A large data semi-global existence and convergence theorem for vacuum Einstein's equations

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Abstract

I prove a convergence theorem in the context of $3+1$ dimensional vacuum Einstein's equations including a positive cosmological constant on spacetimes $\widetilde{M}$ of topological type $M \times \mathbb{R}$, where M is a closed connected oriented Riemannian $3-$manifold of negative Yamabe type. The theorem concerns the global existence of the solutions of the vacuum Einstein$-\Lambda$ equations for arbitrary large initial data and convergence of the spatial geometry to a constant negative scalar curvature metric in a constant mean extrinsic curvature (CMC) transported spatial gauge. This theorem could be interpreted as a weak cosmic censorship result in the cosmological framework under *generic* initial data. An important corollary of the theorem is the failure of cosmological principle as a derived consequence of Einstein's equations including a positive cosmological constant as well as the failure of geometrization of spatial slice in the sense of Thurston. A similar theorem is stated for positive Yamabe slices under a technical assumption.

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1. Introduction

We will study the vacuum Einstein's equations including a positive cosmological constant Λ. We are given a $n+1$-dimensional ($n=3$ in this article) C^∞ globally hyperbolic connected Lorentzian manifold $(\widetilde{M}, \widehat{g})$ with signature $-+++$. The einstein's equations on $(\widetilde{M}, \widehat{g})$ reads

$$\text{Ric}[\widehat{g}] - \frac{1}{2}R[\widehat{g}]\widehat{g} + \Lambda\widehat{g} = 0. \tag{1.1}$$

Global hyperbolicity implies the existence of a Cauchy hypersurface and in particular $\widetilde{M} = M \times \mathbb{R}$ with M being diffeomorphic to a Cauchy hypersurface. In this article, we will choose M to be of closed negative Yamabe type (a topological invariant the so-called σ constant $\sigma(M) \leq 0$). By definition, these admit no Riemannian metric g having scalar curvature $R(g) \geq 0$ everywhere. A closed 3-manifold M is of negative Yamabe type if and only if it lies in one of the following three mutually exclusive subsets:

(1) M is hyperbolizable (that admits a hyperbolic metric),

(2) M is a non-hyperbolizable $K(\pi,1)$ manifold of non-flat type (the six flat $K(\pi,1)$ manifolds are of zero Yamabe type), here $K(\pi,1)$ is of Eilenberg-MacLane type and by definition $\pi_1(M) = \pi$ and $\pi_i = 0,\ i \geq 2$-the universal cover of $K(\pi,1)$ is contractible and known to be diffeomorphic to $\mathbb{R}^3$ [39], and

(3) M has a nontrivial connected sum decomposition (i.e., M is composite)

$$M \approx (\mathbf{S}^3/\Gamma)_1\#..\#(\mathbf{S}^3/\Gamma)_k\#(\mathbf{S}^2 \times \mathbf{S}^1)_1\#..\#(\mathbf{S}^2 \times \mathbf{S}^1)_l\#\mathbf{K}(\pi,1)_1\#..\#\mathbf{K}(\pi,1)_m \tag{1.2}$$

in which at least one factor is a $K(\pi,1)$ manifold i.e, $m \geq 1$ [41]. In this case the $K(\pi,1)$ factor may be either of flat type or hyperbolizable or non-hyperbolizable non-flat type.

The six flat manifolds comprise by themselves the subset of zero Yamabe type. These admit metrics having vanishing scalar curvature (the flat ones) but no metrics having strictly positive scalar curvature. Here $\Gamma \subset SO(4)$ acts freely and properly discontinuously on $\mathbb{S}^3$. Also note that $M\#\mathbb{S}^3 \approx M$ for any 3-manifold M. It is known that every prime $K(\pi,1)$ manifold is decomposable into a (possibly trivial but always finite) collection of (complete, finite volume) hyperbolic and graph manifold [2] components.

Our goal is to study the long-term behavior of Einstein's equations 1.1 on spacetimes $\widetilde{M} = M \times \mathbb{R}$ with M being general $3-$ manifolds of negative Yamabe type as in 1.2. An issue closely related to Penrose's weak cosmic censorship [34] in the cosmological context (compact spatial topology where the notion of null infinity is not defined) may be phrased as follows

[2]A compact oriented $3-$manifold M is a graph manifold if $\exists$ a finite disjoint collection of embedded $2-$tori $\{T_j\} \subset M$ such that each connected component of $M - \cup_j T_j$ is the total space of circle bundle over surfaces

Conjecture 1.1 *Let $(\widetilde{M}, \widehat{g})$ be a $3+1$ dimensional connected oriented smooth globally hyperbolic Lorentzian manifold resembling a cosmological spacetime i.e., $\widetilde{M} = M \times \mathbb{R}$ with M closed. A solution to Einstein's equations (vacuum or coupled with physically reasonable matter and radiation sources) with generic initial data on M can not break down (via blow up of geometric invariants) in finite time in the future forming naked singularity, that is, a singularity that future time-like observers can access.*

1.1. Background Literature

In the context of this cosmological spacetimes, several results exist in the regime of small data on spatial manifolds of type (1) (i.e., M admits a hyperbolic metric) in the list of negative Yamabe categories. The first breakthrough result is that by Andersson and Moncrief [2] who proved that spacetimes of the type $\mathbb{H}^3/\Gamma \times \mathbb{R}$ ($\Gamma \subset SO^+(1,3)$ acts freely and properly discontinuously on $\mathbb{H}^3$-this model is called the *Milne* model) with metric

$$\widehat{g} = -dt^2 + \frac{t^2}{9}\eta, \ \mathrm{Ric}[\eta] = -\frac{2}{9}\eta \tag{1.3}$$

are future globally stable as solutions to the vacuum Einstein's equations (under a topological constraint on the spatial manifold to exclude the possibility of a non–trivial moduli space of flat spacetime perturbations). The resulting spacetime is flat and realized as the quotient of the interior of the future light cone in the Minkowski space $\mathbb{R}^{1,3}$. Later the same authors extended the result to any spacetime dimensions ≥ 4 where the spatial manifold is assumed to admit a stable negative Einstein metric [3]. This proof uses the monotonicity of a generalized energy functional, a small initial data assumption, and the existence of a stable Einstein metric.

Later several authors generalized this to include matter and radiation sources. Andersson and Fajman [4] proved the future stability of the Milne model 1.3 as a vacuum solution to the Einstein-massive Vlasov equations. Branding, Fajman, and Kröncke [21] proved the future stability of the Kaluza-Klain type spacetimes of dimensions $d+4$ with the macroscopic manifold being the Milne model and the internal manifold d dimensional flat Torus $\mathbb{T}^d$. Under the Kaluza-Klein reduction procedure, the perturbed system reduces to a coupled Einstein-Maxwell-wave map system. Therefore such matter model is also taken care of by this study. In addition to these, there are several results including a positive cosmological constant [13, 12, 29], relativistic fluid [32, 33, 20, 24, 31], Klein-Gordon field [45, 14]. We will not discuss these results here in detail for brevity. In a nutshell, the answer to the conjecture 1.1 is in the affirmative for the spatial manifold M of type (1) and a small data assumption (with respect to a well-defined background solution that exists if and only if M is of type (1)). There are also a plethora of results [25, 23, 15, 44, 30] in the regime of asymptotically flat spacetimes following the breakthrough work by Christodoulou and Klainerman [10] on the stability of Minkowski spacetime.

