The Thurston boundary of Teichmüller space is the space of big bang singularities of 2+1 gravity

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Abstract. We study the asymptotic behaviour of the solution curves of the dynamics of spacetimes of the topological type $\Sigma_g \times \mathbb{R}$, $g > 1$, where Σ_g is a closed Riemann surface of genus g, in the regime of $2+1$ classical general relativity. The configuration space of the gauge fixed dynamics is identified with the Teichmüller space ($\mathcal{T}\Sigma_g \approx \mathbb{R}^{6g-6}$) of Σ_g. Utilizing the properties of the Dirichlet energy of certain harmonic maps, estimates derived from the associated elliptic equations in conjunction with a few standard results of surface theory, we show that every non-trivial solution curve runs off the edge of the Teichmüller space at the limit of the big bang singularity and approaches the space of projective measured laminations/foliations ($\mathcal{PML}\ \mathcal{PMF}$), the Thurston boundary of the Teichmüller space. This result which identifies the complete solution space of the Einstein equations on flat spacetimes of the type $\Sigma_g \times \mathbb{R}$, also yields yet another way to compactify the Teichmüller space.

1. Introduction

2+1 gravity formulated for spacetimes of the type $\Sigma_g \times \mathbb{R}$, where Σ_g is the closed (compact without boundary) Riemann surface of genus $g > 1$, is of considerable interest in mathematical relativity despite the fact that it does not allow for gravitational wave degrees of freedom and as such is devoid of straightforward physical significance. However, it becomes extremely important while studying '$3+1$' gravity on spacetimes of a certain topological type. [9] studied the Einstein's equations for $U(1)$ symmetric vacuum spacetimes with spatial topology being a circle bundle over $\mathbb{S}^2$. Later [6, 7, 8] studied the vacuum Einstein equations for $U(1)$ symmetric spacetimes with spatial topology being circle bundles over higher genus Riemann surfaces ($g > 1$), where $3+1$ gravity is reduced to $2+1$ gravity coupled to a wave map which has the hyperbolic plane as its target space. In addition to these classical analyses, considerable attention has been paid quantum mechanically [4, 5, 10, 11], where $2+1$ gravity is essentially treated as a toy model for $3+1$ quantum gravity.

Despite such physical motivations to study $2+1$ gravity as a tool for studying physically interesting $3+1$ gravity, $2+1$ gravity is itself a mathematically rich topic with several open issues even at the purely classical level. A considerable amount of work has been done on purely classical $2+1$ gravity. Moncrief [1] reduced the Einstein equations in $2+1$ dimensions to a Hamiltonian system over Teichmüller space, where the phase space of the dynamics was identified with the co-tangent bundle of Teichmüller space ($\approx \mathbb{R}^{12g-12}$). Later [12] proved the global existence of the Einstein equations on spacetimes of the topological type $\Sigma_g \times \mathbb{R}, g > 1$ by controlling the Dirichlet energy (a proper function on Teichmüller space) of an associated harmonic map. Moncrief's extensive analysis of $2+1$ gravity (using constant mean curvature spatial harmonic gauge) in [13] led to several classical results of Teichmüller theory, which were obtained by means of purely relativistic/Riemannian geometric analysis. This included, e.g., the homeomorphism between the Teichmüller space and the space of holomorphic quadratic differentials (transverse-traceless tensors in the context of relativity) etc. In the same paper, the term '*Relativistic Teichmüller theory*' was coined. Through studying a Hamilton Jacobi equation whose complete solution determines all the solution curves of the reduced Einstein equations and a Monge-Ampere type equation which allows for a more explicit characterization of these solution curves, he defined a family of ray structures on the Teichmüller space of Σ_g. Studying the behaviour of the associated Dirichlet energy, Moncrief [13] conjectured that *each of these non-trivial solution curves runs off the edge of Teichmüller space at the limit of the big-bang singularity and attaches to the Thurston boundary of the Teichmüller space, that is, the space of projective measured laminations or foliations ($\mathcal{PML}$, $\mathcal{PMF}$).* This, in principle, if it holds true, then classifies the big bang singularities of '$2+1$' gravity as the points on the Thurston boundary and serves as another means to compactify Teichmüller space.

[14] studied the spacetimes of simplicial type (a dense subset in the space of all such spacetimes) in cosmological time gauge and obtained a similar result that the past singularity corresponds to the isometric action of the fundamental group of Σ_g on a certain real tree, in other words, that a point on the Thurston boundary is associated to the initial singularity. Later, based on the work of [14], [15] used barrier arguments to control the constant mean curvature slices relative to the cosmic time ones near the big bang singularities and thereby to show that Thurston boundary points are attained in the limit, by the former as well as the latter. Despite the fact that these results conform to the conjecture of Moncrief to a large extent, they lack direct arguments and also differ in the choice of gauge. Whether this result is gauge invariant is currentsly unknown. Therefore, it is worth proving the conjecture by a direct analysis of the Einstein evolution and constraint equations in CMCSH gauge.

In addition to the general relativistic perspective, M. Wolf [30] established the homeomorphism between the space of holomorphic quadratic differential and the Teichmüller space of Σ_g by utilizing the complex analytic properties such as the Beltrami differential (stretching) of the associated harmonic map. One may naively expect that Wolf's result might be directly applicable to the relativistic case since the transverse-traceless tensor of GR may be associated to a holomorphic quadratic differential. However, in Wolf's case, the domain is kept fixed while the dynamics occurs on the target surface and therefore the available machinery from complex analysis became useful. But, in the relativistic case, the domain (conformal structure) is varying while the target is fixed (an interior point of the Teichmüller space). Therefore, the traditional machinery becomes useless and we are left with tools which are only seemingly accessible through GR.

In this paper, we aim to study the '2 + 1' gravity on vacuum spacetimes of topological type $\Sigma_g \times \mathbb{R}$ in constant mean extrinsic curvature spatial harmonic gauge (CMCSH). Utilizing the direct estimates from the Einstein evolution and constraint equations in conjunction with a few established results from [13] and the theory of Riemann surfaces, we show via a direct argument that indeed Moncrief's conjecture holds true, that is, at the limit of the big-bang singularity, the conformal geometry degenerates and every corresponding non-trivial solution curve attaches to the Thurston boundary. The structure of the paper is as follows. We begin with introducing necessary background for the theory of Riemann surfaces such as harmonic maps, holomorphic quadratic differentials, the associated measured foliations and their transverse measures etc. Then we study the reduced Einstein equations through a conformal technique and obtain the estimates necessary from the associated elliptic PDEs. Finally, we state the relativistic interpretation of the concepts introduced from surface theory and show using the estimates obtained that the conjecture holds true, that is, at the limit of big-bang singularity, every non-trivial solution curve runs off the edge of the Teichmüller space and attaches to the space of projective measured foliations/laminations and exhausts these spaces. We conclude by discussing the potential validity of the conjecture with the inclusion of a cosmological constant and suitable matter sources.

2. Notations and facts

We denote the '2 + 1' spacetime by $\tilde{M}$ with its topology being $\Sigma_g \times \mathbb{R}$. Here, Σ_g is a closed (compact without boundary) Riemann surface with genus $g > 1$. The space of Riemannian metrics on Σ_g is denoted by $\mathcal{M}$ and its closed submanifold $\mathcal{M}_{-1}$ is defined as follows

$$\mathcal{M}_{-1} = \{\gamma \in \mathcal{M} | R(\gamma) = -1\}, \tag{1}$$

where $R(\gamma)$ is the scalar curvature of the metric γ. The space of symmetric 2-tensor fields is denoted by $S^0_2(\Sigma_g)$. The L^2 inner product with respect to the metric $\gamma \in \mathcal{M}$ between any two elements A and B of $S^0_2(\Sigma_g)$ is defined as

$$< A, B >_{L^2} := \int_{\Sigma_g} A_{ij} B_{kl} \gamma^{ik} \gamma^{jl} \mu_\gamma, \tag{2}$$

where $\mu_\gamma = \sqrt{\det(\gamma_{ij})} dx^1 \wedge dx^2$ is the volume form on Σ_g. Abusing notation we will use μ_γ for both $\sqrt{\det(\gamma_{ij})}$ and the volume form. Unless otherwise stated, we will consider an element of $\mathcal{M}$ in isothermal coordinates that is $\mathcal{M} \ni \gamma := e^{\eta(z)}|dz|^2, \eta : \Sigma_g \to \mathbb{R}$. The Laplacian Δ_γ is defined so as to have non-negative spectrum on Σ_g, that is, $\Delta_\gamma := -\gamma^{ij}\nabla_i\nabla_j$. For, $a, b > 0$, $a \leq Cb$ ($\geq$ resp.) for some $\infty > C > 0$ is denoted by $a \lesssim b$ ($a \gtrsim b$ resp.). $C_1 b \leq a \leq C_2 b$, $0 < C_1 < C_2 < \infty$ is denoted by $a \approx b$ (this essentially denotes that one is controlled by the other or two entities are bounded by one another). By a nontrivial element of $\pi_1(\Sigma_g)$, we will always mean a non-trivial closed curve since, there is a one to one correspondence between the homotopy classes of essential (not homotopic to a point or neighbourhood of a puncture) closed curve and the conjugacy classes of non-trivial elements in $\pi_1(\Sigma_g)$. The group of diffeomorphisms (of Σ_g) and its identity component are denoted by $\mathcal{D}$ and $\mathcal{D}_0$, respectively.

3. Background on Teichmüller space

Teichmüller space is studied from an algebraic topologic perspective [17, 16], a complex analytic perspective [18, 16], and a Riemannian geometric perspective[19]. Here, we will focus mainly on the latter as the Teichmüller space while viewed from a Riemannian geometric perspective naturally appears as the configuration space of vacuum Einstein gravity (with or without a positive cosmological constant) on $\Sigma_g \times R$. Nevertheless, we will state the algebraic topologic definition of Teichmüller space and show how this is connected to Einstein gravity. The Teichmüller space of Σ_g is defined as the space of homomorphisms (more accurately the discrete and faithful representations) of the fundamental group of Σ_g into the isometry group of its universal cover that is the hyperbolic plane modulo the action of the isometry group by conjugation. If the Poincaŕe disk model of the hyperbolic plane is assumed, then the Teichmüller space is defined to be

$$\begin{aligned}\mathcal{T}\Sigma_g &:= \rho(\pi_1(\Sigma_g), PSL_2R)/PSL_2Rconj \\ &\subset Hom(\pi_1(\Sigma_g), PSL_2R)/PSL_2Rconj,\end{aligned} \tag{3}$$

where ρ denotes the space of discrete and faithful representations (sometimes representations abusing notation). Here note that $Hom(\pi_1(\Sigma_g), PSL_2R)/PSL_2Rconj$ is also called character variety and $\mathcal{T}\Sigma_g$ is a connected component of this character variety. Dimension of $\mathcal{T}\Sigma_g$ may be calculated as follows. The space of homomorphisms $Hom(\pi_1(\Sigma_g), PSL_2R)$ is moded out by the PSL_2R conjugation so as to remove the base point of the homotopy (at the level of loops). This definition precisely identifies the ways to equip Σ_g with distinct conformal structures (or hyperbolic structures). The fundamental group $\pi_1(\Sigma_g)$ is to be viewed as a discrete and faithful subgroup of PSL_2R and as such is finitely generated ($2g$ generators). The dimension of PSL_2R is 3 and action by conjugation by an element of PSL_2R produces equivalence classes (with respect to gauge transformation in physics terminology). In addition, the generators $(A_i, B_i)_{i=1}^{g}$ satisfy the commutation relation $\prod_{i=1}^{g} A_iB_iA_i^{-1}B_i^{-1} = id$ implying the representation $\rho \in Hom(\pi_1(\Sigma_g), PSL_2R)/PS_2R\ conj$ would satisfy $\prod_{i=1}^{g} \rho(A_i)\rho(B_i)\rho(A_i)^{-1}\rho(B_i)^{-1} = id$ as well. Therefore we lose $3 + 3 = 6$ degrees of freedom out of $2g \times 3 = 6g$ and the dimension of the Teichmuller space turns out to be $6g - 6$. Let us now show how this is related to vacuum Einstein dynamics. The vacuum Einstein equations in $2 + 1$ dimension reads

$$R_{\mu\nu} = 0, \tag{4}$$

where (μ, ν) correspond to the spacetime indices. Now, in $2 + 1$ dimension, vanishing of the Ricci tensor ($R_{\mu\nu}$) implies vanishing of the full Riemann tensor (or the sectional curvature) and therefore, the solutions of the Einstein equations are necessarily the flat spacetimes and consequently are locally isometric to the Minkowski spacetime. Now we are interested in flat spacetimes foliated by Σ_g. In order to obtain the solution space, we therefore need to identify the space of homomorphisms (once again space of discrete and faithful representations to be precise) of $\pi_1(\Sigma_g \times R)$ into the isometry group of the flat spacetimes, which in this case is the full Poincare group $ISO(2,1)$. Now $\pi_1(\Sigma_g \times R) \approx \pi_1(\Sigma_g)$ and therefore the solution space is described as

$$Ein_S = \rho(\pi_1(\Sigma_g), ISO(2,1))/ISO(2,1)conj, \tag{5}$$

where Ein_S is the space of solutions of the equation (4). In the similar way, we may compute the dimension of Ein_S. Note that now the isometry group $ISO(2,1)$ has dimension 6 and therefore following the exact same procedure, we obtain the dimension of Ein_S to be $12g-12$. Therefore, the full solution space is twice the dimension of the Teichmüller space. One immediate guess would be that the co-tangent bundle $T^*\mathcal{T}\Sigma_g$ of the Teichmüller space acts as the full solution space, which is precisely the case as shown in [1, 13]. $T^*\mathcal{T}\Sigma_g$ is indeed the phase space of the reduced dynamics. We will get back to this point in detail later. Let us first develop the concepts of geodesic currents, measured laminations and foliations, which will be required to prove the conjecture.

Let us now introduce a few elementary concepts from the theory of Riemann surfaces. From elementary hyperbolic geometry, we know that there exists a unique geodesic between any two distinct points lying on the boundary of the Poincaŕe disc (in this model of the hyperbolic 2-space). Therefore, we define the set of all un-oriented geodesics on $\tilde{\Sigma}_g$ (lift of Σ_g to its universal cover) as the $\mathbb{Z}_2$ graded double boundary of $\tilde{\Sigma}$ i.e., $G(\tilde{\Sigma}_g)$={The set of all un-oriented geodesics on $\tilde{\Sigma}$} $\approx (\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty - \Delta)/\mathbb{Z}_2$, where Δ represents the diagonal. A geodesic current is a radon measure on $G(\tilde{\Sigma})$ which is invariant under the $\pi_1(\Sigma_g)$ action (see [25, 24] for more details and see [22] for details about radon measures). The property of a radon measure which would be of particular interest to us is that it is locally finite. In a sense, a geodesic currents is essentially an assignment of a radon measure to the open sets of $G(\tilde{\Sigma})$, which remain invariant under the action of the fundamental group $\pi_1(\Sigma_g)$. This $\pi_1(\Sigma_g)$ invariance property of the geodesic currents allows one to define it on the space of geodesics on Σ_g i.e., $G(\Sigma_g) = G(\tilde{\Sigma}_g)/\pi_1(\Sigma_g)$ (note that the action of $\pi_1(\Sigma_g)$ extends continuously to $\partial\tilde{\Sigma}_g$). Now, for a closed hyperbolic surface of genus greater than 1, $\pi_1(\Sigma_g)$ while viewed as a proper discrete subgroup of the isometry group of the hyperbolic plane that is PSL_2R, consists of hyperbolic (also called loxodromic) elements only (see [17, 23] for a detailed classification of the types of isometries of $\mathbb{H}^2$). Each element of $\pi_1(\Sigma_g)$ has an axis geodesic along which it acts by translation and in general it has two fixed points: one attracting, one repelling. Therefore each element of $\pi_1(\Sigma_g)$, a homotopy class of nontrivial loops (rectifiable), has a unique geodesic representative. Whenever we will consider the length of a non-trivial closed curve on Σ_g we will always mean the length of the geodesic in its homotopy class. A geodesic lamination is a closed subset of Σ_g which is the union of disjoint geodesics. A measured lamination is defined as a geodesic lamination equipped with a transverse measure (invariant under translations along the leaves of the lamination). Clearly, the space of measured laminations is a subset of the space of geodesic currents. A geodesic foliation may be thought of as the union of the geodesics which are also integral curves of a vector field. Zeros of the vector field correspond to the singularities of the foliation. One may similarly assign a transverse measure to the foliation promoting it to a measured foliation. There is a natural homeomorphism between the space of measured laminations and measured foliations (via a straightening map; see Fig [1]). This homeomorphism persists at the level of corresponding projective spaces. This projective space (projective measured laminations or foliations) is the Thurston boundary of the Teichmüller space. At this point, it suffices to know this fact and therefore, we do not dwell on this matter further rather provide a small detail in the appendix. Interested readers are referred to the same.

3.1. Homeomorphism between $\mathcal{ML}$, $\mathcal{MF}$, and $\mathcal{QD}$

Let us first define a holomorphic quadratic differential on a Riemann surface Σ_g. A holomorphic quadratic differential is a holomorphic section of the symmetric square of the holomorphic cotangent bundle of Σ_g. It may be defined locally as follows. Let $\{z_a : U_a \to \mathbb{C}\}$ be an atlas for Σ_g. A holomorphic quadratic differential Φ on Σ_g is locally expressible on the chart z_a as $\Phi_a(z_a)dz_a^2$ with the following properties: [1] $\Phi_a : z_a(U_a) \to \mathbb{C}$ is holomorphic, i.e., $\frac{\partial \Phi_a}{\partial \bar{z}_a} = 0$, and [2] $\Phi_a(z_a)(\frac{dz_a}{dz_b})^2 = \Phi_b(z_b)$ for two different overlapping charts $z_a : U_a \to \mathbb{C}$ and $z_b : U_b \to \mathbb{C}$. The second condition precisely states the invariance of Φdz^2 under coordinate transformations. Let us denote the space of holomorphic quadratic differentials on Σ_g by $\mathcal{QD}$. By the famous theorem of Hubbard and Masur [31], there is a homeomorphism between the space of holomorphic quadratic differential $\mathcal{QD}$ and the space of measured foliations $\mathcal{MF}$ on Σ_g. One may simply associate a vertical or horizontal foliation with $\Phi \in \mathcal{QD}$ (up to isotopy and Whitehead moves; see [30] for details about Whitehead moves). For details, the reader is referred to [32]. For now we will only need this homeomorphism property. Given a holomorphic quadratic differential $\Phi(z)dz^2$ in some chart, the transverse measures of a non-trivial element $\mathcal{A}$ of $\pi_1(\Sigma_g)$ (except at the zeros of Φ, which correspond to the singularities of the foliation) with respect to the vertical foliation and the horizontal foliation associated with Φ are defined as follows

$$\mu_{vert}(\mathcal{A}) := \int_{\mathcal{A}} |\mathcal{R}\left(\sqrt{\Phi(z)}dz\right)|, \tag{6}$$

$$\mu_{hor}(\mathcal{A}) := \int_{\mathcal{A}} |\mathcal{I}\left(\sqrt{\Phi(z)}dz\right)|, \tag{7}$$

where $\mathcal{R}$ and $\mathcal{I}$ denote the real and imaginary parts, respectively. We will use these definitions later while considering the Einstein flow on Σ_g exclusively. Given a measured foliation, one may obtain a measured lamination via a suitable straightening map [29, 33] (or collapsing a lamination yields a foliation). Therefore, there is a homeomorphism between $\mathcal{MF}$ and $\mathcal{ML}$. Figure (1) depicts the mechanism of yielding a lamination from a foliation. For our purposes, we will only use the homeomorphism between $\mathcal{QD}$ and

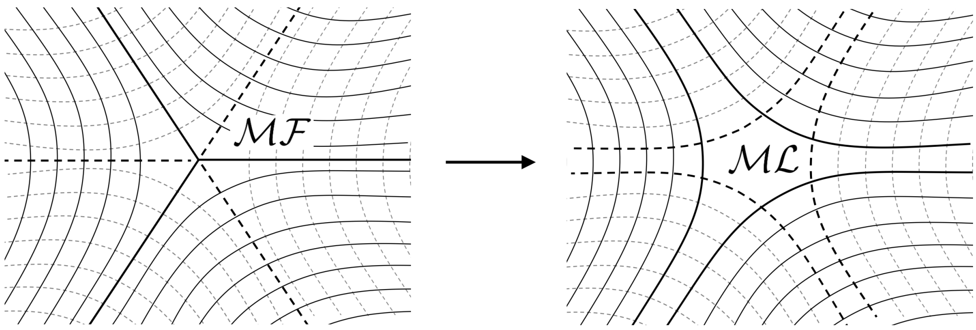

Figure 1. The Straightening map which transforms a singular measured foliation (and the transverse one) to a measured lamination (and respective transverse one).

