{d}{ds}\left[1 + s\,\mathrm{tr}(\hat g_0^{-1}h) + \mathcal{O}(s^2)\right]\big|_{s=0}$$
$$= \frac{-\det(\hat g_0)}{2\sqrt{-\det(\hat g_0)}}\mathrm{tr}(\hat g_0^{-1}h) = \frac{1}{2}\sqrt{-\det(\hat g_0)}\,\mathrm{tr}(\hat g_0^{-1}h) = \frac{1}{2}\mu_{\hat g_0}\mathrm{tr}(\hat g_0^{-1}h) \quad \boxtimes$$

$A \cdot B := A^{\mu\nu} B_{\nu\mu}$

just like for vectors

$A \cdot B := A_\mu B^\mu$

$tr(g_0^{-1} h) = tr(g_0^{\mu\nu} h_{\nu\lambda}) = g_0^{\mu\nu} h_{\mu\nu} = g_0^{-1} \cdot h$

h symmetric as $L_X g$ and g symmetric

Continuing calc:

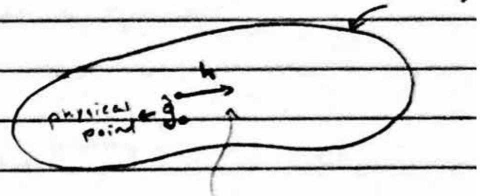
$$DS(\hat{g}_0, \Psi_0) \cdot (h, \delta\varphi) = \int_M \{ D_{\hat{g}_0} L[\hat{g}_0, \Psi_0] \cdot h \, \mu_{\hat{g}_0} + \tfrac{1}{2} L[\hat{g}_0, \Psi_0] \mu_{\hat{g}_0} \, tr(\hat{g}_0^{-1} h)$$

$$= \int_M \{ D_{\hat{g}_0} L[\hat{g}_0, \Psi_0] + \tfrac{1}{2} L[\hat{g}_0, \Psi_0] g_0^{-1} \} \cdot h \, \mu_{\hat{g}_0}$$

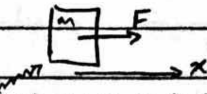
Locally, $\overbrace{\{ D_{\hat{g}_0} L[\hat{g}_0, \Psi_0] + \tfrac{1}{2} L[\hat{g}_0, \Psi_0] g_0^{-1} \}}^{T} \cdot h = T^{\mu\nu} h_{\mu\nu}$

$\Rightarrow DS(g_0, \Psi_0) \cdot (h, \delta\varphi) = \int_M \underbrace{T^{\mu\nu} h_{\mu\nu}}_{\text{analogous} \rightarrow} \mu_{g_0}$

moduli space / space of metrics / C^∞(sections)



Classically,



: Work = F · x

object of mass m is the 'classical field'

moving by h gives a change in action

⇒ change in free energy

⇒ doing work, where T is like the force

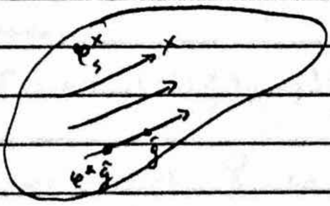
Call T the "Stress-Energy tensor of the field Ψ_0"

Thm (Stress-Energy Thm)

Let Ψ be a section of a suitable vector bundle on $(M, \hat{g})$ describing a physical phenomena (i.e. satisfying the equation of motion $DS \cdot \delta\varphi = 0$). Then $\exists$ a stress-energy tensor (energy-momentum tensor) given by:

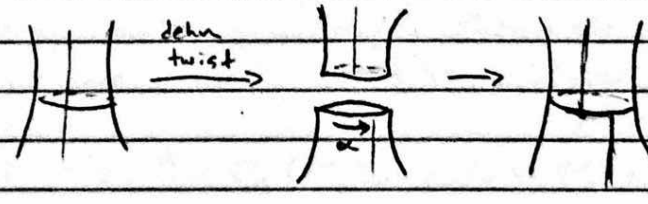
$$T := D_{\hat{g}} L[\hat{g}, \Psi] + \tfrac{1}{2} L[\hat{g}, \Psi] \hat{g}^{-1}.$$

Now want to think about h:



$h = L_X \hat{g}_0$

continuous operation, but did not need in $\underbrace{\int_M T^{\mu\nu} h_{\mu\nu} \mu_{\hat{g}}}_{\text{holds if } \|h\|_{L^\infty} < \text{const}}$



discontinuous and valid deformation, so $\int_M T^{\mu\nu} h_{\mu\nu} \mu_{\hat{g}}$ still makes sense, but $h_{\mu\nu}$ can't come from flow of vector field

⇒ Only consider continuous / smooth deformations of metric (small diffeos ~~do~~ manifold)

Physical Postulate: the action functional should remain invariant under a diffeomorphism that is continuously connected to the identity.

So consider $\underbrace{\varphi_s^X \in \mathcal{D}_0(M)}$ group of diffeos continuously connected to identity

$X \in \Gamma^{C^\infty}(M)$ → focus on h generated by $X \in \Gamma^{C^\infty}_c(M)$

Then $h = L_X \hat{g}_0 \Rightarrow h_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu$

So $DS_{\hat{g}}[\hat{g}_0, \Psi_0] \cdot (L_X \hat{g}_0) = 0$

$\Rightarrow \int_M T^{\mu\nu} (\nabla_\mu X_\nu + \nabla_\nu X_\mu)\, \mu_{\hat{g}} = 0$

$\Rightarrow 2 \int_M T^{\mu\nu} \nabla_\mu X_\nu\, \mu_{\hat{g}} = 0$ b/c both terms symmetric

Integrate by parts: $\int_M T^{\mu\nu} \nabla_\mu X_\nu\, \mu_{\hat{g}} = 0$

$$\underbrace{\int_M \nabla_\mu (T^{\mu\nu} X_\nu)\, \mu_{\hat{g}}}_{\text{use 'Schwartz Function trick' (compactly supported } C^\infty \text{ functions)} \Rightarrow\, = 0} - \int_M (\nabla_\mu T^{\mu\nu}) X_\nu\, \mu_{\hat{g}} = 0$$

Left with $\int_M (\nabla_\mu T^{\mu\nu}) X_\nu\, \mu_{\hat{g}} = 0$ for all $X \in \Gamma^{C^\infty}(M)$

$\Rightarrow \boxed{\nabla_\mu T^{\mu\nu} = 0}$

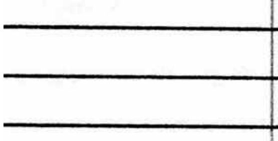
Th$^{\underline{m}}$ (Divergence th$^{\underline{m}}$) Let Ψ be a section of any suitable vector bundle on $(M, \hat{g})$ with action $S[\hat{g}, \Psi] = \int_M L[\hat{g}, \Psi]\, \mu_{\hat{g}}$ describing a physical phenomenon (i.e. $D_\Psi S[\hat{g}, \Psi] \cdot \delta\Psi = 0$). Then the associated stress-energy tensor $T := T^{\mu\nu} \partial_\mu \otimes \partial_\nu$ satisfies the following equation: $\nabla_\mu T^{\mu\nu} = 0$.

Assume $(M, \hat{g})$ admits a Killing field Y (i.e. $L_Y \hat{g} = 0 \Rightarrow \nabla_\mu Y_\nu + \nabla_\nu Y_\mu = 0$)

Lemma Let Y be a Killing vector field of $(M, \hat{g})$ and $T = T^{\mu\nu} \partial_\mu \otimes \partial_\nu$ be the stress-energy tensor associated to a section Ψ of a suitable vector bundle on $(M, \hat{g})$. Then the vector field $J^\mu := T^{\mu\nu} Y_\nu$ is divergence free ($J = J^\mu \partial_\mu$ is called "current" vector field)

Pf $\nabla_\mu J^\mu = \nabla_\mu (T^{\mu\nu} Y_\nu) = Y_\nu \underbrace{\cancel{\nabla_\mu T^{\mu\nu}}}_{0 \text{ by divergence th}^{\underline{m}}} + T^{\mu\nu} \nabla_\mu Y_\nu = \frac{1}{2} T^{\mu\nu} \underbrace{\cancel{(\nabla_\mu Y_\nu + \nabla_\nu Y_\mu)}}_{0 \text{ by Killing field}} = 0$

Th$^{\underline{m}}$ (Noether's th$^{\underline{m}}$)



Let Y be Killing field, ..., with current $J^\mu = T^{\mu\nu} Y_\nu$. Then

$\underset{\text{"Charge"}}{Q} := \int_{\substack{t = \text{constant} \\ \text{hypersurface}}} J^0\, \mu_{\hat{g}}$ is conserved along the flow of the $\underbrace{\text{time vector field}}_{(dt)^\#}$

$\Sigma_t = \{ t = \text{const. hypersurface in } (M, \hat{g}) \}$

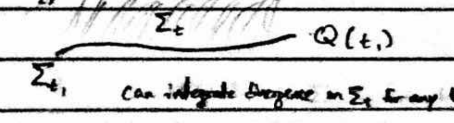
Pf

Recall identity: $\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu{}_{\mu\nu} J^\nu = \frac{1}{\mu_{\hat{g}}} \partial_\mu (\mu_{\hat{g}} J^\mu)$

$Y = Y^\mu \partial_\mu$

$Y_\mu = Y^\flat$

$Y^\mu = (Y^\flat)^\#$



can integrate divergence on Σ_t for any t

$\int_{\Sigma_t} \nabla_\mu J^\mu\, \mu_{\hat{g}}\, dx^1 \wedge \dots \wedge dx^n = 0 \Rightarrow \int_{\Sigma_t} \frac{1}{\cancel{\mu_{\hat{g}}}} \partial_\mu (\mu_{\hat{g}} J^\mu) \cancel{\mu_{\hat{g}}}\, dx^1 \wedge \dots \wedge dx^n = 0$

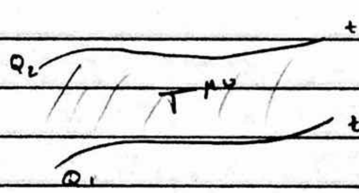
$\Rightarrow \int_{\Sigma_t} \partial_t (\mu_{\hat{g}} J^0)\, dx^1 \wedge \dots \wedge dx^n + \int_{\Sigma_t} \partial_i (\mu_{\hat{g}} J^i)\, dx^1 \wedge \dots \wedge dx^n = 0$

cont. →

$\mu_{\hat g} J^i \equiv$ vector density (so $\nabla \leftrightarrow \partial$ doesn't matter).

So $\int_{\Sigma_t} \partial_i (\mu_{\hat g} J^i)\, dx^1 \wedge \dots \wedge dx^n = \int_{\partial\Sigma_t} \mu_{\hat g}\, \hat g(J, dt)\, \mu_{\partial\Sigma_t} = 0$ if Σ_t compact w/o boundary or through Schwartz function trick

$\Rightarrow d_t \underbrace{\int \mu_g J^0\, dx^1 \wedge \dots \wedge dx^n}_{Q} = 0 \Rightarrow Q$ conserved ■



$Q = T^{0\nu} \overset{\text{killing}}{Y_\nu} = T(dt, Y)$

$\nabla_\mu T^{\mu\nu} = 0$ says field doesn't do work if $\mathcal{L}_Y g = 0$

↳ if nothing happens to metric, shouldn't do any work

Since no work done, $Q_1 = Q_2$

Heuristically: $T^{\mu\nu}$ is a "source" that satisfies $\nabla_\mu T^{\mu\nu} = 0$.

To generate gravity, need (something geometric / dependent on curvature) = (source)

↳ needs to satisfy similar condition to $\nabla_\mu T^{\mu\nu} = 0$

(i.e. $G = T \Rightarrow \nabla_\mu G^{\mu\nu} = 0$)

- Recall Bianchi identities of curvature

Saw before gravity physical manifestation of spacetime curvature Riem (or R)

↳ aim: Construct theory of gravity out of Riem

$\Rightarrow$ construct action functional S for gravity that satisfies diffeo invariant, etc., using Riem to do it.

$\Rightarrow S = \int$ (scalar function of Riem)

$$\mathrm{Riem} = R^\alpha{}_{\beta\mu\nu} \frac{\partial}{\partial x^\alpha} \otimes dx^\beta \otimes dx^\mu \otimes dx^\nu$$

↳ can get multiple scalar functions: $R^\alpha{}_{\beta\mu\nu} R_\alpha{}^{\beta\mu\nu}$, $\hat g^{\alpha\beta} \mathrm{Ric}_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}\, \hat g^{\alpha\beta}$, etc.

↳ no unique choice, but need to match physical observations

Einstein-Hilbert Postulate

Let $(M, \hat g)$ be a Lorentzian manifold. The pure gravity is the critical points of the following action functional:

$$S_{EH}[\hat g] = \int_M R[\hat g]\, \mu_{\hat g}\, dx^0 \wedge \dots \wedge dx^n$$

Define $\mathcal{M} := \{$space of Lorentzian metrics on M with prescribed regularity$\} \equiv$ moduli space

↳ $S_{EH} : \mathcal{M} \to \mathbb{R}$.

To find critical points, need Fréchet derivative of S to vanish $\left(\frac{d}{ds} S[\hat{g}+sh]\big|_{s=0} = 0\right)$

Recall: Tensors $\Longleftrightarrow$ sections of bundle

$\hookrightarrow$ local object (e.g. if vanishes in one coord., vanishes in all)

$h \in T_{\hat{g}}\mathcal{M}$

$$\frac{d}{ds} S[\hat{g}+sh] = \frac{d}{ds}\int_M R[\hat{g}+sh]\,\mu_{\hat{g}+sh}\, dx^0\wedge\dots dx^3\big|_{s=0}$$
$$= \int_M \left(\frac{d}{ds} R[\hat{g}+sh]\big|_{s=0}\,\mu_{\hat{g}} + R[\hat{g}]\frac{d}{ds}\mu_{\hat{g}+sh}\big|_{s=0}\right) dx^0\wedge\dots\wedge dx^3$$

$(\Gamma^\alpha{}_\beta)_\nu \in \Omega^1$(matrix on M) — matrix-valued 1-form

$(R^\alpha{}_\beta)_{\mu\nu} = (d(\Gamma^\alpha{}_\beta))_{\mu\nu} + [\Gamma_\mu, \Gamma_\nu]^\alpha{}_\beta$

$$R[\hat{g}] = \hat{g}^{\alpha\beta}\,Ric_{\alpha\beta} = \hat{g}^{\alpha\beta} R^\mu{}_{\alpha\mu\beta} \qquad \nabla_{\partial_\beta}\partial_\nu = \Gamma^\alpha_{\beta\nu}\,\partial_\alpha$$

$\hookrightarrow R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\beta\nu} - \Gamma^\alpha_{\lambda\nu}\Gamma^\lambda_{\beta\mu}$

$\hookrightarrow Ric_{\alpha\beta} = \partial_\mu\Gamma^\mu_{\alpha\beta} - \partial_\beta\Gamma^\mu_{\alpha\mu} + \Gamma^\mu_{\lambda\mu}\Gamma^\lambda_{\alpha\beta} - \Gamma^\mu_{\lambda\beta}\Gamma^\lambda_{\alpha\mu}$

$$\Rightarrow \int_M \left(\underbrace{\frac{d}{ds}\left(Ric[\hat{g}+sh]_{\alpha\beta}((\hat{g}+sh)^{-1})^{\alpha\beta}\right)\big|_{s=0}}\right)\mu_{\hat{g}} + R[\hat{g}]\frac{d}{ds}\mu_{\hat{g}+sh}\big|_{s=0}\; dx^0\wedge\dots dx^3$$

$$\underbrace{\frac{d}{ds}(Ric[\hat{g}+sh]_{\alpha\beta})\big|_{s=0}}(\hat{g}^{-1})^{\alpha\beta} + Ric[\hat{g}]_{\alpha\beta}\frac{d}{ds}((\hat{g}+sh)^{-1})^{\alpha\beta}\big|_{s=0}$$

to evaluate, use geodesic normal coords.