1.2. Main result of this article

In this article, we consider the equations 1.1 and assume M to be negative Yamabe but consider all of the three types i.e., we do not assume that M admits an Einstein metric. In particular in connected sum decomposition 1.2, M may contain graph manifolds that do not admit any Einstein metrics. Secondly, we remove the assumption of the smallness of the initial data. Before we proceed to the main theorem, let us introduce the basic notions. Vacuum Einstein equations, written in their conventional form, are an autonomous system of partial differential equations for the spacetime metric. When the gauge is fixed by constant mean curvature (CMC) slicing however this autonomous character is broken since both the constraint and the evolution equations, as well as the elliptic equation for the lapse function all depend explicitly upon the mean extrinsic curvature τ (of the manifold M in the spacetime $\widetilde{M}$) which is now playing the role of time. In the presence of a positive cosmological constant and negative Yamabe spatial manifold, the solutions are of expanding type (if we assume they are initially expanding which is of enormous interest to the physics community), it only makes sense to study the dynamics after re-scaling the equations by appropriate powers of $\varphi^2 := \tau^2 - 3\Lambda$ (this entity is positive as claimed in 1). By appropriate scaling, we consider the reduced Einstein's equations 3.24-3.28 for the data (g, Σ) in CMC gauge. A *Newtonian* like CMC time $T \in (-\infty, \infty)$ is chosen. Here Σ is the transverse-traceless (with respect to g) second fundamental form representing the physical gravitational wave degrees of freedom. Instead of working with (g, Σ), we define a new entity

$$\mathfrak{T}[g] := \mathrm{Ric} + \frac{n-1}{n^2} g,\ n = 3 \tag{1.4}$$

which we call it the *obstruction* tensor. Clearly, $\mathfrak{T} = 0$ would imply the existence of a negative Einstein metric (in 3 dimensions hyperbolic metric by means of Mostow rigidity). For a negative Yamabe M of generic type 1.2, $\mathfrak{T}[g]$ will never be zero for any g admitted by M. For the spatial coordinate, we choose *transported spatial coordinates* (see 3 for the definition and its consequences) where the pair $(\Sigma, \mathfrak{T})$ form a manifestly hyperbolic system supplemented by an elliptic equation for the lapse function. The main theorem is as follows

Theorem 1.1 (Semi-global existence, $\boldsymbol{\Lambda > 0, \sigma(M) \leq 0}$) *Let $(\widehat{M}, \widehat{g})$ be a 3+1 dimensional globally hyperbolic spacetime. Suppose the Cauchy slice (M, g) is negative Yamabe type i.e., $\sigma(M) \leq 0$. Given a large $\mathcal{I}^0$, there exists a sufficiently large $a = a(\mathcal{I}^0) > 0$. Let the initial data set (g_0, Σ_0) for the reduced Einstein-Λ equations 3.24-3.28 in CMC transported spatial gauge verify the constraint equations together with the estimates*

$$C^{-1}\xi_0 \leq g_0 \leq C\xi_0,$$

$$\left(\sum_{0\leq I\leq 4} ||\nabla^I \Sigma_0||_{L^2(M)} + \sum_{0\leq I\leq 3} ||\nabla^I \mathfrak{T}[g_0]||_{L^2(M)}\right) + \sum_{0\leq I\leq 3} ||e^a \nabla^I \Sigma_0||_{L^2(M)} \leq \mathcal{I}^0$$

for a constant $C > 0$, a smooth background Riemannian metric ξ_0. Then a unique solution to the Einstein-Λ equation with the initial data (M, g_0, Σ_0) at CMC Newtonian time $T_0 = a$ exists for all $T \in [T_0, \infty)$ and the following estimates hold

$$\left(\sum_{0\leq I\leq 4} ||\nabla^I \Sigma(T)||_{L^2(M)} + \sum_{0\leq I\leq 3} ||\nabla^I \mathfrak{T}[g(T)]||_{L^2(M)}\right) + \sum_{0\leq I\leq 3} ||e^T \nabla^I \Sigma||_{L^2(M)} \leq C_1(1 + \mathcal{I}^0),$$

$$C_2^{-1} g_0 \leq g(T) \leq C_2 g_0\ \forall T \in [T_0, \infty)$$

for numerical constants C_1, C_2. Moreover, every solution $T \mapsto (g(T), \Sigma(T))$ with initial data solving constraints 3.27-3.28 and verifying the estimates 5.1-5.1 converges to $(\widetilde{g}, 0)$ for a metric $\widetilde{g}$ with point-wise constant negative scalar curvature as $T \to \infty$.

Remark 1 *One may easily check that the largeness condition 5.1 can be translated to the largeness of the electric part E_{ij} of the Weyl curvature of the spacetime. Structure equation implies (see 3.2 for a detailed computation)*

$$E_{ij} = \mathfrak{T}_{ij} - \Sigma_{im}\Sigma^m_j - \frac{tr_g k}{2\sqrt{\tau^2 - 3\Lambda}} \Sigma_{ij} \tag{1.5}$$

which implies the largeness of the re-scaled electric field E of the Weyl curvature of the spacetime $\mathbb{R} \times M$. Vanishing of E together with the vanishing of Σ would imply the spacetime being the de-sitter type with spatial manifold being negative Einstein (hyperbolic in $3+1$ dimensional case). However, the type of manifolds we are considering E may never be zero (a similar estimate works for the magnetic part of the Weyl curvature using the Codazzi equation). Therefore, the largeness condition assumed in the main theorem implies the spacetime is far away in an appropriate norm from a de-Sitter-type background (which may not exist if spatial manifold M is not hyperbolizable).