$\mathcal{MF}$. All of these spaces remain homeomorphic to each other at the level of projective spaces.

3.2. Harmonic Maps

Let us now introduce another essential ingredient of our analysis: the harmonic maps. These will be crucial later in studying the Einsteinian dynamics. Let us consider a map $\mathcal{E} : (M, g) \to (N, \rho)$ (where M and N are considered to be two closed Riemann surfaces) and define the Dirichlet energy

$$E[\mathcal{E}; g, \rho] = \frac{1}{2} \int_M \rho_{\alpha\beta} \frac{\partial \mathcal{E}^\alpha}{\partial x^i} \frac{\partial \mathcal{E}^\beta}{\partial x^j} g^{ij} \mu_g. \tag{8}$$

From the expression of the Dirichlet energy, it is obvious that it only depends on the conformal structure of the domain, that is, a conformal transformation $g_{ij} \mapsto e^{2\delta} g_{ij}, \delta : M \to \mathbb{R}$ leaves E invariant. Harmonic maps are defined to be the critical points of this Dirichlet energy functional in the space of $\mathcal{E}$. The critical points of E may be computed as follows. On the bundle $T^*M \otimes \mathcal{E}_*^{-1}TN$ (while restricted to the image), one has the following connection

$$\nabla_i A^\alpha_j := \partial_i A^\alpha_j +^N \Gamma^\alpha_{\beta\gamma} A^\beta_j \frac{\partial \xi^\gamma}{\partial x^i} -^M \Gamma^k_{ij} A^\beta_k, \tag{9}$$

for $A \in \{\text{sections}(T^*M \otimes \mathcal{E}_*^{-1}TN)\}$. Using this definition of the connection, a few lines of calculation yields the harmonicity condition

$$g^{ij} \partial_i \partial_j \xi^\alpha - g^{ij} \; {}^M\Gamma^k_{ij} \partial_k \xi^\alpha +^N \Gamma^\alpha_{\beta\gamma} \partial_i \xi^\beta \partial_j \xi^\gamma g^{ij} = 0. \tag{10}$$

From [34, 35], we know that there is a harmonic map homotopic to the identity i.e., $\mathcal{E} \in \mathcal{D}_0$ and in fact such a map is an orientation preserving diffeomorphism. If we take $\mathcal{E}$ to be the identity map, then the harmonicity condition reduces to the following

$$-g^{ij} \left(\Gamma[g]^\alpha_{ij} - \Gamma[\rho]^\alpha_{ij} \right) = 0. \tag{11}$$

This condition will be of extreme importance when we fix the spatial gauge of the Einstein equations and also in the later part of the analysis. The Dirichlet energy of this identity map is computed to be

$$E[id; g, \rho] = \frac{1}{2} \int_{\Sigma_g} \rho_{ij} g^{ij} \mu_g. \tag{12}$$

Note that the conformal and diffeomorphism invariance of $E[id; g, \rho]$ allow it to be a function on the Teichmüller space of Σ_g and in particular a proper function (that is the inverse images of the compact sets are compact)[19, 30, 35, 36].

4. Einstein flow on $\Sigma_g \times \mathbb{R}$

We will use the ADM formalism of general relativity in order to obtain a Cauchy problem for '$2+1$' gravity. The ADM formalism of '2+1' gravity splits the spacetime described by a '2+1' dimensional Lorentzian manifold $\tilde{M}$ into $\mathbb{R} \times \Sigma_g$ with each level set $\{t\} \times \Sigma_g$ of the time function t being an orientable 2-manifold diffeomorphic to a Cauchy hypersurface (assuming the spacetime admits a Cauchy hypersurface) and equipped with a Riemannian metric. Such a split may be executed by introducing a lapse function N and shift vector field X belonging to suitable function spaces and defined such that

$$\partial_t = N\hat{n} + X \tag{13}$$

with t and $\hat{n}$ being time and a hypersurface orthogonal future directed timelike unit vector i.e., $\tilde{g}(\hat{n}, \hat{n}) = -1$, respectively. The above splitting writes the spacetime metric $\tilde{g}$ in local coordinates $\{x^\alpha\}_{\alpha=0}^2 = \{t, x^1, x^2\}$ as

$$\tilde{g} = -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt) \tag{14}$$

where $g_{ij} dx^i \otimes dx^j$ is the induced Riemannian metric on Σ_g. In order to describe the embedding of the Cauchy hypersurface Σ_g into the spacetime $\tilde{M}$, one needs the information about how the hypersurface is curved in the ambient spacetime. Thus, one needs the second fundamental form k defined as

$$K_{ij} = -\frac{1}{2N}(\partial_t g_{ij} - (L_X g)_{ij}), \tag{15}$$

the trace of which ($tr_g K = \tau = g^{ij} K_{ij}$, $g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} := g^{-1}$) is the mean extrinsic curvature of Σ_g in $\tilde{M}$ and L denotes the Lie derivative operator. The vacuum Einstein equations

$$R_{\mu\nu}(\tilde{g}) - \frac{1}{2} R(\tilde{g}) \tilde{g}_{\mu\nu} = 0 \tag{16}$$

may now be expressed as the evolution and constraint (Gauss and Codazzi equations) equations of g and k

$$\begin{aligned}
\partial_t g_{ij} &= -2NK_{ij} + (L_X g)_{ij}, && (17)\\
\partial_t K_{ij} &= -\nabla_i \nabla_j N + N(R_{ij} + \tau K_{ij} - 2k_i^k k_{jk}) + (L_X k)_{ij}, && (18)\\
0 &= R(g) - |K|^2 + (tr_g K)^2, &&\\
0 &= \nabla^i K_{ij} - \nabla_j tr_g K. && (19)
\end{aligned}$$

Note that there is no canonical way way to split the spacetimes, that is, the choice of a spacelike hypersurface is not unique. In order to choose a slice and study its evolution under the Einstein flow, we must fix the gauge. In our case, the most convenient choice is the constant mean extrinsic curvature spatial harmonic gauge used by [37]. In this gauge, $\tau = tr_g K$ is constant throughout the hypersurface ($\partial_i \tau = 0$) and therefore is chosen to play the role of time

$$t = monotonic\ function\ of\ \tau. \tag{20}$$

Spatial harmonic gauge is precisely the vanishing of the tension vector field $-g^{ij}\left(\Gamma[g]^k_{ij} - \Gamma[\hat{g}]^k_{ij}\right)$, where $\hat{g}$ is an arbitrary background metric or in other words, the harmonicity of the identity map defined in the previous section. This choice of gauge yields the following two elliptic equations for the lapse function and the shift vector field, respectively

$$\Delta_g N + N(|K^{TT}|^2_g + \frac{\tau^2}{2}) = \partial_t \tau, \tag{21}$$

$$\Delta_g X^i - R^i_j X^j = (\nabla^i N)\tau - 2\nabla^j N K^i_j + (2NK^{jk} - 2\nabla^j X^k) \tag{22}$$
$$(\Gamma[g]^i_{jk} - \hat{\Gamma}[\hat{g}]^i_{jk}).$$

This Cauchy problem (with initial data (g_0, k_0)) with constant mean extrinsic curvature and spatially harmonic gauge is referred to as CMCSH Cauchy problem.

4.1. ***Well-posedness:***

[37] proved a local well posedness theorem for the Cauchy problem for a family of elliptic-hyperbolic systems that included the '$n+1$' dimensional vacuum Einstein equations in CMCSH gauge, $n \geq 2$. They also proved the conservation of gauges and constraints. In addition to the local well-posedness, [12] proved a global existence theorem for the expanding solutions in the same gauge through controlling the Dirichlet energy of an associated harmonic map for any $\tau \in (-\infty, 0)$. Therefore, the well-posedness of the Cauchy problem is established and we do not wish to repeat the same here. Interested readers are referred to these articles.

4.2. ***Reduced Dynamics***

Given a scalar function $\varphi : \Sigma_g \to \mathbb{R}$, we define a set of conformal variables (γ, k^{TT}) (k^{TT} is transverse-traceless with respect to the metric γ) in terms of the physical variables (g, k^{TT}) by setting

$$(g_{ij}, K^{TTij}) = (e^{2\varphi}\gamma_{ij}, e^{-4\varphi}k^{TTij}), \tag{23}$$

where $R(\gamma) = -1$ (the Uniformization theorem guarantees that such γ exists if genus$(\Sigma_g) > 1$)) and the second fundamental form is written as follows

$$K = K^{TT} + \frac{\tau}{2} g, \tag{24}$$

by using the momentum constraint with K^{TT} being transverse-traceless with respect to g. Here k^{TT} is transeverse-traceless with respect to γ, that is,

$$\nabla[\gamma]_j k^{TTij} = 0, \tag{25}$$

$$\gamma_{ij} k^{TTij} = 0, \tag{26}$$

if and only if K^{TT} is transverse-traceless with respect to g. Naturally

$$k^{TT}_{ij} = K^{TT}_{ij}, \tag{27}$$

$$k^{TTij} = \gamma^{ik}\gamma^{jl}k^{TT}_{kl}. \tag{28}$$

φ can be found by solving the Hamiltonian constraint which now takes the form of the following semilinear elliptic PDE namely the Lichnerowicz equation

$$-2\Delta_\gamma\varphi + 1 + e^{-2\varphi}|k^{TT}|^2_\gamma - \frac{e^{2\varphi}\tau^2}{2} = 0. \tag{29}$$

Using the sub and super solution technique [38, 39], it is established that there is a unique solution $\varphi[\gamma, k^{TT}, \tau]$ of the Lichnerowicz equation. Indeed, this equation will be crucial to our analysis towards proving the main theorem. The phase space of the reduced dynamics now may be defined as $\{(\gamma_{ij}, k^{TTij})|\gamma \in \mathcal{M}_{-1}, tr_\gamma k^{TT} = 0 = \nabla[\gamma]_j k^{TTij}\}$. In reality, the true dynamics assumes a metric lying in the orbit space $\mathcal{M}_{-1}/D_0$, D_0 being the group of diffeomorphisms (of Σ_g) isotopic to identity. This is a consequence of the fact that if $\gamma_{ij} \in \mathcal{M}_{-1}, k^{TTij}, \varphi, N$, and X^i solve the Einstein equations, so do $((\phi^{-1})^*\gamma)_{ij}, (\phi_* k^{TT})^{ij}, (\phi^{-1})^*\varphi = \varphi\circ\phi^{-1}, (\phi^{-1})^*N = N\circ\phi^{-1}$, and $(\phi_*X)^i$, where $\phi \in D_0$ and *, and $_*$ denote the pullback and push-forward operations (time independent) on the cotangent and tangent bundles of M, respectively. Let us now consider a time dependent $\phi_t \in \mathcal{D}_0$ and go back to the un-scaled dynamical equation (17) (note that the un-scaled fields (g, K, N, X)) solve the Einstein's dynamical and constraint equations (17-19) iff $(\gamma, k, \varphi, N, X)$ solve the reduced equations)

$$\begin{aligned}
&\partial_t((\phi_t^{-1})^*g)_{ij} = -2(\phi_t^{-1})^*)(NK_{ij}) + (L_{\phi_{t*}X}(\phi_t^{-1})^*g)_{ij}, \\
&(\phi_t^{-1})^*\partial_t g_{ij} + (\partial_t(\phi_t^{-1})^*)g_{ij} = -2(\phi_t^{-1})^*(NK_{ij}) + \frac{\partial}{\partial s}((\phi_t^{-1}\varphi_s^X\phi_t)^*(\phi_t^{-1})^*g)|_{s=0}, \\
&(\phi_t^{-1})^*\partial_t g_{ij} + (\partial_s(\phi_{t+s}^{-1})^*)g_{ij}|_{s=0} = -2(\phi_t^{-1})^*(NK_{ij}) + (\phi_t^{-1})^*(L_X g)_{ij}, \\
&(\phi_t^{-1})^*\partial_t g_{ij} + (\phi_t^{-1})^*(L_Y g)_{ij} = -2(\phi_t^{-1})^*(NK_{ij}) + (\phi_t^{-1})^*(L_X g)_{ij}, \\
&(\phi_t^{-1})^*\left\{\partial_t g_{ij} = -2NK_{ij} + (L_{X-Y}g)_{ij}\right\}.
\end{aligned} \tag{30}$$

Here Y is the vector field associated with the flow ϕ_t and φ_s^X is the flow of the shift vector field X. A similar calculation for the evolution equation for the second fundamental form shows that if we make a trasformation $X \mapsto X + Y$, the Einstein evolution and constraint (due to their natural spatial covariance nature) equations are satisfied by the transformed fields. Action of ϕ_t on the un-scaled fields naturally extends to the conformally scaled fields. Therefore, the true reduced dynamics occurs on the quotient space $\mathcal{M}_{-1}/\mathcal{D}_0$. Now, $\mathcal{M}_{-1}/D_0$ is precisely the Teichmüller space of Σ_g and following [19], the transverse-traceless tensor k^{TT} models the tangent space at γ. Therefore, we obtain the Teichmüller space ($6g-6$ dimensional) $\mathcal{T}\Sigma_g$ as the configuration space, while the cotangent bundle ($12g-12$ dimensional) of $\mathcal{T}\Sigma_g$ serves as the phase space of the reduced dynamics. This is precisely what was stated previously in section 2 while relating the full solution space of the vacuum Einstein equations and the Teichmüller space through its algebraic topologic definition.

Now we will obtain a series of estimates which will be useful for the later analysis. The following lemma provides a point-wise estimate for the conformal factor $e^{2\varphi}$.

Lemma 1: *Let $\varphi:\Sigma_g\to\mathbb{R}$ solves the Lichnerowicz equation (29). Then $e^{2\varphi}:=e^{2\varphi(\tau,k^{TT},\gamma)}$ verifies the following point-wise estimate*

$$\frac{2}{\tau^2}\leq e^{2\varphi}\leq\frac{1+\sqrt{1+2\tau^2\sup_{\Sigma_g}|k^{TT}|^2_\gamma(\tau)}}{\tau^2}\qquad\forall\ \tau\in(-\infty,0). \tag{31}$$

Proof: A standard maximum principle argument while applied to the Lichnerowicz equation (29), yields the following

$$\tau^2e^{4\varphi}-2e^{2\varphi}-2\sup_{\Sigma_g}|k^{TT}|^2_\gamma(\tau)\leq 0. \tag{32}$$

Noting that the discriminant of the quadratic form $\tau^2e^{4\varphi}-2e^{2\varphi}-2\sup_{\Sigma_g}|k^{TT}|^2_\gamma(\tau)$, $4+8\tau^2\sup_{\Sigma_g}|k^{TT}|^2_\gamma(\tau)$, is strictly positive, the inequality is satisfied only for a specific range of $e^{2\varphi}$ i.e.,

$$\left(e^{2\varphi}-\frac{1+\sqrt{1+2\tau^2\sup_{\Sigma_g}|k^{TT}|^2_\gamma(\tau)}}{\tau^2}\right) \tag{33}$$
$$\left(e^{2\varphi}-\frac{1-\sqrt{1+2\tau^2\sup_{\Sigma_g}|k^{TT}|^2_\gamma(\tau)}}{\tau^2}\right)\leq 0.$$

But, $e^{2\varphi}>0$ and therefore, we must have

$$e^{2\varphi}\leq\frac{1+\sqrt{1+2\tau^2\sup_{\Sigma_g}|k^{TT}|^2_\gamma(\tau)}}{\tau^2}. \tag{34}$$

Similarly, at a minimum, the following holds

$$\tau^2e^{4\varphi}-2e^{2\varphi}\geq 0, \tag{35}$$

that is,

$$e^{2\varphi}\geq\frac{2}{\tau^2}, \tag{36}$$

where the equality holds if and only if

$$k^{TT}\equiv 0. \tag{37}$$

In summary, we have the following estimate of the conformal factor from the Lichnerowicz equation

$$\frac{2}{\tau^2}\leq e^{2\varphi}\leq\frac{1+\sqrt{1+2\tau^2\sup_{\Sigma_g}|k^{TT}|^2_\gamma(\tau)}}{\tau^2}, \tag{38}$$

which will be useful later. This concludes the proof of the lemma. □

Now we will obtain an estimate for $|K^{TT}|_g^2 = e^{-4\varphi}|k^{TT}|_\gamma^2$. In order to do so, we first obtain an elliptic equation for $|K^{TT}|_g^2$. The following lemma provides the necessary elliptic equation $|K^{TT}|_g^2$.

Lemma 2: *Let $K := K^{TT} + \frac{1}{2}\operatorname{tr}_g K g$ solves the momentum constraint (19) in CMC gauge $\partial_i\tau := \partial_i \operatorname{tr}_g K = 0$, then $|K^{TT}|_g^2$ satisfies the following quasi-linear elliptic equation on a constant time hypersurface*

$$-\Delta_g(|K^{TT}|_g^2) - 2|K^{TT}|_g^2(|K^{TT}|_g^2 - \frac{1}{2}\tau^2) = 2\nabla[g]_k(K^{TT}_{ij})\nabla[g]^k(K^{TTij}). \quad (39)$$

Proof: Note that in 2 dimensions, the momentum constraint

$$\nabla[g]_j K^j_i - \nabla_i tr_g K = 0 \quad (40)$$

implies that K is a Codazzi tensor [13, 12] i.e.,

$$\nabla[g]_j K^i_k - \nabla[g]_k K^i_j = 0. \quad (41)$$

After substituting the decomposition $K = K^{TT} + \frac{\operatorname{tr}_g K}{2} g$ in the Codazzi equation, Covariant divergence yields

$$\nabla[g]^j\nabla[g]_j K^{TTi}_k - \nabla[g]^j\nabla[g]_k K^{TTi}_j = 0, \quad (42)$$

$$\nabla[g]^j\nabla[g]_j K^{TTi}_k - \nabla[g]_k\nabla[g]_j K^{TTij} - R[g]^i{}_{mjk}K^{TTmj} - R[g]^j{}_{mjk}K^{TTim} = 0,$$

which upon utilizing $\nabla[g]_j K^{TTij} = 0$ and $R[g]^i{}_{mjk} = \frac{R(g)}{2}(\delta^i_j g_{mk} - \delta^i_k g_{mj})$ reduces to

$$\nabla[g]^j\nabla[g]_j K^{TTi}_k = R(g)K^{TTi}_k. \quad (43)$$

$\Delta_g(|K^{TT}|_g^2)$ may be evaluated as follows

$$\begin{aligned}\Delta_g(|K^{TT}|_g^2) &= -\nabla[g]^j\nabla[g]_j|K^{TT}|_g^2 = -\nabla[g]^j\nabla[g]_j(K^{TTi}_k K^{TTk}_i) \quad (44)\\ &= -2(\nabla[g]^j\nabla[g]_j K^{TTi}_k)K^{TTk}_i - 2\nabla[g]^j K^{TTi}_k\nabla[g]_j K^{TTk}_i,\\ &= -2R(g)|K^{TT}|_g^2 - 2\nabla[g]^j K^{TTi}_k\nabla[g]_j K^{TTk}_i,\\ &= -2(|K^{TT}|_g^2 - \frac{\tau^2}{2})|K^{TT}|_g^2 - 2\nabla[g]^j K^{TTi}_k\nabla[g]_j K^{TTk}_i,\end{aligned}$$

i.e.,

$$-\Delta_g(|K^{TT}|_g^2) - 2|K^{TT}|_g^2(|K^{TT}|_g^2 - \frac{1}{2}\tau^2) = 2\nabla[g]_k(K^{TT}_{ij})\nabla[g]^k(K^{TTij}). \quad (45)$$

Here, we have used the Hamiltonian constraint (19) $|K^{TT}|_g^2 = R(g) + \frac{\tau^2}{2}$. This concludes the proof of the lemma. □

Remarkably, the quasi-linear term appearing in the right hand side of the elliptic equation (38) has a favourable sign that is conducive to an application of a standard maximum principle.

Lemma 3: *$|K^{TT}|_g^2$ satisfies the estimate*

$$|K^{TT}|_g^2 \le \frac{\tau^2}{2} \quad (46)$$

for all $\tau \in (-\infty, 0)$.