$\hookrightarrow$  at p, $\hat{g}|_p = \eta$, $\Gamma|_p = 0$ — b/c Ric is a tensor, can compute in any coord and get same result after contracting indices

$$\Rightarrow \frac{d}{ds}Ric[\hat{g}+sh]_{\alpha\beta}\big|_{s=0}\, g^{\alpha\beta}$$
$$= \left(\partial_\mu \frac{d}{ds}\Gamma[\hat{g}+sh]^\mu{}_{\alpha\beta}\big|_{s=0} - \partial_\beta\frac{d}{ds}\Gamma[\hat{g}+sh]^\mu{}_{\alpha\mu}\big|_{s=0}\right)g^{\alpha\beta}$$

$\Rightarrow \frac{d}{ds}\Gamma[\hat{g}+sh]\big|_{s=0} = \lim_{s\to 0}\frac{\Gamma[g+sh]-\Gamma[g]}{s}$ $\leftarrow$ difference of two Γ's $\Rightarrow$ tensor

$\Rightarrow \frac{d}{ds}\Gamma$ transforms as section under diffeo, write as $J^\mu{}_{\alpha\beta}$

$$\Rightarrow \frac{d}{ds}Ric[\hat{g}+sh]_{\alpha\beta}\big|_{s=0}\, g^{\alpha\beta}$$
$$= (\partial_\mu J^\mu{}_{\alpha\beta} - \partial_\beta J^\mu{}_{\alpha\mu})\,g^{\alpha\beta}$$
$$= (\nabla_\mu J^\mu{}_{\alpha\beta} - \cancel{\Gamma J} - \dots - \nabla_\beta J^\mu{}_{\alpha\mu} - \cancel{\Gamma J} - \dots)\,g^{\alpha\beta}$$

0 b/c normal coords

Also need $\frac{d}{ds}((g+sh)^{-1})^{\alpha\beta}\big|_{s=0}$

$\hookrightarrow \frac{d}{ds}(g+sh)^{-1}(g+sh)\big|_{s=0} = 0$

$\frac{d}{ds}(g+sh)^{-1}\big|_{s=0}\, g + g^{-1}\frac{d}{ds}(g+sh)\big|_{s=0} = 0$

$\frac{d}{ds}(g+sh)^{-1}\big|_{s=0}\, g + g^{-1}h = 0 \qquad \Rightarrow \frac{d}{ds}(g+sh)^{-1}\big|_{s=0} = -g^{-1}hg^{-1}$

Recall $\frac{d}{ds}\mu_{\hat{g}+sh} = \frac{1}{2}\mu_{\hat{g}}\, g^{\alpha\beta}h_{\alpha\beta}$

$$\Rightarrow \frac{d}{ds}S[\hat{g}+sh] = \int_M\left[(\nabla_\mu J^\mu{}_{\alpha\beta} - \nabla_\beta J^\mu{}_{\alpha\mu})g^{\alpha\beta} - \underbrace{Ric[\hat{g}]_{\alpha\beta}\,g^{\alpha\mu}h_{\mu\nu}g^{\nu\beta}}_{Ric[\hat{g}]^{\mu\nu}h_{\mu\nu}} + \frac{1}{2}R[\hat{g}]\,g^{\alpha\beta}h_{\alpha\beta}\right]\mu_{\hat{g}}\; dx^0\wedge\dots\wedge dx^3$$

$[\nabla, g] = 0$

$$\Rightarrow \quad = \int_M\left\{\nabla_\mu(\underbrace{J^\mu{}_{\alpha\beta}\,g^{\alpha\beta}}_{A^\mu}) - \nabla_\beta(\underbrace{J^{\mu\beta}{}_\mu}_{B^\beta})\right\}\mu_{\hat{g}}\, dx^0\wedge\dots + \int_M\left\{-Ric[\hat{g}]^{\mu\nu}h_{\mu\nu} + \frac{1}{2}R[\hat{g}]\,g^{\alpha\beta}h_{\alpha\beta}\right\}\mu_{\hat{g}}$$

$$= \int_M(\cancel{\nabla_\mu A^\mu} - \cancel{\nabla_\beta B^\beta})\,\mu_{\hat{g}}\, dx^0\wedge\dots + \underbrace{\int_M\left\{-Ric[\hat{g}]^{\mu\nu} + \frac{1}{2}R[\hat{g}]^{\mu\nu}\right\}h_{\mu\nu}\,\mu_{\hat{g}}\, dx^0\wedge\dots dx^3}_{0\ \forall h}$$

0 by Schwartz function technique + Stoke's thm

Einstein Field Equations

So integrand $= 0 \Rightarrow \boxed{Ric[\hat{g}]^{\alpha\beta} - \frac{1}{2} R[\hat{g}]\, \hat{g}^{\alpha\beta} = 0}$

Th$^{\underline{m}}$ Let M be a C^∞-manifold and $\mathcal{M}$ be the space of Lorentzian metrics on M. The vaccum gravity is described by the critical points of the Einstein-Hilbert action functional $S : \mathcal{M} \to \mathbb{R}$, $S[\hat{g}] = \int_M R[\hat{g}]\, \mu_{\hat{g}}$. These are given by the following set of equations:

$$Ricci[\hat{g}] - \frac{1}{2} R[\hat{g}]\, \hat{g} = 0.$$

also write $G = 0$, with $G = Ricci - \frac{1}{2} R \hat{g}$

Also saw for other fields have $DS \cdot h = \int T^{\mu\nu} h_{\mu\nu}$ so when $h = L_X \hat{g}$ (X generator of $D_0(M)$)

we need D_0-invariance, so need divergence thm: $\underline{\nabla_\mu (Ric^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0}$

in pure gravity, this is identically satisfied b/c of Bianchi identity

The dynamics (sol$^{\underline{n}}$ of $G = 0$) occurs on $\mathcal{M}/D_0$

$\Rightarrow$ large redundancy in solution space

Eq$^{\underline{n}}$ of motion for gravity: $G^{\mu\nu} = 0 \quad \longleftrightarrow \quad DS[\psi + \delta\psi] \cdot \delta\psi = 0$

Conservation law: $\nabla_\mu G^{\mu\nu} = 0 \quad \longleftrightarrow \quad \nabla_\mu T^{\mu\nu} = 0$

but this is identity; satisfied for all $\hat{g} \in \mathcal{M}$

so want to find symmetries for $G^{\mu\nu}$

Lemma The 2-tensor $G^{\mu\nu} \partial_\mu \otimes \partial_\nu = (Ric^{\mu\nu} - \frac{1}{2} R g^{\mu\nu})\, \partial_\mu \otimes \partial_\nu$ is divergence-free for any $\hat{g} \in \mathcal{M}$ (with prescribed regularity, i.e. finite $\partial^3 \hat{g}$)

If $(A_{\mu\nu})^\alpha{}_\beta = R^\alpha{}_{\beta\mu\nu}$, Bianchi identity comes from $DA = 0$
$\Downarrow$
gauge covariant exterior deriv. (satisfies $D \cdot D \cdot B = 0$)
$\Downarrow$
Gravity $= (Y\text{-}M)^2$

Pf Recall Bianchi identity: $\nabla_\mu R^\alpha{}_{\beta\nu\lambda} + \nabla_\nu R^\alpha{}_{\beta\lambda\mu} + \nabla_\lambda R^\alpha{}_{\beta\mu\nu} = 0.$

$\hookrightarrow$ take traces to get $Ric_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. Contract α, ν (can do b/c ∇ metric compat.)

$\Rightarrow \nabla_\mu R^\alpha{}_{\beta\alpha\lambda} + \nabla_\alpha R^\alpha{}_{\beta\lambda\mu} + \nabla_\lambda R^\alpha{}_{\beta\mu\alpha} = 0$

$\nabla_\mu Ric_{\beta\lambda} + \nabla_\alpha R^\alpha{}_{\beta\lambda\mu} - \nabla_\lambda Ric_{\beta\mu} = 0$

Contract β, μ:

$\Rightarrow \nabla_\mu Ric^\mu{}_\lambda + \nabla_\alpha R^{\alpha\mu}{}_{\lambda\mu} - \nabla_\lambda R = 0$

$\nabla_\mu Ric^\mu{}_\lambda + \nabla_\alpha R^\alpha{}_\lambda - \nabla_\lambda R = 0$

$\nabla_\mu Ric^\mu{}_\lambda - \frac{1}{2} \nabla_\lambda R = 0$

$\Rightarrow \nabla_\mu (Ric^{\mu\lambda} - \frac{1}{2} R g^{\mu\lambda}) = 0.$ ▨

Suppose $G[\hat{g}]^{\mu\nu} = 0 \Rightarrow$ gives $\hat{g} \in \mathcal{M}$. Can do $\hat{g} + h = \hat{g} + L_X \hat{g}$ to also get sol$^{\underline{n}}$

b/c can do int. by parts and use $\nabla_\mu G^{\mu\nu} = 0$ identity

$\hookrightarrow$ can perturb critical point and get critical point

$\hookrightarrow$ there are directions in ∞-dim $\mathcal{M}$ which are insensitive to perturbing $\hat{g}$.

$L_X \hat{g}$ can be thought of as the diffeo; $\underline{\hat{g} \text{ describes physics}}$, $\underline{L_X \hat{g} \text{ describes coordinate system}}$

Corr Let $\hat{g} \in \mathcal{M}$ be a sol^{n} for $G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 0$. Then

$\hat{g} + L_X \hat{g}$, $\forall X \in \mathfrak{X}(M)$ is also a sol^{n} to $G[\hat{g} + L_X \hat{g}] = 0$

$\Rightarrow$ obstruction to uniqueness $\qquad \hat{g} \sim \varphi_s^* \hat{g} = \hat{g} + s L_X \hat{g} + \mathcal{O}(s^2)$ two metrics equiv. if related by the pullback of diffeo

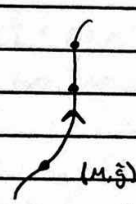
$\Rightarrow$ to fix, look at $\mathcal{M}/D_0$

↳ instead of considering $\mathcal{M}$ as the "configuration space", consider $\mathcal{M}/D_0$

Def^n The redundancy in $\mathcal{M}$ associated with the eq^n $G = 0$ is called "gauge freedom"

Def^n Descending to $\mathcal{M}/D_0$ is called "gauge fixing"

↳ getting rid of non-physical degrees of freedom



Before studied geodesics in finite space

$\mathcal{M}/D_0$

Point particle $\Longleftrightarrow$ $\hat{g}$ tensor field in ∞-dim

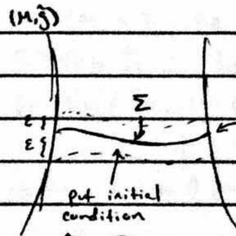
Solved geodesic eq^n by writing as first order system of diff eqs by defining $v = \frac{dx}{d\lambda}$ $\longrightarrow$ analogously: $x \equiv \hat{g}$

$Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$

$Ric = tr(Riem) \sim \partial \Gamma \sim \partial^2 \hat{g} \quad \Rightarrow \quad G^{\mu\nu} = 0$ like geodesic eqn.

$\partial^2 \hat{g} + F(\hat{g}, \partial \hat{g}) = 0 \xrightarrow{\text{will define}} \underbrace{\partial_t \hat{g} = K}_{\text{"position"}}, \ \underbrace{\partial_t K = F(\hat{g}, K, \dots)}_{\text{"velocity"}}$

$(M, \hat{g})$



fix a constant 't' slice (∂t is normal to Σ) $\longrightarrow$ Locally, splitting $M = \Sigma \times (-\varepsilon, \varepsilon)$

put initial condition

$\left. \begin{array}{l} \hat{g}|_\Sigma \\ \partial_t \hat{g}|_\Sigma \end{array} \right\}$ and try to evolve it according to $G_{\mu\nu} = 0$

IF $i : \Sigma \hookrightarrow M$, then $i^* \hat{g}$ is a Riemannian metric (where $\hat{g}$ Lorentzian metric)

⚠ Abuse of notation
$X \in TM \longleftrightarrow X \in \Gamma^{0,1}$

Σ_t "horizontal space"

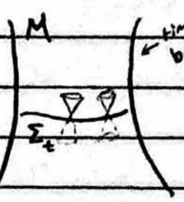
Geometry of Submanifold in Lorentzian Manifold

Let $i : \Sigma_t \hookrightarrow M$ a submfld

$\pi : M \to \Sigma_t$

$y \mapsto \pi(y)$

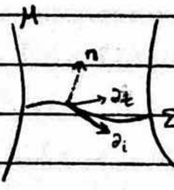
$(t, x^1, ..) \mapsto (x^1, ..)$



- On $(M, \hat{g})$ have $\nabla[\hat{g}]$, $Riem[\hat{g}]$, ...
- On Σ_t have $g = i^* \hat{g}$ Riemannian metric, $\nabla[g]$, $Riem[g]$, ...

At each $y \in M$, $T_y M = T_{\pi(y)} \Sigma_t \oplus \mathbb{R}$

~~do not fix~~ coord. here yet



$dt \perp T^* \Sigma_t \not\Rightarrow \frac{\partial}{\partial t} \perp T\Sigma_t$ (b/c $\hat{g}$ not necessarily diagonal)

↳ rather $\partial_t = N n + X^i \partial_i$ by $T_y M$ decomposition above

(where $n \perp T_{\pi(y)} \Sigma_t$, and n unit normal wlog)

↳ i.e. $g(n,n) = -1$

So $\boxed{n = \frac{1}{N}(\partial_t - X^i \partial_i)}$

$$\underset{\substack{\uparrow \\ \text{Lapse function}}}{\frac{\partial}{\partial t} = N} \, n + \underset{\substack{\nwarrow \\ \text{shift vector field}}}{X}$$

$\hat{g}(n,n) = -1$

$\partial_i \in T\Sigma_t$

$n \perp T\Sigma_t$

$X \in T\Sigma_t$

etc.