Remark 2 *The result of the convergence of an arbitrary metric to the constant negative scalar curvature metric can be compared to the* ***Yamabe flow*** *on negative Yamabe manifolds. Of course, the convergence of Yamabe flow on manifolds with $\sigma(M) \leq 0$ is much easier than its positive counterpart. Contrary to the Einstein$-\Lambda$ flow, Yamabe flow is of energy-critical parabolic nature and endowed with monotonic entities (e.g., similarly heat flow [46] is of parabolic nature). Moreover, the limit constant scalar curvature metric obtained in the framework of Einstein's equations is not conformal to the initial metric.*

An intuitive interpretation of our result shows that the entity Σ (physical gravitational wave degrees of freedom) that is driving the flow is damped out too quickly to reach the Einstein metric even if M admits an Einstein metric. This could be attributed to the rapid expansion induced by the positive cosmological spacetime. Next, we discuss the *genericity* of the initial data. Notice that the manifold M under consideration does not admit any Riemannian metric whatsoever that has non-trivial, global Killing symmetries (i.e., admits a nonvanishing globally defined Killing field). Any compact manifold that supports a metric with a continuous symmetry group (which of necessity would be a compact Lie group) must necessarily admit an $SO(2)$ action and the (compact, oriented, connected) $3-$manifolds that do admit such actions have been fully classified [17]. To summarize the results discussed therein no such composite $3-$manifold (i.e., non-trivial connected sum) of negative Yamabe type admits an $SO(2)$ action, and the only prime such manifolds (again $\sigma(M) \leq 0$) that can admit such actions are 'standalone' $K(\pi, 1)$ manifolds of non-flat type whose fundamental group π has infinite cyclic center $\approx \mathbb{Z}$ (in which case $M \to M/SO(2)$ is a Seifert fibered space determined by its numerical Seifert invariants). Even for these exceptional cases, the *generic* metrics will not support any Killing symmetry. Therefore our data can be considered to be fully generic in the cosmological context. An obvious corollary to the theorem 1.1 is as follows

Corollary 1.1 *The answer to the conjecture 1.1 is affirmative in the framework of Einstein-Λ, $\Lambda > 0$ equations with the spatially compact topology of negative Yamabe type with generic arbitrary large initial data.*

1.3. Novelty and the main difficulty

Let us discuss the novelty of our study very briefly. We will not elongate this section unnecessarily since section 3 provides the essential detail to the reader. Firstly, this is the first study that deals with generic large data in the cosmological framework. Instead of studying the hyperbolic system for (g, Σ) in CMC and spatial harmonic gauge, we consider the manifestly hyperbolic system for $(\Sigma, \mathfrak{T})$ supplemented by an elliptic equation for the lapse function 3.43 in CMC transported spatial gauge. Our choice of this system is motivated by the earlier study of Choquet-Bruhat and York [1] who considered the hyperbolic system for (Σ, Ric). The equation for the metric 3.24 is treated as a transport equation by the flow of the time vector field ∂_T. In particular, the characteristic of the metric is the integral curves of ∂_T orthogonal to the Cauchy slice contrary to the spatial harmonic gauge where the characteristic of the metric is the null cone. In addition, the spatial harmonic gauge is not suitable for large data problems due to the presence of the Gribov horizon [17]. In particular, recall from [13, 29] that the preservation of the spatial harmonic gauge condition in time yields the following equation for the shift vector field X

$$-\Delta_g X^i - \mathrm{Ric}[g]^i_j X^j + (2\nabla^j X^k)(\Gamma[g]^i_{jk} - \Gamma[\xi]^i_{jk}) =$$
$$(2N\Sigma^{jk})(\Gamma[g]^i_{jk} - \Gamma[\xi]^i_{jk}) + \frac{\tau}{\sqrt{\tau^2 - 3\Lambda}}(2-n)\nabla[g]^i(\frac{N}{n} - 1)$$
$$-2\nabla[g]^j N\Sigma^i_j,$$

where ξ is a background Riemannian metric on the Cauchy slice M. Our formulation has the natural advantage of allowing for large data since we can avoid handling the elliptic equation for the shift vector field. The potential problem is the first order term $(2\nabla^j X^k)(\Gamma[g]^i_{jk} - \Gamma[\xi]^i_{jk})$ that does not have a definite sign. In addition, the Ricci curvature is not necessarily negative definite for the class of manifolds ($\sigma(M) \leq 0$) that we are considering here. In the usual stability problems concerning the background solution with spatially hyperbolic geometry, oftentimes ξ is assumed to be the hyperbolic metric. Since, $||g-\xi||$ is sufficiently small in an appropriate norm, the associated elliptic operator is an isomorphism between the appropriate Sobolev spaces. However, in the current context, we do not have the luxury of the existence of a background solution whose spatial geometry is hyperbolic since the most generic metrics on M with $\sigma(M) \leq 0$ will never admit a hyperbolic metric. This turns out to be a significant difficulty in using the spatial harmonic gauge in the context of large data problems. It may be possible to circumvent this issue but that seems to be technically challenging. The CMC-transported spatial gauge does not have this issue since $X = 0$ in this gauge. This can be compared with the non-abelian Yang-Mills equations where zero shift condition parallels temporal gauge while spatial harmonic condition resembles Coulomb gauge which once again suffers from Gribov ambiguity.

Secondly, we introduce a hierarchy of energy norms that exhibit different decay rates and carefully exploit the decay of the lower order norm of the transverse-traceless second fundamental form Σ caused by the expansion of the geometry (notice that in the main theorem 1.1 the top order norm of Σ does not exhibit decay). This hierarchy of estimates is essential to our proof since we need to improve the estimates at each step in order to close the bootstrap assumption. In addition, the structure of Einstein's equations is utilized crucially. The most dangerous terms that do not exhibit decay at the top order are always multiplied with a decay factor e^{-T}. While this is not a null structure, it remains to be understood the deeper nature of such structures. In addition, the renormalized lapse function does not exhibit decay at the topmost order causing potential trouble. However, for the terms that do not appear multiplied with the universal decay factor e^{-T}, the normalized lapse always appears differentiated in lower order producing necessary decay. Lastly, our norms are defined with respect to the dynamical metric g. Therefore, we need a priori estimates for the metric coefficients. To this end, we employ a delicate boot-strap machinery that is closed through the hierarchy of elliptic and energy estimates.

Secondly, the lapse equation can not be used for obtaining estimates directly due to the regularity problem. This could be understood as follows. Recall the Lapse equation

$$-\Delta_g(\frac{N}{n} - 1) + (|\Sigma|^2 + \frac{1}{n})(\frac{N}{n} - 1) = -|\Sigma|^2 \tag{1.6}$$

and notice that fact that the operator $-\Delta_g + (|\Sigma|^2 + \frac{1}{n})$ is an isomorphism between appropriate Sobolev spaces. If one defines the Sobolev norms with respect to a background metric ξ_0 (as is usually done in the stability problems e.g., [3]), then in principle, one expects an estimate of the type

$$||\frac{N}{n}-1||_{H^{I+2}(M)} \lesssim ||\Sigma||^2_{H^I(M)},\ I>\frac{3}{2}. \tag{1.7}$$

Note this requires by standard Schauder's estimate that $g \in H^{I+1}(M)$. However, in the framework of treating the wave-transport system for $(\Sigma, \mathfrak{T}, g)$, we only have control over $H^I(M)$ norm of g by integrating the transport equation 3.24 in CMC transported spatial gauge

$$\frac{\partial g_{ij}}{\partial T} = -\frac{2\varphi}{\tau}N\Sigma_{ij} - 2(1-\frac{N}{n})g_{ij}. \tag{1.8}$$