Proof: The quasi-linear term satisfies $\nabla[g]_k(K^{TT}_{ij})\nabla[g]^k(K^{TTij}) \ge 0$. Application of a standard maximum principle argument yields

$$|K^{TT}|_g^2 \le \frac{\tau^2}{2}. \quad □ \quad (47)$$

Lastly, we will obtain an estimate for the lapse function after choosing the following time coordinate

$$t := -\frac{1}{\tau}. \tag{48}$$

The allowed time range in this coordinate is $(0, \infty)$. The lapse equation (21) now reads

$$\Delta_g N + N(|K^{TT}|_g^2 + \frac{\tau^2}{2}) = \tau^2. \tag{49}$$

Once again, a standard maximum principle argument applied to the lapse equation together with the estimate (47) yields the following estimate of N

$$1 \leq N \leq 2. \tag{50}$$

Now we will describe Moncrief's ray structure [13] of the Teichmüller space, which will be of crucial in obtaining the main result. The ray structure defined by Moncrief is the following equation

$$\begin{aligned}\rho_{ij} &= |K|_g^2 g_{ij} + 2\tau(K_{ij} - \frac{1}{2}\tau g_{ij}) \\ &= (|K^{TT}|_g^2 + \frac{\tau^2}{2})g_{ij} + 2\tau K_{ij}^{TT} \\ &= (e^{-4\varphi}|k^{TT}|_\gamma^2 + \frac{\tau^2}{2})e^{2\varphi}\gamma_{ij} + 2\tau k_{ij}^{TT}\end{aligned} \tag{51}$$

together with an associated Hamilton-Jacobi equation. Here ρ is a fixed metric satisfying $R(\rho) = -1$ (and therefore lies inside the Teichmüller space) and g_{ij} is solved in terms of ρ_{ij}. This computes the end point of a ray in terms of the data along the ray. For the detailed derivation of this expression, one may consult the relevant section of [13]. This is designated in [13] as the '**Gauss**' map equation. For our purpose, the derivation of this map is tangential and hence, we do not wish to repeat the same here. The vital question is whether such (g_{ij}, K^{TTij}, N, X) actually solves the Einstein equations for all τ given an initial $(g_{ij0}, K_0^{TTij}, N_0, X_0)$ satisfying the constraint equations. This is equivalent to solving for conformal variables $(\gamma_{ij}, k^{TTij}, \varphi)$ and associated lapse function N and shift vector field X. This is exactly shown in [13] through studying the associated Hamilton Jacobi equation for the reduced dynamics. When this lagrangian formulation is cast into a more natural Hamiltonian one, one clearly sees that the original Einstein-Hilbert action may be written as follows

$$S = \int_{I\subset\mathbb{R}} \int_{\Sigma_g} \left(\mu_g(-K^{ij} + \tau g^{ij})\frac{\partial g_{ij}}{\partial t} - N\mathcal{H} - X^i\mathcal{P}_i \right) d^2x dt, \tag{52}$$

where $\mathcal{H} := \mu_g K_{ij}^{TT} K^{TTij} - \frac{1}{2}\tau^2\mu_g - \mu_g R(g)$, and $\mathcal{P}_i := 2\nabla[g]_j(\mu_g K_i^j - \tau\mu_g\delta_i^j)$. Note that vanishing of $\mathcal{H}$ and $\mathcal{P}_i$ is precisely equivalent to (g_{ij}, K^{ij}) satisfying the Hamiltonian and momentum constraints. When both of these constraints are satisfied we obtain the reduced action

$$S_{reduced} = \int_{I\subset\mathbb{R}} \int_{\Sigma_g} \mu_g(-K^{ij} + \tau g^{ij})\frac{\partial g_{ij}}{\partial t} d^2x dt, \tag{53}$$

which through the conformal transformation (23) becomes

$$S_{reduced} = \int_{I\subset\mathbb{R}} \left(\int_{\Sigma_g} (-\mu_\gamma k^{TTij} \frac{\partial \gamma_{ij}}{\partial t} - \frac{\partial \tau}{\partial t} \mu_g) d^2x \right) dt, \tag{54}$$

where the boundary in time terms are ignored, because, they do not contribute to the equations of motions at the classical level. The Hamiltonian of this reduced dynamics can be read off as follows from the expression of the previous action

$$H_{reduced} = \int_{\Sigma_g} \frac{\partial \tau}{\partial t} \mu_g. \tag{55}$$

Substituting the time coordinate from equation (48) into the expression of the reduced Hamiltonian together with the Hamiltonian constraint yields

$$H_{reduced} = 2 \int_{\Sigma_g} |K^{TT}|_g^2 - 8\pi\chi, \tag{56}$$

where $\chi = 2(1-g) < 0$ is the Euler characteristics of Σ_g. This reduced Hamiltonian can be related to the Dirichlet energy of the Gauss map. The Dirichlet energy (conformally invariant on the domain) associated to the Gauss map (51) is given as

$$\begin{aligned} E[id; g, \rho] &= \frac{1}{2} \int_\Sigma \mu_g g^{ij} \rho_{ij} = \frac{1}{2} \int_\Sigma \mu_\gamma \gamma^{ij} \rho_{ij} = E[id; \gamma, \rho] \\ &= 2 \int_\Sigma |K^{TT}|_g^2 \mu_g - 4\pi\chi. \end{aligned} \tag{57}$$

Therefore, we have the following relation between the Dirichlet energy of the Gauss map and the reduced Hamiltonian of the dynamics

$$H_{reduced} = E[id; \gamma, \rho] - 4\pi\chi. \tag{58}$$

Let us consider that the Teichmüller space $\mathcal{T}\Sigma_g$ is parametrized by $\{q_\alpha\}_{alpha=1}^{6g-6}$, which may be of the Fenchel-Neilsen type (see [17] for details about Fenchel-Neilsen parametrization). Now we observe the following

$$\frac{\partial E[id; \gamma(q), \rho]}{\partial q_\alpha} = \frac{1}{4} \int_{\Sigma_g} \mu_\gamma \left(\gamma^{mn} \gamma^{ij} \rho_{ij} - 2\gamma^{im} \gamma^{jn} \rho_{ij} \right) \frac{\partial \gamma_{mn}}{\partial q_\alpha}, \tag{59}$$

which after substituting $\rho_{ij} = (|K^{TT}|_g^2 + \frac{\tau^2}{2}) g_{ij} + 2\tau K_{ij}^{TT} = (e^{-4\varphi} |k^{TT}|_\gamma^2 + \frac{\tau^2}{2}) e^{2\varphi} \gamma_{ij} + 2\tau k_{ij}^{TT}$ yields

$$\frac{\partial E[id; \gamma(q), \rho]}{\partial q_\alpha} = -\tau \int_{\Sigma_g} \mu_\gamma k^{TTij} \frac{\partial \gamma^{mn}}{\partial q_\alpha}. \tag{60}$$

Now let us go back to equation (54) and substitute $\gamma = \gamma(q)$. We immediately obtain

$$\begin{aligned} S_{reduced} &= \int_{I\subset\mathbb{R}} \left((\int_{\Sigma_g} -\mu_\gamma k^{TTij} \frac{\partial \gamma_{ij}}{\partial q_\alpha}) \dot{q}_\alpha - H_{reduced}(\gamma(q), p, \rho) \right) dt \\ &= \int_{I\subset\mathbb{R}} (p^\alpha \dot{q}_\alpha - H_{reduced}(\gamma(q), p, \rho))\, dt, \end{aligned} \tag{61}$$

which upon utilizing equations (58) and (60) leads to

$$\frac{\partial H_{reduced}(\gamma(q), p, \rho)}{\partial q_\alpha} = \tau p^\alpha. \tag{62}$$

Here $\{(q_\alpha, p^\alpha)\}_{\alpha=1}^{6g-6}$ parametrizes the phase space i.e., the co-tangent bundle of $\mathcal{T}\Sigma_g$. Now using the time defined in (48), we may construct a principle functional after substituting $T = -\frac{1}{\tau}$

$$\mathcal{S}(q, \gamma(q), \rho) = -T(E[id; \gamma(q), \rho] - 4\pi\chi) \tag{63}$$

which then clearly satisfies

$$p^\alpha = \frac{\partial \mathcal{S}}{\partial q_\alpha}, \tag{64}$$

$$-\frac{\partial \mathcal{S}}{\partial T} = E[id; \gamma(q), \rho] - 4\pi\chi = H_{reduced}(q, p, \gamma(q)), \tag{65}$$

that is, $\mathcal{S}$ satisfies the Hamilton-Jacobi equation

$$-\frac{\partial \mathcal{S}}{\partial T} = H_{reduced}(q, p, \gamma(q)) \tag{66}$$

for all $T \in (0, \infty)$. In other words $\mathcal{S}$ is dynamically complete. For detailed analysis (arguments underlying dynamical completeness of $\mathcal{S}$), the reader is referred to the relevant sections of [13]. Here we only require the fact that through the solution of this Hamilton-Jacobi equation, the Gauss map equation defined in (51) solves the Einstein equation for all $T \in (0, \infty)$ or equivalently for all $\tau \in (-\infty, 0)$ and defines a ray-structure based at ρ of the Teichmüller space parametrized by the transverse-traceless conformally invariant 2-tensor k^{TT}_{ij}. The conformal metric $\gamma_{ij} = e^{-2\varphi} g_{ij} \in \mathcal{T}\Sigma_g$ indeed approaches to ρ_{ij} in the limit $\tau \to 0^-$. Therefore, if we run the Einstein flow in the reverse direction, then the expression of γ_{ij} in terms of ρ_{ij} and k^{TT}_{ij} obtained from the Gauss map equation defines a ray-structure of the Teichmüller space parametrized by k^{TT}_{ij} i.e., for a fixed ρ, two different k^{TT}_{ij} correspond to two different rays. An implicit solution [13] of the Gauss map equation (51) gives

$$\gamma^{ij} = e^{2\varphi} g^{ij} = e^{2\varphi}\left(\frac{2\tau^3}{\mu_\rho}\frac{\rho^{ik}\mu_\gamma \gamma^{jl} k^{TT}_{kl}}{1 + \sqrt{1 + \frac{2\tau^2\mu_\gamma^2 |k^{TT}|^2_\gamma}{\mu_\rho^2}}} + \tau^2 \frac{\sqrt{1 + \frac{2\tau^2\mu_\gamma^2 |k^{TT}|^2_\gamma}{\mu_\rho^2}}}{1 + \sqrt{1 + \frac{2\tau^2\mu_\gamma^2 |k^{TT}|^2_\gamma}{\mu_\rho^2}}}\rho^{ij}\right).$$

Using this equation (which is effectively the same as the Gauss map equation), [13] constructed a fully non-linear elliptic equation of Monge-Ampere type and showed that a unique solution of such equation exists. Recently [41] showed using a direct analytic technique that such a unique solution exists for all $\tau \in (-\infty, 0)$. Essentially, these analyses are in a sense complementary to the Hamilton-Jacobi theory and provide a more explicit description of the ray structure of the Teichmüller space. Analyzing the associated Monge-Ampere equation, [13] explicitly showed that every non-trivial solution curve of the reduced dynamics in the configuration space ($\mathcal{T}\Sigma_g$) approaches a point (ρ) lying in the interior of the Teichmüller space, that is,

$$\lim_{\tau \to 0^-} \gamma^{ij} = \rho^{ij}. \tag{67}$$

Note that the choice of ρ is arbitrary as long as it does not leave the compact sets of $\mathcal{T}\Sigma_g$, and therefore, one may vary ρ over $\mathcal{T}\Sigma_g$ to obtain the full ray-structure of the

Teichmüller space. We do not provide the complete calculations regarding the $\tau \to 0^-$ behavior of the solution curve as it is derived and described in detail by Moncrief in [13]. Readers are referred to the relevant sections of the same. We only need the information that the Gauss map equation together with the Hamilton-Jacobi equation indeed describes ray structures of the Teichmüller space and every such ray solves the reduced Einstein equation. Forward time asymptotics of each such ray corresponds to an interior point which also realizes the infimum of the Dirichlet energy (and the reduced Hamiltonian). Each member of a family of rays which asymptotically approach the point $\rho \in \mathcal{T}\Sigma_g$ corresponds to a unique choice of k^{TT} and none of the two rays of a same family intersect each other (except at ρ, where they approach as $\tau \to 0^-$).

The forward in time limit of the solution curves is well studied in [13]. Therefore, without repeating the same here, we will proceed to study the other limit, that is, the $\tau \to -\infty$ limit which corresponds to the big bang singularity. In this limit the solution curve leaves every compact set of the Teichmüller space, a conclusion which may be obtained through studying the time evolution of the Dirichlet energy (a proper function on $\mathcal{T}\Sigma_g$) of the Gauss map. The time is chosen to be $t = -\frac{1}{\tau}$ (48). From equation (57), the time derivative of the $|K^{TT}|_g^2$ reads

$$\frac{d}{dt}\int_\Sigma |K^{TT}|_g^2\mu_g = \frac{d}{dt}\int_\Sigma (\frac{\tau^2}{2} + R(g))\mu_g, \tag{68}$$

$$= \tau^2\frac{d}{d\tau}\int_\Sigma (\frac{\tau^2}{2} + R(g))\mu_g, \tag{69}$$

$$= \tau^3\int_\Sigma \mu_g + \frac{\tau^2}{2}\int_\Sigma \mu_g(-2N\tau), \tag{70}$$

$$= \tau\int_\Sigma N|K^{TT}|_g^2\mu_g,$$

$$= -\frac{1}{t}\int_\Sigma N|K^{TT}|_g^2\mu_g,$$

where, we have used the lapse equation $\Delta_g N + N(|K^{TT}|_g^2 + \frac{\tau^2}{2}) = \tau^2$, the Hamiltonian constraint $|K^{TT}|_g^2 = \frac{\tau^2}{2} + R(g)$, and the evolution equation $\frac{\partial g_{ij}}{\partial t} = -2NK_{ij} + (L_X g)_{ij}$. Utilizing the estimate of the lapse function (50), we immediately obtain

$$-\frac{2}{t}\int_\Sigma |K^{TT}|_g^2\mu_g \le \frac{d}{dt}\int_\Sigma |K^{TT}|_g^2\mu_g \le -\frac{1}{t}\int_\Sigma |K^{TT}|_g^2\mu_g, \tag{71}$$

integration of which yields at the $t \to 0$ limit

$$\frac{const.}{t} \le \int_\Sigma |K^{TT}|_g^2\mu_g \le \frac{const.}{t^2}. \tag{72}$$

Using the expression of the Dirichlet energy $E[id;\gamma,\rho]$ from equation (57), the following estimate is obtained in the limit $\tau \to -\infty$ i.e., $t \to 0$

$$\frac{2C_2}{t} - 4\pi\chi \le E_\gamma \le \frac{2C_3}{t^2} - 4\pi\chi, \tag{73}$$

which clearly implies that the Dirichlet energy blows up at the limit of the big bang singularity i.e., in the limit $t \to 0$ or equivalently $\tau \to -\infty$ unless $\int_{\Sigma_g} |K^{TT}|_g^2 \equiv 0$,

$0 < C_2, C_3 < \infty$. An immediate interpretation of such limiting behavior would be that the corresponding Einstein solution curve leaves every compact set in the Teichmüller space (configuration space). This is precisely a consequence of the fact that the Dirichlet energy is a proper function on the Teichmüller space (see [19] for the detailed proof of the properness of the Dirichlet energy). Therefore, every non-trivial solution curve leaves the Teichmüller space at the limit of the big-bang. However, we do not know where they converge in the space of projective currents. Note that the space of projective currents is compact and therefore every sequence has a convergent subsequence (since for a metric space, compactness and sequential compactness are equivalent). However, in our context, convergence is a bit more subtle since we are necessarily dealing with curves. We have to extract a sequence $\{l_{\tau_i}\}_{i=1}^{\infty}$ and show that it converges in the space of projective currents in the limit $i \to \infty$ ($\lim_{i\to\infty} \tau_i = -\infty$) and that the limit does not depend on the choice of the sequence. We want to identify this limit set in the space of projective currents. In fact we would like to show in the following sections that every non-trivial solution curve indeed attaches to the Thruston boundary of the Teichmüller space. In addition to the backward in time asymptotic behavior of the Dirichlet energy, we also observe the monotonic decay of the same in the time forward direction

$$\begin{aligned}\frac{d}{dt}E[id;\gamma,\rho] &= 2\frac{d}{dt}\int_{\Sigma}|K^{TT}|_g^2\mu_g, \\ &= 2\tau\int_{\Sigma}N|K^{TT}|_g^2\mu_g < 0.\end{aligned} \tag{74}$$

$\frac{d}{dt}E[id;\gamma,\rho] \equiv 0$ if and only if $K^{TT} \equiv 0$ (or $k^{TT} \equiv 0$) and $\frac{d}{dt}E[id;\gamma,\rho] \to 0$ at the limit $\tau \to 0$. The solutions corresponding to $k^{TT} \equiv 0$ are nothing but the fixed points of the reduced Einstein evolution equations (i.e., the trivial solutions). Substituting $k^{TT} = 0$ into the Lichnerowicz equation (29) yields

$$-2\Delta_\gamma\varphi + 1 - \frac{e^{2\varphi}\tau^2}{2} = 0, \tag{75}$$

which has a unique solution

$$e^{2\varphi} = \frac{2}{\tau^2}. \tag{76}$$

The reduced evoluion equation reads

$$\frac{\partial\gamma_{ij}}{\partial t} = e^{-2\varphi}\left(-(\partial_t e^{2\varphi}\gamma_{ij} - 2Nk_{ij}^{TT} - e^{2\varphi}N\tau\gamma_{ij} + (L_X e^{2\varphi}\gamma)_{ij}\right), \tag{77}$$

which, upon substituting $e^{2\varphi} = \frac{2}{\tau^2}$, $k^{TT} = 0$ and utilizing the lapse equation, shift equation, and Hamiltonian constraint yields

$$\frac{\partial\gamma_{ij}}{\partial t} = 0. \tag{78}$$

A few lines of simple calculation yields $\partial_t k_{ij}^{TT} = 0$ as well. These fixed points characterized by $(\gamma_{ij}, k_{ij}^{TT} = 0, N = 2, X^i = 0)$, $R(\gamma) = -1$, are indeed fixed points for arbitrary large data (even though the Dirichlet energy controls the $(H^1 \times L^2)$ norm of the data (γ, k^{TT}), finite dimensionality of the phase space implies that control on this norm is sufficient). This is precisely a consequence of the monotonic decay of the Dirichlet energy and $d_t E[id; \gamma, \rho] \equiv 0$ precisely at these fixed points (the Dirichlet energy acts as a Lyapunov function). Every solution curve approaches one of this fixed points in forward infinite time t. This point even though it is described in detail in [13, 12], is extremely important and will be of use in obtaining the main result. Summarizing this section together with the results of [13], we have the following theorem

Theorem 1: *Let Σ_g be a closed (compact without boundary) Riemann surface of genus $g > 1$. The data $(\gamma, k^{TT}, \tau, N, X)$ defined through the Gauss map equation (51) and elliptic equations (21-22) solve the Einstein Einstein dynamical equations iff they also solve the constraints and the associated Hamilton-Jacobi equation (66) is satisfied. Such a solution asymptotically approaches the fixed point solution $(R(\gamma) = -1, k^{TT} = 0, N = 2, X^i = 0)$ of the dynamical equations in the limit $\tau \to 0$ and every such non-trivial solution curve runs off the edge of the configuration space (Teichmüller space) in the limit of the big-bang singularity $(\tau \to -\infty)$.*

Now we enter into the final phase where we utilize available results stated in the previous sections and obtain the main result.

5. Asymptotic behavior of the solution curve at big-bang and Thurston boundary

In the previous section, we have established that every non-trivial solution curve runs off the edge of the Teichmüller space. However, we do not apriori know whether they actually attach to the Thurston boundary. However, when realizing the Teichmüller space as a subset of the space of projective currents (which is compact), if we extract a sequence from the solution curve, this must converge somewhere at the limit $\tau \to -\infty$ (after passing to a subsequence and the limit should not depend on the choice of the sequence). Here, we will show that this limit set will be characterized by $\int_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}\mu_\gamma = C$, $C < \infty$ is an uniform constant. Let us designate this boundary as the 'Einstein boundary' of the Teichmüller space and denote it by Ein_g. Our goal in this section is to show that this boundary is indeed equivalent to the Thurston boundary that is $\bar{\mathcal{T}\Sigma_g}^{Th} = \mathcal{T}\Sigma_g \cup \partial\mathcal{T}\Sigma_g^{Th} \approx \mathcal{T}\Sigma_g \cup Ein_g$. Note that Michael Wolf [30] obtained a compactification of Teichmüller space through the use of holomorphic quadratic differentials and he proved that his compactification is indeed equivalent to the Thurston compactification. In our case, we are automatically equipped with a holomorphic quadratic differential k^{TT} (the transverse-traceless tensor). However, importantly, Wolf's analysis is quite different from ours (and complementary in nature) in a sense that the Einsteinian dynamics occurs in the domain of the associated harmonic map while Wolf's dynamics materializes in the target space.