Aim: Use available information to split $\hat{g}$ as Σ_t parallel & Σ_t orthogonal

↳ from $n = \frac{1}{N}(\partial_t - X^i \partial_i)$, $n^0 = 1/N$, $n^i = -X^i/N$

$= n^\mu \partial_\mu = n_\mu dx^\mu$

Now $\hat{g}(n,n) = -1 \Rightarrow \hat{g}_{\mu\nu} n^\mu n^\nu = -1 \Rightarrow n_\mu n^\mu = -1 \to n_0 n^0 + \cancel{n_i n^i} = -1$

Since $n \perp \Sigma_t \Rightarrow \hat{g}(n, \partial_i) = 0 \Rightarrow n_i = 0 \Rightarrow n = n_0 dt$; $\boxed{n_0 = -N}$

$\Rightarrow \boxed{n = -N dt}$

Lemma (ADM Decomposition)

Let $(M, \hat{g})$ be a Lorentzian manifold which can be locally written as $[-\varepsilon, \varepsilon] \times \Sigma$. Then the Lorentzian metric $\hat{g}$ admits the following decomposition in local chart (t, x^i)

$$\hat{g} := -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt)$$

where $g_{ij} := \hat{g}(\partial_i, \partial_j)$ is the induced Riemannian metric on Σ

g_{ij} has 6 components ⤳
X^i has 3
N has 1
$\Rightarrow$ 10 indep components, as needed

Pf Using n and dx^i, construct basis for $Sym(T^*M \otimes T^*M)$:

$n \otimes n$, $n \otimes dx^i$, $dx^i \otimes n$, $dx^i \otimes dx^j$ $\forall i,j \in \{1,2,3\}$

With basis, $\hat{g} = A\, n \otimes n + B_i (n \otimes dx^i + dx^i \otimes n) + C_{ij}\, dx^i \otimes dx^j$

↳ $\hat{g}(\partial_k, \partial_l) = g(\partial_k, \partial_l) = 0 + 0 + C_{ij} \delta^i{}_k \delta^j{}_l = \boxed{C_{kl} = g_{kl}}$

↳ $\hat{g}(n, \partial_k) = 0 = 0 + B_i \hat{g}(n,n)\hat{g}(\partial_k, dx^i) + B_i \hat{g}(n, dx^i) \cancel{\hat{g}(\partial_k, n)} + g_{ij} \hat{g}(n, dx^i) \hat{g}(\partial_k, dx^j)$

$= -B_i \delta^i{}_k + g_{ij} n^i \delta^j{}_k \Rightarrow B_k = g_{ki} n^i = -\frac{1}{N} g_{ki} X^i$

↳ $\hat{g}(n,n) = -1 = A - 2B_i n^i + g_{ij} n^i n^j = A + \frac{2}{N} g_{ik} X^k \left(\frac{-X^i}{N}\right) + g_{ij}\left(\frac{-X^i}{N}\right)\left(\frac{-X^j}{N}\right) \Rightarrow A = -1 + \frac{g_{ij} X^i X^k}{N^2}$ → cont.

using $n = -N\,dt$

$$\hat g = \left(-1 + \frac{g_{kl}X^k X^l}{N^2}\right) N^2\, dt\otimes dt - \frac{1}{N} g_{ik} X^k \left(-N\, dt\otimes dx^i - N\, dx^i \otimes dt\right) + g_{ij}\, dx^i \otimes dx^j$$
$$= -N^2\, dt\otimes dt + g_{ij}\left(dx^i + X^i dt\right)\left(dx^j + X^j dt\right). \qquad \blacksquare$$

$$\hat g = -N^2\, dt \otimes dt + g_{ij}\,(dx^i + \underbrace{X^i}\, dt)(dx^j + \underbrace{X^j}\, dt)$$
$X \in T\Sigma_t$

↳ X element of Diff_0

$\mathrm{Diff}_0(M \times [-\varepsilon, \varepsilon]) = \mathrm{Diff}_0(M) \times \mathrm{Diff}_0([-\varepsilon,\varepsilon])$

$\dim(\mathrm{Diff}_0(M)) = \infty = 4 \times \infty \longrightarrow X^1, X^2, X^3 \longrightarrow N$

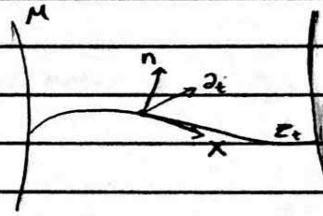
↳ $(t, x^1, x^2, x^3) \to \varphi^0, \varphi^1, \varphi^2, \varphi^3$

When going to $\mathcal{M}/\mathrm{Diff}_0$ lets you fix $\partial_t N = \partial_t X^i = 0$

↳ gauge fixing $\equiv \mathcal{M} \to \mathcal{M}/\mathrm{Diff}_0 \equiv$ fixing N & X

We have $g \equiv$ (position of geodesic motion), need "velocity"

↳ need extrinsic curvature of Σ_t in M



Notion of 2nd Fundamental Form

Define

$$\hat\nabla_P Q := \nabla_P Q + \Pi(P,Q)\, n$$

$\langle \cdot, \cdot \rangle \equiv \hat g$

$$\Rightarrow \langle \hat\nabla_P Q, n\rangle = \cancelto{0}{\langle \nabla_P Q, n\rangle} - \Pi(P,Q)$$
$$\Rightarrow \Pi(P,Q) = -\langle \hat\nabla_P Q, n\rangle$$
$$= -\hat g(\hat\nabla_P Q, n) \quad \text{b/c } Q \in T\Sigma_t$$
$$= \hat g(Q, \hat\nabla_P n) - \hat\nabla_P \cancelto{0}{\hat g(Q,n)}$$

on M: $M, \hat g, \nabla[\hat g] \equiv \hat\nabla$

on Σ_t: $\Sigma, g, \nabla[g] \equiv \nabla$

$P, Q \in T\Sigma_t \subset TM$

↳ $\underset{TM}{\hat\nabla_P Q} \neq \underset{T\Sigma_t}{\nabla_P Q}$

Def$^{\underline{n}}$ (2nd Fundamental Form)

Let $P, Q \in T\Sigma_t$, $\hat\nabla \equiv \hat g$, $\nabla \equiv g$. Then $K(P,Q) := -\hat g(\hat\nabla_P n, Q)$

$(K \in T^*\Sigma_t \otimes T^*\Sigma_t)$.

Claim: $K \in \mathrm{Sym}(T^*\Sigma_t \otimes T^*\Sigma_t)$

Pf| $K(P,Q) = -\hat g(\hat\nabla_P n, Q) = \hat g(\hat\nabla_P Q, n) = \hat g(\hat\nabla_Q P + [P,Q], n)$

$= \hat g(\hat\nabla_Q P, n) + \hat g([P,Q], n) = K(Q,P) + \hat g([P,Q], n).$

NB| Frobenius Th$^{\underline{m}}$: $E \subset TM$. Then E is integrable iff E is involutive

$E = TN$, $N \subset M$ $\qquad P, Q \in E \Rightarrow [P,Q] \in E$

Since Σ_t submfld, $T\Sigma_t$ is integrable $\Rightarrow [P,Q] \in \Sigma_t \Rightarrow \hat g([P,Q], n) = 0$

$\Rightarrow K(P,Q) = K(Q,P)$. $\blacksquare$

Want to study time evolution of g:

$$\partial_t g_{ij} = (\partial_t g)(\partial_i, \partial_j) = (\partial_t \hat{g})(\partial_i, \partial_j) = (\mathcal{L}_{\partial_t} \hat{g})(\partial_i, \partial_j)$$
$$= \mathcal{L}_{\partial_t}(\hat{g}(\partial_i, \partial_j)) - \hat{g}(\mathcal{L}_{\partial_t}\partial_i, \partial_j) - \hat{g}(\partial_i, \mathcal{L}_{\partial_t}\partial_j)$$
$$= \hat{\nabla}_{\partial_t}\hat{g}(\partial_i, \partial_j) - \hat{g}([\partial_t, \partial_i], \partial_j) - \hat{g}(\partial_i, [\partial_t, \partial_j])$$

using torsion free property: $\nabla_X Y = \nabla_Y X + [X,Y]$

$$= \hat{g}(\hat{\nabla}_{\partial_t}\partial_i, \partial_j) + \hat{g}(\partial_i, \hat{\nabla}_{\partial_t}\partial_j) - \hat{g}([\partial_t, \partial_i], \partial_j) - \hat{g}(\partial_i, [\partial_t, \partial_j])$$
$$= \hat{g}(\hat{\nabla}_{\partial_i}\partial_t, \partial_j) + \hat{g}(\partial_i, \hat{\nabla}_{\partial_j}\partial_t) + \cancel{\hat{g}([\partial_t, \partial_i], \partial_j)} + \cancel{\cdots} - \cancel{\hat{g}([\partial_t, \partial_i], \partial_j)} - \cdots$$
$$= \hat{g}(\hat{\nabla}_{\partial_i}(Nn + X), \partial_j) + \hat{g}(\partial_i, \hat{\nabla}_{\partial_j}(Nn + X))$$

$\hat{g}(\hat{\nabla}_{\partial_i}(Nn), \partial_j)$
$= \hat{g}(N\hat{\nabla}_{\partial_i}n, \partial_j)$
$+ \underbrace{\hat{g}(n\hat{\nabla}_{\partial_i}N, \partial_j)}$
$n \perp \partial_j \Rightarrow = 0$

$$= \hat{g}(\hat{\nabla}_{\partial_i}(Nn), \partial_j) + \hat{g}(\hat{\nabla}_{\partial_i}X, \partial_j) + \hat{g}(\partial_i, \hat{\nabla}_{\partial_j}(Nn)) + \hat{g}(\partial_i, \hat{\nabla}_j X)$$
$$= N\hat{g}(\hat{\nabla}_i n, \partial_j) + N\hat{g}(\partial_i, \hat{\nabla}_i n) + \hat{g}(\hat{\nabla}_i X, \partial_j) + \hat{g}(\partial_i, \hat{\nabla}_i X)$$
$$= \underbrace{-2NK(\partial_i, \partial_j)} + \underbrace{(\mathcal{L}_X g)(\partial_i, \partial_j)}$$

$\hat{\nabla}_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i - K(\partial_i, \partial_j)\, n \Rightarrow \hat{g}(\hat{\nabla}_{\partial_i}\partial_j, n) = K(\partial_i, \partial_j) = -\hat{g}(\hat{\nabla}_{\partial_i} n, \partial_j)$

$\hat{g}(\nabla_{\partial_i}X - \cancel{K(\partial_i, X)\, n}^{0}, \partial_j) = \hat{g}(\nabla_{\partial_i}X, \partial_j) = g(\nabla_{\partial_i}X, \partial_j)$

$\Rightarrow g(\nabla_{\partial_i}X, \partial_j) + g(\partial_i, \nabla_{\partial_j}X) = (\mathcal{L}_X g)(\partial_i, \partial_j)$

1st Evolution Lemma

Let Σ_t be a spacelike submanifold of M. The induced metric g on Σ_t satisfies the following evolution equation in (t, x^i):

$$\partial_t g_{ij} = -2NK_{ij} + (\mathcal{L}_X g)_{ij}$$

↳ Purely kinematic analog to $\frac{dx^\mu}{d\lambda} = v$ for geodesic; no forces b/c no information from Einstein eq's yet.

2nd Evolution Lemma

" . . . . . " following evolution equation:

$$\partial_t K_{ij} = -\nabla_i \nabla_j N + N\left(\hat{R}(\partial_i, n, \partial_j, n) - K_i{}^a K_{ja}\right) + (\mathcal{L}_X K)_{ij}$$

Pf sketch $(\partial_t K)(\partial_i, \partial_j) = \nabla_{\partial_t} K(\partial_i, \partial_j) - K(\ldots$

$= -\nabla_{\partial_t}\langle \nabla_{\partial_i} n, \partial_j \rangle + \cdots$

$= -\langle \nabla_{\partial_t}\nabla_{\partial_i} n, \partial_j \rangle - \langle \nabla_{\partial_i} n, \nabla_{\partial_t}\partial_j \rangle + \cdots$

$\vdots$

Want to impose $G = 0$ $\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0\right)$ on evolution eq^{n}s

↳ Want to understand the relation between $\hat{R}$ and R

Take $X, Y, Z, W \in T\Sigma_t \subset TM$, and assume $[Z, W] = 0$

$\hat{R}(X, Y, Z, W) = \langle \hat{\nabla}_Z \hat{\nabla}_W Y - \hat{\nabla}_W \hat{\nabla}_Z Y, X \rangle.$

Use $\hat{\nabla}_X Y = \nabla_X Y - K(X, Y)\, n$

cont.

$$\hat{R}(X,Y,Z,W) = \langle \hat{\nabla}_Z (\overset{T\Sigma_t}{\overset{\cup}{\nabla_W Y}} - K(W,Y)n) - (W \leftrightarrow Z), X \rangle$$

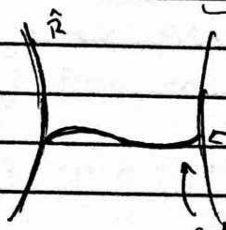
$$= \langle \nabla_Z \nabla_W Y - \underbrace{K(Z, \cancel{\nabla_W Y})n}_{0} - \underbrace{\hat{\nabla}_Z(K(\cancel{W,Y}))n}_{0 \text{ b/c } n \perp X} - K(W,Y)\hat{\nabla}_Z n - (W \leftrightarrow Z), X \rangle$$

$$= \langle \nabla_Z \nabla_W Y - K(W,Y)\hat{\nabla}_Z n - (W \leftrightarrow Z), X \rangle$$

$$= \langle \nabla_Z \nabla_W Y - \nabla_W \nabla_Z Y, X \rangle - \langle K(W,Y)\hat{\nabla}_Z n, X \rangle + \langle K(Z,Y)\hat{\nabla}_W n, X \rangle$$

$$= R(X,Y,Z,W) - K(W,Y)\langle \hat{\nabla}_Z n, X \rangle + K(Z,Y)\langle \hat{\nabla}_W n, X \rangle$$

$$= \underbrace{R(X,Y,Z,W)}_{\text{intrinsic to } \Sigma_t} + \underbrace{K(W,Y)K(X,Z) - K(Z,Y)K(X,W)}_{\text{relative curvature of } \Sigma_t \text{ in ambient space}}$$



$\hat{R}$ get contribution from intrinsic curvature R

and also from extrinsic curvature on how spacelike mfld embedded in ambient space

Gauss Lemma

Σ_t embedded submanifold of M with induced metric g and second fund. form K, the projection of $\hat{R}$ on to Σ_t satisfies the following eq$^{\underline{n}}$:

$$\hat{R}(X,Y,Z,W) = R(X,Y,Z,W) + K(W,Y)K(X,Z) - K(Z,Y)K(X,W).$$

Codazzi Eq$^{\underline{n}}$

Take $X, Y, Z \in T\Sigma_t \subset TM$, $n \perp T\Sigma_t$. (assume $[X,Y]=0$)

$$\hat{R}(n,Z,X,Y) = \langle \hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z, n \rangle$$

$$= \langle \hat{\nabla}_X \nabla_Y Z - \hat{\nabla}_X (K(Y,Z))n - K(Y,Z)\hat{\nabla}_X n - (X \leftrightarrow Y), n \rangle$$

$$= \langle \underbrace{\nabla_X \cancel{\nabla_Y Z}}_{0 \text{ b/c } \nabla_X \nabla_Y Z \perp n} - K(X, \nabla_Y Z)n - \hat{\nabla}_X(K(Y,Z))n - \underbrace{K(Y,Z)\cancel{\hat{\nabla}_X n}}_{0 \text{ b/c } K(Y,Z)\langle \hat{\nabla}_X n, n \rangle = K(Y,Z)\hat{\nabla}_X \cancel{\langle n,n \rangle}^{0}} - (X \leftrightarrow Y), n \rangle$$

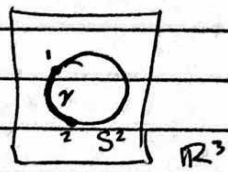
$$= K(X, \cancel{\nabla_Y Z}) + \underbrace{X(K(Y,Z))}_{\substack{\| \\ (\nabla_X K)(Y,Z) + K(\nabla_X Y, Z) + \cancel{K(Y,\nabla_X Z)}}} - \cancel{K(Y, \nabla_X Z)} - \underbrace{Y(K(X,Z))}_{\substack{\| \\ -(\nabla_Y K)(X,Z) - K(\nabla_Y X, Z) - \cancel{K(X,\nabla_Y Z)}}}.$$

$$= (\nabla_X K)(Y,Z) - (\nabla_Y K)(X,Z) + K(\underbrace{\nabla_X Y - \nabla_Y X}_{=[X,Y]=0}, Z)$$

$$\boxed{\hat{R}(n,Z,X,Y) = (\nabla_X K)(Y,Z) - (\nabla_Y K)(X,Z)}$$

Codazzi eq$^{\underline{n}}$.

Geometric Significance of K



γ geodesic on S^2 $\to$ not geodesic on $\mathbb{R}^3$
as result of extrinsic curvature

Def$^{\underline{n}}$ (totally geodesic submanifold)

Let $\Sigma_t \hookrightarrow M$ be an embedded submanifold. Σ_t is called a totally geodesic submanifold of M if every geodesic on Σ_t is also a geodesic on M

Ex) $\mathbb{R}^n \subset \mathbb{R}^{n+k}$

Lemma

$\Sigma_t \hookrightarrow M$ is totally geodesic iff $K = 0$.

Pf $X \in T\Sigma_t \subset TM$, X any geodesic vector field on Σ_t.
Then $\nabla_X X = 0$. For X to be geodesic vector field on M, $\hat{\nabla}_X X = 0$
$\Rightarrow \hat{\nabla}_X X = \nabla_X X - K(X,X)\, n = -K(X,X)\, n = 0 \Rightarrow K(X,X) = 0$
$\forall$ geodesic vector field $X \in T\Sigma_t$. $\Rightarrow \underset{\text{(almost everywhere)}}{K=0}$ (since geodesic vec. fields form dense subset)

Saw $\partial_t g = \cdots L_X g$
$\partial_t K = \cdots L_X K$
$\Rightarrow$ diff eq$^{\underline{n}}$s like tangent spaces in infinite dim space
X is generator of diffeo $\to$ acts on sections as Lie deriv. (by def$^{\underline{n}}$ $\frac{d}{ds}(\varphi_s^*(\text{section}))|_{s=0}$) $\Rightarrow$ X must appear as Lie derivative.