This seems to be a potential obstruction in working with the transported spatial gauge. In addition, since we are not in spatial harmonic gauge, we do not have an estimate for the metric in H^{I+1} given the H^{I-1} regularity of $\mathfrak{T}$ and H^I regularity of Σ by means of elliptic estimates. To circumvent this, we define the norms with respect to the dynamical metric g and commute higher order covariant derivatives (with respect to $g-$Levi Civita connection). Naturally, the higher order commutation yields Riemann curvature of g and in the resulting elliptic problem, to apply elliptic regularity one needs control of $I-1$ derivatives of the Riemann curvature while controlling $I+1$ derivatives of $\frac{N}{n}-1$ in $L^2(M)$. But this is possible by the algebraic Ricci decomposition of the Riemann tensor in terms of $\mathfrak{T}$ and the scalar curvature (which can further be expressed in terms of the transverse-traceless shear Σ by means of the Hamiltonian constraint 3.27). This closes the regularity argument since for the wave system we need I and $I-1$ derivatives of Σ and $\mathfrak{T}$, respectively. This regularity is sufficient to control $I-1$ derivatives of spatial Riemann curvature in L^2 norm. Therefore, we obtain the desired elliptic estimates 1.7 for the normalized lapse $\frac{N}{n}-1$.
Another novelty of our study is that we consider every closed-connected oriented manifold that falls under the category of negative Yamabe type. In other words, our theorem holds for all the classes 1-3 in the list of manifolds. Prior PDE-type analysis of the full Einstein's equations with such a generic class of spatial manifolds is not known to me.
Our method has potential applications to Einstein-$\Lambda-$Maxwell, Einstein-$\Lambda-$Euler, and other matter models. It would be interesting to study such systems and deduce whether such large global data results hold.

Remark 3 *I want to point out the heuristic study by Wald [43] that addresses the late-time behavior of initially expanding homogeneous cosmological models satisfying Einstein's equations with a cosmological constant* $\Lambda > 0$.

Remark 4 *Our technique can be applied to the case when* M *is positive Yamabe i.e.,* $\sigma(M) > 0$ *including* $\Lambda > 0$ *under a technical condition on the spectra of the Laplacian-this ensures the uniqueness of a solution to the elliptic equation for the lapse function. In such case, the metric converges to a constant positive scalar curvature metric. Generically, such a limit will be non-unique if one removes the spectral condition reminiscent of the outcome of Brendle's proof [6] of the convergence of the Yamabe flow of arbitrary initial energy to a metric of constant positive scalar curvature. In our context, non-uniqueness is caused by the non-uniqueness of the lapse function. For completeness, we state the following theorem whose proof follows essentially in a similar fashion (see [42, 41] on results related to non-negative scalar curvature geometry). The limit metric, however, is* ***not*** *conformal to the initial metric.*

First, note that in case of $\sigma(M) > 0$, the renormalized lapse equation takes the following

$$-\Delta_g(1-\frac{N}{n}) + (|\Sigma|^2 - \frac{1}{n})(1-\frac{N}{n}) = -|\Sigma|^2, \ \Delta_g := g^{ij}\nabla_i\nabla_j. \tag{1.9}$$

It follows from this equation that if a solution exists, it may not be unique due to the fact that the elliptic operator $-\Delta_g + n(|\Sigma|^2 - \frac{1}{n})$ does have *finite* dimensional kernel. Therefore, we need to make the technical assumption that 1.9 admits a solution (which is true if $|\Sigma|^2$ is L^2 orthogonal to the kernel: such a solution may not be unique: de-Sitter spacetime admits slicing for which the lapse *is* non-unique). Then the remaining analysis goes through in a similar fashion. But since the renormalized lapse $1 - \frac{N}{n}$ sources the evolution equations for $(\Sigma, \mathfrak{T})$, the solution to the reduced Einstein equations becomes non-unique. Therefore, the limit metric will not be unique.

Theorem 1.2 (Global existence, $\Lambda > 0$, $\sigma(M) > 0$) *Let $(\widehat{M}, \widehat{g})$ be a 3+1 dimensional globally hyperbolic spacetime. Suppose the Cauchy slice (M, g) is positive Yamabe type. Given $\mathcal{I}^0$, there exists a sufficiently large $a = a(\mathcal{I}^0) > 0$. Let the initial data set (g_0, Σ_0) for the reduced Einstein-Λ equations for $\sigma(M) > 0$ case in CMC transported spatial gauge verify the constraint equations together with the estimates*

$$C^{-1}\xi_0 \leq g_0 \leq C\xi_0, \tag{1.10}$$

$$\left(\sum_{0\leq I\leq 4} ||\nabla^I\Sigma_0||_{L^2(M)} + \sum_{0\leq I\leq 3} ||\nabla^I\mathfrak{T}[g_0]||_{L^2(M)}\right) + \sum_{0\leq I\leq 3} ||e^a\nabla^I\Sigma_0||_{L^2(M)} \leq \mathcal{I}^0 \tag{1.11}$$

for a constant $C > 0$, a smooth background Riemannian metric ξ_0. Then, if there exists a solution to the lapse equation 1.9, then the corresponding solution to the Einstein-Λ equation with the initial data (M, g_0, Σ_0) at CMC Newtonian time $T_0 = a$ exists for all $T \in [T_0, \infty)$. Moreover, every solution $T \mapsto (g(T), \Sigma(T))$ with initial data verifying the estimates 5.1-5.1 converges to $(\widetilde{g}, 0)$ for a metric $\widetilde{g}$ with point-wise constant negative scalar curvature as $T \to \infty$. Moreover, this limit metric $\widetilde{g}$ is non-unique if the lapse equation yields a non-unique solution.

1.4. Consequences for geometrization

One important point that I want to discuss is the Thurston geometrization in the context of Einstein flow. A major open problem in mathematical general relativity is understanding if Einstein's equations on spatially compact cosmological spacetimes can geometrize the spatial manifold. Fischer-Moncrief initiated this program in the context of vacuum cosmological spacetime [18, 19]. Also, see works of Anderson [5] on the long-time evolution in general relativity and geometrization of 3−manifolds. In the optimistic scenario, the induced asymptotic geometry on the spatial manifold M would implement the geometrization of 3−manifolds. Following [5], one may consider the following definition of geometrization

Definition 1 (geometrization) *Let M be a closed, oriented, and connected 3−manifold with $\sigma(M) \leq 0$. A weak geometrization of M is a following decomposition of M*

$$M = H \cup G \tag{1.12}$$

where H is a finite collection of complete connected finite volume hyperbolic manifolds embedded in M and G is a finite collection of connected graph manifolds embedded in M. The union is along a finite collection of embedded tori $\cup_i T_i$ such that $\cup_i T_i = \partial G = \partial H$. A strong geometrization is a weak geometrization for which each T_i is incompressible i.e., the inclusion of T_i into M induces an injection of the fundamental group.