Now we will show the boundedness of $|k^{TT}|^2_\gamma$ in the limit $\tau \to -\infty$.

Lemma 4: *Let (k^{TT}, γ) solve the reduced Einstein equations after imposing the constraints and gauge conditions. Then the following estimates hold for $|k^{TT}|^2_\gamma$ at the* *limit* $\tau \to -\infty$

$$0 < \lim_{t \to 0} \int_{\Sigma} \sqrt{|k^{TT}|^2_{\gamma}} \mu_{\gamma} \leq C < \infty. \tag{79}$$

Proof: Note that the following entity is conformally invariant

$$P = \int_{\Sigma_g} \sqrt{|K^{TT}|^2_g} \mu_g = \int_{\Sigma_g} \sqrt{|k^{TT}|^2_{\gamma}} \mu_{\gamma}. \tag{80}$$

Applying the Cauchy-Swartz inequality, Hamiltonian constraint $|K^{TT}|^2_g = \frac{\tau^2}{2} + R(g)$, and time defined in (48), we immediately obtain

$$\left(\int_{\Sigma_g} \sqrt{|k^{TT}|^2_{\gamma}} \mu_{\gamma}\right)^2 = \left(\int_{\Sigma_g} \sqrt{|K^{TT}|^2_g} \mu_g\right)^2$$

$$\left(\int_{\Sigma_g} \sqrt{|k^{TT}|^2_{\gamma}} \mu_{\gamma}\right)^2 \leq \left(\int_{\Sigma_g} |K^{TT}|^2_g \mu_g\right) \left(\int_{\Sigma_g} \mu_g\right)$$

$$= \left(\int_{\Sigma_g} (\frac{\tau^2}{2} + R(g)) \mu_g\right) \left(\int_{\Sigma_g} \mu_g\right)$$

$$= \frac{\tau^2}{2} \left(\int_{\Sigma_g} \mu_g\right)^2 + 4\pi\chi \int_{\Sigma_g} \mu_g \leq \frac{1}{2t^2} V(g)^2,$$

where we have the used Gauss-Bonet theorem $\int_{\Sigma_g} R(g)\mu_g = 4\pi\chi$, where $\chi = 2(1-g) < 0$ is the Euler characteristic. On the other hand, we know that the volume $V(g)$ of (Σ_g, g) approaches zero at the big-bang. However, we will study the evolution of $V(g)$ and obtain a more precise estimate in terms of $|\tau|$. Time differentiating $V(g) = \int_{\Sigma_g} \mu_g$ (here 'g' in Σ_g denotes genus while g in μ_g denotes the volume form associated to metric g) yields

$$\frac{dV(g)}{dt} = \frac{1}{2} \int_{\Sigma_g} g^{ij} \partial_t g_{ij} \mu_g, \tag{81}$$

which together with the evolution equation $\partial_t g_{ij} = -2N(K^{TT}_{ij} + \frac{\tau}{2} g_{ij}) + (L_X g)_{ij}$ yields

$$\frac{dV(g)}{dt} = \int_{\Sigma_g} (-N\tau + \nabla[g]_i X^i) \mu_g = -\tau \int_{\Sigma_g} N \mu_g, \tag{82}$$

where the total covariant divergence term is dropped following Stokes' theorem. Utilizing the estimate of the lapse function $1 \leq N \leq 2$ (50) and $t = -\frac{1}{\tau}$ (48), we immediately achieve the following bound for the time derivative of the volume $V(g)$

$$\frac{1}{t} V(g) \leq \frac{dV(g)}{dt} \leq \frac{2}{t} V(g), \tag{83}$$

integration of which yields the following at the limit $\tau \to -\infty$ or $t \to 0$

$$constant_1 \cdot \; t^2 \leq V(g(t)) \leq \; constant_2 \cdot \; t \tag{84}$$

Therefore, by using the inequality $0 < \left(\int_{\Sigma_g} \sqrt{|k^{TT}|^2_{\gamma}} \mu_g\right)^2 \leq \frac{1}{2t^2} (V(g))^2$, we obtain

$$0 < \lim_{t \to 0} \int_{\Sigma} \sqrt{|k^{TT}|^2_{\gamma}} \mu_{\gamma} \leq C < \infty, \tag{85}$$

for a uniform constant C (uniform over the conformal structure). Since, $k^{TT} \equiv 0$ implies convergence to a point lying in the interior of $\mathcal{T}\Sigma_g$, the left inequality in (85) is strict (by the blow up of Dirichlet energy). This concludes the proof of the lemma. $\square$

More importantly, $\sup_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}(\tau)$ appears explicitly in a later part where we analyze the Gauss-map equation. In that particular analysis, we require a point-wise control of $\sqrt{|k^{TT}|^2_\gamma}$. Notice that this is the L^∞ norm of the holomorphic quadratic differential $\phi_\tau := (k^{TT}_{11} - ik^{TT}_{12})dz^2$ with respect to the metric γ. The obvious problem is that ϕ_τ may have singularities on measure zero sets. Remarkably, an integrable holomorphic quadratic differential enjoys the property of possessing at most simple poles at punctures of Σ_g. Now, even though Σ_g in question does not have punctures, it forms $\delta-$thin regions in the limit of $\tau \to -\infty$. This is precisely the consequence of the blow up of the Dirichlet energy. If we parameterize the Teichmüller space in the space of projective currents by the lengths of $9g-9$ nontrivial elements of $\pi_1(\Sigma_g)$ ($9g-9$ theorem), then the properness of the Dirichlet energy yields geodesics with large hyperbolic length. As a consequence of the Collar lemma [19], a geodesic transverse to such a long geodesic shrinks (again relative to hyperbolic length) leading to development of a $\delta-$thin region. We will shortly show via the thick-thin decomposition of Σ_g that an integrable holomorphic quadratic differential is bounded (in the sense of L^∞ norm with respect to the metric γ) even if Σ_g develops "bad" parts. Let us first define the norms we are interested in. The L^1 norm and the L^∞ norm (with respect to $\gamma := e^{2\eta}(dx \otimes dx + dy \otimes dy)$) or Ber's supremum norm of ϕ_τ are defined as follows

$$||\phi_\tau||_{L^1(\Sigma_g)} := \frac{1}{\sqrt{2}} \int_{\Sigma_g} |\phi_\tau| = \int_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma}\mu_\gamma, \tag{86}$$

$$\begin{aligned} ||\phi_\tau||_{L^\infty(\Sigma_g)} &:= \sup_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma} \\ &:= \sup_{\Sigma_g} \sqrt{\gamma^{ik}\gamma^{jl}k^{TT}_{ij}k^{TT}_{kl}} \\ &= \sup_{\Sigma_g} \sqrt{e^{-4\eta}\delta^{ik}\delta^{jl}k^{TT}_{ij}k^{TT}_{kl}} \\ &= \sqrt{2}\sup_{\Sigma_g} e^{-2\eta}\sqrt{(k^{TT}_{11})^2 + (k^{TT}_{12})^2}. \end{aligned} \tag{87}$$

Here, we have used the symmetry and traceless property of k^{TT} i.e., $k^{TT}_{12} = k^{TT}_{21}$, and $k^{TT}_{11} + k^{TT}_{22} = 0$. These norms are the natural ones defined for sections of vector bundles defined on Σ_g. Both norms make the space of holomorphic quadratic differentials on Σ_g to be a Banach space. Since, the dimension (real) of this space is $6g-6$ (therefore, finite), the L^1 norm is equivalent to Ber's supremum norm. However, let us explicitly establish the equivalence between L^1 and L^∞ in the case when Σ_g contains "bad" parts by invoking the thick-thin decomposition of Σ_g.

Let Σ_g be a hyperbolic Riemann surface. We will think of $\pi_1(\Sigma_g)$ as the set of the non-trivial loops up to homotopy. For $\delta > 0$, the thin and thick parts of Σ_g are defined as follows

$$\Sigma_{g(0,\delta]} := \{x \in \Sigma_g : \exists <\alpha> \in \pi_1(\Sigma_g) |\ l_\gamma(\alpha) \leq \delta\} \tag{88}$$
$$\Sigma_{g[\delta,\infty)} := \{x \in \Sigma_g : \exists <\alpha> \in \pi_1(\Sigma_g) |\ l_\gamma(\alpha) \geq \delta\}.$$

Here, $l_\gamma(\alpha)$ indicates the length of the geodesic in the homotopy class $<\alpha>$ with respect to the hyperbolic metric γ. The thin part may consist of cusps and Margulis tubes. Since, we are dealing with the compact case, the thin part contains a Margulis tube ($\mathbb{S}^1 \times I,\ I \subset \mathbb{R}$) only. The obvious problem arises in the $\delta-$thin region since, in this region, the length of a geodesic decreases without bound as we approach the big-bang. Here we fix a $\delta > 0$ and focus on the behaviour of the L^∞ norm (w.r.t γ_τ) of the holomorphic quadratic differential (ϕ_τ) in the $\delta-$thin region since, in the $\delta-$thick region, the L^∞ norm is always controlled by the L^1 norm. We now state two lemmas which conclude the business of controlling the L^∞ norm (with respect to γ) in terms of the L^1 norm of ϕ_τ. Note that an integrable holomorphic quadratic differential on a closed (no punctures, no boundary components) Riemann surface does not have poles and therefore has zero principle part. For such an integrable holomorphic quadratic differential on Σ_g (since it is compact without boundary), the following lemma holds. Proof of this lemma uses results from elementary complex analysis such as the maximum principle for holomorphic functions.

Lemma 5a:[40] *For $\delta > 0$ and any closed Riemann surface Σ_g, there exists a constant $C < \infty$ depending only on the genus g of Σ_g and independent of δ such that for every hyperbolic metric γ on Σ_g the following holds for the holomorphic quadratic differential in the $\delta-$thin region*

$$||\phi||_{L^\infty(M_{(0,\delta]})} \leq Ce^{-\pi/\delta}/\delta^2||\phi||_{L^1(\Sigma_g)}. \tag{89}$$

Of course the boundedness follows from the boundedness of $e^{-\pi/\delta}/\delta^2$. In the $\delta-$thick region, L^∞ (with respect to the metric γ) control in terms of L^1 (with respect to the metric γ) is trivial and follows from the following lemma.

Lemma 5b:[40] *For any $\delta > 0$ and any closed Riemann surface Σ_g, there exists a constant $C_\delta < \infty$ depending only on δ and the genus g of Σ_g such that for every hyperbolic metric γ on Σ_g the following holds for the holomorphic quadratic differential in the $\delta-$thick region*

$$||\phi||_{L^\infty(M_{[\delta,\infty)})} \leq C_\delta||\phi||_{L^1(\Sigma_g)}. \tag{90}$$

Now consider the case when Σ_g completely degenerates and forms punctures (From the Deligne-Mumford compactness theorem, punctured surfaces are achieved as a limit of a Riemann surface degenerating via collapsing non-trivial simply closed geodesics). Now, the analysis becomes a little more subtle since integrable holomorphic quadratic differentials may have a simple pole at a puncture (at worst). A neighbourhood of this puncture corresponds to a cusp and is equivalent to a punctured open disc (punctured at 0) equipped with the metric $e^{2\eta}(dx^2 + dy^2) = \frac{1}{|z|^2(\log(|z|))^2}|dz|^2$. Now, roughly it is clear that the simple pole of ϕ_τ cancels in the norm $||\phi||_{L^\infty} := e^{-2\eta}\sqrt{k_{11}^2 + k_{12}^2}$. But for completeness we state the following lemma from [40].

Lemma 5c: *Let (Σ_g, γ) be a hyperbolic Riemann surface with finite area. Then for any holomorphic quadratic differential ϕ of Σ_g the following are equivalent*

1. $||\phi||_{L^1(\Sigma_g)} \lesssim C, C < \infty$

2.$||\phi||_{L^\infty(\Sigma_g)} \lesssim C, C < \infty$

3. At each of the punctures of Σ_g the differential ϕ has at worst a simple pole.

Now let us consider a sequence $\{(\gamma_{\tau_j}, \phi_{\tau_j}, \tau_j\}_{i=1}^{\infty}$ lying on the solution curve of the Einstein flow (on the phase space) with $\lim_{j\to\infty} \tau_j = -\infty$. If each member of the sequence satisfies $||\phi_{\tau_j}||_{L^1(\Sigma_g)} \leq C$ with the limit satisfying $\lim_{i\to\infty} ||\phi_{\tau_j}||_{L^1(\Sigma_g)} \leq C$ (this is precisely what we have derived in (85)), then from the lemmas 1a,b,c, we conclude that the L^∞ norm of the limit is also bounded, that is, the following is satisfied

$$\lim_{j\to\infty} \sup_{\Sigma_g} \sqrt{|k^{TT}|^2_{\gamma_{\tau_j}}} \leq C, C < \infty. \tag{91}$$

Since $||\phi_{\tau_j}||_{L^1(\Sigma_g)}$ can only increase as $\tau_j \to -\infty$, we may set C as the uniform upper bound for the L^1 norm of each element of the sequence. We may therefore conclude the following boundedness of the L^∞ norm of the limit of the sequence $\{\phi_{\tau_j}\}$ (w.r.t $\{\gamma_{\tau_j}\}$) i.e.,

$$\lim_{\tau_j\to-\infty} ||\phi_{\tau_j}||_{L^\infty} \leq \sqrt{2}C_\infty \tag{92}$$

or

$$\lim_{\tau_j\to-\infty} \sup_{\Sigma_g} \sqrt{|k^{TT}|^2_{\gamma_{\tau_j}}} \leq C_\infty, \tag{93}$$

for some uniform $C_\infty < \infty$. Now we go back to the following point-wise inequality (31)

$$\frac{2}{\tau^2} \leq e^{2\varphi} \leq \frac{1 + \sqrt{1 + 2\tau^2 \sup_{\Sigma_g} |k^{TT}|^2_\gamma(\tau)}}{\tau^2}. \tag{94}$$

Using the fact that $\sup_{\Sigma_g} \sqrt{|k^{TT}|^2_\gamma} \leq C_\infty < \infty$, we may conclude that the following estimate of $\sup_{\Sigma_g} e^{2\varphi}$ holds in the limit $\tau \to -\infty$

$$\frac{2}{\tau^2} \leq e^{2\varphi} \leq \frac{C_\varphi}{|\tau|}, \tag{95}$$

for a suitable constant $0 < C_\varphi < \infty$.

In summary, we have obtained the two following crucial estimate which will be utilized later

$$\sup_{\Sigma_g} |k^{TT}|^2_\gamma \leq C^2_\infty, \tag{96}$$

$$\frac{2}{\tau^2} \leq e^{2\varphi} \leq \frac{C_\varphi}{|\tau|}, \tag{97}$$

as $\tau \to -\infty$. We have now obtained the necessary estimates from Einstein's equations in CMCSH gauge. Utilizing these estimates we want to establish a relation between the hyperbolic length of a nontrivial element of $\pi_1(\Sigma_g)$ and its transverse measure against the measured foliation associated with the holomorphic quadratic differential. As mentioned previously, we have a natural holomorphic quadratic differential associated to the Einstein flow due to the fact that corresponding to each transverse-traceless tensor k^{TT}, we may associate a holomorphic quadratic differential. Here, we define the following quadratic differential

$$
\begin{aligned}
\phi_\tau &= (k^{TT}_{11} - ik^{TT}_{12})dz^2 && (98)\\
&= \phi_\tau(z)dz^2, && (99)\\
&= k^{TT}_{11}(dx^2 - dy^2) + 2k^{TT}_{12}dxdy + \\
&\quad - i(k^{TT}_{12}(dx^2 - dy^2) - 2k^{TT}_{11}dxdy),\\
&= k + i\xi,
\end{aligned}
$$

where $i = \sqrt{-1}$. Note that the transverse-traceless tensor k^{TT} may be recovered as follows

$$k^{TT} = \mathcal{R}\left(\phi_\tau(z)dz^2\right). \qquad (100)$$

The transverse-traceless property of k^{TT}_{ij} precisely implies $\frac{\partial\phi_\tau}{\partial\bar{z}} = 0$ i.e., ϕ_τ is holomorphic. This establishes the well known homeomorphism between the space of holomorphic quadratic differentials and the space of transverse-traceless tensors. In addition, we have a natural homeomorphism between the space of transverse traceless tensors on (Σ_g, γ) and the Teichmüller space from the Einstein flow (for detailed analysis see [13]). Once we have a quadratic differential we immediately obtain horizontal and vertical measured foliations associated with this holomorphic quadratic differential. The transverse measures of the vertical measured foliation and horizontal measured foliation are (as follows from (6) and (7))

$$\mu_{vert}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{\frac{k + \sqrt{k^2 + \xi^2}}{2}}, \qquad (101)$$

$$\mu_{hor}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{\frac{\sqrt{k^2 + \xi^2} - k}{2}}, \qquad (102)$$

respectively. Let us consider that the tangent vector field to the curve $\mathcal{C}$ be $u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$ and denote this by $(u^1, u^2)^T$. The term k^2 may be written as the bi-linear form $k^{TT}_{ij}u^iu^j(d\lambda)^2$, where λ is the parameter along $\mathcal{C}$. Similarly, the term ξ may be written as $k^{TT}_{im}J^m_j u^iu^j = k^{TT}_{im}u^iv^m$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $v^m = J^m_j u^j$ that is $v = (-u^2, u^1)^T$. More importantly, we see that the following holds in isothermal coordinates ($\gamma = \gamma(z)|dz|^2, \gamma(z) = e^{\delta(z)}, \delta(z) : \Sigma_g \to \mathbb{R}$)

$$\gamma(u, v) = \gamma(z)(-u^1u^2 + u^2u^1) = 0, \qquad (103)$$

that is, u and v are orthogonal to each other with respect to the metric γ. This is precisely a consequence of the existence of an isothermal chart around any point on Σ_g and since $\gamma(u, v)$ is a scalar, vanishing in one coordinate chart implies vanishing in every coordinate chart (as mentioned in the begining, we use isothermal coordinates throughout). The transverse measure to the vertical foliation may be written as follows

$$\mu_{vert}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{|\frac{k^{TT}_{ij}u^iu^j + \sqrt{(k^{TT}_{ij}u^iu^j)^2 + (k^{TT}_{ij}u^iv^j)^2}}{2}|} d\lambda. \tag{104}$$

Let us now compute the $\gamma-$length of a geodesic in the homotopy class $[\mathcal{C}]$ and relate it to its transverse measure associated to the measured foliation of the holomorphic quadratic differential ϕ_τ. Through the unique solution of the Monge-Ampere equation, the Gauss map equation defines a ray structure of the Einstein equations. Therefore analyzing the asymptotic behaviour of the Monge-Ampere equation is in principle the same as analysing the Gauss map equation while satisfying the Einstein's equations through the associated Hamilton-Jacobi equation. In addition, analysis of the Gauss map equation seems more tractable (and relevant) because we have a handful of estimates from the elliptic equations associated with the Einstein dynamics. Using the Gauss map equation, we obtain

$$\begin{aligned} \rho_{ij}u^iu^j &= |K|^2_g g_{ij}u^iu^j + 2\tau K_{ij}u^iu^j \\ &\quad - \tau^2 g_{ij}u^iu^j, \\ &= (|K^{TT}|^2_g + \frac{\tau^2}{2}) g_{ij}u^iu^j + 2\tau K^{TT}_{ij}u^iu^j \\ &= (e^{-4\varphi}|k^{TT}|^2_\gamma + \frac{\tau^2}{2}) e^{2\varphi}\gamma_{ij}u^iu^j \\ &\quad + 2\tau k^{TT}_{ij}u^iu^j. \end{aligned} \tag{105}$$