Back to Gauss + Codazzi:

Take $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$, $W = \partial_l$:

$$\hat{R}(\partial_i, \partial_j, \partial_k, \partial_l) = R(\partial_i, \partial_j, \partial_k, \partial_l) + K_{lj} K_{ik} - K_{il} K_{jk}$$

↳ trace to relate to Einstein eq$^{\underline{n}}$

$$\Rightarrow g^{ik} \hat{R}(\partial_i, \partial_j, \partial_k, \partial_l) = g^{ik} R(\partial_i, \partial_j, \partial_k, \partial_l) + K_{lj} \overbrace{tr_g(K)}^{= g^{ik} K_{ik}} - K_l{}^m K_{jm}$$

not $\hat{Ric}$ b/c only spacelike metric (not full $\hat{g}$) $\rightsquigarrow$ $$g^{ik} \hat{R}(\partial_i, \partial_j, \partial_k, \partial_l) = Ric(\partial_j, \partial_l) + K_{lj}\, tr_g(K) - K_l{}^m K_{jm}$$

Recall $\hat{R}_{\mu\nu} = \hat{g}^{\alpha\beta} \hat{R}(\partial_\alpha, \partial_\mu, \partial_\beta, \partial_\nu)$

$$\Rightarrow \hat{R}_{jl} = Ric(\partial_j, \partial_l) = \hat{g}^{\alpha\beta} \hat{R}(\partial_\alpha, \partial_j, \partial_\beta, \partial_l)$$
$$= \hat{g}^{nn} \hat{R}(n, \partial_j, n, \partial_l) + \{\text{mixed } g^{in}, g^{ni}\}$$
$$+ \underbrace{g^{ik} \hat{R}(\partial_i, \partial_j, \partial_k, \partial_l)}_{\text{denote } (I)_{jl}}$$

can be made to vanish by choosing shift $X = 0$ $\sim X^i$

$\longrightarrow$

$$\Rightarrow (I)_{j\ell} = \hat{R}_{j\ell} - \hat{g}^{nn}\,\hat{R}(n,\partial_j,n,\partial_\ell) = \hat{R}_{j\ell} + \hat{R}(n,\partial_j,n,\partial_\ell)$$

$$\Rightarrow \underline{\hat{R}_{j\ell}} + \underbrace{\hat{R}(n,\partial_j,n,\partial_\ell)} = R_{j\ell} + k_{\ell j}\,(tr_g k) - k_\ell{}^m k_{jm}$$

known from Einstein eq^n — can substitute into k evolution eq^n

Use this in k evolution eq^n:

$$\partial_t k_{ij} = -\nabla_i \nabla_j N + N\left(Ric_{ij} - \hat{R}ic_{ij} - k_{i\ell}k^\ell{}_j + k_{ij}\,(tr_g k) - k_{i\ell}k^\ell{}_j\right) + (L_X k)_{ij}$$

$$\partial_t g_{ij} = -2N k_{ij} + (L_X g)_{ij}$$

others found by differentiating or integrating

$\Rightarrow \hat{R}ic$ from Einstein eq^n $\Rightarrow$ Einstein evolution eq^n

↳ have 6 eq^{n}s total $\big(6\ \partial_t k_{ij},\ \underbrace{6\ \partial_t g_{ij}}$ $\Rightarrow$ only 6 eq^{n}s to study$\big)$

but these are just des^{n}s for k_{ij} $\Rightarrow$ nothing new

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Einstein vacuum eq^n: $\hat{R}ic_{\mu\nu} - \frac{1}{2}\hat{R}\,\hat{g}^{\mu\nu} = 0$

$$\Rightarrow \hat{R} - \tfrac{1}{2}\hat{R}\cdot 4 = 0 \Rightarrow \hat{R} = 0 \Rightarrow \hat{R}ic_{\mu\nu} = 0 \Rightarrow 10 \text{ eq}^n\text{s}$$

↳ have 6 evolution eq^{n}s; still need 4 related to orthogonal / normal part

which give Ric_{ij} projection to Σ using $\hat{R}_{ij} = 0$

↳ will use $\hat{R}ic(n,\partial_i) = \hat{R}ic(n,n) = 0$

Take Codazzi eq^n: $\hat{R}(n,\partial_k,\partial_i,\partial_j) = \nabla_i k_{jk} - \nabla_j k_{ik}$

trace k, j: $g^{kj}\,\hat{R}(n,\partial_k,\partial_i,\partial_j) = \nabla_i (tr_g k) - \nabla^k k_{ik}$

$$\hat{R}(n,\partial_i)\{1 + \text{terms w/ } k\} = \nabla_i (tr_g k) - \nabla^k k_{ik}$$

Impose Einstein eq^n $\to \hat{R}(n,\partial_i) = 0 \Rightarrow$ $\boxed{\nabla_i (tr_g k) - \nabla^k k_{ik} = 0}$ "momentum constraint"

Use $\hat{R}(n,n) = 0$:

$$\hat{R}_{j\ell} + \hat{R}(n,\partial_j,n,\partial_\ell) = R_{j\ell} + k_{\ell j}\,(tr_g k) - k_\ell{}^m k_{jm}$$

trace $\Rightarrow \hat{R}_{j\ell}\, g^{j\ell} + \hat{R}(n,n) = R + (tr_g k)^2 - k_{j\ell}k^{j\ell}$

Einstein eq^n $\Rightarrow$ $\underbrace{\boxed{0 = R[g] + (tr_g k)^2 - |k|_g^2}}_{\text{roughly free energy of system}}$ "energy / Hamiltonian constraint"

Evolution + Constraint Eq^{n}s (Vaccum)

$$\partial_t g_{ij} = -2N k_{ij} + (L_X g)_{ij}$$
$$\partial_t k_{ij} = -\nabla_i \nabla_j N + N\big(Ric_{ij} + k_{ij}\,(tr_g k) - 2k_i{}^\ell k_{j\ell}$$

} evolution

must be satisfied at every Σ_t {

$$\nabla_i (tr_g k) - \nabla^k k_{ik} = 0$$
$$R[g] + (tr_g k)^2 - k_{ij}k^{ij} = 0$$

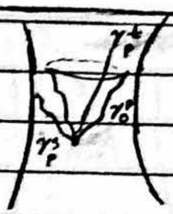
} constraints

↳ equations describing 'Formal' $T(M/D_0)$

Constrained evolution: the particle / field is constrained to move on a "submanifold" of the config. space

$E \overset{\text{formally}}{=} T(M/D_0)$

Th^m | In 2+1 dimensions, the distribution E is integrable
$\Rightarrow$ Teichmüller space

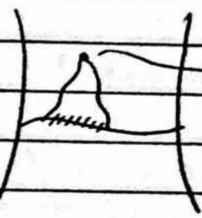


Can look at 'deformed' lightcones in arbitrary spacetimes

Denote $C^+(p)$ = mantle of future lightcone
$\mathcal{J}^+(p)$ = interior " " "
$D^+(p) = C^+(p) \cup \mathcal{J}^+(p)$

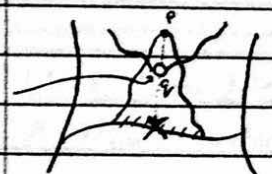
- $\{\gamma_p^t : I \subset \mathbb{R} \to M \mid \gamma_p^t \in \mathcal{J}^+(p)\} \to$ chronological curves
- $\{\gamma_p : I \subset \mathbb{R} \to M \mid \gamma_p \in D^+(p)\} \to$ causal curves

Newtonian gravity fails in this case: would say every object effects every other object regardless of separation



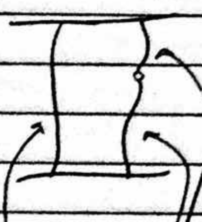
$\to$ can only be influenced by certain region of spacelike mfld b/c speed of light $= 1$
$\Rightarrow$ domain of influence is everything lying in lightcone
$\Rightarrow$ given data on submfld, can uniquely determine point in future
$\hookrightarrow$ "Cauchy problem" (initial value problem)

removed point



Information that propagates through q is lost
$\Rightarrow$ no longer unique determination of p

Def^n (Cauchy Horizon) | The null hypersurface beyond which the causality of Einstein's eq^{n}s breaks down



inextendible causal curves

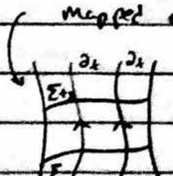
Def^n (Inextendible Causal Curve) | A causal curve that exists as long as the spacetime exists

Def^n (Cauchy hypersurface) | A spacelike hypersurface that meets every inextendible causal curve exactly once.

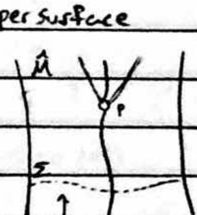
Def^n (Globally hyperbolic spacetime) | A spacetime that admits a Cauchy hypersurface

$\Sigma_1 \cup \Sigma_2$ also Cauchy hypersurface on $\hat{M} \setminus D^+(p)$

Th^m | Let M be a globally hyperbolic spacetime and time orientable. Then two Cauchy hypersurfaces Σ_1, Σ_2 can be mapped onto eachother by a homeomorphism



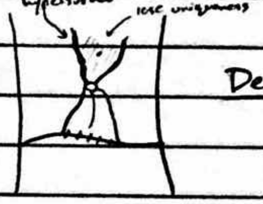
Pf | Take φ_{∂_t} as the flow of ∂_t. Then map $x \in \Sigma_{t_1}$, $x \mapsto \varphi_{\partial_t}(x)$. ∎



Σ not Cauchy hypersurface b/c does not intersect other causal curve
But $\hat{M} \setminus D^+(p)$ is globally hyperbolic b/c Σ becomes a Cauchy hypersurface (as other causal curve is now removed from spacetime)

whole purpose of Cauchy hypersurface is to do things in a unique way
$\Rightarrow$ no obstruction to uniqueness

Corr A globally hyperbolic Spacetime M is necessarily of the topological type $\Sigma \times \mathbb{R}$, where Σ is a Cauchy hypersurface.



Def$^{\underline{n}}$ (Cauchy Horizon) The null hypersurface beyond which the Predictive property of GR fails

Rough def$^{\underline{n}}$ of Singularity: blow up of geometric entities constructed in a coordinate invariant way

First indication of singularity formation is geodesic incompleteness

$\underbrace{\text{characteristics}}$ of the Einstein eq$^{\underline{n}}$s
"a null hypersurface"

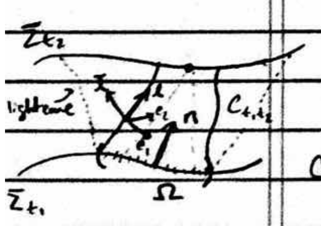
Wave Operator

A wave operator on a Lorentzian spacetime $(M, \hat{g})$ is a second order differential operator given as follows: $\Box_{\hat{g}} = \hat{g}^{\mu\nu}\hat{\nabla}_\mu \hat{\nabla}_\nu$

Recall Scalar field: $S = -\frac{1}{2}\int \hat{g}^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \cdots \; \mu_{\hat{g}}$

$$DS\cdot h = 0 \Rightarrow \hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu\varphi = \Box_{\hat{g}}\varphi = 0.$$

$$2T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\hat{g}_{\mu\nu}\hat{g}(\partial\varphi, \partial\varphi)$$

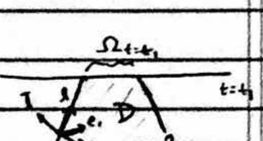


Construct a basis $\{\underbrace{l, \bar{l}}_{\text{null}}, \underbrace{e_1, e_2}_{\text{spacelike}}\}$ s.t. $\hat{g}(l,l) = \hat{g}(\bar{l},\bar{l}) = 0$, $g(l,\bar{l}) = -1$.

$\Omega \subset \Sigma_{t_1}$

l is both tangent and normal to C_{t_1,t_2}

- energy at point only influenced by region in Ω
  - $\hookrightarrow$ Some energy in Ω 'leaks' out of region on Σ_{t_1} though



Consider Minkowski space: $\hat{g} = -dt^2 + (dx^1)^2 + (dx^2)^2 + \cdots = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

- $T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\hat{g}_{\mu\nu}\hat{g}(\partial\varphi, \partial\varphi)$
- Claim: $\partial_t - \partial_r$, $\partial_t + \partial_r$ are both null

Pf $\eta(\partial_t - \partial_r, \partial_t - \partial_r) = \eta_{tt} - 2\eta_{tr} + \eta_{rr} = -1 - 0 + 1 = 0$
$\eta(\partial_t + \partial_r, \partial_t + \partial_r) = \text{- - - - - -} = 0$ ∎

Choose $n = \partial_t = \frac{1}{2}(l + \bar{l})$

Choose $l = \partial_t - \partial_r$, $\bar{l} = \partial_t + \partial_r$,
$\eta(e_1, e_2) = 0$
$\eta(e_1, e_1) = 1 = \eta(e_2, e_2)$
$\eta(e_i, l) = 0 = \eta(e_i, \bar{l})$

Define energy: $E_{\Sigma_t} = Q_{\Sigma_t} = \int_{\Sigma_t} T(\partial_t, \partial_t)\, \mu_\eta$

Th^{m} | $E_{\Omega_{t=t_1}} \leq E_{\Omega_{t=0}}$

Pf | $T(\overset{n}{\partial_t}, \cdot) = J$ current $\Rightarrow J^\mu = T^{\mu\nu} n_\nu$

$\Rightarrow \nabla_\mu J^\mu = (\overset{0}{\cancel{\nabla_\mu T^{\mu\nu}}})\, n_\nu + T^{\mu\nu} \nabla_\mu n_\nu$

$= \frac{1}{2} T^{\mu\nu} (\underbrace{\nabla_\mu n_\nu + \nabla_\nu n_\mu}_{\to\, 0 \text{ b/c } n = \partial_t \text{ killing}})$

$= 0.$

$\partial D = C_{0t_1} \cup \Omega_{t=0} \cup \Omega_{t=t_1}$

$\int_D \nabla_\mu J^\mu = 0 \Rightarrow -\int_{\Omega_{t=0}} J^\mu n_\mu + \int_{\Omega_{t=t_1}} J^\mu n_\mu + \underbrace{\int_{C_{0t_1}} J^\mu \ell_\mu}_{\text{Flux out of cone}} = 0$

Since $J^\mu n_\mu = T^{\mu\nu} n_\mu n_\nu = T(n,n) = T(\partial_t, \partial_t)$ $\Rightarrow -E_{\Omega_{t=0}} + E_{\Omega_{t=t_1}} + \int_{C_{0t_1}} J(\ell) = 0$

$\Rightarrow E_{\Omega_{t=t_1}} = E_{\Omega_{t=0}} - \int_{C_{0t_1}} J(\ell)$ $\to$ remains to show $\int_{C_{0t_1}} J(\ell) \geq 0$

Claim: $T(n, \ell) = J(\ell) \geq 0$ $\Rightarrow$ "dominant energy condition"

Pf | $T(n, \ell) = \frac{1}{2} T(\ell + \bar{\ell}, \ell) = \frac{1}{2} [T(\ell, \ell) + T(\bar{\ell}, \ell)]$

$\ell(\varphi) = \ell^\mu \partial_\mu \varphi = \hat{\eta}(\ell, \partial\varphi)$

$\partial\varphi := \hat{\eta}^{\mu\nu} \partial_\nu \varphi\, \partial_\mu \in TM \Rightarrow \partial\varphi = -\frac{1}{2} \ell(\varphi) \bar{\ell} - \frac{1}{2} \bar{\ell}(\varphi) \ell + e_A(\varphi) e_A$

$\Rightarrow T(\ell, \ell) = \ell(\varphi) \ell(\varphi)$ (b/c $\hat{\eta}(\ell, \ell) = 0$)