Unfortunately, the addition of a positive cosmological constant $\Lambda > 0$ destroys any hope of such a geometrization by Einstein flow. We discuss a rough idea behind such a conclusion below. Recall the theorem of Cheeger and Gromov [7, 8] which states an equivalence between the volume collapse with two-sided bounded curvature and the existence of an $\mathcal{F}-$structure of positive rank. A direct corollary of the Cheeger-Gromov theorem is that if the sequence of Riemannian manifolds $\{M_j\}$ that collapse with two-sided bounded curvature, then M_j is a graph manifold for large j. Therefore, in order for geometrization to occur through Einstein$-\Lambda$ flow, unit balls in the graph part G of M should collapse with bounded curvature. Our main theorem 1.1 already proves the uniform boundedness of the curvature. But the volume of unit balls remains uniformly bounded in terms of the initial volume for all time and this bound does not depend on G or H in addition to uniform boundedness of the diameter. In particular, the following corollary holds

Corollary 1.2 (No-collapse theorem) *Let $B(1) \subset M$ be a geodesic ball of unit radius. The volume $Vol(B)_T$ of $B(1)$ at a CMC time $T \in [T_0, \infty)$ remains uniformly bounded by the initial volume $Vol(B)_{T_0}$ along the Einstein-Λ flow.*

We present the proof in section 5.2 which is a simple consequence of the main theorem 1.1. This could be attributed to the fact that the addition of a positive cosmological constant results in rapid expansion of the physical volume that the graph and hyperbolic components become indistinguishable in the limit of $T \to \infty$. Of course, it is still open whether the geometrization can be achieved for the $\Lambda = 0$ case. It seems much more difficult to obtain such a result using Einstein's equations than its parabolic counterpart the Ricci flow [35, 36, 37]. Nevertheless, with the development of novel recent techniques in hyperbolic PDEs in the large data regime [9] (in different contexts, but hopefully some of the ideas can be imported to handle large data problems in cosmological spacetimes)

1.5. Cosmological interpretation

One of the important open problems in mathematical relativity is to understand whether the so-called *cosmological principle* can be derived consequence of Einstein's equations. The cosmological principle roughly states that the spatial universe is homogeneous and isotropic while viewed at a sufficiently large scale. Oftentimes in cosmology, this assumption is used to obtain analytical solutions of Einstein's equations (e.g., FLRW models are spatially homogeneous and isotropic). A mathematically interesting question would be whether the geometry of the spatial slice can converge in a long time to homogeneous and isotropic geometry (i.e., hyperbolic geometry) even if one starts the evolution with a geometry that is far away. In [28], It was conjectured that such a homogeneous and isotropic limit may be obtained as an end result of Einstein-Λ flow. In the present article, we give a negative answer to this question based on the main theorem 1.1. The topology of the spatial slice that we consider is the most general one possible in three dimensions.

Corollary 1.3 (Failure of Cosmological principle as a derived consequence) *The rescaled geometry of the spatial universe does not converge to the hyperbolic geometry as a solution of Einstein's equation with a positive cosmological constant Λ as the CMC time approaches infinity in any appropriate topology. In particular, the limiting spatial geometry is* ***not*** *homogeneous and isotropic.*

For a short proof of this corollary, see remark 11. In this study, we do not include a matter source. But we conjecture that in the presence of a positive cosmological constant, irrespective of matter sources, the spatial geometry will never converge to a homogeneous and isotropic one.

Acknowledgement

I thank Professor S-T Yau for many discussions regarding the scalar curvature geometry and Professor V. Moncrief for numerous discussions regarding Einstein flow. This work was supported by the Center of Mathematical Sciences and Applications, Department of Mathematics at Harvard University.

2. Preliminaries

We use $n+1$ or ADM formalism. The ADM formalism splits the spacetime described by an '$n+1$' dimensional Lorentzian manifold $\widetilde{M}$ into $\mathbf{R}\times M$ with each level set $\{t\}\times M$ of the time function t being an orientable n-manifold diffeomorphic to a Cauchy hypersurface (assuming the spacetime to be globally hyperbolic) and equipped with a Riemannian metric. Here we will focus on $n = 3$. Such a split may be implemented by introducing a lapse function N and shift vector field X belonging to suitable function spaces and defined such that

$$\partial_t \quad = \quad N\widehat{\mathbf{n}} + X \tag{2.1}$$

with t and $\widehat{\mathbf{n}}$ being time and a hypersurface orthogonal future-directed timelike unit vector i.e., $\widehat{g}(\widehat{\mathbf{n}}, \widehat{\mathbf{n}}) = -1$, respectively. The above splitting puts the spacetime metric $\widehat{g}$ in local coordinates $\{x^\alpha\}_{\alpha=0}^n = \{t, x^1, x^2, ..., x^n\}$ into the form

$$\widehat{g} \quad = \quad -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt) \tag{2.2}$$

where $g_{ij}dx^i \otimes dx^j$ is the induced Riemannian metric on M. In order to describe the embedding of the Cauchy hypersurface M into the spacetime $\widetilde{M}$, one needs the information about how the hypersurface is curved in the ambient spacetime. Thus, one needs the second fundamental form k defined as

$$k_{ij} := \langle \nabla_{\partial_i}\widehat{\mathbf{n}}, \partial_j \rangle = -\frac{1}{2N}(\partial_t g_{ij} - (L_X g)_{ij}), \tag{2.3}$$

the trace of which ($\tau = g^{ij}k_{ij}$, $g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} := g^{-1}$) is the mean extrinsic curvature of M in $\widetilde{M}$. Here L denotes the Lie derivative operator. The vacuum Einstein equations with a cosmological constant Λ

$$R_{\mu\nu}(\widehat{g}) - \frac{1}{2}R(\widehat{g})\widehat{g}_{\mu\nu} + \Lambda\widehat{g}_{\mu\nu} = 0 \tag{2.4}$$

may now be expressed as the evolution and Gauss and Codazzi constraint equations of g and k

$$\partial_t g_{ij} = -2Nk_{ij} + (L_X g)_{ij}, \tag{2.5}$$

$$\partial_t k_{ij} = -\nabla_i\nabla_j N + N(R_{ij} + \tau k_{ij} - 2k_i^k k_{jk} - \frac{2\Lambda}{n-1}g_{ij}) + (L_X k)_{ij}, \tag{2.6}$$

$$2\Lambda = R(g) - |k|_g^2 + (\tau)^2, \tag{2.7}$$

$$0 = \nabla^i k_{ij} - \nabla_j \tau, \tag{2.8}$$

where $\tau = \mathrm{tr}_g k$. We denote the connection coefficients of metric g by $\Gamma[g]$. The Riemann and Ricci curvatures are denoted by Riem and Ric, respectively. We define the ordinary Laplacian of g in the following way such that it has a non-positive spectrum i.e.,

$$\Delta_g \equiv g^{ij}\nabla_i\nabla_j. \tag{2.9}$$

We also define a Lichnerowicz type Laplacian $\mathcal{L}_g$ acting on sections of a symmetric rank-2 vector bundle, which will play a crucial role in our analysis

$$\mathcal{L}_g u_{ij} = -\Delta_g u_{ij} - (\text{Riem}_i{}^k{}_j{}^l + \text{Riem}_j{}^k{}_i{}^l) u_{kl}. \tag{2.10}$$