We do know the fact that $\rho \in \mathcal{T}\Sigma_g$ is fixed and the $\rho-$length of $\mathcal{C}$ is bounded (due to the properness of the Dirichlet energy which remains finite in the interior of the Teichmüller space). Following the Gauss map equation, we have the following

$$\begin{aligned} |\rho_{ij}u^iu^j| &= |\left\{|K^{TT}|^2_g + \frac{\tau^2}{2}\right\} g_{ij}u^iu^j + 2\tau k^{TT}_{ij}u^iu^j|, \\ &\geq |\left\{|K^{TT}|^2_g + \frac{\tau^2}{2}\right\} g_{ij}u^iu^j| - 2|\tau k^{TT}_{ij}u^iu^j|, \\ &= ||k^{TT}|^2_\gamma e^{-2\varphi}\gamma_{ij}u^iu^j + \frac{\tau^2}{2}e^{2\varphi}\gamma_{ij}u^iu^j| - 2|\tau k^{TT}_{ij}u^iu^j|, \\ &\geq 2|\sqrt{\frac{|k^{TT}|^2_\gamma \tau^2}{2}(\gamma_{ij}u^iu^j)^2}| - 2|\tau k^{TT}_{ij}u^iu^j|, \end{aligned} \tag{106}$$

that is,

$$\frac{1}{\sqrt{2}}\sqrt{|k^{TT}|^2_\gamma}\gamma_{ij}u^iu^j \leq |k^{TT}_{ij}u^iu^j| + \frac{1}{2|\tau|}\rho_{ij}u^iu^j. \tag{107}$$

Here, we have used $a^2 + b^2 \geq 2ab$ for $a, b \in \mathbb{R}$. Now point-wise norm of $\sqrt{|k^{TT}|^2_\gamma}$ satisfies

$$0 \leq |k^{TT}|^2_\gamma \leq C^2_\infty. \tag{108}$$

Notice that the infimum of $|k^{TT}|^2_\gamma$ may be zero since the holomorphic quadratic differential $\phi_\tau := (k^{TT}_{11} - ik^{TT}_{12})dz^2$ has finite number of zeros. Let us consider that the infimum of $\sqrt{|k^{TT}|^2_\gamma}$ be C_f which is strictly positive provided that we stay away from the zeros (finite number) of the quadratic differential ϕ_τ (which correspond to the singularities of the associated measured foliation). Let us consider that the quadratic differential has zeros at $(z_1, z_2, ..., z_n)$, $n < \infty$. Consider ϵ disks $D_\epsilon(z_i)$ around each of the zeros. As these zeros correspond to the singularities of the associated measured foliation, we will consider the trasnverse measure on $\Sigma_g^{'} = \Sigma_g - \{\cup_{i=1}^n D_\epsilon(z_i)\}$ (a detailed rationale is sketched in Wolf's paper and therefore we do not repeat the same here). On $\Sigma_g^{'}$, the previous inequality becomes

$$\frac{|C_f|}{\sqrt{2}}\gamma_{ij}u^iu^j \leq |k^{TT}_{ij}u^iu^j| + \frac{1}{2|\tau|}\rho_{ij}u^iu^j, \tag{109}$$

$$\lim_{\tau\to-\infty}\frac{|C_f|}{\sqrt{2}}\gamma_{ij}u^iu^j \leq \lim_{\tau\to-\infty}|k^{TT}_{ij}u^iu^j| + \lim_{\tau\to-\infty}\frac{1}{2|\tau|}\rho_{ij}u^iu^j. \tag{110}$$

Now notice the fact that $\rho-$length of $\mathcal{C}$ is finite and independent of τ since ρ lies in the interior of the Teichmüller space and therefore

$$\lim_{\tau\to-\infty}\frac{1}{2|\tau|}\rho_{ij}u^iu^j = 0. \tag{111}$$

We obtain the following inequality

$$\lim_{\tau\to-\infty}\frac{|C_f|}{\sqrt{2}}\gamma_{ij}u^iu^j \leq \lim_{\tau\to-\infty}|k^{TT}_{ij}u^iu^j|. \tag{112}$$

Let us analyze the Gauss-map equation in a different way

$$|\rho_{ij}u^iu^j - 2\tau k^{TT}_{ij}u^iu^j| = |\left\{|K^{TT}|^2_g + \frac{\tau^2}{2}\right\}g_{ij}u^iu^j. \tag{113}$$

Now utilizng the estimate of $|K^{TT}|^2_g$ from (31), we obtain

$$|2\tau k^{TT}_{ij}u^iu^j| - |\rho_{ij}u^iu^j| \leq \tau^2e^{2\varphi}\gamma_{ij}u^iu^j, \tag{114}$$

which utilizing the estimate (31) yields

$$|2\tau k^{TT}_{ij}u^iu^j| - |\rho_{ij}u^iu^j| \leq \left(1 + \sqrt{1 + 2\tau^2\sup_{x\in\Sigma_g^{'}}|k^{TT}|^2_\gamma(\tau)}\right)\gamma_{ij}u^iu^j. \tag{115}$$

Substituting the estimate (96) into the previous inequality leads to

$$|k^{TT}_{ij}u^iu^j| \leq \left(\frac{1}{2\tau} + \frac{\sqrt{1+2C_\infty^2\tau^2}}{2\tau}\right)\gamma_{ij}u^iu^j + \frac{1}{2|\tau|}\rho_{ij}u^iu^j,$$

that is,

$$|k^{TT}_{ij}u^iu^j| \leq \left(\frac{1}{2\tau} + \frac{\sqrt{1+2C_\infty^2\tau^2}}{2\tau}\right)\gamma_{ij}u^iu^j + \frac{1}{2|\tau|}\rho_{ij}u^iu^j \tag{116}$$

and therefore, in the limit $\tau \to -\infty$

$$\lim_{\tau\to-\infty}\frac{|k^{TT}_{ij}u^iu^j|}{\gamma_{ij}u^iu^j} \leq \lim_{\tau\to-\infty}\left(\frac{1}{2\tau} + \frac{\sqrt{1+2C_\infty^2\tau^2}}{2\tau}\right) = \frac{|C_\infty|}{\sqrt{2}}. \tag{117}$$

In a sense, we have as $\tau \to -\infty$

$$\frac{|C_f|}{\sqrt{2}}\gamma_{ij}u^iu^j \leq |k^{TT}_{ij}u^iu^j| \leq \frac{|C_\infty|}{\sqrt{2}}\gamma_{ij}u^iu^j, \tag{118}$$

with $0 < C_f^2 < C_\infty^2 < \infty$. This is an important expression obtained at the limit of the big-bang ($\tau \to -\infty$). On the other hand, the expression for the transverse measure of the vertical foliation reads (104)

$$\mu_{vert}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{|\frac{k^{TT}_{ij}u^iu^j + \sqrt{(k^{TT}_{ij}u^iu^j)^2 + (k^{TT}_{ij}u^iv^j)^2}}{2}|}d\lambda. \tag{119}$$

We still need to obtain an estimate for the term $k^{TT}u^iv^j$. In addition to the transverse measure to the vertical foliation, we also have the following transverse measure to the horizontal foliation of the holomorphic quadratic differential ϕ_τ

$$\mu_{hor}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{|\frac{\sqrt{(k^{TT}_{ij}u^iu^j)^2 + (k^{TT}_{ij}u^iv^j)^2} - k^{TT}_{ij}u^iu^j}{2}|}d\lambda. \tag{120}$$

In the analysis of Wolf [30], it is shown that this transverse measure associated to the horizontal foliation collapses asymptotically. In Wolf's [30] construction, the domain is fixed while the target is varied, that is the dynamics occurs in the target space. In our case, the dynamics takes place in the domain. Therefore, we can not utilize the available machinery such as the Beltrami differential $\nu := \frac{|W_{\bar{z}}|}{|W_z|}$ ($W : \Sigma_g(\gamma) \to \Sigma_g(\rho)$ and harmonic) or the associated Bochner equation controlling the behaviour of ν to show that μ_{hor} vanishes and therefore, $k^{TT}_{ij}u^iv^j$ approaches zero asymptotically. Once again the Gauss map equation (51) together with the Lichnerowicz equation (relativistic version of the Bochner equation) come to the rescue and notably they are of purely relativistic origin. The Gauss-map equation reads

$$\rho_{ij} = (e^{-4\varphi}|k^{TT}|^2_\gamma + \frac{\tau^2}{2})e^{2\varphi}\gamma_{ij} + 2\tau k^{TT}_{ij}, \tag{121}$$

which upon contracting with ζ and η yields

$$\rho_{ij}\zeta^i\eta^j = (e^{-4\varphi}|k^{TT}|^2_\gamma + \frac{\tau^2}{2})e^{2\varphi}\gamma_{ij}\zeta^i\eta^j + 2\tau k^{TT}_{ij}\zeta^i\eta^j. \tag{122}$$

Performing an exactly similar analysis as before, we may arrive without much difficulty to the following relation in the limit of the big-bang ($\tau \to -\infty$)

$$C^a|\gamma_{ij}\zeta^i\eta^j| \leq |k^{TT}_{ij}\zeta^i\eta^j| \leq C^b|\gamma_{ij}\zeta^i\eta^j|, \tag{123}$$

for $0 < C^a < C^b < \infty$. Now using (104), we immedeately obtain the following at the limit when τ approaches $-\infty$

$$\begin{aligned} &\oint_{\mathcal{C}} \sqrt{|\frac{k^{TT}_{ij}u^iu^j + \sqrt{(k^{TT}_{ij}u^iu^j)^2 + C^{a2}(\gamma_{ij}u^iv^j)^2}}{2}|}d\lambda \\ &\leq \mu_{vert}(\mathcal{C}) \leq \\ &\oint_{\mathcal{C}} \sqrt{|\frac{k^{TT}_{ij}u^iu^j + \sqrt{(k^{TT}_{ij}u^iu^j)^2 + C^{b2}(\gamma_{ij}u^iv^j)^2}}{2}|}d\lambda. \end{aligned} \tag{124}$$

But, from the orthogonality of u and v i.e., $\gamma_{ij}u^i v^j = 0$ (103) and using the bound $C^a|\gamma_{ij}u^i v^j| \leq |k^{TT}_{ij}u^i v^j| \leq C^b|\gamma_{ij}u^i v^j|$, we immediately observe that $|k^{TT}_{ij}u^i v^j| = 0$ which leads to the following expression for the transverse measure of curve $\mathcal{C}$ with respect to the vertical foliation defined by the holomorphic quadratic differential ϕ_τ

$$\mu_{vert}(\mathcal{C}) = \oint_{\mathcal{C}} \sqrt{|k^{TT}_{ij}u^i u^j|}d\lambda. \tag{125}$$

The asymptotic vanishing of the term $\xi = k^{TT}_{ij}u^i v^j$ precisely implies that the transverse measure of the associated horizontal folitaion vanishes i.e.,

$$\begin{aligned}\mu_{hor}(\mathcal{C}) &= \oint_{\mathcal{C}} \sqrt{|\frac{\sqrt{(k^{TT}_{ij}u^i u^j)^2 + (k^{TT}_{ij}u^i v^j)^2} - k^{TT}_{ij}u^i u^j}{2}|}d\lambda \\ &= 0.\end{aligned} \tag{126}$$

Thus, the high Dirichlet energy limit (while viewed as a proper function on the Teichmüller space of the domain) precisely indicates that the transverse measure to the horizontal foliation associated to the quadratic differential ϕ_τ defined in terms of k^{TT} (or equivalently k^{TT}) vanishes. Note that the metric γ, the quadratic differential ϕ_τ, and the dynamics of the associated measured foliations are related to each other via the Einstein flow. In a sense, the Einstein flow drives the solution curve in such a way that the measured foliation behaves in this way at the limit of the big-bang singularity.

Therefore, we obtain the following crucial relation in the big-bang limit $(\tau \to -\infty)$

$$\mu_{vert}(\mathcal{C}) = C\oint_{\mathcal{C}} \sqrt{\gamma_{ij}u^i u^j}d\lambda = Cl_\gamma(\mathcal{C}), \tag{127}$$

for a suitable constant $\frac{|C_f|^{1/2}}{2^{1/4}} \leq C \leq \frac{|C_\infty|^{1/2}}{2^{1/4}}$. An important point to notice is that the constants C_f and C_∞ are uniform in a sense that they do not depend on the chosen homotopy class $[\mathcal{C}]$. Now this does not imply that C is independent of the homotopy class chosen. In fact we need to show that C does not depend on the homotopy class of loops at the limit $\tau \to -\infty$. This follows since we will show that $|k^{TT}|^2_\gamma$ behaves as a constant modulo factors involving inverse power of the mean extrinsic curvature τ as τ approaches $-\infty$ (i.e., big bang). We claim the following.

Lemma 6: *The following is a solution of the Lichnerowicz equation (29) as $\tau \to -\infty$ i.e.,*

$$e^{2\varphi} = \frac{\sqrt{2}a}{|\tau|} + \frac{2}{\tau^2} + O(\frac{1}{|\tau|^3}), \tag{128}$$

where $a^2 := \lim_{\tau\to-\infty}\sup_\Sigma |k^{TT}|^2_\gamma = C_f$ (as shown previously by the equivalence of norm property) if

$$|k^{TT}|^2_\gamma = a^2 + O(\frac{1}{|\tau|^2}) \text{ a.e on } \Sigma_g \text{ as } \tau \to -\infty. \tag{129}$$

The corresponding lapse function N and the shift vector field X satisfy

$$N = 1 + \frac{\sqrt{2}}{|\tau|a} + O(\frac{1}{\tau^2}) \text{ a.e. on } \Sigma_g \text{ as } \tau \to -\infty. \tag{130}$$

and

$$\gamma(X,X)=O(\frac{1}{|\tau|})\ a.e\ on\ \Sigma_g\ as\ \tau\to-\infty. \tag{131}$$

Proof: Substitute this form of $e^{2\varphi}$ in the Lichnerowicz equation (29) to yield

$$-2\Delta_\gamma\varphi+e^{-2\varphi}(|k^{TT}|^2_\gamma-a^2)=O(\frac{1}{|\tau|}) \tag{132}$$

integration of which yields at $\tau\to-\infty$

$$\int_\Sigma e^{-2\varphi}(|k^{TT}|^2_\gamma-a^2)\mu_\gamma=O(\frac{1}{|\tau|}). \tag{133}$$

However, by definition, $\sup_{\Sigma_g}|k^{TT}|^2_\gamma=a^2\geq|k^{TT}|^2_\gamma$ everywhere on Σ_g and therefore

$$|k^{TT}|^2_\gamma=a^2+O(\frac{1}{|\tau|^2})\ a.e\ on\ \Sigma\ as\ \tau\to-\infty. \tag{134}$$

Substituting $|k^{TT}|^2_\gamma=a^2+O(\frac{1}{|\tau|^2})$ *a.e on* Σ *as* $\tau\to-\infty$ into the lapse equation yields

$$N=1+\frac{\sqrt{2}}{|\tau|a}+O(\frac{1}{\tau^2})\ as\ \tau\to-\infty \tag{135}$$

which yields through the shift equation and an integration by parts argument (22)

$$\int_{\Sigma_g}(\nabla[\gamma]_iX^j\nabla[\gamma]_kX^l\gamma^{ik}\gamma_{jl}+\gamma_{ij}X^iX^j)\mu_\gamma=O(\frac{1}{|\tau|}) \tag{136}$$

yielding

$$\gamma(X,X)=O(\frac{1}{|\tau|})\ a.e\ on\ \Sigma_g\ as\ \tau\to-\infty.\qquad\square \tag{137}$$

Now we have to show that $|k^{TT}|^2_\gamma$ obtained in (134) satisfies the evolution equation at the limit $\tau\to-\infty$.

Lemma 7: *$g-$ norm of the transverse-traceless tensor k^{TT} i.e., $|k^{TT}|^2_g$ satisfies the following evolution equation*

$$\partial_t|k^{TT}|^2_g=L_X|k^{TT}|^2_g+2N\tau|k^{TT}|^2_g+2\nabla[g]^i\nabla[g]^jNk^{TT}_{ij}. \tag{138}$$

Moreover $|k^{TT}|^2_\gamma=e^{4\varphi}|k^{TT}|^2_g=a^2+O(\frac{1}{\tau^2})$ satisfies this evolution equation almost everywhere on Σ_g up to $O(\frac{1}{|\tau|})$ as $\tau\to-\infty$.

Proof: Explicit calculation using $|k^{TT}|^2_g=g^{ij}g^{kl}k^{TT}_{ik}k^{TT}_{jl}$ and the Einstein evolution equations (17-18) yields the desired evolution equation for $|k^{TT}|^2_g$. Now utilizing the conformal transformation $g_{ij}=e^{2\varphi}\gamma_{ij}$ and noting k^{TT}_{ij} is conformally invariant, the evolution equation for $|k^{TT}|^2_g$ may be transformed into an evolution equation for $|k^{TT}|^2_\gamma$

$$\begin{aligned}\partial_t|k^{TT}|^2_\gamma+e^{4\varphi}|k^{TT}|^2_\gamma\partial_te^{-4\varphi}=2N\tau|k^{TT}|^2_\gamma+L_X|k^{TT}|^2_\gamma-e^{4\varphi}L_Xe^{-4\varphi}\\+\gamma^{ik}\gamma^{jl}\nabla[\gamma]_i\nabla_jNk^{TT}_{kl}-\frac{1}{2}e^{-2\varphi}(\gamma^{ik}\gamma^{ml}\partial_ie^{2\varphi}+\gamma^{mk}\gamma^{jl}\partial_je^{2\varphi})\nabla_mNk^{TT}_{kl},\end{aligned} \tag{139}$$

where we have utilized the fact $\nabla[g]_i\nabla_jN=\nabla[\gamma]_i\nabla_jN-\frac{1}{2}g^{mn}(\nabla[\gamma]_ig_{nj}+\nabla[\gamma]_jg_{in}-\nabla[\gamma]_ng_{il})$. Using $|k^{TT}|^2_\gamma=a^2+O(\frac{1}{\tau^2})$, $N=1+\frac{\sqrt{2}}{|\tau|a}+O(\frac{1}{\tau^2})$, $e^{2\varphi}=\frac{\sqrt{2}a}{|\tau|}+\frac{2}{\tau^2}+O(\frac{1}{|\tau|^3})$,

$e^{-2\varphi}=\frac{|\tau|}{\sqrt{2}a}-\frac{1}{a^2}+O(\frac{1}{|\tau|})$, and $\gamma(X,X)=O(\frac{1}{|\tau|})$, we may compute each term of the evolution equation (139) as follows

$$e^{4\varphi}|k^{TT}|^2_\gamma\partial_t e^{-4\varphi} \tag{140}$$
$$=\left((\frac{2a^2}{\tau^2}+\frac{4}{\tau^4}+\frac{4\sqrt{2}a}{\tau^3})+O(\frac{1}{\tau^4})\right)(a^2+O(\frac{1}{\tau^2}))(\frac{\tau^3}{a^2}-\frac{\sqrt{2}\tau^2}{a^3}+O(|\tau|))$$
$$=2a^2\tau+2\sqrt{2}a+O(\frac{1}{|\tau|}),$$
$$2N\tau|k^{TT}|^2_\gamma=2\tau a^2+2\sqrt{2}a+O(\frac{1}{|\tau|}) \tag{141}$$

$$e^{4\varphi}L_X e^{-4\varphi}=O(\frac{1}{\tau^{5/2}}), \tag{142}$$
$$\gamma^{ik}\gamma^{jl}\nabla[\gamma]_i\nabla_j N k^{TT}_{kl}=O(\frac{1}{\tau^2}), \tag{143}$$
$$\frac{1}{2}e^{-2\varphi}(\gamma^{ik}\gamma^{ml}\partial_i e^{2\varphi}+\gamma^{mk}\gamma^{jl}\partial_j e^{2\varphi})\nabla_m N k^{TT}_{kl}=O(\frac{1}{\tau^4}). \tag{144}$$

After substituting into the evolution equation (139), we observe that the dangerous terms ($O(|\tau|)$ and $O(1)$ terms) are precisely cancelled with their respective negative counterparts. Therefore we observe that the evolution equation for $|k^{TT}|^2_\gamma$ is solved by $|k^{TT}|^2_\gamma=a^2+O(\frac{1}{\tau^2})$ almost everywhere on Σ_g as $\tau\to-\infty$. □

This property is extremely important and indicates an asymptotically velocity dominated behaviour i.e., the evolution equations are effective ordinary differential equations in time as one approaches the big-bang since the spatial parts are weighted by the inverse power of the mean curvature. Velocity term dominated behaviour (VTD) has also been previously noted in the context of big-bang singularity [45]. Since $|k^{TT}|^2_\gamma$ asymptotically approaches a constant a^2, we obtain at the limit $\tau\to-\infty$, $C_\infty=C_f=a$ and therefore the constant C in equation (127) is independent of the homotopy class of curve i.e., $C=\frac{a}{2^{1/4}}$ =constant on Σ_g as $\tau\to-\infty$.