Recall $T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} g(\partial\varphi, \partial\varphi)$

$T(\ell, \bar{\ell}) = \ell(\varphi) \bar{\ell}(\varphi) - \hat{\eta}(\ell, \bar{\ell})\, \hat{\eta}(\partial\varphi, \partial\varphi) = \ell(\varphi) \bar{\ell}(\varphi) + \hat{\eta}(\partial\varphi, \partial\varphi)$

$\hat{\eta}(\partial\varphi, \partial\varphi) = \hat{\eta}(-\frac{1}{2} \ell(\varphi) \bar{\ell} - \frac{1}{2} \bar{\ell}(\varphi) \ell + e_A(\varphi) e_A,\ -\frac{1}{2} \ell(\varphi) \bar{\ell} - \frac{1}{2} \bar{\ell}(\varphi) \ell + e_B(\varphi) e_B)$

$= \frac{1}{4} \ell(\varphi) \bar{\ell}(\varphi)\, \hat{\eta}(\ell, \bar{\ell}) + \frac{1}{4} \ell(\varphi) \bar{\ell}(\varphi)\, \hat{\eta}(\bar{\ell}, \ell) + e_A(\varphi) e_B(\varphi)\, \hat{\eta}(e_A, e_B)$

$= -\ell(\varphi) \bar{\ell}(\varphi) + |e_A(\varphi)|^2$

$\Rightarrow T(\ell, \bar{\ell}) = |e_A(\varphi)|^2$

$\Rightarrow T(n, \ell) = \frac{1}{2} \Big[\underbrace{|\ell(\varphi)|^2}_{\text{Flux along cone}} + \underbrace{|e_A(\varphi)|^2}_{\text{Flux across cone (out of cone)}} \Big] \geq 0$ ☒

$\Rightarrow E_{\Omega_{t_1}} = E_{\Omega_{t_0}} - \int_{C_{0t_1}} (|\ell(\varphi)|^2 + |e_A(\varphi)|^2) \Rightarrow E_{\Omega_{t_1}} \leq E_{\Omega_{t_0}}$ ☒

EoM: $\hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu \varphi = 0 \Rightarrow \Box \varphi = 0 \longleftrightarrow T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu}\, \hat{g}(\partial\varphi, \partial\varphi)$

↳ any different op. would give different $T_{\mu\nu}$ and above th^{m} would not work

$E_{\Omega_t} = \int_{\Omega_t} T(n,n)\, \mu_\eta|_{\Omega_t}$ with $T(n,n) = n(\varphi)n(\varphi) - \frac{1}{2}\eta(n,n)\, \eta(\partial\varphi, \partial\varphi)$

$$= |\partial_t\varphi|^2 + \frac{1}{2}\left(-(\partial_t\varphi)^2 + \Sigma_{i=1}^3 (\partial_i\varphi)^2\right)$$

$$= \frac{1}{2}\left((\partial_t\varphi)^2 + \Sigma_{i=1}^3 (\partial_i\varphi)^2\right)$$

$$\Rightarrow E_{\Omega_t} = \frac{1}{2}\int (\partial_t\varphi)^2 + \Sigma_{i=1}^3 (\partial_i\varphi)^2\, \mu_\eta$$
$$= \frac{1}{2}\left(\|\partial_t\varphi\|^2_{L^2(\Omega_t)} + \|\nabla\varphi\|^2_{L^2(\Omega_t)}\right)$$

$\Rightarrow E_{\Omega_t}$ controls $L^2 \times H^1$ norm of $(\partial_t\varphi, \varphi)$

Banach space of the sol^n of $\Box\varphi = 0$ is $(L^2 \times H^1)$

analogous to existence of geodesic equation $\Rightarrow$ need norm to be finite for all time to do contraction mapping.

$\Rightarrow$ Singularity in φ field $\Rightarrow$ blow up of $L^2 \times H^1$ norm of $(\partial_t\varphi, \varphi)$.

$\Rightarrow$ b/c of bound $E_{\Omega_t} \le E_{\Omega_{t_0}}$ (initial finite data), on Minkowski space, there cannot be blow up in finite time

$\Rightarrow$ no singularity in Minkowski

$\hat{g} = -N^2 dt^2 + g_{ij}(dx^i + x^i dt)(dx^j + x^j dt)$

Wave op. can be written as

$$\hat{g}^{\mu\nu}\nabla_\mu\nabla_\nu = \hat{g}^{00}\partial_t^2 + \hat{g}^{ij}\partial_i\partial_j + (\text{first derivatives})$$
$$= -\frac{1}{N^2}\partial_t^2 + \hat{g}^{ij}\partial_i\partial_j + (\text{first derivs})$$

Claim: $Ric_{ij} = -g^{k\ell}\partial_k\partial_\ell g_{ij} + \underbrace{\overbrace{(\partial\Gamma - \partial\Gamma)}^{\partial(g^{-1}(\partial g + \cdots)),\ \text{not of form in wave op.}}}_{(\text{Bad})} + (\text{terms of type } (\partial g)^2)$

$$\left.\begin{aligned}\partial_t g_{ij} &= -2N k_{ij} + \cdots \\ \partial_t k_{ij} &= N\, Ric_{ij} + (k)^2 + \cdots\end{aligned}\right\} \Rightarrow \partial_t^2 g_{ij} = -2N\,\partial_t k_{ij} + \cdots = -2N^2\left(-\tfrac{1}{2}g^{k\ell}\partial_k\partial_\ell g_{ij} + (\text{Bad}) + (\partial g)^2\right)$$

$$\Rightarrow \underbrace{\frac{1}{N^2}\partial_t^2 g_{ij} - g^{k\ell}\partial_k\partial_\ell g_{ij}}_{\text{wave op. form}} = (\text{Bad}) + (\partial g)^2 + \cdots + k^2$$

Claim: Gravity does satisfy a wave eq^n. modulo a gauge choice

Above working only on M. Descending from $\underbrace{M \to M/\mathcal{D}_0}_{\text{gauge fixing}}$ cancels out (Bad)

$$\frac{1}{N^2}\partial_t^2 g_{ij} = g^{k\ell}\partial_k\partial_\ell g_{ij} + (\partial g)^2 + \underbrace{(k)^2}_{\sim (\partial_t g)^2}$$

Consider $\partial_\ell g = 0$ solution:

$$\frac{1}{N^2}\partial_t^2 g_{ij} = (\partial_t g_{ij})^2 + \cdots \xRightarrow{x = \partial_t g_{ij}} d_t x = x^2 \Rightarrow \int_{x_0}^{x} \frac{1}{x^2}dx = \int_0^t dt' \Rightarrow \frac{1}{x_0} - \frac{1}{x} = t$$

$$\Rightarrow x = \frac{x_0}{1 - x_0 t}$$

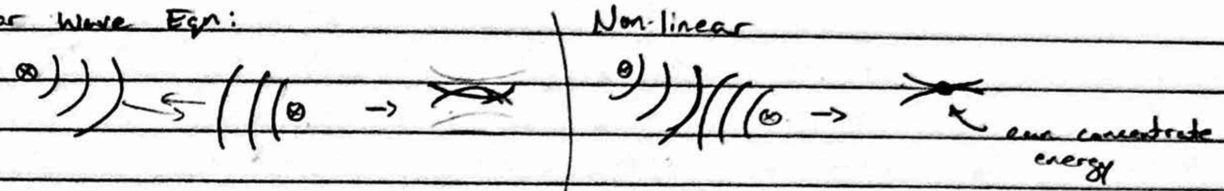
$\Rightarrow$ blow up at $t = 1/x_0 < \infty$

$\Rightarrow$ have singularities

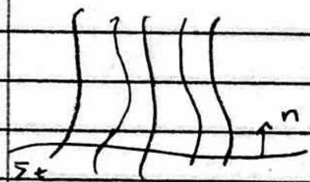
Such singularities are "naked singularities"

$\nexists$ relation to QFT divergences

Linear Wave Eqn: | Non-linear



Kinematics and Dynamics of Timelike Geodesics



consider $n \perp \Sigma_t$ is a timelike geodesic

$\hat{\nabla}_n n = 0 \Rightarrow n^\mu \hat{\nabla}_\mu n^\nu = 0$

$\hat{g}(n,n) = \hat{g}_{\mu\nu} n^\mu n^\nu = n_\mu n^\mu = -1$

describes any mode of deformation

Quantity $\hat{\nabla}_\alpha n_\beta = (\nabla n)(\partial_\alpha, \partial_\beta)$ measures how timelike geodesic changes when moving along Σ_t.

Consider elastic body. Can deform by: (1) compress
(2) twist by rotation
(3) sheer

↳ can split $\underbrace{\hat{\nabla}_\alpha n_\beta}_{\text{"deformation tensor"}}$ into same modes of deformations

Claim: $n^\beta \hat{\nabla}_\alpha n_\beta = 0$

Pf| $n_\mu n^\mu = -1 \to n_\mu \hat{\nabla}_\alpha n^\mu = 0$

geodesic bundle

$\Rightarrow$ can project $\hat{\nabla}_\alpha n_\beta$ on Σ_t to understand deformations

↳ project $\hat{\nabla}_\alpha n_\beta$ onto Σ_t and decompose into fundamental modes

Need Π : Sections $(TM \otimes \cdots) \to$ Sections $(T\Sigma \otimes \cdots)$ with $\Pi^2 = \Pi$.

Claim: Π : sections $(TM \otimes \cdots) \to$ sections $(T\Sigma \otimes \cdots)$ given by $\Pi^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$ is a projection operator

Pf| WTS: $\Pi^2 = \Pi$: $\Pi^\mu{}_\alpha \Pi^\alpha{}_\nu = (\delta^\mu{}_\alpha + n^\mu n_\alpha)(\delta^\alpha{}_\nu + n^\alpha n_\nu)$

$= \delta^\mu{}_\nu + n^\mu n_\nu + \cancel{n^\mu n_\nu} + \cancel{n^\mu n_\nu \overbrace{n^\alpha n_\alpha}^{-1}} = \Pi^\mu{}_\nu$ ✓

$\Pi(n) = 0$: $\Pi^\mu{}_\nu n^\nu = (\delta^\mu{}_\nu + n^\mu n_\nu) n^\nu = \cancel{n^\mu} + \cancel{n^\mu \underbrace{n_\nu n^\nu}_{-1}} = 0$ ✓

∎

Define $\Pi(\underset{TM \otimes TM}{\hat{\nabla}_\alpha n_\beta}) := \Pi^\alpha{}_\mu \Pi^\beta{}_\nu \hat{\nabla}_\alpha n_\beta$ ⊛

Claim: ⊛ is a well-defined projection of ∇n onto $T\Sigma \otimes T\Sigma$.

Pf| $(\delta^\alpha{}_\mu + n^\alpha n_\mu)(\delta^\beta{}_\nu + n^\beta n_\nu)\hat{\nabla}_\alpha n_\beta = (\delta^\alpha{}_\mu + n^\alpha n_\mu)(\hat{\nabla}_\alpha n_\nu + n_\nu \overbrace{\cancel{n^\beta \hat{\nabla}_\alpha n_\beta}}^{0})$

$= \hat{\nabla}_\mu n_\nu + n_\mu n^\alpha \hat{\nabla}_\alpha n_\nu$

Then $\langle \Pi(\hat{\nabla} n), n \rangle = \underbrace{\cancel{n^\nu \hat{\nabla}_\mu n_\nu}}_{0} + \underbrace{\cancel{n_\mu n^\alpha n^\nu \hat{\nabla}_\alpha n_\nu}}_{0} = 0 = \cancel{n^\mu \hat{\nabla}_\mu n_\nu} + \cancel{\underbrace{n^\mu n_\mu}_{-1} n^\alpha \hat{\nabla}_\alpha n_\nu}$ ✓

Rotations → anti-symmetric
Compression → trace
Shear → traceless

(1+3 dim)

Lemma The deformation field $\hat{\nabla}_\alpha n_\beta$ admits the following decomposition: (projection of the ... onto Σ_t)

$$\Pi(\hat{\nabla} n)_{\alpha\beta} = \hat{\nabla}_\alpha n_\beta + n_\alpha n^\mu \hat{\nabla}_\mu n_\beta = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \tfrac{1}{3}\Theta \Pi_{\alpha\beta}$$

where $\omega_{\alpha\beta} = \frac{1}{2}\Pi^\mu{}_\alpha \Pi^\nu{}_\beta (\hat{\nabla}_\mu n_\nu - \hat{\nabla}_\nu n_\mu) \sim \Pi(dn)$ ← "projection of curl"

$\sigma_{\alpha\beta} = \frac{1}{2}\Pi^\mu{}_\alpha \Pi^\nu{}_\beta (\hat{\nabla}_\mu n_\nu + \hat{\nabla}_\nu n_\mu) - \frac{1}{3}\Theta \Pi_{\alpha\beta}$ ← "projection of Lie derivative ~ projection of strain"

$\Theta = \hat{\nabla}_\alpha n^\alpha$ ← divergence = 'controlling volume'

Pf Sketch $\omega_{\alpha\beta} = \frac{1}{2}\Pi^\mu{}_\alpha \Pi^\nu{}_\beta (\hat{\nabla}_\mu n_\nu - \hat{\nabla}_\nu n_\mu)$

$$= \tfrac{1}{2}\left(\hat{\nabla}_\alpha n_\beta + n_\alpha n^\mu \hat{\nabla}_\mu n_\beta - \hat{\nabla}_\beta n_\alpha - n_\beta n^\mu \hat{\nabla}_\mu n_\alpha\right)$$

$$\sigma_{\alpha\beta} = \tfrac{1}{2}(\hat{\nabla}_\alpha n_\beta + n_\alpha n^\mu \hat{\nabla}_\mu n_\beta + \hat{\nabla}_\beta n_\alpha + n_\beta n^\mu \hat{\nabla}_\mu n_\alpha) - \tfrac{1}{3}\hat{\nabla}_\mu n^\mu (g_{\alpha\beta} + n_\alpha n_\beta)$$

→ Sum together and everything cancels

Since ω, σ, Π are parallel to Σ_t: consider only $\omega_{ij} = \omega(\partial_i, \partial_j), \dots$

Now take n as geodesic vector field: $n^\mu \hat{\nabla}_\mu n_\beta = 0$

$n_i = \hat{g}(n, \partial_i) = 0$

$$\Rightarrow \quad \omega_{ij} = \tfrac{1}{2}(\hat{\nabla}_i n_j - \hat{\nabla}_j n_i)$$

$$\sigma_{ij} = \tfrac{1}{2}(\underbrace{\hat{\nabla}_i n_j}_{\langle \hat{\nabla}_{\partial_i} n, \partial_j\rangle = -k_{ij}} + \hat{\nabla}_j n_i) - \tfrac{1}{3}\Theta g_{ij} \qquad \Pi_{ij} = \hat{g}_{ij} = g_{ij}$$

$\Rightarrow \omega_{ij} = \frac{1}{2}(-k_{ij} + k_{ji}) = 0$ b/c k's symmetric ← with geodesic, can't have rotation b/c also orthogonal to hypersurface ⇒ could intersect more than once

$\sigma_{ij} = \frac{1}{2}(-k_{ij} - k_{ji}) - \frac{1}{3}\Theta g_{ij} = -k_{ij} - \frac{1}{3}\Theta g_{ij} = -k_{ij} + \frac{1}{3}g_{ij}\,\mathrm{Tr}_g[k]$

Check
$\Theta = \hat{\nabla}_\alpha n^\alpha$
$= -\mathrm{tr}_g k$

Look at evolution of $\underbrace{\Theta = \hat{\nabla}_\alpha n^\alpha}$ along n