For stable Einstein metrics, the spectra of this operator are positive.
Lastly, for functions $f(t) \in \mathbb{R}_{\geq 0}$ and $h(t) \in \mathbb{R}_{\geq 0}$ $f(t) \lesssim h(t)$ means $f(t) \leq Ch(t)$ and $f(t) \gtrsim h(t)$ means $f(t) \geq Ch(t)$, and $f(t) \approx h(t)$ implies $C_1 h(t) \leq f(t) \leq C_2 h(t)$ for constant C_1 and C_2 that *do not* depend on time t and only depend on the dimension n. The involved constants C_1, C_2 may depend only on a fixed background geometry i.e., the initial metric. The spaces of symmetric covariant 2-tensors and vector fields on M are denoted by $S_2^0(M)$ and $\mathfrak{X}(M)$, respectively. By $\widehat{A}$ and A^{TT}, we will denote symmetric trace-less and transverse-trace-less 2-tensors, respectively. Often times we denote the Cauchy slice M at CMC time T by $M(T)$. We denote the spatial dimensions by n which is always 3 in this article.

3. Gauge fixed Einstein's equations

We study the Cauchy problem in constant mean extrinsic curvature (CMC) gauge. To this end, we assume that the Lorentzian spacetime diffeomorphic to $M \times \mathbb{R}$ admits a constant mean curvature Cauchy slice.
Now recall the vacuum Einstein's field equations with a cosmological constant Λ expressed in $3+1$ formulation

$$\partial_T g_{ij} = -2N k_{ij} + (L_X g)_{ij}, \tag{3.1}$$

$$\partial_T k_{ij} = -\nabla_i \nabla_j N + N(R_{ij} + \text{tr}_g k k_{ij} - 2k_i^k k_{jk} - \frac{2\Lambda}{n-1} g_{ij}) + (L_X k)_{ij}, \tag{3.2}$$

$$2\Lambda = R(g) - |k|_g^2 + (\text{tr}_g k)^2, \tag{3.3}$$

$$0 = \nabla^i k_{ij} - \nabla_j \text{tr}_g k. \tag{3.4}$$

Now we define the CMC gauge. First, recall that

Definition 2 (CMC gauge) *CMC or constant mean extrinsic curvature gauge is the choice of the time coordinate function t where it is equal to the mean extrinsic curvature of its level sets*

The definition automatically requires that the mean extrinsic curvature of level sets of the time be constants on the level sets i.e., $\nabla \tau = 0$. More precisely the mean extrinsic curvature $\tau = \text{tr}_g k$ of M verifies the following

$$\tau = \text{tr}_g k = \text{monotonic function of } t \text{ alone}. \tag{3.5}$$

In particular, we choose

$$t = \tau. \tag{3.6}$$

The first non-trivial question to address would be the existence of a constant mean curvature slice M in the spacetime $M \times \mathbb{R}$. A fundamental difference with the stability problem is that we possess a background solution in the case of stability problems where the spatial slices are negative Einstein and such slices are CMC. A rigidity argument shows that small perturbations of the CMC slice still produce a CMC slice. However, we do not have the same luxury in this problem where apriori, we don't know any particular background solution.

We are trying to investigate if there is any and if every generic solution converges to these solutions and if there are any obstructions from doing so. Therefore we need to make the assumption that the spacetime admits a CMC slice. Since the model that we are considering consists of a big-bang singularity in the past beyond which past-directed causal geodesics can not be extended (here essentially we assume a strong version (C^∞) of strong cosmic censorship in cosmological vacuum), the following theorem of [26] applies in this context related to the existence of CMC slice.

Theorem ([26], Theorem 6) *For a cosmological spacetime, if there exists a Cauchy surface from which all orthogonal (future or past) timelike geodesics end in a strong curvature singularity, then that singularity is crushing, and so there is a CMC slice.*

In the CMC gauge, the first notice a few basic important points.

Claim 1 *Let $M \times \mathbb{R}$ be a globally hyperbolic spacetime and a Cauchy slice M is of negative Yamabe type. Then initially expanding solutions in CMC gauge do not develop a maximal hypersurface i.e., the initially expanding solutions can only continue to expand as long as they exist.*

Proof. The goal is to prove the impossibility of the formation of a maximal hypersurface during evolution. The proof is a simple consequence of the Hamiltonian constraint. First, recall that a symmetric $\binom{0}{2}$ tensor can be decomposed as follows [22]

$$k = k^{TT} + \frac{\tau}{3}g + (\mathscr{L}_Z g - \frac{2}{3}\text{div}Zg), \tag{3.7}$$

where k^{TT} is trace-free and divergence-free with respect to g and Z is a C^∞ vector field. First, we prove that the momentum constraint

$$\nabla^j k_{ij} = 0 \tag{3.8}$$

implies

$$\mathscr{L}_Z g - \frac{2}{3}\text{div}Zg = 0. \tag{3.9}$$

This follows by L^2 pairing of momentum constraint with the vector field Z and an application of Stokes' theorem

$$\int_M (Z^i \nabla^j k_{ij} + Z^j \nabla^i k_{ji})\mu_g = 0 \tag{3.10}$$

$$\int_M (\nabla^i Z^j + \nabla^j Z^i)(\nabla_i Z_j + \nabla_j Z_i - \frac{2}{3}\text{div } Z g_{ij})\mu_g = 0 \tag{3.11}$$

$$\int_M (\nabla^i Z^j + \nabla^j Z^i - \frac{2}{3}\text{div } Z g^{ij})(\nabla_i Z_j + \nabla_j Z_i - \frac{2}{3}\text{div } Z g_{ij})\mu_g = 0$$

$$\int_M |\mathscr{L}_Z g - \frac{2}{3}\text{div } Z g|^2 \mu_g = 0$$

implying

$$\mathscr{L}_Z g - \frac{2}{3}\text{div } Z g = 0. \tag{3.12}$$

We conclude k solving the momentum constraint $\nabla^j k_{ij} = 0$ can be written as

$$k_{ij} = k^{TT}_{ij} + \frac{\tau}{3} g_{ij}. \tag{3.13}$$

Momentum constraint reads on a $t-$ level set M

$$R(g) - |k^{TT} + \frac{\tau}{3} g|^2 + \tau^2 = 2\Lambda \tag{3.14}$$

$$R(g) - |k^{TT}|^2 = 2(\Lambda - \frac{\tau^2}{3}). \tag{3.15}$$

M being negative Yamabe implies that in dimension ≥ 3, $R(g)$ can never be non-negative everywhere on M. This implies

$$\tau^2 > 3\Lambda \tag{3.16}$$

since τ is constant on M. Therefore for expanding solutions

$$-\infty < \tau < \sqrt{3\Lambda}. \tag{3.17}$$

Since τ is a monotonically increasing function of time by our gauge choice 3.5, initially expanding solutions can only continue to expand. This completes the proof of the claim. $\square$