In the proof of the compactification, we will need to choose a sequence $(\{\gamma_{\tau_j},k^{TT}_{\tau_j},\tau_j\})$ from the Einstein solution curve (here $f_\tau\equiv f(\tau)$ and both of these notations are used interchangeably). However, the limit should not depend on the sequence chosen as long as $\lim_{j\to\infty}\tau_j=-\infty$. In other words, the ratio of the transverse measure of a curve $\lambda\mapsto\mathcal{C}(\lambda)$ with respect to the vertical foliation corresponding to k^{TT} and its length with respect to the hyperbolic metric γ does not depend on the chosen sequence on the solution curve as one approaches the big-bang singularity. The following lemma establishes such a property.

Lemma 8: *Let $\{\gamma_{\tau_j},k^{TT}_{\tau_j},\tau_j\}$ be any sequence on the solution curve $\tau\mapsto(\gamma_\tau,k^{TT}_\tau,\tau)$ such that $\lim_{j\to\infty}=-\infty$. Let $\mathcal{C}$ be a curve in any homotopy class. Then the following holds*

$$\lim_{j\to\infty}\mu_{vert,\tau_j}(\mathcal{C})=\frac{C^{1/2}}{2^{1/4}}\lim_{j\to\infty}l_{\gamma_{\tau_j}}(\mathcal{C}), \tag{145}$$

where the constant $C=\lim_{\tau\to\infty}|k^{TT}|^2_\gamma$ and therefore is universal.

Proof: Let us consider a slightly general case where $\tau \in (-\infty, 0)$ instead of $\tau \to -\infty$. Let us define the following entities

$$\mathcal{I}n(\tau) := \inf_{\Sigma'_g} \sqrt{|k^{TT}|^2_{\gamma}(\tau)} \tag{146}$$

$$\mathcal{S}(\tau) := \sup_{\Sigma'_g} \sqrt{|k^{TT}|^2_{\gamma}(\tau)}. \tag{147}$$

Both $\mathcal{I}n(\tau)$ and $\mathcal{S}(\tau)$ are continuous functions of τ by existence-uniqueness-continuity (or well-posedness) of the Einstein's equations [12]. Clearly the following holds by continuity and the result of lemma 7

$$\lim_{\tau \to -\infty} \mathcal{I}n(\tau) = C_f = a \tag{148}$$

$$\lim_{\tau \to -\infty} \mathcal{S}(\tau) \quad = C_{\infty} = a. \tag{149}$$

Performing a similar analysis on the Gauss-map equation as previously, we obtain

$$\frac{1}{\sqrt{2}} \mathcal{I}n(\tau) \gamma_{\tau}(u, u) \leq |k^{TT}_{\tau}(u, u)| + \frac{1}{|\tau|} \rho(u, u) \tag{150}$$

$$|k^{TT}_{\tau}(u, u)| \leq \left(\frac{1}{2|\tau|} + \frac{\sqrt{1 + 2\mathcal{S}(\tau)^2 \tau^2}}{2|\tau|} \right) \gamma_{\tau}(u, u) + \frac{1}{|\tau|} \rho(u, u) \tag{151}$$

$$|k^{TT}_{\tau}(u, v)| \leq \frac{1}{|\tau|} |\rho(u, v)|, \tag{152}$$

where the metric ρ is fixed i.e., independent of τ and lies in the interior of the Teichmüller space. Let us now define

$$\mathcal{A}(\tau) := \frac{1}{2|\tau|} + \frac{\sqrt{1 + 2\mathcal{S}(\tau)^2 \tau^2}}{2|\tau|}. \tag{153}$$

Thus, the second inequality of the three inequalities stated above reads

$$|k^{TT}_{\tau}(u, u)| \leq \mathcal{A}(\tau) \gamma_{\tau}(u, u) + \frac{1}{|\tau|} \rho(u, u) \tag{154}$$

Now, we go back to the formula for the transverse measure to the vertical foliation and obtain the following inequality for $\tau \in (-\infty, 0)$

$$\oint_{\mathcal{C}} \sqrt{k^{TT}_{ij}(\tau) u^i u^j} d\lambda \leq \mu_{vert,\tau}(\mathcal{C}) = \tag{155}$$

$$\oint_{\mathcal{C}} \sqrt{\left| \frac{|k^{TT}_{ij}(\tau) u^i u^j| + \sqrt{(k^{TT}_{ij}(\tau) u^i u^j)^2 + (k^{TT}_{ij}(\tau) u^i v^j)^2}}{2} \right|} d\lambda$$

since $(k^{TT}_{ij}(\tau) u^i v^j)^2 \geq 0$. Now utilizing (152), we obtain

$$\oint_{\mathcal{C}} \sqrt{|k^{TT}_{ij}(\tau) u^i u^j|} d\lambda \leq \mu_{vert,\tau}(\mathcal{C}) \leq \tag{156}$$

$$\oint_{\mathcal{C}} \sqrt{\left| \frac{|k^{TT}_{ij}(\tau) u^i u^j| + \sqrt{(k^{TT}_{ij}(\tau) u^i u^j)^2 + (\frac{1}{|\tau|} \rho_{ij} u^i u^j)^2}}{2} \right|} d\lambda.$$

Now consider any sequence $\{\gamma_{\tau_j}, k^{TT}_{\tau_j}, \tau_j\}$ on the solution curve $\tau \mapsto (\gamma_\tau, k^{TT}_\tau, \tau)$ such that $\lim_{j\to\infty}\tau_j = -\infty$. We have the following limits as $j \to \infty$

$$\lim_{j\to\infty} A(\tau_j) = \lim_{j\to\infty}\left(\frac{1}{2|\tau_j|} + \frac{\sqrt{1+2\mathcal{S}(\tau_j)^2\tau_j^2}}{2|\tau_j|}\right) = \frac{1}{\sqrt{2}}\lim_{j\to\infty}\mathcal{S}(\tau_j) = \frac{C_\infty}{\sqrt{2}}$$

$$\lim_{j\to\infty}\frac{1}{|\tau_j|}\rho(u,u) = 0, \tag{157}$$

$$\frac{C_f}{\sqrt{2}}\lim_{j\to\infty}\gamma_{\tau j}(u,u) \leq \lim_{j\to\infty}|k^{TT}_{\tau_j}(u,u)| \leq \frac{C_\infty}{\sqrt{2}}\lim_{j\to\infty}\gamma_{\tau_j}(u,u) \tag{158}$$

and therefore

$$\lim_{j\to\infty}\mu_{vert,\tau_j} = \lim_{j\to\infty}\oint_{\mathcal{C}}\sqrt{|k^{TT}_{ik}(\tau_j)u^iu^k|}d\lambda \tag{159}$$

(The last equality may be obtained from (156) more formally as

$$\oint_{\mathcal{C}}\sqrt{|\frac{|k^{TT}_{ij}(\tau_k)u^iu^j|+\sqrt{(k^{TT}_{ij}(\tau_k)u^iu^j)^2+(\frac{1}{|\tau_k|}\rho_{ij}u^iu^j)^2}}{2}}|d\lambda = \oint_{\mathcal{C}}\sqrt{|k^{TT}_{ij}(\tau_k)u^iu^j|}\sqrt{|\frac{1+\sqrt{1+\frac{|\rho_{ij}u^iu^j|^2}{|\tau_k||k^{TT}_{ij}(\tau_k)u^iu^j|^2}}}{2}}|d\lambda$$

$\leq \sup_{\mathcal{C}}\sqrt{|\frac{1+\sqrt{1+\frac{|\rho_{ij}u^iu^j|^2}{|\tau_k||k^{TT}_{ij}(\tau_k)u^iu^j|^2}}}{2}}|\oint_{\mathcal{C}}\sqrt{|k^{TT}_{ij}(\tau_k)u^iu^j|}$ and noting that $\sup_{\mathcal{C}}\sqrt{|\frac{1+\sqrt{1+\frac{|\rho_{ij}u^iu^j|^2}{|\tau_k||k^{TT}_{ij}(\tau_k)u^iu^j|^2}}}{2}}|$ approaches 1 as $\tau_k \to -\infty$). Finally, we have the following result as $j \to \infty$

$$\frac{|C_f|^{1/2}}{2^{1/4}}\lim_{j\to\infty}l_{\gamma_{\tau_j}}(\mathcal{C}) \leq \lim_{j\to\infty}\mu_{vert,\tau_j} \leq \frac{|C_\infty|^{1/2}}{2^{1/4}}\lim_{j\to\infty}l_{\gamma_{\tau_j}}(\mathcal{C}). \tag{160}$$

However $C_f = C_\infty = C$ yielding

$$\lim_{j\to\infty}\mu_{vert,\tau_j} = \frac{C^{1/2}}{2^{1/4}}\lim_{j\to\infty}l_{\gamma_{\tau_j}}(\mathcal{C}). \tag{161}$$

and clearly both of these entities have the same limit at the level of projective space as they are proportional to each other by finite constant (as will be discussed in detail in the next section). This completes the analysis that the limiting behaviour does not depend on the chosen sequence as long as $\lim_{\mathrm{j}\to\infty}\tau_j = -\infty$. □

Summarizing this section, we state the following theorem.

Theorem 2: *Let Σ_g be a closed (compact without boundary) Riemann surface of genus $g > 1$ and the data $(\gamma, k^{TT}, \tau, e^\varphi, N, X)$ defined by the solution of the Gauss map equation (51), Lichnerowicz equation (29), and the elliptic equations (21-22) solve the reduced Einstein equations via the associated Hamilton-Jacobi equation. The ratio of the transverse measure of any non-trivial element $\mathcal{C}$ of $\pi_1(\Sigma_g)$ with respect to the vertical measured foliation of the natural holomorphic quadratic differential $\phi_\tau := (k^{TT}_{11} - ik^{TT}_{12})dz^2$, and its hyperbolic length that is the length with respect to the metric γ approaches to a finite constant independent of any homotopy class in the limit of the big-bang singularity i.e., $\tau \to -\infty$ along every sequence on the solution curve. The transverse measure associated with the horizontal foliation collapses to zero in the same limit.*

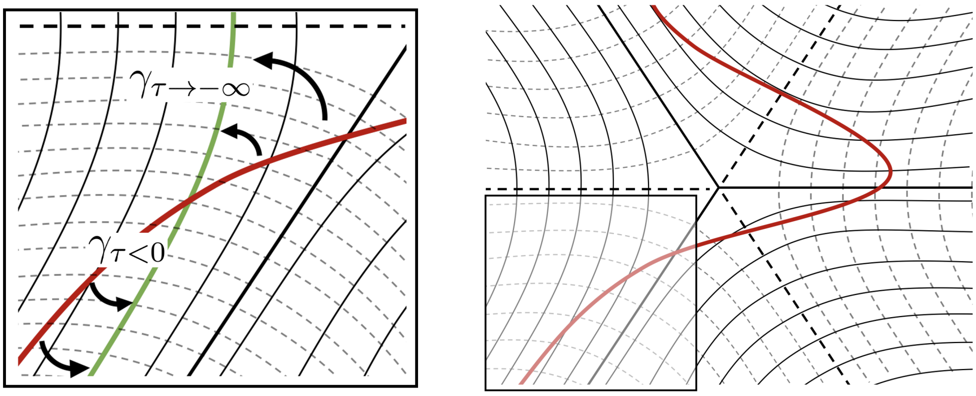

Figure 2. In the limit $\tau \to -\infty$, a non-trivial element γ of $\pi_1(\Sigma_g)$ aligns itself (unbroken pieces) with a leaf of horizontal foliation and thus its transverse measure with respect to the horizontal foliation collapses.

6. Compactification

In this section we claim that the Thurston compactification of the Teichmüller space is equivalent to our relativistic compactification. Let us denote the Einstein compactification of $\mathcal{T}\Sigma_g$ by $\bar{\mathcal{T}\Sigma_g}^{Ein}$. In this section, we claim that the following theorem holds

Theorem 3: $\bar{\mathcal{T}\Sigma}_g^{Th} \approx \bar{\mathcal{T}\Sigma}_g^{Ein}$.

Before proving this theorem, we need a few additional concepts and two lemmas. Let us consider the functional space $\Omega = \mathbb{R}_{>0}^{G(\Sigma_g)}$, where the space of geodesics on Σ_g is denoted as $G(\Sigma_g)$, which may be obtained by the $\pi_1(\Sigma_g)$ action on the space of geodesics on $\mathbb{H}^2$, that is $\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty - \Delta$, Δ being the diagonal. Essentially Ω consists of functionals which take elements of $G(\Sigma_g)$ and associate a positive number to each (in this case, length of the geodesic representative of each homotopy class to be precise). It can essentially be viewed as the space of geodesic currents given that the association of a measure (Radon measure to be precise) $G(\Sigma_g)$ is $\pi_1(\Sigma_g)$ invariant. We may construct the following map

$$l : \mathcal{T}\Sigma_g \to \Omega \tag{162}$$
$$\gamma \mapsto l_\gamma : G(\Sigma_g) \to R_{>0}.$$

We may projectivize the space Ω as follows

$$\mathcal{P}\Omega = \Omega/(\beta \sim t\beta, t > 0, \beta \in \Omega) \tag{163}$$

and subsequently obtain the following map

$$\pi \circ l : \mathcal{T}\Sigma_g \to \mathcal{P}\Omega. \tag{164}$$

Clearly, the map l can be identified with the Liouville currents (see appendix) (or the '$9g-9$' map). The injectivity of $\pi \circ l$ follows from the injectivity of the map L of section (2). Similarly, we may construct the following map from the space of measured laminations to Ω

$$\begin{aligned}\nu &: \mathcal{MF} \to \Omega \\ \mathcal{F} &\mapsto (i(\mathcal{F},\gamma) = \oint_{[\gamma]\in G(\Sigma_g)} \mu_{\mathcal{F}}).\end{aligned} \tag{165}$$

Here, $\mu_{\mathcal{F}}$ corresponds to the transverse measure associated with $\mathcal{F} \in \mathcal{MF}$ and $[\gamma]$ is a homotopy class. Clearly the space of measured geodesic foliations is a subset of the space of all geodesics and therefore, we have the following

$$\begin{aligned}\mathcal{PMF} &:= (\mathcal{MF} - \{0\})/(\mathcal{F} \sim t\mathcal{F}, t > 0, \mathcal{F} \in \mathcal{MF}) \\ &= \pi \circ \nu(\mathcal{MF}) \subset \mathcal{P}\Omega.\end{aligned} \tag{166}$$

Note that $\pi \circ \nu$ is injective. Let the space of holomorphic quadratic differentials with respect to the conformal structure (M,γ) and (M,ρ) be defined as $\mathcal{HQD}(\gamma)$ and $\mathcal{HQD}(\rho)$, respectively. Now we define another important map namely the Hubbard-Masur homeomorphism

$$\mathcal{F} : \mathcal{HQD}(\gamma) \to \mathcal{MF} \tag{167}$$

$$\phi \mapsto \mathcal{F}(\phi). \tag{168}$$

Recall that $\mathcal{PMF}$ and $\mathcal{T}\Sigma_g$ are disjoint in the space of projective currents i.e., $\mathcal{P}\Omega$. The Thurston compactification, essentially, is given as $\bar{\mathcal{T}\Sigma_g}^{Th} = \mathcal{T}\Sigma_g \cup \mathcal{PMF}$. Now we state the following crucial lemma.

Lemma 9: *Let the sequence $\{\gamma_{\tau_j}\}$ leave all the compact sets in $\mathcal{T}\Sigma_g$ at the limit of big-bang i.e., $j \to \infty$ ($\tau_j \to -\infty$). Then $\pi\circ l(\gamma_{\tau_j})$ converges if and only if $\pi\circ\nu(\mathcal{F}_{\tau_j})$ converges and subsequently both have the same limit in $\mathcal{P}\Omega$. Here $\mathcal{F}_{\tau_j}$ is the vertical measured foliation associated to the holomorphic quadratic differential $\phi_{\tau_j} = (k^{TT}_{11\tau_j} - ik^{TT}_{12\tau_j})dz^2$ through the Hubbard-Masur homeomorphism $\mathcal{F}$.*

Proof: $\{\gamma_{\tau_j}\}$ diverges in $\mathcal{T}\Sigma_g$ (identified with its image in $\mathcal{P}\Omega$ under the map $\pi \circ l$) and therefore $\lim_{j\to\infty} l_{\gamma_{\tau_j}}(\mathcal{C}) = \infty$ for some $\mathcal{C} \in G(\Sigma_g)$. Now, It must converge to $\mathcal{P}\Omega$ due to the fact that the later is compact (passing to the level of a subsequence). Therefore, $\exists$ $\{\lambda_{\tau_j}\}$ with $\lim_{j\to\infty} \lambda_{\tau_j} = 0$ such that $\lim_{j\to\infty} \lambda_{\tau_j} l_{\gamma_{\tau_j}}(\mathcal{C}) = \mathcal{L} < \infty$. Now, utilizing the theorem 2 or the following equality (145) derived in lemma 8, we have in the limit $j \to \infty$

$$l_\gamma(\mathcal{C}) = \frac{2^{1/4}}{C^{1/2}} i(\mathcal{F},\mathcal{C}) = i(\frac{2^{1/4}}{C^{1/2}}\mathcal{F},\mathcal{C}), \tag{169}$$

and since the constant C does not depend on the homotopy class of curves $\mathcal{C}$, we may immediately obtain $\lim_{j\to\infty} \lambda_{\tau_j} i(\frac{2^{1/4}}{C^{1/2}}\mathcal{F}_{\tau_j},\alpha) = \mathcal{L}$ i.e., equal to the limit of $\lambda_{\tau_j} l_{\gamma_{\tau_j}}$. Moreover, $\lambda_{\tau_j}\frac{2^{1/4}}{C^{1/2}}\mathcal{F}_{\tau_j}$ converges in $\mathcal{P}\Omega$ or $\frac{2^{1/4}}{C^{1/2}}\mathcal{F}_{\tau_j}$ converges in $\mathcal{PMF}$ (the image of $\mathcal{PMF}$ in $\mathcal{P}\Omega$ under $\pi \circ \nu$ is identified with $\mathcal{PMF}$ and multiplication of a measured foliation by an overall constant yields the same foliation only measure gets scaled. But it makes no difference at the level of projective space by definition). The reverse may be obtained in a similar way. This lemma essentially tells us that a sequence on solution curve (of Einstein's reduced equations) diverging (leaving every compact set) in the configuration space ($\mathcal{T}\Sigma_g$), converges in the space of projective measured foliations as $\tau_j \to -\infty$ (big-bang limit). The space of projective measured foliations is well known to be the Thurston boundary of the Teichmüller space. This establishes the fact that any solution curve approaches the Thurston boundary at the big-bang singularity.

Proof of Theorem 3: We now want to establish the homeomorphism between the Thurston compactified Teichmüller space $\mathcal{T}\bar{\Sigma}_g{}^{Th}$ and the Einstein compactified Teichmüller space $\mathcal{T}\bar{\Sigma}_g{}^{Ein}$. Let us first define the space of holomorphic quadratic differential in our setting. Recall that the holomorphic quadratic differential $\phi_\tau := (k^{TT}_{11} - ik^{TT}_{12})dz^2$ is defined with respect to the metric γ_τ. Using ϕ_τ, we may obtain a holomorphic quadratic differential $\Phi(\phi_\tau)$ with respect to the fixed metric ρ as follows. First, decompose k^{TT} into $\rho-$transverse-traceless part and $\rho-$Lie derivative part in the following $\rho - L^2$ orthogonal sense

$$k^{TT}_{ij} = \zeta^{TT}_{ij} + (L_Y\rho)_{ij}, \tag{170}$$

where $\nabla[\rho]^j\zeta^{TT}_{ij} = 0 = \rho^{ij}\zeta^{TT}_{ij}$ and $Y \in \mathfrak{X}(\Sigma_g)$. Define the homolorphic quadratic differential with respect to ρ as $\Phi(\phi_\tau) := (\zeta_{11} - i\zeta_{12})dz^2$. This identification is not obvious in the current context since k^{TT} is transverse-traceless with respect to γ where as ζ^{TT} is transverse-traceless with respect to ρ. Therefore, one may simply add $Y^{'}$ to Y to yield another $\gamma-$transverse-traceless tensor $k^{TT^{'}}$ while keeping ζ^{TT} fixed. Therefore, two different $\gamma-$transverse-traceless tensor can have same $\rho-$transverse-traceless part. However, in the current context, this decomposition is not arbitrary. Through imposition of the spatial harmonic gauge condition '$id : (\Sigma,\gamma) \to (\Sigma,\rho)$ is harmonic', Y and ζ^{TT} are related to each other through a truly non-linear elliptic PDE (Monge-Ampere type equation). Moncrief [13] proved the existence of a unique solution of this Monge-ampere type equation (see sections 6 through 8 of [13]). Given a ζ^{TT}, one may retrieve k^{TT} through solving Moncrief's Monge-Ampere equation. k^{TT} always have a unique ζ^{TT} due to the fact that (Σ_g,ρ) does not have Killing/conformal Killing vector fields. Through the well-posedness of Moncrief's Monge-ampere equation, k^{TT} depends continuously on ζ^{TT} or ζ^{TT} depends continuously on k^{TT}. In other words, we may also write $\phi_\tau = \phi_\tau(\Phi) := (k^{TT}_{11}(\zeta^{TT}) - ik^{TT}_{12}(\zeta^{TT}))dz^2$. In summary

$$\Phi(\phi_\tau) := (\zeta_{11} - i\zeta_{12})dz^2, \phi_\tau = \phi_\tau(\Phi) := (k^{TT}_{11}(\zeta^{TT}) - ik^{TT}_{12}(\zeta^{TT}))dz^2. \tag{171}$$

This identification yields a curve in the space of holomorphic quadratic differentials with respect to ρ corresponding to a solution curve $\tau \mapsto (\gamma_\tau, k^{TT}_\tau, \tau)$ and also allows us to talk about the boundary of the space of holomorphic quadratic differentials without any ambiguity .