⇒ convergence or divergence of geodesics

$$\Rightarrow \frac{d\Theta}{ds} = n^\mu \hat{\nabla}_\mu \Theta = n^\mu \hat{\nabla}_\mu \hat{\nabla}_\alpha n^\alpha = n^\mu\left[\hat{\nabla}_\alpha \hat{\nabla}_\mu n^\alpha + \hat{R}^\alpha{}_{\beta\mu\alpha} n^\beta\right]$$

s affine param of n

$$= n^\mu \hat{\nabla}_\alpha \hat{\nabla}_\mu n^\alpha - Ric_{\beta\mu} n^\mu n^\beta$$

$$= \hat{\nabla}_\alpha (\cancel{n^\mu \hat{\nabla}_\mu n^\alpha}) - (\hat{\nabla}_\alpha n^\mu)(\hat{\nabla}_\mu n^\alpha) - Ric_{\beta\mu} n^\mu n^\beta$$

$$= -(\hat{\nabla}_\alpha n_\mu)(\hat{\nabla}^\mu n^\alpha) - Ric_{\beta\mu} n^\mu n^\beta$$

Recall $\Pi(\hat{\nabla} n)_{\alpha\beta} = \hat{\nabla}_\alpha n_\beta + \cancel{n_\alpha n^\mu \hat{\nabla}_\mu n_\beta}^{\,0 \text{ b/c geodesic}} = -K_{ij}\,\delta^\alpha{}_i\,\delta^\beta{}_j$

$\sigma = -k^{\text{traceless}} = -k^{tr}$
$\Theta = -\mathrm{tr}_g k$

Thus:

$$\frac{d\Theta}{ds} = -(\sigma_{\alpha\mu} + \tfrac{1}{3}\Theta g_{\alpha\mu})(\sigma^{\mu\alpha} + \tfrac{1}{3}\Theta g^{\mu\alpha}) - Ric_{\beta\mu} n^\mu n^\beta$$

Lemma The expansion $\Theta = \hat{\nabla}_\alpha n^\alpha$ satisfies the following evolution eq^n:

$$\frac{d\Theta}{ds} = -(k^{tr}_{ij} - \tfrac{1}{3}g_{ij}(\mathrm{tr}_g k))(k^{tr\,ij} - \tfrac{1}{3}g^{ij}(\mathrm{tr}_g k)) - \hat{R}_{\mu\nu} n^\mu n^\nu$$

$$= -|k^{tr}|_g^2 - \tfrac{1}{3}(\mathrm{tr}_g k)^2 - \hat{R}ic(n,n)$$

$$= -|k^{tr}|_g^2 - \tfrac{1}{3}\Theta^2 - \hat{R}ic(n,n)$$

Where $|k^{tr}|_g^2 = k^{tr}{}_{ij}\,k^{tr\,ij}$

In vacuum, $\hat{Ric}_{\mu\nu} = 0 \Rightarrow \frac{d\Theta}{ds} = -\frac{\Theta^2}{3} - |k^{tr}|^2_g \leq -\frac{\Theta^2}{3}$

Define $\tilde{\Theta} = -\Theta = tr_g K \Rightarrow \frac{d\tilde{\Theta}}{ds} \geq \frac{\tilde{\Theta}^2}{3} \Rightarrow \int_{\tilde{\Theta}_0}^{\tilde{\Theta}(s)} \frac{d\tilde{\Theta}}{\tilde{\Theta}^2} \geq \int_0^s \frac{ds'}{3} \Rightarrow \tilde{\Theta}(s) \geq \frac{3\tilde{\Theta}_0}{3 - s\tilde{\Theta}_0} \to \infty$ at $s = 3/\tilde{\Theta}_0$

$\Rightarrow$ in finite time, energy concentrates

Intuition: $\underbrace{\int \nabla \cdot E \, dV}_{\sim \# \text{ density}} = \oint E \cdot ds \to$ blowing up $\Rightarrow$ # density blowing up

Lemma In a vacuum spacetime, timelike geodesics converge in a finite time

Pf

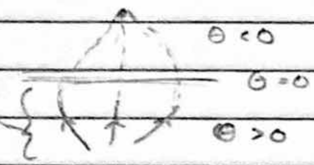
Case 1: If $\tilde{\Theta}_0$ were < 0, then $\frac{3\tilde{\Theta}_0}{3 - s\tilde{\Theta}_0}$ is bounded $\Rightarrow$ regular

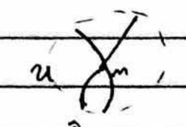
Case 2: $\tilde{\Theta}_0 > 0 \Rightarrow 3 - s\tilde{\Theta}_0$ at $s = 3/\tilde{\Theta}_0 \Rightarrow \tilde{\Theta}(s) \to \infty$ as $s \to 3/\tilde{\Theta}_0$.

↳ $\tilde{\Theta}_0 > 0 \Rightarrow \Theta_0 < 0 \Rightarrow$ geodesics already contracting

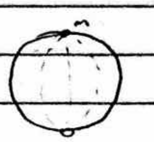
Why $\Theta(0) = \Theta_0 < 0$? Or, how Θ becomes negative?

- Suppose have initial condition w/ expanding geodesics
  - ↳ how does gravity make Θ go from positive to negative?





geodesics don't intersect in $\mathcal{U}$

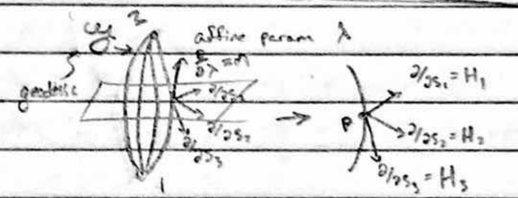


Recall: $\exp_m : T_m M \to \mathcal{U} \subset M$ s.t. $\exp_m$ is a local diffeo

$v \mapsto \exp_m(tv)$ (ie $\exists$ a minimum $d > 0$ s.t. radius of $\mathcal{U}$ is d and $\exp_m$ is diffeo on $\mathcal{U}$)

$d = \inf$ (radius of $\mathcal{U} \ni m$ s.t. $\exp_m$ is diffeo on $\mathcal{U}$)

Def$^{\underline{n}}$ (Jacobi Vector Field) A vector field orthogonal to the the geodesic γ vanishes at 1 & 2 and satisfies the geodesic deviation eq$^{\underline{n}}$. (with 2 the conjugate point of 1)



- Once two geodesics intersect, they are no longer geodesics (no longer unique, local distance maximizers)

- $\{H_1, H_2, H_3\}$ span the tangent space $T_p\Omega$ $(\Omega \subset \Sigma_t)$. Moreover can take $\{H_1, H_2, H_3\}$ to be the basis of deviation vector fields.

WTS: $H_i \to 0$ at 2 $\Rightarrow \Theta \to -\infty$ at 2

↳ $\frac{dH^j_{(i)}}{d\lambda} = n^\lambda \hat{\nabla}_\lambda H^j_{(i)} = H^k_{(i)} \hat{\nabla}_k n^j$

$H^j_{(i)} = \begin{pmatrix} H_1^1 & H_1^2 & H_1^3 \\ \vdots & \vdots & \vdots \end{pmatrix}$

Define $H^k_{(i)} = A_{3\times3}$, $\hat{\nabla}_k n^j = B_{3\times3} \Rightarrow tr(B) = \Theta$

$\Rightarrow \frac{dA}{d\lambda} = AB \Rightarrow B = A^{-1}\frac{dA}{d\lambda}$

$\Rightarrow tr(B) = tr\left(A^{-1}\frac{dA}{d\lambda}\right) = \frac{1}{\det A}\frac{d(\det A)}{d\lambda}$

$\Rightarrow \boxed{\Theta = \frac{d \ln(\det A)}{d\lambda}}$ ⟵ @ $H_{(i)}(2) = 0 \Rightarrow \ln(\det A) = \ln(0) \to -\infty$ ✓

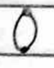
- Have $\left[\frac{\partial}{\partial s_i}, \frac{\partial}{\partial \lambda}\right] = 0 \Rightarrow [H_i, n] = 0$

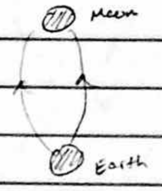
$\Rightarrow \boxed{H^i_{(i)} \hat{\nabla}_i n^k - n^\mu \hat{\nabla}_\mu H^k_{(i)} = 0.}$ ⊛

as well as $\frac{d^2 H_{(i)}}{d\lambda^2} = \text{Riem}(H_{(i)}, n, H_{(i)}, \cdot)$.

- $H_{(i)}(1) = H_{(i)}(2) = 0$.

uniquely defined

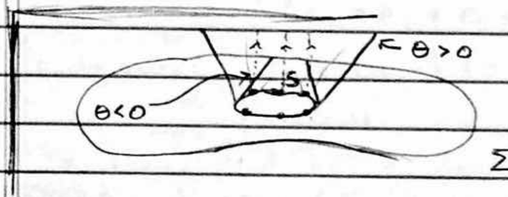
On a spacetime where exponential map is global, cannot have  geodesics, so if $\Theta > 0$ becomes $\Theta < 0$ $\Rightarrow$ geodesics *must* terminate 



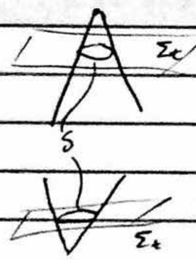
really just a geometric artifact $\to$ nothing deep.

Th$^{\underline{m}}$ (Weak Singularity Th$^{\underline{m}}$) If $\Theta > 0$ becomes $\Theta < 0$ after some p in a globally hyperbolic spacetime, then timelike geodesics must terminate at a finite affine length.

To get from weak (singularity thm) $\to$ strong (singularity thm), replace η by L where $\hat{g}(L,L)=0$, $\hat{\nabla}_L L = 0$



Assume Σ_t spacelike, $S \subset \Sigma_t$ a topological 2-sphere. Have two families of null geodesics (,). Assume S bounds a ball B.



Formation of Trapped Surfaces

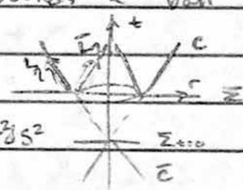
Consider Minkowski space:

↳ in (t,r,θ,φ) coords, $\eta = -dt^2 + dr^2 + r^2 dS^2$

$C = \{t - r = 0\}$, $\bar{C} = \{t + r = 0\}$

↳ outgoing null cones are the level sets of $u: t-r$

incoming " " " " $\bar{u}: t+r$



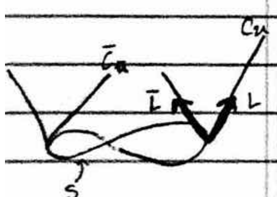
Claim: on $(\mathbb{R}^{1,3}, \eta)$, $L_\eta := -\eta^{\mu\nu}\partial_\mu u \, \partial_\nu$, $\bar{L}_\eta := -\eta^{\mu\nu}\partial_\mu \bar{u} \, \partial_\nu$ are null vector fields tangent (normal) to C and $\bar{C}$, respectively.

Pf $L_\eta := -(\eta^{tt}\partial_t u)\partial_t - (\eta^{rr}\partial_r u)\partial_r = \partial_t + \partial_r$ $\quad \eta(L_\eta, L_\eta) = 0$

$\bar{L}_\eta = \partial_t - \partial_r$ $\quad \Rightarrow \eta(\bar{L}_\eta, \bar{L}_\eta) = 0$ ∎

B/c $L, \bar{L}$ null, $\eta^{\mu\nu}\partial_\mu u \, \partial_\nu u = 0$ and $\eta^{\mu\nu}\partial_\mu \bar{u} \, \partial_\nu \bar{u} = 0$ $\to$ "Eikonal eq$^{\underline{n}}$"

$\Rightarrow$ **Demand:** Find $u: M \to \mathbb{R}$ and $\bar{u}: M \to \mathbb{R}$ s.t. $\hat{g}^{\mu\nu}\partial_\mu u \, \partial_\nu u = 0$, $\hat{g}^{\mu\nu}\partial_\mu \bar{u} \, \partial_\nu \bar{u} = 0$



Claim: on $(M, \hat{g})$, $L := -\hat{g}^{\mu\nu}\partial_\mu u \, \partial_\nu$, $\bar{L} := -\hat{g}^{\mu\nu}\partial_\mu \bar{u} \, \partial_\nu$ are null

Lemma L and $\bar{L}$ are null geodesic generators of C_u and $\bar{C}_{\bar{u}}$

Pf Want to show $\hat{\nabla}_L L = 0$: Consider α component $\hat{\nabla}_L L^\alpha = 0$

$\hat{\nabla}_{-\hat{g}^{\mu\nu}\partial_\mu u \partial_\nu} L^\alpha = 0$

$\Rightarrow -\hat{g}^{\mu\nu}\partial_\mu u \, \hat{\nabla}_\nu(-g^{\alpha\beta}\partial_\beta u) = 0$

$\Rightarrow (\hat{\nabla}^\nu u)\hat{\nabla}_\nu \hat{\nabla}^\alpha u = 0$

$\hat{g}(L,L) = 0$

$\Rightarrow \hat{g}^{\mu\nu}\partial_\mu u \, \partial_\nu u = 0$ (apply $\hat{\nabla}_\alpha$)

$\Rightarrow \hat{\nabla}^\nu u \, \partial_\nu u = 0 \Rightarrow (\hat{\nabla}_\alpha \hat{\nabla}^\nu u)\partial_\nu u + (\hat{\nabla}^\nu u)(\hat{\nabla}_\alpha \partial_\nu u)$

$\Rightarrow 2(\hat{\nabla}^\nu \nabla_\alpha u)\partial_\nu u = 0$ ∎

In Minkowski space, took $e_1, e_2 \perp$ to $L_\eta, \bar{L}_\eta$ and tangential to S

$\Rightarrow$ On general spacetime, construct a frame $(L, \bar{L}, e_1, e_2)$ s.t.

$$\hat{g}(L, e_i) = \hat{g}(\bar{L}, e_i) = 0, \quad \hat{g}(e_i, e_j) = \gamma_{ij} = \delta_{ij}$$ $\rightarrow$ but can't fix $\hat{g}(L, \bar{L}) = -1$ b/c norm of $L, \bar{L}$ fixed by geodesic eq^n

can flow along null hypersurface and foliate spacetime in this way. (similar to ADM w/ flowing along a time vector field)

Strategy: Find (e_3, e_4, e_1, e_2) satisfying:

$$\hat{g}(e_3, e_4) = -2, \quad \hat{g}(e_3, e_3) = 0, \quad \hat{g}(e_4, e_4) = 0$$

$$\Rightarrow e_3 = \Omega(x)\bar{L}, \quad e_4 = \Omega(x) L \qquad \text{"}\Omega \equiv \text{null lapse"}$$

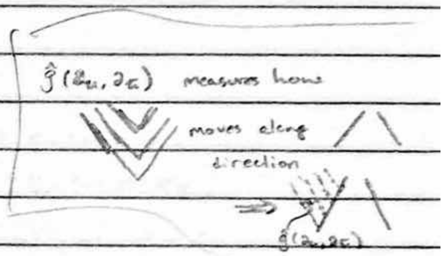
Fact: $$\hat{g} = -\tfrac{1}{2}(e_3 \otimes e_4 + e_4 \otimes e_3) + \overbrace{(e_1 \otimes e_1 + e_2 \otimes e_2)}^{\gamma}$$

$\hookrightarrow$ check: $g(e_3, e_4) = -\tfrac{1}{2}(0 + (-2)(-2)) = -2$ ✓

$L = -\partial_\mu u \, dx^\mu = -du$

To flow along vector field, need affine parameters of e_3, e_4:

$$\hat{g} = -\tfrac{1}{2}\Omega^2(x)(\bar{L} \otimes L + L \otimes \bar{L}) + \gamma$$
$$= -\tfrac{1}{2}\Omega^2(x)(d\bar{u} \otimes du + du \otimes d\bar{u}) + \gamma$$



Know that $e_3 \sim \frac{\partial}{\partial \bar{u}}$, $e_4 \sim \frac{\partial}{\partial u}$ $\rightarrow$