The following scaling is crucial. Consider the entity $\varphi^2 := \tau^2 - 3\Lambda > 0$ that monotonically decays in the time forward direction and $\sqrt{\tau^2 - 3\Lambda}$ acts as a scale. To obtain a sensible dynamical system, one ought to re-scale the vacuum Einstein's equations. This needs to be done in a dimensionally consistent manner. Recall the analysis of [3]. We choose that the coordinates $\{x^i\}$ on M is dimensionless and the metric components g_{ij} has dimensions of $[\text{length}]^2$. In particular, the following holds

$$[g_{ij}] = [\text{length}]^2,\ [k_{ij}] = \text{length},\ [X^i] = \text{length},\ [N] = \text{length}^2. \tag{3.18}$$

Moreover, the mean extrinsic curvature τ has the dimension of length^{-1}. In particular $\varphi^2 = \tau^2 - 3\Lambda$ has dimension of length^{-2}.
We denote the dimensional entities by a $\widetilde{}$ sign, while dimensionless entities are written simply without $\widetilde{}$ sign for convenience. The re-scaled entities are given as follows

$$\widetilde{g}_{ij} = \frac{1}{\varphi^2} g_{ij},\ \widetilde{N} = \frac{1}{\varphi^2} N,\ \widetilde{X}^i = \frac{1}{\varphi} X^i,\ \widetilde{k}^{TT}_{ij} = \frac{1}{\varphi} \Sigma_{ij}, \tag{3.19}$$

where $\varphi = -\sqrt{\tau^2 - 3\Lambda}$ such that $\frac{\varphi}{\tau} > 0$. Now we make the following time coordinate transformation $\tau \mapsto T(\tau)$

$$\partial_T = -\frac{\varphi^2}{\tau} \partial_\tau \tag{3.20}$$

integration of which yields

$$\varphi(T) = \varphi(T_0) e^{T_0} e^{-T}. \tag{3.21}$$

Set $\varphi(T_0) e^{T_0} = -1$. which fixes $\varphi(T)$ and $\tau(T)$ as follows

$$\varphi(T) = -e^{-T},\ \tau(T) = -\sqrt{e^{-2T} + 3\Lambda} \tag{3.22}$$

and in particular $T \in (-\infty, \infty)$ and acts as a *Newtonian* like time. Note that

$$\frac{\varphi(T)}{\tau(T)} = \frac{e^{-T}}{\sqrt{e^{-2T} + 3\Lambda}} \leq (3\Lambda)^{-\frac{1}{2}} e^{-T} \approx e^{-T}. \tag{3.23}$$

The rescaled Einstein's equations in the CMC gauge can be cast into the following system

$$\frac{\partial g_{ij}}{\partial T} = -\frac{2\varphi}{\tau} N\Sigma_{ij} - 2(1 - \frac{N}{n}) g_{ij} - \frac{\varphi}{\tau}(\mathscr{L}_X g)_{ij} \tag{3.24}$$

$$\frac{\partial \Sigma_{ij}}{\partial T} = -(n-1)\Sigma_{ij} - \frac{\varphi}{\tau} N(\mathrm{Ric}_{ij} + \frac{n-1}{n^2} g_{ij}) + \frac{\varphi}{\tau}\nabla_i\nabla_j(\frac{N}{n} - 1) + \frac{2\varphi}{\tau} N\Sigma_{ik}\Sigma^k_j \tag{3.25}$$

$$-\frac{\varphi}{n\tau}(\frac{N}{n} - 1) g_{ij} - (n-2)(\frac{N}{n} - 1)\Sigma_{ij} - \frac{\varphi}{\tau}(\mathscr{L}_X \Sigma)_{ij} \tag{3.26}$$

$$R + \frac{n-1}{n} - |\Sigma|^2 = 0, \tag{3.27}$$

$$\nabla_j \Sigma^j_i = 0. \tag{3.28}$$

For the purpose of our study, this system is not suitable. Instead of treating this system as a coupled wave system (which would inevitably require invoking spatial harmonic gauge to turn the system into a strong hyperbolic system), we will treat the equation for the metric components g_{ij} as transport equations. We define the following new entity that we shall study

$$\mathfrak{T}_{ij} := \mathrm{Ric}_{ij} + \frac{n-1}{n^2} g_{ij}. \tag{3.29}$$

We obtain the following manifestly wave system for $(\Sigma, \mathfrak{T})$.

Remark 5 *We note that Choquet-Bruhat and York [1] considered a similar coupled system for Σ and Ric.*

Lemma 3.1 (wave pair) *Let (g, Σ) verify the reduced Einstein's equations 3.24-3.28. Then the pair $(\Sigma, \mathfrak{T})$ verifies the following system of coupled wave equations modulo spatial diffeomorphism Ψ generated by the vector field X.*

$$\begin{aligned}
\frac{\partial \Sigma_{ij}}{\partial T} &= -\frac{\varphi}{\tau}(\mathscr{L}_X\Sigma)_{ij} - (n-1)\Sigma_{ij} - \frac{\varphi}{\tau}N\mathfrak{T}_{ij} + \frac{\varphi}{\tau}\nabla_i\nabla_j(\frac{N}{n} - 1) + \frac{2\varphi}{\tau}N\Sigma_{ik}\Sigma^k_j && (3.30)\\
&\quad -\frac{\varphi}{n\tau}(\frac{N}{n} - 1)g_{ij} - (n-2)(\frac{N}{n} - 1)\Sigma_{ij}, && (3.31)\\
\frac{\partial \mathfrak{T}_{ij}}{\partial T} &= -\frac{\varphi}{\tau}(\mathscr{L}_X\mathfrak{T})_{ij} - N\frac{\varphi}{\tau}(\Delta_g\Sigma_{ij} - R^m{}_{jli}\Sigma^l_m - R^m{}_{ilj}\Sigma^l_m) + n\frac{\varphi}{\tau}\nabla^l\nabla_i(\frac{N}{n} - 1)\Sigma_{jl}\\
&\quad +n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n} - 1)\nabla_i\Sigma_{jl} + (2-n)\nabla_i\nabla_j(\frac{N}{n} - 1) + n\frac{\varphi}{\tau}\nabla^l\nabla_j(\frac{N}{n} - 1)\Sigma_{il}\\
&\quad +n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n} - 1)\nabla_j\Sigma_{il} - n\frac{\varphi}{\tau}\Delta_g(\frac{N}{n} - 1)\Sigma_{ij} - 2n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n} - 1)\nabla_l\Sigma_{ij}\\
&\quad -\Delta_g(\frac{N}{n} - 1)g_{ij} + N\frac{\varphi}{\tau}(\mathfrak{T}_{ki}\Sigma^k_j + \mathfrak{T}_{kj}\Sigma^k_i) + \frac{2(n-1)}{n^2}(\frac{N}{n} - 1)g_{ij}
\end{aligned}$$