Now let $\mathcal{QD}(\rho)$ be the space of homlomorphic quadratic differentials with respect to ρ. Let $\{\gamma_{\tau_j}, k^{TT}_{\tau j}, \tau_j\}$ be a sequence. Through the correspondence (171), this identifies a sequence $\{\zeta^{TT}_{\tau_j}\}$ in $\mathcal{QD}(\rho)$. In the limit $\tau_j \to -\infty$, we have established in theorem 1 that the Dirichlet energy corresponding to the associated harmonic map blows up and subsequently every sequence $\{\gamma_{\tau_j}\}$ leaves all the compact sets of $\mathcal{T}\Sigma_g$. On the other hand, we also have the associated "velocity" variable, the transverse-traceless tensor $\{k^{TT}_{\tau_j}\}$. It was shown in section 5 that its L^1 norm remains bounded as $\tau_j \to -\infty$ and moroever

$$\lim_{\tau_j\to-\infty}\int_{\Sigma_g}\sqrt{|k^{TT}_{\tau_j}|^2_{\gamma_{\tau_j}}}\mu_{\gamma_{\tau_j}}=C. \tag{172}$$

Note that the constant C is universal and does not depend on the chosen solution ray. Following the boundedness of the L^1 norm of $k^{TT}_{\tau_j}$ at the limit τ_j, we immediately obtain that the L^1 norm of the associated quadratic differential ϕ_{τ_j} remains bounded as well (norms of ϕ were defined in section 5 and considerable detail was presented there) i.e.,

$$\lim_{\tau_j\to-\infty}||\phi_{\tau_j}||=C \tag{173}$$

which essentially means that the space of $\gamma_\tau-$holomorphic quadratic differential is $\mathbb{S}^{6g-7}$ at the limit $\tau\to-\infty$ and through the homeomorphic correspondence (171), this can be identified with a boundary of a ball $\bar{B}$ in $\mathcal{QD}(\rho)$ radius of which may be set to 1 after a suitable re-normalization i.e., $||\Phi||=1$. Let us denote this unit ball in $\mathcal{QD}(\rho)$ by $\bar{B}(1)$. In order to prove theorem 3, it is sufficient to establish the homeomorphism between the ball $\bar{B}(1)$ and the Thurston compactified Teichmüller space $\mathcal{T}\bar{\Sigma}_g{}^{Th}$. The Einstein boundary of the Teichmüller space is therefore $\mathcal{PQD}(\rho)$ and the Einstein compactification of the Teichmüller space is $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$ (naturally through the Einstein flow and the correspondence (171)). The procedure to demonstrate the homeomorphism between $\mathcal{T}\bar{\Sigma}_g{}^{Th}$ and $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$ is routine and similar to the one described in Wolf's work. Nevertheless, we sketch the proof for the sake of completeness. More specifically, let us explicitly define the following spaces

$$\mathcal{HQD}(\rho):=\{\Phi\in QD(\rho)|\ ||\Phi||<1\} \tag{174}$$

$$\mathcal{PQD}(\rho):=\{\Phi\in QD(\rho)|\ ||\Phi||=1\} \tag{175}$$

$$\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho):=\mathcal{HQD}(\rho)\cup\mathcal{PQD}(\rho). \tag{176}$$

Following the homeomorphism between $\mathcal{T}\Sigma_g^{Ein}$ and $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$, our problem reduces to establishing the homeomorphism between $\mathcal{T}\Sigma_g^{Th}$ and $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$. Let us define the following map after using polar coordinates (r,θ) for the space $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)$

$$\begin{aligned}\varphi:\mathcal{T}\bar{\Sigma}_g^{Th}\subset\mathcal{P}\Omega &\to \mathcal{HQD}(\rho)\cup\mathcal{PQD}(\rho)\approx\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}(\rho)\approx\mathcal{T}\Sigma_g^{Ein}\subset\mathcal{P}\Omega \\ x &\mapsto \left(||\Phi(x)||,\frac{\Phi(x)}{||\Phi(x)||}\right),\forall\pi\circ l(x)\in\mathcal{T}\Sigma_g\subset\mathcal{P}\Omega \\ &\mapsto \left(1,\lim_{n\to\infty}\frac{\Phi(x_n)}{||\Phi(x_n)||}\right),\ \pi\circ l(x_n)\to\partial\mathcal{T}\Sigma_g^{Th}\subset\mathcal{P}\Omega.\end{aligned} \tag{177}$$

Here $\mathcal{T}\Sigma_g$ and $\partial\mathcal{T}\Sigma_g$ are realized as the image of $\pi\circ l$ in $\mathcal{P}\Omega$. In addition, through the Hubbard-Masur homeomorphism $\mathcal{F}$ and $\pi\circ\nu$, $\mathcal{HQD}(\rho)$ and $\mathcal{PQD}(\rho)$ may also be realized as their images in $\mathcal{P}\Omega$. First we want to show that this map is well defined. Consider two solutions $x_n,x_n^{'}\in\mathcal{T}\Sigma_g$ that approach the boundary

$$\lim_{n\to\infty}\pi\circ l(x_n)=\lim_{n\to\infty}\pi\circ l(x_n^{'})=y\in\partial\mathcal{T}\Sigma_g. \tag{178}$$

But, then following lemma 9 and the correspondence (171), we immediately have

$$\lim_{n\to\infty}\pi\circ\nu\circ\mathcal{F}\circ\phi_{\tau_n}\circ\Phi(x_n)=\lim_{n\to\infty}\pi\circ\nu\circ\mathcal{F}\circ\phi_{\tau_n}\circ\Phi(x_n^{'}) \tag{179}$$

and following the Hubbard-Masur homeomorphism

$$\lim_{n\to\infty}\frac{\Phi(x_n)}{||\Phi(x_n)||}=\lim_{n\to\infty}\frac{\Phi(x_n^{'})}{||\Phi(x_n^{'})||}. \tag{180}$$

Now we establish the injectivity, surjectivity and continuity of Ψ as well as continuity of Ψ^{-1}. Here, by a sequence, we will mean a sequence chosen from a solution curve.

Continuity of Ψ: Since, $\lim_{n\to\infty}||\Phi(x_n)||=1$, the continuity in the first component follows. Continuity in the second component is obvious.

Injectivity of Ψ: Injectivity of Ψ on $\mathcal{T}\Sigma_g$ is clear (corresponding to each metric, Φ assigns exactly one holomorphic quadratic differential or transverse-traceless tensor). Now, suppose $x_n, x_n^{'}\in\mathcal{T}\Sigma_g$ such that $\pi\circ l(x_n)\to\partial\mathcal{T}\Sigma_g, \pi\circ l(x_n^{'})\to\partial\mathcal{T}\Sigma_g$ and $\Psi(x)=\Psi(x^{'})$. We want to show that $\lim_{n\to\infty}\pi\circ l(x_n)=\lim_{n\to\infty}\pi\circ l(x_n^{'})\in\partial\mathcal{T}\Sigma_g$. Following $\Psi(x)=\Psi(x^{'})$, we have

$$\lim_{n\to\infty}\frac{\Phi(x_n)}{||\Phi(x_n)||}=\lim_{n\to\infty}\frac{\Phi(x_n^{'})}{||\Phi(x_n^{'})||} \tag{181}$$

which following the Hubbard-Masur homeomorphism implies

$$\lim_{n\to\infty}\pi\circ\nu\circ\mathcal{F}\circ\phi_{\tau_n}\circ\Phi(x_n)=\lim_{n\to\infty}\pi\circ\nu\circ\mathcal{F}\circ\phi_{\tau_n}\circ\Phi(x_n^{'}). \tag{182}$$

But, then lemma 9 (together with the correspondence (171)) implies

$$\lim_{n\to\infty}\pi\circ l(x_n)=\lim_{n\to\infty}\pi\circ l(x_n^{'})\in\partial\mathcal{T}\Sigma_g \tag{183}$$

concluding injectivity of Ψ.

Surjectivity of Ψ: Obviously, Ψ is onto from $\mathcal{T}\Sigma_g$ to $\mathcal{HQD}$. Let $(1,\alpha)\in\mathcal{PQD}$ and $a_n\alpha\to\alpha$. Now, following lemma 9, since, $\pi\circ\nu\circ\mathcal{F}\circ\phi_{\tau_n}(a_n\alpha)=constant$, $\pi\circ l(\phi^{-1}(a_n\alpha))$ converges to $y\in\partial\mathcal{T}\Sigma_g$. Therefore, using continuity of Ψ

$$\Psi(y)=\left(1,\lim_{n\to\infty}\frac{\Phi(\Phi^{-1}(a_n\alpha))}{||\Phi(\Phi^{-1}(a_n\alpha))||}\right)=(1,\alpha), \tag{184}$$

which concludes surjectivity.

Continuity of Ψ^{-1}: For, the continuity of Ψ^{-1}, we only need to verify the continuity on $\mathcal{PQD}$. Let us consider that $(a_n,\alpha_n)\to(1,\alpha)$. Following the Hubbard-Masur homeomorphism, $\pi\circ\nu\circ\mathcal{F}\circ\phi_{\tau_n}(a_n\alpha_n)$ converges. On the other hand, lemma 9 implies

$$\lim_{n\to\infty}\pi\circ\nu\circ\mathcal{F}\circ\phi_{\tau_n}(a_n\alpha_n)=\lim_{n\to\infty}\pi\circ l(\Phi^{-1}(a_n\alpha_n))\in\partial\mathcal{T}\Sigma_g. \tag{185}$$

But, from the definition of Ψ, we obtain

$$\Psi(\pi\circ l(\Phi^{-1}(a_n\alpha_n)))=\left(||\Phi(\Phi^{-1}(a_n\alpha_n))||,\frac{\Phi(\Phi^{-1}(a_n\alpha_n))}{||\Phi(\Phi^{-1}(a_n\alpha_n))||}\right)=(a_n,\alpha_n),$$

since $\lim_{n\to\infty}||\alpha_n||=1$. On the other hand, we also have

$$\Psi(\lim_{n\to\infty}\pi\circ l(\Phi^{-1}(a_n\alpha_n)))=\left(1,\lim_{n\to\infty}\frac{\Phi(\Phi^{-1}(a_n\alpha_n))}{||\Phi(\Phi^{-1}(a_n\alpha_n))||}\right)=(1,\alpha). \tag{186}$$

Therefore, we finally have

$$\Psi^{-1}(1,\alpha)=\lim_{n\to\infty}\pi\circ l(\Phi^{-1}(a_n\alpha_n))=\lim_{n\to\infty}\Psi^{-1}(a_n,\alpha_n) \tag{187}$$

which concludes the continuity of Ψ^{-1}. which completes the proof of

$$\bar{\mathcal{T}\Sigma}_g^{Th} \approx \bar{\mathcal{T}\Sigma}_g^{Ein}. \qquad \square \tag{188}$$

$$\begin{array}{ccc} \partial\mathcal{T}\Sigma_g^{Ein} & \longleftrightarrow & \partial\mathcal{T}\Sigma_g^{Th} \\ {\scriptstyle \mathcal{TT}}\Big\downarrow & & {\scriptstyle \mathcal{Q}}\Big\downarrow \\ \mathcal{PQD} & \longleftrightarrow & \mathcal{PQD} \\ {\scriptstyle \mathcal{F}}\Big\downarrow & & {\scriptstyle \mathcal{F}}\Big\downarrow \\ \mathcal{PMF} & \longleftrightarrow & \mathcal{PMF} \end{array}$$

An important observation would be that we are essentially showing the homoemorphism between $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}$ and $\bar{\mathcal{T}\Sigma_g}^{Th}$. Simultaneously, Einsteinian dynamics provides a natural homeomorphism between $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}$ and $\bar{\mathcal{T}\Sigma_g}^{Ein}$. Therefore, one might naively expect that Wolf's analysis would be directly applicable to obtain the desired result. However, as mentioned previously, Wolf's dynamics occurs in the target space while ours does in the domain. In addition, to establish the homeomorphism between $\mathcal{H}\bar{\mathcal{Q}}\mathcal{D}$ and $\bar{\mathcal{T}\Sigma_g}^{Th}$, lemma 6 and the correspondence (171) play the most important role. However, lemma 6 is obtained by completely relativistic means i.e., by utilizing the Gauss map and Hamilton-Jacobi equation in addition to the estimates derived from the elliptic equations associated to the Gauge and constraints of the Einstein's equations. Similarly the correspondence (171) is made solely through solving Moncrief's Monge-Ampere equation which once again is of purely relativistic origin. Let us explain the mechanism in a little less technical way. Notice the diagram above. Moncrief has shown in [13] that no two solution rays originating at an interior point ρ (defined by the Gauss-map, satisfying constraint and gauges, and uniquely satisfying the reduced Einstein equations through the Hamilton-Jacobi equation) intersect each other (except at the limit $\tau \to 0$, where they may asymptotically approach each other). This gives a homoemorphism between the Teichmüller space $\mathcal{T}\Sigma_g$ and the space of transverse-traceless tensors. However, each such transverse-traceless tensor k^{TT} has a holomorphic quadratic differential ϕ_τ associated to it. Morover, each such holomorphic quadratic differential represents a measured foliation (with zeros of the quadratic differential being the singularities of the foliation), which follows from the classical result of Hubbard and Masur [31]. Now let us consider that $\{\gamma_{\tau_j}\}$ leaves every compact set in the Teichmüller space and converges to the $\partial\mathcal{T}\Sigma_g^{Ein}$. Associated with the sequence $\{\gamma_{\tau_j}\}$, there is a sequence of quadratic differentials $\{\phi_{\tau_j}dz^2\}$ (defined in 98) from relativistic dynamics and such a unique sequence satisfies

$$\lim_{\tau_j \to -\infty} ||\phi_{\tau_j}||_{L^1} = C, \tag{189}$$

for some suitable constant $0 < C < \infty$. Through the correspondence (171), this is precisely equivalent to saying that as the sequence $\{\gamma_{\tau_j}\}$ converges in $\partial\mathcal{T}\Sigma_g^{Ein}$, $\{\Phi(\phi_{\tau_j})\}$ approaches the $6g-7$ dimensional sphere $\mathcal{PQD}(\rho)$ in the space of holomorphic quadratic differential $\mathcal{HQD}(\rho)$ and is defined by

$$||\Phi||_{L^1} = 1 \tag{190}$$

after suitable re-scaling. Now associated to the sequence $\{\phi_{\tau_j}\}$, there exists a unique sequence of measured foliations $\{\mathcal{F}_{\tau_j}\}$ corresponidng to $\Phi(\phi_{\tau_j})$. Now, the lemma 6 enters into the picture. lemma 6 precisely states that the limits of the sequence $\{\gamma_{\tau_j}\}$ and the sequence $\{\mathcal{F}_{\tau_j}\}$ are the same in the space of projective currents ($\mathbb{P}$Curr) and lie in the space of projective measured foliations ($\mathcal{PMF}$). Therefore, the space $\partial\mathcal{T}\Sigma_g^{Ein}$ is precisely the space of projective measured foliations $\mathcal{PMF}$. But, $\mathcal{PMF}$ is nothing but the Thurston boundary of the Teichmüller space in $\mathbb{P}$Curr. Therefore

$$\partial\mathcal{T}\Sigma_g^{Ein} \approx \partial\mathcal{T}\Sigma_g^{Th}. \tag{191}$$

In addition note that each of $\partial\mathcal{T}\Sigma_g^{Ein}$ and $\mathcal{PMF}$ are homeomorphic to $\mathcal{PQD}$. Is a sense the maps $\mathcal{TT}$, $\mathcal{F}$, $\mathcal{F}\circ\mathcal{TT}$ are all homeomorphisms. In a sense, we thus obtained a proof of Moncrief's conjecture that *each of the non-trivial solution curves of the reduced Einsteinian dynamics runs off the edge of the Teichmüller space at the limit of big-bang singularity and attaches to the Thurston boundary of the Teichmüller space, that is, the space of projective measured laminations or foliations ($\mathcal{PML}$, $\mathcal{PMF}$).* As a bonus, we also have in this relativistic setting that the space $\mathcal{PQD} \subset \mathcal{QD}$ is homoemorphic to $\partial\mathcal{T}\Sigma_g^{Ein}$ and therefore $\mathcal{PMF}$. In a sense, we also recover Wolf's result. Now we will describe the possible two mechanisms of approaching the boundary of the Teichmüller space in the next section.

7. Approaching $\partial\mathcal{T}\Sigma_g$

Let us consider the Fenchel Neilsen coordinates of the Teichmüller space. Figure (5) shows the pants decomposition of the Teichmüller space and the associated Fenchel-Neilen co-ordinates (see [32] for the details of the Fenchel-Neilsen parametrization and pants decomposition). Such parametrization is given by the lengths of $3g-3$ nontrivial (nontrivial in $\pi_1(\Sigma_g)$) geodesics $\{l_i\}_{i=1}^{3g-3}$ along with $3g-3$ associated twist parameters $\{\theta_{i=1}^{3g-3}\}$ (twist is performed about the same geodesic). The two possible mechanisms of attaining the boundary of the Teichmüller space are descried below.

7.1. Pinching of Σ

Let $\gamma(l_i^n)$ denotes a sequence of hyperbolic metrics and let $\theta_i = 0\ \forall i = 1, 2, 3, ..., 3g-3$. Letting any one of the l_i tend to infinity i.e., $\lim_{n\to\infty} l_i^n = \infty$ implies approaching the boundary $\partial\mathcal{T}\Sigma$. Using the collar lemma (see [19] for the detailed proof of the collar lemma), we immediately obtain that there is a non-trivial geodesic transverse to l_i^n with length $\approx \lim_{n\to\infty} e^{-l_i^n}$. This is the pinching mechanism described in figure

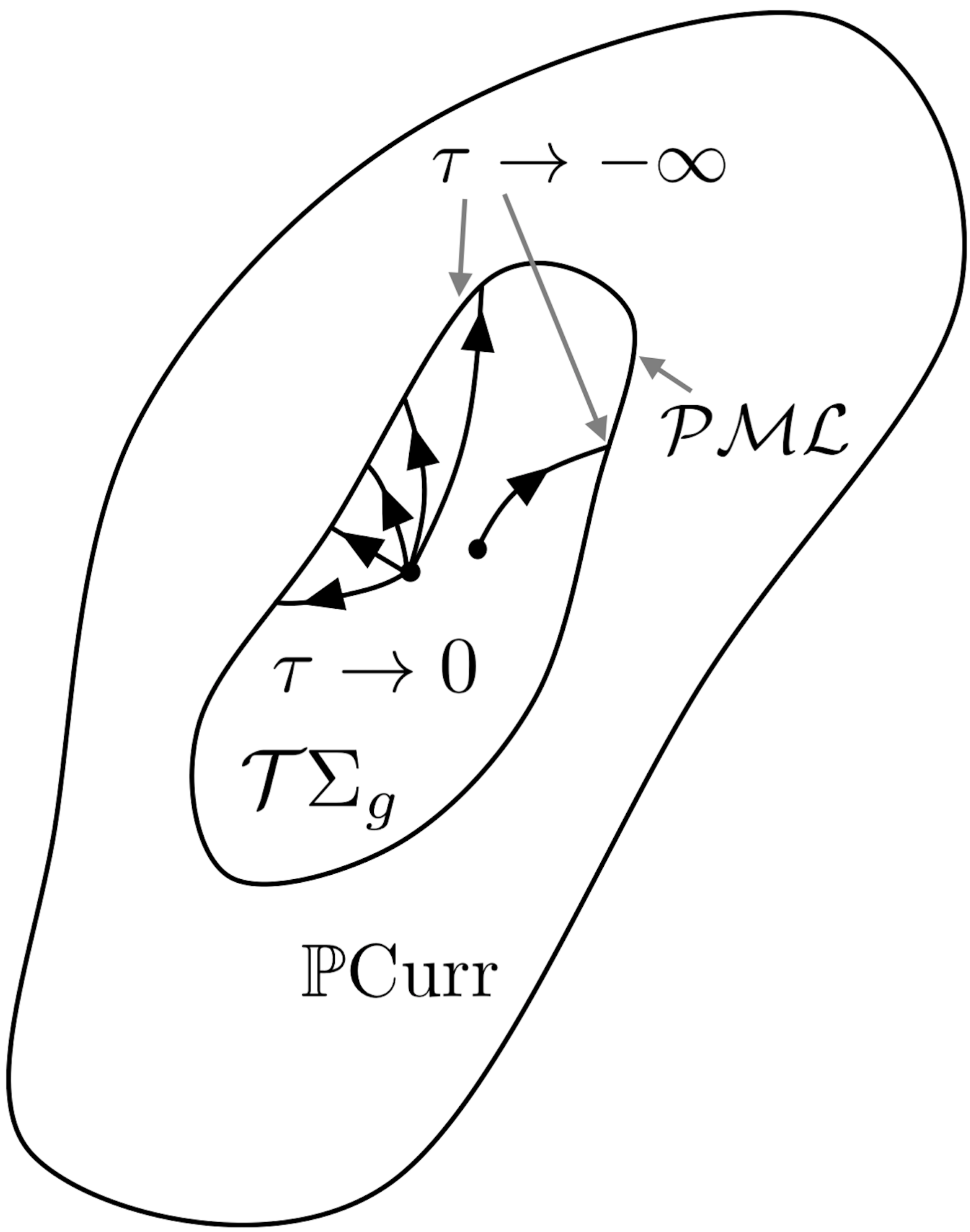

Figure 3. The schematics of the reduced dynamics on the configuration space $\mathcal{T}\Sigma_g$ ($\approx \mathbb{R}^{6g-6}$). Each solution curve starts at $\tau \to 0$ and approaches the Thurston boundary of $\mathcal{T}\Sigma_g$ in $\mathbb{P}\mathrm{Curr}$ (or equivalently $\mathbb{P}\Omega$) in the limit of big-bang.