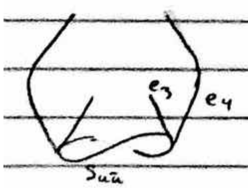
$$\hat{g}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial \bar{u}}\right) = -\tfrac{1}{2}\Omega^2 \Rightarrow \hat{g}\left(\frac{1}{\Omega}\frac{\partial}{\partial u}, \frac{1}{\Omega}\frac{\partial}{\partial \bar{u}}\right) = -\tfrac{1}{2}$$
$$\Rightarrow \hat{g}\left(\frac{2}{\Omega}\frac{\partial}{\partial u}, \frac{2}{\Omega}\frac{\partial}{\partial \bar{u}}\right) = -2 \Rightarrow \boxed{e_3 = \frac{2}{\Omega}\frac{\partial}{\partial \bar{u}}, \quad e_4 = \frac{2}{\Omega}\frac{\partial}{\partial u}}$$

Have two ways to evolve S – flow along e_3 or along e_4 $\Rightarrow$ need two 2^{nd} fundamental forms

Let $\{e_A\}_{A=1,2}$ be horizontal vector fields ($\in TS_{u\bar{u}}$), (e_3, e_4) transverse vec. fields, $\hat{\nabla}$ spacetime cov. deriv, ∇ $S_{u\bar{u}}$ cov. deriv

$$\hookrightarrow TM \ni \hat{\nabla}_{e_A} e_B = \nabla_{e_A} e_B - \tfrac{1}{2}\langle \hat{\nabla}_{e_A} e_B, e_3 \rangle e_4 - \tfrac{1}{2}\langle \hat{\nabla}_{e_A} e_B, e_4 \rangle e_3$$
$$= \nabla_{e_A} e_B + \tfrac{1}{2}\underbrace{\langle \hat{\nabla}_{e_A} e_3, e_B \rangle}_{:= \bar{\chi}_{AB}} e_4 + \tfrac{1}{2}\underbrace{\langle \hat{\nabla}_{e_A} e_4, e_B \rangle}_{:= \chi_{AB}} e_3$$
$$= \nabla_{e_A} e_B + \tfrac{1}{2}\bar{\chi}_{AB} e_4 + \tfrac{1}{2}\chi_{AB} e_3.$$

$\leftarrow$ null 2^{nd} fundamental forms



Lemma Evolution of area of $S_{u\bar{u}}$ along e_4 is controlled by the trace of χ_{AB}.

$$\chi_{AB} = \chi^{tr}_{AB} + \tfrac{1}{2}\gamma_{AB}(\operatorname{tr}\chi) \quad \text{where} \quad \gamma^{AB}\chi^{tr}_{AB} = 0$$

same for $e_3, \bar{\chi}$

Pf $$\frac{d}{d\bar{u}}\int_{S_{u\bar{u}}} \mu_\gamma = \int_{S_{u\bar{u}}} \frac{d}{d\bar{u}}\mu_\gamma = \tfrac{1}{2}\int \tfrac{1}{2}\gamma^{AB}\frac{d}{d\bar{u}}\gamma_{AB}\,\mu_\gamma = \tfrac{1}{4}\int \gamma^{AB}\frac{d}{d\bar{u}}\langle e_A, e_B\rangle_{\hat{g}}\,\mu_\gamma$$
$$= \tfrac{1}{4}\int \gamma^{AB}\frac{\Omega}{2} e_4 \langle e_A, e_B\rangle_{\hat{g}}\,\mu_\gamma = \tfrac{1}{8}\int \gamma^{AB}\Omega\left(\langle \hat{\nabla}_{e_4} e_A, e_B\rangle_{\hat{g}} + \langle e_A, \hat{\nabla}_{e_4} e_B\rangle_{\hat{g}}\right)\mu_\gamma$$
$$= \tfrac{1}{8}\int \gamma^{AB}\Omega\left(\langle \hat{\nabla}_{e_A} e_4 + [e_4, e_A], e_B\rangle + (A \leftrightarrow B)\right)\mu_\gamma = \tfrac{1}{8}\int \gamma^{AB}\Omega\left(\langle \hat{\nabla}_{e_A} e_4, e_B\rangle + (A \leftrightarrow B)\right)\mu_\gamma$$

$= (e_A \ln\Omega) e_4$ from next page $\Rightarrow \langle e_4, e_B\rangle_{\hat{g}} = 0$ (also $\Rightarrow \chi_{AB} = \chi_{BA}$)

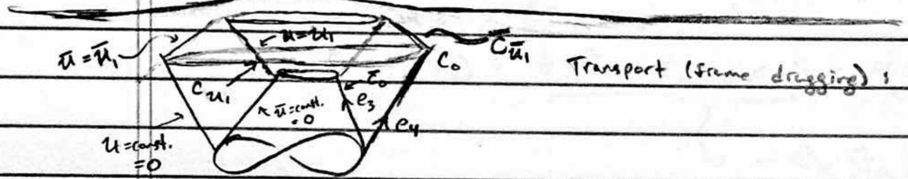
$$= \tfrac{1}{4}\int \gamma^{AB}\Omega\,\chi_{AB}\,\mu_\gamma = \tfrac{1}{4}\int \underset{\sim}{\Omega}(\operatorname{tr}\chi)\,\mu_\gamma$$

$\Omega > 0 \Rightarrow \operatorname{tr}\chi$ controls evolution of area ∎

To have uniqueness, need double null foliation (not just one null cone).

① Flow of frames: given a frame on C_0 or $\bar{C}_0$, want to evolve the frame along e_3 or e_4 s.t. they remain orthonormal frame ("frame dragging")
↳ by their geodesic nature, (e_3, e_4) remain fixed
↳ want to drag (e_1, e_2).

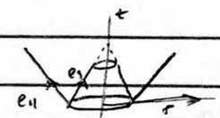
$\Rightarrow$ 1. drag $\{e_A\}_{A=1}^{2}$ along e_3 or e_4 s.t. $\mathcal{L}_{\Omega e_4} e_A = \mathcal{L}_{\Omega e_3} e_A = 0$.
$\Rightarrow [\Omega e_4, e_A] = 0 \Rightarrow \Omega [e_4, e_A] - e_A(\Omega) e_4 = 0$
$\Rightarrow [e_4, e_A] = e_A(\ln \Omega)\, e_4$
↳ integrable $\to$ have C_0, $\bar{C}_0$ submflds



Transport (frame dragging): $[e_4, e_A] = e_A(\ln \Omega)\, e_4$
$[e_3, e_A] = e_A(\ln \Omega)\, e_3$.

Have $\frac{d}{d\bar{u}} A_S = \frac{1}{4} \int_{S_{u\bar{u}}} \Omega \operatorname{tr} \bar{\chi}\ \mu_\gamma$, $\frac{d}{du} A_S = \frac{1}{4} \int_{S_{u\bar{u}}} \Omega \operatorname{tr} \chi\ \mu_\gamma$.

On $(\mathbb{R}^{1+3}, \eta)$:

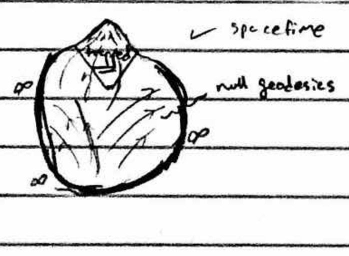


$\Rightarrow \operatorname{tr} \chi = \frac{2}{r} \implies A_S \to \infty$
$\operatorname{tr} \bar{\chi} = -\frac{2}{r}$ $\quad A_S \to 0$

Suppose $\frac{d}{d\bar{u}} A_S < 0 \Rightarrow$

Defn | A topological 2-sphere $S_{u\bar{u}}$ is called trapped if for every measurable subset $\mathcal{U} \subset S_{u\bar{u}}$, then $\operatorname{vol}(\mathcal{U})$ decreases in the outward null direction (i.e. e_4 or $\frac{d}{d\bar{u}}$ direction)
↳ i.e. $\operatorname{tr} \chi < 0$ point-wise.

$S_{u\bar{u}}$ on which $\operatorname{tr} \chi = 0$ is called "apparent horizon"



Defn (black hole) Let M be a spacetime...
A black hole region is the compliment of the past of the null geodesics that have ∞ affine lengths

Define connection coefficients: $\underline{\eta} = \langle \hat\nabla_{e_4} e_A, e_3 \rangle$, $\eta = \langle \hat\nabla_{e_3} e_A, e_4 \rangle$

$\underline{\omega} = \langle \hat\nabla_{e_4} e_3, e_4 \rangle$, $\omega = \langle \hat\nabla_{e_3} e_4, e_3 \rangle$, ... other coefficients

---

$\chi \in$ symm co-frame bundle

∇ projected ⇝ cov. deriv. onto $S_{u,\bar u}$

$\hookrightarrow (\nabla_4 \chi)(e_A, e_B) = (\hat\nabla_4 \chi)(e_A, e_B)$

$= \hat\nabla_{e_4}(\chi(e_A, e_B)) - \chi(\hat\nabla_4 e_A, e_B) - \chi(e_A, \hat\nabla_4 e_B)$

$= \frac{1}{2} \hat\nabla_4 (\langle \hat\nabla_{e_A} e_4, e_B \rangle) - \chi(\hat\nabla_{e_A} e_4 + [e_4, e_A], e_B) - (A \leftrightarrow B)$

$= \frac{1}{2} \hat\nabla_{e_4} \langle \hat\nabla_{e_A} e_4, e_B \rangle - \underbrace{\chi(\hat\nabla_{e_A} e_4, e_B)} - \chi(\overbrace{e_A(\ln a) e_4}^{0}, e_B) - (A \leftrightarrow B)$

b/c χ horizontal

$\hookrightarrow \hat\nabla_{e_A} e_4 = \underbrace{\langle \hat\nabla_{e_A} e_4, e_C \rangle}_{2\chi_{AC}} e_C - \underbrace{\tfrac{1}{2} \langle \hat\nabla_{e_A} e_4, e_3 \rangle e_4}_{\text{will} \to 0 \text{ b/c } \chi \text{ is horizontal}} - \frac{1}{2} \underbrace{\langle \hat\nabla_{e_A} e_4, e_4 \rangle}_{= \frac{1}{2} \hat\nabla_{e_A} \langle e_4, e_4 \rangle = 0} e_3$

$\Rightarrow \chi(\hat\nabla_{e_A} e_4, e_B) = \chi(2\chi_{AC} e_C, e_B)$

$\Rightarrow (\nabla_4 \chi)(e_A, e_B) = \underbrace{\tfrac{1}{2} \hat\nabla_4 \langle \hat\nabla_{e_A} e_4, e_B \rangle} - 4 \underbrace{\chi_{AC} \chi_{CB}}_{\text{squared-terms} \to \text{finite time blow-up}}$

$\hookrightarrow \sim \hat\nabla_{e_A} \hat\nabla_{e_4} e_4 + \hat R(e_4, e_A, e_4, e_B) + \cdots$

$W(e_4, e_A, e_4, e_B)$ in vacuum

$\boxed{\text{Null Evolution Lemma}}$ $(\nabla_4 \chi)(e_A, e_B) = -\chi_{AC}\chi_{CB} - \hat R(e_4, e_A, e_4, e_B)$ − (terms involving connection coeffs. ω)

$\Rightarrow (\nabla_4 \operatorname{tr} \chi) = -\chi_{AB}\chi_{AB} - 0 + (\text{some } \omega)$

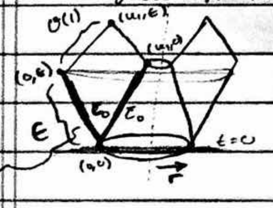
$= -(\hat\chi_{AB} + \frac{1}{2} \operatorname{tr}\chi \, \gamma_{AB})(\hat\chi_{AB} + \frac{1}{2} \operatorname{tr}\chi \, \gamma_{AB})$

$= -|\hat\chi|^2 - \frac{1}{4} \cdot 2 (\operatorname{tr}\chi)^2 + \cdots$

$\Rightarrow \nabla_4 \operatorname{tr}\chi + \frac{1}{2} |\operatorname{tr}\chi|^2 = -|\hat\chi|^2 + E_1$ — hard to control these in full pf.

$\hookrightarrow$ also get $\nabla_3 \hat{\underline\chi} + \frac{1}{2} (\operatorname{tr}\underline\chi) \hat{\underline\chi} = E_2$

Aim: show $\operatorname{tr}\chi < 0$ at some $(u, \bar u)$ given suitable regular initial conditions.



take initial data on C_0 regular w/ $\operatorname{tr}\chi > 0$, $\operatorname{tr}\underline\chi < 0$

Assume data on $\bar C_0$ is Minkowski: $u = \frac{1}{2}(t - r + r_0)$, $\bar u = \frac{1}{2}(t + r - r_0) = 0$

and also assume $u \approx \frac{1}{2}(t - r + r_0)$, $\bar u \approx \frac{1}{2}(t + r - r_0)$

$\Rightarrow r \approx \bar u - u$, $t \approx \bar u + u$

Assume $\operatorname{tr}\underline\chi \approx -\frac{2}{r}$.

Now use $\frac{d}{d\bar u} \operatorname{tr}\chi + \frac{1}{2} |\operatorname{tr}\chi|^2 = -|\hat\chi|^2 + E_1 \Rightarrow \frac{d \operatorname{tr}\chi}{d\bar u} \leq -|\hat\chi|^2 + E_1$

$\frac{d}{du} \hat{\underline\chi} + \frac{1}{2} (\operatorname{tr}\underline\chi) \hat{\underline\chi} = E_2 \Rightarrow \hat{\underline\chi} \frac{d\hat{\underline\chi}}{du} + \frac{1}{2} (\operatorname{tr}\underline\chi) |\hat{\underline\chi}|^2 = E_2 \hat{\underline\chi} \Rightarrow \frac{d|\hat{\underline\chi}|^2}{du} + (\operatorname{tr}\underline\chi)|\hat{\underline\chi}|^2 = \underbrace{2E_2\hat{\underline\chi}}_{F_2}$

$\Rightarrow \operatorname{tr}\chi \leq \operatorname{tr}\chi|_{\bar u = 0} - \int_0^{\bar u} |\hat\chi|^2 \, d\bar u' + F_1$

Study $\frac{d}{du}(r^2 |\hat{\underline\chi}|^2) = r^2 \frac{d|\hat{\underline\chi}|^2}{du} + 2r \frac{dr}{du} |\hat{\underline\chi}|^2 = r^2(-\operatorname{tr}\underline\chi \, |\hat{\underline\chi}|^2 + F_2) + 2r \frac{dr}{du} |\hat{\underline\chi}|^2$

$= r^2 |\hat{\underline\chi}|^2 (-\operatorname{tr}\underline\chi + \frac{2}{r} \frac{dr}{du}) + r^2 F_2 \approx r^2 |\hat{\underline\chi}|^2 (-\operatorname{tr}\underline\chi - \frac{2}{r}) + r^2 F_2 = O(\epsilon)$

$\Rightarrow \frac{d}{du}(r^2 |\hat{\underline\chi}|^2) = O(\epsilon) \Rightarrow r^2(u, \bar u) |\hat{\underline\chi}|^2(u, \bar u) \approx r^2(0, \bar u) |\hat{\underline\chi}|^2(0, \bar u)$

→ cont.