(4). Note that the nontrivial (in $\pi_1(\Sigma_g)$) geodesic γ_2 collapses while the hyperbolic length l_1 of γ_1 approaches infinity. Now, the Dirichlet energy of the harmonic map $id : (\Sigma_g, \gamma(l_i^n)) \to (\Sigma_g, \rho)$ i.e., between fixed domain (with metric ρ) and the varying target (with metric $\gamma(l_i)$) defined is a continuous proper function on the Teichmüller space. Therefore, the sequence of Dirichlet energies associated with the diverging sequence of metrics (or degenerating to be precise) $\gamma(l_i^n)$ can not stay in a compact set; that is the sequence blows up. Therefore we have the following correspondence

$$\lim_{n\to\infty} l_i^n \to \infty => \lim_{n\to\infty} E_{\gamma(l_i^n)} \to \infty. \tag{192}$$

Notice that multiple non-trivial geodesics γ_i (and the corresponding transverse ones) may show the pinching behavior at once and each such limit corresponds to distinct points on $\partial\mathcal{T}\Sigma_g$.

7.2. Wringing of Σ_g by its neck

In order to explain the approach to $\partial\mathcal{T}\Sigma_g$ through wringing of Σ_g, we need to introduce the symplectic geometry of the Teichmuller space [42, 43]. Using the parametrization $(l_i, \theta_i)_{i=1}^{3g-3}$ of Teichmuller space, define the symplectic form

$$\omega = \sum_{i=1}^{3g-3} dl_i \wedge d\theta_i, \tag{193}$$

which is preserved under the flow of the vector field $v = -\frac{\partial}{\partial\theta_i}$ and satisfies

$$\omega(-\frac{\partial}{\partial\theta_i}, \cdot) = dl_i. \tag{194}$$

The conserved Hamiltonian is nothing but the length l_i. Here, θ_i is the twist parameter about the ith geodesic. Therefore, flow of the vector field $-\frac{\partial}{\partial\theta_i}$ preserves the length l_i of the geodesic about which Σ_g is twisted. After n such twists, the length of the geodesic transverse to the ith geodesic increases by nl_i. The wringing of Σ_g about the ith geodesic corresponds to the limit $n \to \infty$. Let the length of the transverse geodesic before the twist be L^T. After performing n twists, the length becomes $\sim L^T + nl_i$ and therefore, the wringing corresponds to the fact that $\lim_{n\to\infty} \frac{l_i}{L_T + nl_i} = 0$. This is the other mechanism to approach the boundary of the Teichmüller space. Note that every point on the boundary $\partial\mathcal{T}\Sigma_g$ can be obtained through a combination of these two basic operations and in every situation, the Dirichlet energy approaches infinity.

8. Conclusion

Despite the fact that '2+1' gravity is devoid of a straightforward physical significance due to the lack of gravitational wave degrees of freedom, it is of extreme importance while studying '3+1' gravity on spacetimes of certain topological type ($\mathbb{S}^2 \times \mathbb{S}^1 \times \mathbb{R}, \mathbb{T}^2 \times \mathbb{S}^1 \times \mathbb{R}$, and $\Sigma_g \times \mathbb{S}^1 \times \mathbb{R}$, non-trivial $\mathbb{S}^1$ bundles over $\Sigma_g \times \mathbb{R}$). As mentioned in the introduction, several studies have been done of this topic where the '$3+1$' gravity has been realized as the '$2+1$' gravity coupled to a wave map,and where the Teichmüller space of Σ_g plays a crucial role. In the $2+1$ case, the configuration space is the Teichmüller space and we've shown here that the space of big-bang singularities is realized as the Thurston boundary of Teichmüller space. At the big-bang, the conformal geometry degenerates via pinching and wringing of (Σ_g, γ). This result essentially characterizes the complete

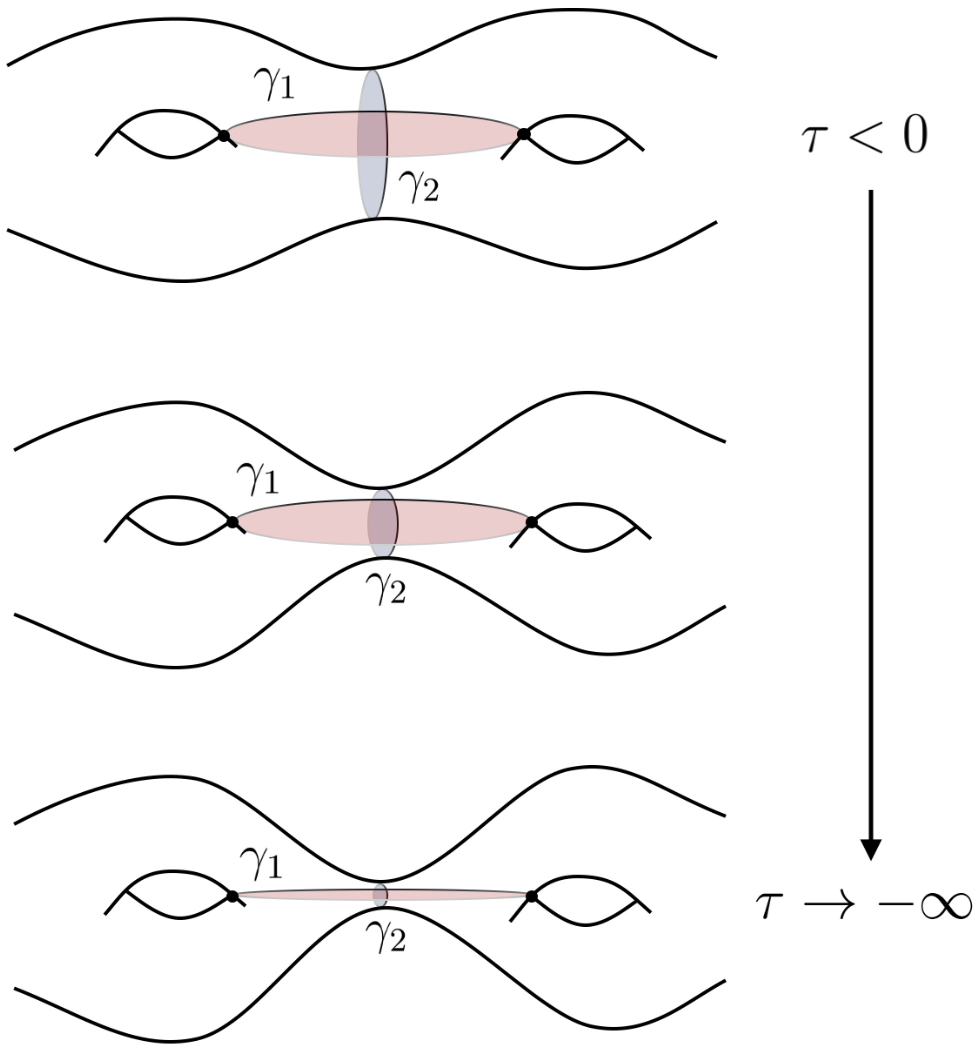

Figure 4. Pinching mechanism collapsing the hyperbolic length of γ_2, while hyperbolic length of γ_1 approaches infinity.

solution space as well as verifies that the reduced Einstein flow can naturally be used to compactify Teichmüller space. While such a result is obtained by studying puerly vacuum gravity, a natural question arises whether inclusion of a positive cosmological constant might yield the same result. [12, 44, 46] studied vacuum GR with a positive cosmological constant in $2+1$ case, where the future in time behavior seems to persist. Therefore, it would be interesting to include a positive cosmological constant and check whether the Thurston boundary is approached in the big-bang limit. In addition, if one includes matter source and focuses on the evolution of the gravitational degrees of freedom (due to the presence of matter sources, the configuration space is now infinite dimensional), can the big-bang limit be realized as the Thurston boundary? Could the Teichmüller degrees of freedom of '3+1' gravity on $U(1)$ symmetric $\mathbb{S}^1$ bundles over $\Sigma_g \times \mathbb{R}$ realize the Thurston boundary in the same limit? What is the implication of

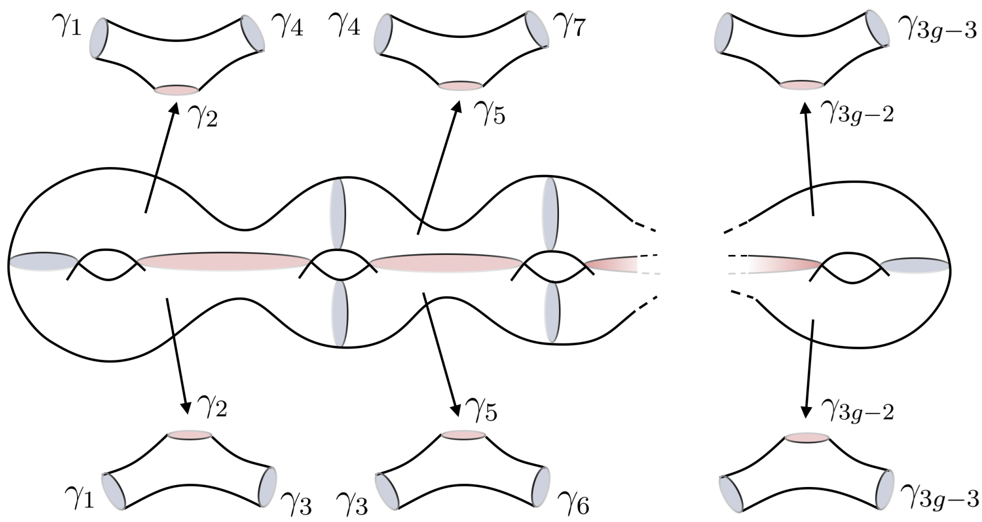

Figure 5. Pants decomposition of the hyperbolic surface Σ_g: hyperbolic length of γ_i together with the twist about the same geodesic γ_i parametrizes the Teichmüller space.

such limiting behavior at the classical level in quantizing '2+1' gravity or '3+1' gravity on these special topologies? Can this characterization of the space of singularities be extended to higher dimensional gravity? Can the Einstein flow be used further to study classical Teichmüller theory?

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Appendix

Space of projective laminations as the Thurston boundary of the Teichmüller space

In this section, we provide a rough sketch of the proof of Thurston compactification of the Teichmüller space by the space of projective measured laminations ($\mathcal{PML}$). The details may be found in [26, 27]. Here we show that a sequence diverging in Teichmüller space converges in the space of projective measured laminations ($\mathcal{PML}$), which is a compact subset of the space of geodesic currents. A $\pi_1(\Sigma_g)-$invariant measure on $G(\tilde{\Sigma}_g)$ may be defined as

$$\hat{L}=\frac{d\alpha d\beta}{|e^{i\alpha}-e^{i\beta}|^2}, \tag{195}$$

where $(e^{i\alpha},e^{i\beta})\in(\mathbb{S}^1\times\mathbb{S}^1)\setminus\Delta$ and Δ represents diagonal. This measure is called the Liouville measure.The Liouville measure corresponding to X is denoted by $\hat{L}_X$, which satisfies the following for any $\gamma\in G(\Sigma_g)$

$$i(\gamma,\hat{L}_X)=l_X(\gamma), \tag{196}$$

where i denotes the bilinear function 'intersection number' and $l_X(\gamma)$ denotes the length of γ with respect to the hyperbolic metric on X. The intersection property may be interpreted as follows. Let us consider a closed non-trivial geodesic $\gamma\in G(\Sigma_g)$. Lift γ to the universal cover and consider its intersection with the set of geodesics transverse to its lift $\tilde{\gamma}$ that is $i(\gamma,\hat{L})$ is defined as $\int_E\hat{L}(E\cap\tilde{\gamma})$, where $E\subset G(\tilde{\Sigma}_g)$ is the set of geodesics transverse to $\tilde{\gamma}$. A few lines of calculations show that this integral is indeed the length of γ with respect to the hyperbolic metric (scalar curvature =-1). Note that Liouville measure may be used to define a geodesic currents on $G(\Sigma_g)$ due to its $\pi_1(\Sigma_g)-$invariance property. Now, let (X,f) be a hyperbolic surface (and thus $\in\mathcal{T}\Sigma$) such that $f:\Sigma_g\to X=\mathbb{H}^2/\pi_1(\Sigma_g)$ is a homeomorphism. Liouville measure provides a well defined map from the Teichmüller space $\mathcal{T}\Sigma$ to the space of currents. Here we just provide a brief description of the Thurston compactification of the Teichmüller space, necessary for the currents purpose. For details, the readers are referred to the excellent book [26], where the proof of the stated theorems may be found.

Lemma 0 [26, 27] *The map $(X, f) \to \hat{L}_X$ is a proper embedding of $\mathcal{T}\Sigma_g$ into the space of currents Curr(Σ_g) given by the intersection number i that is, for all closed curves α in Σ_g,*

$$i(\alpha, \hat{L}_X) = l_X(\alpha) \tag{197}$$

defines a proper embedding of $\mathcal{T}\Sigma_g$ into $\mathrm{Curr}(\Sigma_g)$.

Proof: See [26, 27].

We are now ready to establish the Thurston compactification. Let us first state a lemma.

Lemma 1 [26, 27] For any hyperbolic surface Σ_g with the marking (X, f), we have the following result

$$i(\hat{L}_X, \hat{L}_X) = \pi^2 |\chi(\Sigma_g)|, \tag{198}$$

where $\chi(\Sigma_g) = 2(1-g))$ is the Euler characteristics of Σ_g. Remarkably, this is a topological invariant. Let's denote the map $(X, f) \to \hat{L}_X$ by $\hat{L}$. We have the following lemma

Lemma 2:

$$\hat{L} : \mathcal{T}\Sigma_g \to \mathrm{I\!PCurr}(\Sigma_g) = (\mathrm{Curr}(\Sigma_g) - 0)/(\mu \sim \mathrm{t}\mu, \mu \in \mathrm{Curr}(\Sigma_g), \mathrm{t} \in \mathrm{I\!R}_{>0})$$

is injective.

Proof: Let $[f : \Sigma_g \to X]$ and $[h : \Sigma_g \to Y]$ be two elements of $\mathcal{T}\Sigma_g$. Then

$$[\hat{L}_X] = [\hat{L}_Y] => \hat{L}_X = t\hat{L}_Y. \tag{199}$$

Now we use the previous lemma and obtain

$$\begin{aligned} \pi^2|\chi(\Sigma_g)| &= i(\hat{L}_X, \hat{L}_X) = i(t\hat{L}_Y, t\hat{L}_Y) \\ &= t^2 i(\hat{L}_Y, \hat{L}_Y) = t^2\pi^2|\chi(\Sigma_g)|, \end{aligned} \tag{200}$$

i.e.,

$$t = 1, \tag{201}$$

as $t \in \mathrm{I\!R}_{>0}$ and therefore $L_X = L_Y$.

As we have defined earlier, a lamination $\mathcal{L}$ on Σ_g is a closed subset which is the union of disjoint simple geodesics and the geodesics in $\mathcal{L}$ are called the leaves of the lamination. An important property of these leaves is that they do not intersect each other that is if $\lambda, \alpha \in \mathcal{L}$, then the following is satisfied

$$i(\lambda, \alpha) = 0. \tag{202}$$

If we associate a transverse measure to the leaves of $\mathcal{L}$, then we obtain a measured lamination denoted by $\mathcal{ML}$. We may of course construct the projective measured laminations $\mathcal{PML}$ through the following identification

$$\mathcal{PML} = (\mathcal{ML} - \{0\})/(\lambda \sim t\lambda, \lambda \in \mathcal{ML}, t > 0). \tag{203}$$

Clearly the leaves of a measured lamination define a subset in the space of all geodesics and therefore, the projective measured lamination $\mathcal{PML}$ may be identified as a subset of the space of geodesic currents. It is in fact a compact subset, which may be proven utilizing an elementary result from topology namely Tychonof's theorem [28]. Another important observation is to note that the image of the Teichmüller space under the map $\hat{L}$ i.e., $\hat{L}(\mathcal{T}\Sigma_g)$ and $\mathcal{PML}$ are disjoint. This follows from the definition of the geodesic lamination that is $i(\lambda,\lambda)=0\ \forall\lambda\in\mathcal{PML}$, while $i(\hat{L}_X,\hat{L}_X)=\pi^2|\chi(\Sigma_g)|\neq 0,\ \forall X\in\mathcal{T}\Sigma_g$. Now we finish the Thurston compactification

Lemma 3: The closure of $\mathcal{T}\Sigma_g\subset\mathbb{P}\mathrm{Curr}(\Sigma)$ is precisely $\mathcal{T}\Sigma_g\cup\mathbb{P}\mathcal{ML}$.

Proof: Let say $[f_n:\Sigma_g\to X_n]$ is a sequence that diverges in $\mathcal{T}\Sigma_g$. Then obviously, $\{[\hat{L}_{X_n}]\}\subset\mathcal{P}\mathrm{Curr}(\Sigma_g)$ converges to some element of $\mathcal{P}\mathrm{Curr}(\Sigma_g)$ due to the fact that $\mathcal{P}\mathrm{Curr}(\Sigma_g)$ is a compact subset of $\mathrm{Curr}(\Sigma_g)$ (passing to a subsequence). Then $\exists\ t_n$ such that $Lim_{n\to\infty}t_n\hat{L}_{X_n}=\mu\in$PCurr$(\Sigma_g)$. Now from the divergence criteria, there exists a simply closed curve $\alpha\in\Sigma_g$, such that

$$Lim_{n\to\infty}l_{X_n}(\alpha)=\infty. \tag{204}$$

But, $\infty>i(\alpha,\mu)=i(\alpha,t_n\hat{L}_{X_n})=t_nl_{X_n}(\alpha)$ and thus we must have

$$Lim_{n\to\infty}t_n=0. \tag{205}$$

Now we see the following

$$i(\mu,\mu)=i(lim_{n\to\infty}t_n\hat{L}_{X_n},lim_{n\to\infty}t_n\hat{L}_{X_n}), \tag{206}$$

$$=lim_{n\to\infty}t_n^2i(\hat{L}_{X_n},\hat{L}_{X_n}), \tag{207}$$

$$=lim_{n\to\infty}t_n^2\pi^2|\chi(\Sigma)|, \tag{208}$$

$$=0, \tag{209}$$

and therefore, $\mu\in\mathcal{PML}$.

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