Recall $\hat{\chi} \equiv \chi^{\text{trace free}}$

So $|\hat{\chi}|^2(u,\bar{u}) \approx \frac{r^2(0,\bar{u})}{r^2(u,\bar{u})} |\hat{\chi}|^2(0,\bar{u}) \approx \frac{r_0^2}{r_1^2} |\hat{\chi}|^2(0,\bar{u})$

$\Rightarrow \operatorname{tr}\chi \leq \operatorname{tr}\chi|_{\bar{u}=0} - \frac{r_0^2}{r_1^2}\int_0^{u} |\hat{\chi}|^2(0,\bar{u})\, d\bar{u}'$

$\Rightarrow \int_0^{\bar{u}} |\hat{\chi}|^2(0,\bar{u})\, d\bar{u}' > \frac{r_1^2}{r_0^2} \operatorname{tr}\chi(\bar{u}=0)$ Condition for trapped surface

↳ roughly, $\hat{\chi}$ large enough $\Rightarrow$ form trapped surface ⤳ null cones twisting/ shear is very large 

Full pf need to show E_1, F_2 are small $\to$ big mess

Black Holes

Spherically symmetric spacetimes

Def$^{\underline{n}}$ $(M,\hat{g})$ is spherically symmetric if $\operatorname{Isom}(M) \supset SO(3)$

Ex)

(t,r,θ,φ) coords $\to$ $\hat{g} = -f^2(r,t)\,dt^2 + g^2(r,t)\,dr^2 + h^2(r,t)\, r^2 \overbrace{d\$^2}^{\text{metric on } \$^2} + k^2(r,t)\,dr\,dt$

Def$^{\underline{n}}$ Stationary: there exists a timelike Killing field

Ex) $\hat{g} = -f^2(r)\,dt^2 + g^2(r)\,dr^2 + h^2(r)\, r^2 d\$^2 + k^2(r,t)\,dr\,dt$ ← in static spacetime, lose cross terms

Def$^{\underline{n}}$ Static: there exists a timelike Killing field which is orthogonal to $t=$const. hypersurface

$\Rightarrow 0$ shift

Claim: in the chart (t,r,θ,φ), the metric

$\hat{g} := -\left(1-\frac{2M}{r}\right) dt\otimes dt + \left(1-\frac{2M}{r}\right)^{-1} dr\otimes dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

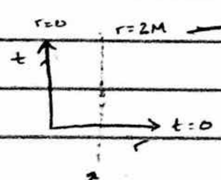
is a static spherically symmetric solution of the vacuum Einstein eq$^{\underline{n}}$ $\hat{R}_{\mu\nu} = 0$

where $M > 0$.

Pf $\underbrace{\cancel{\partial_t g_{ij}}}_{0} = -2N k_{ij} + \underbrace{\cancel{(L_X g)_{ij}}}_{0 \text{ b/c shift } = 0} \Rightarrow k_{ij} = 0$

$\underbrace{\cancel{\partial_t k_{ij}}}_{0} = -\nabla_i\nabla_j N + N(R_{ij} - \underbrace{\cancel{2k_i{}^k k_{kj}}}_{0} + \underbrace{\cancel{(\operatorname{tr}_g k) g_{ij}}}_{0}) + \underbrace{\cancel{(L_X k)_{ij}}}_{0}$

Can simply check $\nabla_i\nabla_j N = N R_{ij}$. ✓



Consider $M/SO(3) \equiv$ 2-dim manifold w/ boundary

seems like singularity at r=2M must come from curvature blowing up

Fact: $\hat{R}_{\mu\nu\alpha\beta}\hat{R}^{\mu\nu\alpha\beta} \sim \mathcal{O}(1)$ at $r=2M$ $\not\Rightarrow$ regularity at $r=2M$

$\hat{R}_{\mu\nu\alpha\beta}\hat{R}^{\mu\nu\alpha\beta} \sim \mathcal{O}(1/r^6)$ at $r=0$. $\Rightarrow$ irregularity at $r=0$

Construct a Riemannian metric from Lorentzian metric: $\mathcal{H} = \hat{g} + 2\, n\otimes n$

$\Rightarrow \mathcal{H}(R,R) \sim \mathcal{O}(1)$ at $r=2M$ $\Rightarrow$ have lower bound now $\Rightarrow$ $r=2M$ 'fake' singularity

Want to find chart where metric is well-behaved at $r=2M$ via real-analytic continuation

Let a null geodesic vector field be $\frac{dx^\mu}{d\lambda}\partial_\mu \Rightarrow g(\dot{x},\dot{x})=0$

$$\Rightarrow 0 = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2$$

$$\Rightarrow dt^2 = \left(1-\frac{2M}{r}\right)^{-2} dr^2$$

$$\Rightarrow dt = \pm \frac{r}{r-2M}\,dr$$

$$\Rightarrow t = \int dt = \pm \int \frac{r-2M+2M}{r-2M}\,dr = \pm\left(r + \int \frac{dr}{r/2M - 1}\right) = \pm\left(r + 2M\ln\left(\frac{r}{2M}-1\right)\right) + C$$

Will foliate by null geodesics

Let $u = t - \left[r + 2M\ln\left(\frac{r}{2M}-1\right)\right]$, $\bar{u} = t + \left[r + 2M\ln\left(\frac{r}{2M}-1\right)\right]$

$= t - r^*$, $= t + r^*$

$$\Rightarrow t = \frac{u+\bar{u}}{2},\quad r^* = \frac{\bar{u}-u}{2} \Rightarrow r + 2M\ln\left(\frac{r}{2M}-1\right) = \frac{\bar{u}-u}{2}$$

$$dr\left(1 + \frac{1}{\frac{r}{2M}-1}\right) = \frac{d\bar{u}-du}{2}$$

$$\frac{r\,dr}{r-2M} = \frac{d\bar{u}-du}{2} \Rightarrow dr = \left(\frac{r-2M}{2r}\right)(d\bar{u}-du)$$

$$\Rightarrow \hat{g}_{SO(3)} = -\frac{\left(1-\frac{2M}{r}\right)}{4}(du+d\bar{u})^2 + \frac{1}{1-\frac{2M}{r}}\left(\frac{r-2M}{2r}\right)^2(d\bar{u}-du)^2$$

$$= -\frac{1}{4}\left(1-\frac{2M}{r}\right)\left[(du+d\bar{u})^2 - (d\bar{u}-du)^2\right]$$

$$= -\left(1-\frac{2M}{r(u,\bar{u})}\right)du\,d\bar{u}$$

↳ Metric now ok, but inverse metric will still be sick at $r=2M$

$$\bar{u}-u = 2r + 4M\ln\left(\frac{r}{2M}-1\right) \Rightarrow \frac{r}{2M}-1 = e^{\frac{\bar{u}-u}{4M}}e^{-\frac{r}{2M}} \Rightarrow 1-\frac{2M}{r} = \frac{2M}{r}e^{-\frac{r}{2M}}e^{\frac{\bar{u}-u}{4M}}$$

$$\Rightarrow \hat{g}_{SO(3)} = -\frac{2M}{r}e^{-\frac{r}{2M}}e^{\frac{\bar{u}-u}{4M}}\,du\,d\bar{u}$$

want to remove $u, \bar{u}$ b/c $\pm\infty$ at $r\to 2M$

$\bar{\mathcal{U}}, \mathcal{U}$ affine params along incoming, outgoing null geodesics

Choose $e^{\bar{u}/4M}\,\bar{u} = \bar{\mathcal{U}}$, $e^{-u/4M}\,u = \mathcal{U}$ as new scaling coords:

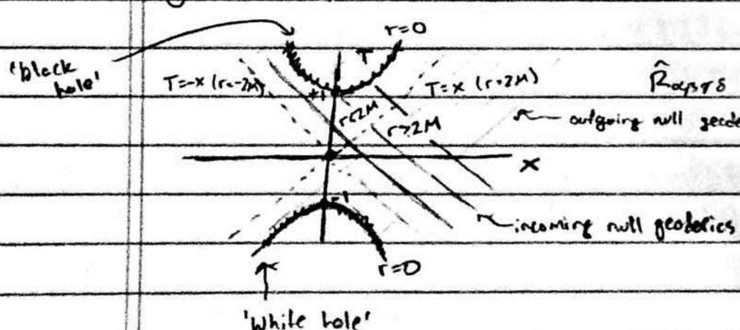
$$\hat{g}^{(1)} = -\frac{32M^3}{r}e^{-\frac{r}{2M}}\,d\mathcal{U}\,d\bar{\mathcal{U}} + r^2 d\mathcal{S}^2,\quad r \text{ function of } \mathcal{U}, \bar{\mathcal{U}}$$

Cor: $\hat{g}^{(1)}$ is regular for all $r>0$.

Claim: $X^2 - T^2 = \left(\frac{r}{2M}-1\right)e^{r/2M}$ by solving earlier eq's and $\mathcal{U} = T-X$, $\bar{\mathcal{U}} = T+X$

$$t/2M = \ln\left|\frac{T-X}{T+X}\right|$$

$$\hat{g}^{(2)} := \frac{32M^3}{r(X,T)}e^{-\frac{r(X,T)}{2M}}\left(-dT^2 + dX^2\right) + r^2 d\mathcal{S}^2$$ ⟿ practically same as Minkowski, except for singularity at origin



$$\hat{R}_{\alpha\beta\gamma\delta}\hat{R}^{\alpha\beta\gamma\delta} = \hat{W}_{\alpha\beta\gamma\delta}\hat{W}^{\alpha\beta\gamma\delta} \sim \mathcal{O}(1/r^6) \text{ @ } r\to 0$$

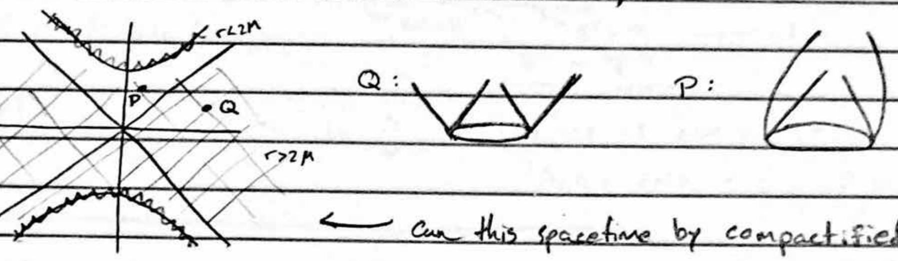
"Kretschen scalar"

⟿ within $r<2M$ region, both incoming + outgoing geodesics are trapped ⇒ black hole

$$\hat{g} := -\frac{16M^3}{r} e^{-r(v,\bar{v})/2M} \left(dv \otimes d\bar{v} + d\bar{v} \otimes dv\right) + r^2 d\mathcal{S}^2$$

$$\Downarrow \quad v \mapsto T - x, \quad \bar{v} \mapsto T + x$$

$$= -\frac{32M^3}{r(T,x)} e^{-r(T,x)/2M} \left(dT \otimes dT - dx \otimes dx\right) + r^2 d\mathcal{S}^2$$



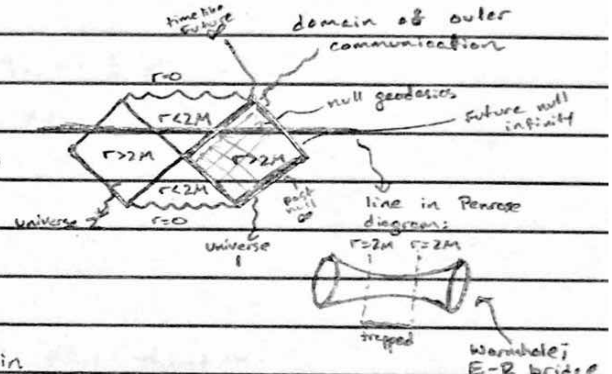
← can this spacetime by compactified : yes

Penrose's Conformal Compactification:

$$\hat{g} = -\frac{32M^3}{r(T,x)} e^{-r(T,x)/2M} (dT^2 - dx^2) + r^2 d\mathcal{S}^2$$

$$= \underbrace{\Omega(\cdots)}_{\text{contains all } \infty\text{'s}} \underbrace{(\text{regular metric})}_{\text{extends over a finite space}} \Rightarrow$$

similar idea to Poincaré disk for $\mathbb{H}^2$

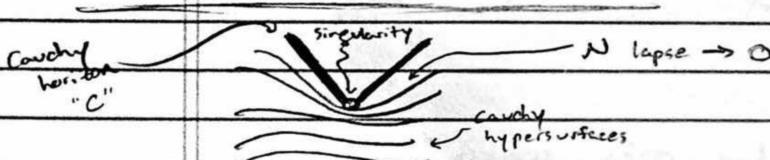


Weak Cosmic Censorship Conjecture

- Naked singularities cannot exist in Einstein gravity (in the presence of physical matter sources)
  - i.e. $T(n,n) \geq 0 \Rightarrow E > 0$

Alternatively stated: all singularities must be hidden behind a horizon

or, future null infinity is complete

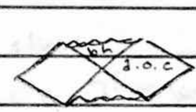


Strong Cosmic Censorship:
Any smooth solution of Einstein's eq$^{\underline{n}}$s cannot be extended beyond C in a suitable sense

Questions

Stability of Schwarzschild black holes:

$$\hat{g} = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\mathcal{S}^2$$



$\|h\|_{H^k} \ll \|\hat{g}\|_{H^k}$

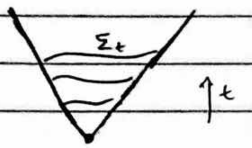
New metric: $g = \hat{g} + h$. with $\text{Ricci}[g] = 0 \Rightarrow$ eq$^{\underline{n}}$ for h.

① Does h remain bounded for all time?

② Does h decay to 0 in suitable sense as $t \to \infty$?

Concern about non-linear evolution eq$^{\underline{n}}$s → even small data can create finite time blowup

"FLRW" solution



Σ_t $\uparrow t$

Claim: A metric of the following type

$$\hat{g} := -dt^2 + a^2(t)\, g_{ij}\, dx^i dx^j \quad \text{with } \lim_{t\to 0} a(t) = 0 \text{ (big bang)},\ \lim_{t\to\infty} a(t) \to \infty \text{ and}$$

where $\underbrace{\mathrm{Ricci}[g]_{ij} = -\tfrac{1}{3} g_{ij}}_{\mathbb{H}^3\times\mathbb{R}},\ \underbrace{0}_{\mathbb{R}^3 \text{ (non-compact) or } \mathbb{T}^3 \text{ (compact)}},\ \underbrace{+\tfrac{1}{3} g_{ij}}_{\mathbb{S}^3\times\mathbb{R} \to \text{unphysical}}$ satisfies Einstein's eq^{n}s with Perfect fluid source

Pf| $N = 1$, $X = 0$ in $\hat{g}$ $\Rightarrow$ get set of diff eqs that can solve from

$$\partial_t g = -2Nk_{ij} + \cdots, \quad \partial_t k_i = \cdots$$

2+1

$$\hat{g} := -dt^2 + a^2(t)\, \gamma_{ij}\, dx^i dx^j, \quad \mathrm{Ricci}[\gamma]_{ij} = -\tfrac{1}{2}\gamma_{ij}$$

$\Rightarrow \mathbb{H}^2/\Gamma$, $\Gamma < PSL\, 2\mathbb{R}$ discrete torsion free co-compact

$\downarrow$

M_g, g genus ≥ 2 (by Gauss-Bonnett)

$\S$

By uniformization, can get M_{-1} structure:

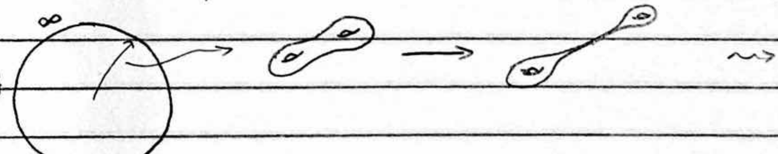
$$M_{-1} : \{\gamma \in M \mid R(\gamma) = -1\}$$

$$\text{Teichmüller space} = M_{-1}/D_0 \cong \mathbb{R}^{6g-6}$$

Thurston Compactification:

∞

TM_g



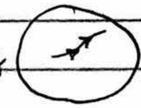
$\rightsquigarrow$ can remove pathological behaviour w/ Thurston compact.

Remarkably, extract $a^2(t)$ and look at γ_{ij} $\rightsquigarrow$ Volume for γ_{ij} finite.

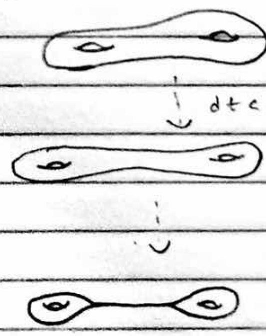
↳ What happens to γ as $t \to 0$

↳ $\gamma(t_i)$ leaves every compact set of TM_g

TM_g



By analyzing Einstein eq^{n}s, can show $t \to 0$ limit of GR is exactly the Thurston boundary of Teichmüller space.



$t = t_0 < \infty$

$dt < 0$

$t \to 0$