Proof. The proof follows directly from Einstein's equations and the variation formula for curvature. Recall the definition $\mathfrak{T} := \mathrm{Ric}[g] + \frac{n-1}{n^2} g$. Now recall the variation formula for the Ricci curvature

$$\frac{\partial R_{ij}}{\partial T} = \frac{1}{2}\left(\nabla^l\nabla_i\partial_T g_{jl} + \nabla^l\nabla_j\partial_T g_{il} - \nabla^l\nabla_l\partial_T g_{ij}\right) - \frac{1}{2}\nabla_i\nabla_j \mathrm{tr}_g \partial_T g_{ij}. \tag{3.32}$$

Compute each term individually using the evolution equation for the metric g_{ij}.

$$\begin{aligned}\nabla^l\nabla_i\partial_T g_{jl} &= -\frac{2\varphi}{\tau}\Big((\nabla^l\nabla_i N)\Sigma_{jl} + \nabla_i N\nabla^l\Sigma_{jl} + \nabla^l N\nabla_i\Sigma_{jl} + N\nabla^l\nabla_i\Sigma_{jl}\Big) \\ &\quad +2\nabla_i\nabla_j(\frac{N}{n}-1) - \frac{\varphi}{\tau}\nabla^l\nabla_i\mathscr{L}_X g_{lj}\end{aligned} \tag{3.33}$$

Similarly $i \leftrightarrow j$, one obtains

$$\begin{aligned}\nabla^l\nabla_j\partial_T g_{il} &= -\frac{2\varphi}{\tau}\Big((\nabla^l\nabla_j N)\Sigma_{il} + \nabla_j N\nabla^l\Sigma_{il} + \nabla^l N\nabla_j\Sigma_{il} + N\nabla^l\nabla_j\Sigma_{il}\Big) \\ &\quad +2\nabla_i\nabla_j(\frac{N}{n}-1) - \frac{\varphi}{\tau}\nabla^l\nabla_j\mathscr{L}_X g_{il}.\end{aligned} \tag{3.34}$$

Now notice that there are two terms $N\nabla^l\nabla_i\Sigma_{jl}$ and $N\nabla^l\nabla_j\Sigma_{il}$ involving the second derivative of Σ that can further be reduced using the momentum constraint $\nabla^j\Sigma_{ij}=0$ i.e., by Ricci identity

$$\nabla_l\nabla_i\Sigma^l_j = \nabla_i\nabla_l\Sigma^l_j + R^l{}_{kli}\Sigma^k_j - R^m{}_{jli}\Sigma^l_m = R_{ki}\Sigma^k_j - R^m{}_{jli}\Sigma^l_m. \tag{3.35}$$

This observation yields the ellipticity of the map $\Sigma \mapsto \partial_T\mathrm{Ric}$ thereby making the evolution system $(\partial_T\Sigma, \partial_T\mathfrak{T})$ manifestly hyperbolic. The last relation that we need is the following identity regarding the Lie derivative and the Fréchet derivative

$$\mathscr{L}_X\mathrm{Ric} = D\mathrm{Ric}\cdot\mathscr{L}_X g. \tag{3.36}$$

Therefore the terms involving shift vector field X combine to form the Lie derivative term $\mathscr{L}_X\mathfrak{T}$. The collection of every term completes the proof. □

Now let us understand the largeness of data in terms of the largeness of the Weyl curvature. Recall that the Weyl curvature $\mathbf{W}$ is the trace-free part of the Riemann curvature. Let $E_{ij} := \mathbf{W}(\partial_i, n, \partial_j, n)$ be the re-scaled electric part of the Weyl curvature $\mathbf{W}$. Then the following holds

Lemma 3.2 *Rescaled Gauss equation for the embedding $M \hookrightarrow \widetilde{M} = M\times\mathbb{R}$ reads*

$$E_{ij} = \mathfrak{T}_{ij} - \Sigma_{im}\Sigma^m_j - \frac{tr_g k}{2\sqrt{\tau^2-3\Lambda}}\Sigma_{ij} \tag{3.37}$$

Proof. The proof is a direct consequence of the Ricci decomposition of the spacetime curvature. First recall the Ricci decomposition of the Riemann curvature **Riem** of the spacetime $\mathbb{R}\times M$

$$\begin{aligned}\mathbf{Riem}_{\alpha\beta\nu\delta} &= \mathbf{W}_{\alpha\beta\nu\delta} + \frac{1}{2}(\widetilde{g}_{\alpha\nu}\mathbf{Ric}_{\beta\delta} + \widetilde{g}_{\beta\delta}\mathbf{Ric}_{\alpha\nu} - \widetilde{g}_{\beta\nu}\mathbf{Ric}_{\alpha\delta} - \widetilde{g}_{\alpha\delta}\mathbf{Ric}_{\beta\nu}) \\ &\quad +\frac{1}{6}(\widetilde{g}_{\alpha\nu}\widetilde{g}_{\beta\delta} - \widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\nu})\mathbf{R}.\end{aligned} \tag{3.38}$$

The electric part E_{ij} of the Weyl tensor $\mathbf{W}$ is defined as $E_{ij} := \mathbf{W}_{i\mu j\nu}n^\mu n^\nu$ for n being the M orthogonal unit time like future directed vector field. use the vacuum Einstein-Λ equations $\mathbf{Ric}_{\mu\nu} = \Lambda\widetilde{g}_{\mu\nu}$ together with the Gauss equation $R_{ij} - k_{im}k^m_j + k_{ij}tr_g k = \mathbf{Riem}_{i\alpha j\beta}n^\alpha n^\beta + \mathbf{Ric}_{ij}$ to obtain

$$R_{ij} - k_{im}k^m_j + k_{ij}tr_g k = E_{ij} + \frac{2}{3}\Lambda g_{ij} \tag{3.39}$$

which while written in the re-scaled form yields the desired result. □

Remark 6 *There are advantages of working with the pair $(\Sigma, \mathfrak{T})$ instead of (g, Σ). Firstly, the equations are manifestly hyperbolic up to a spatial diffeomorphism and one does not need to deal with the Gribov ambiguities associated with the spatial harmonic gauge often used. In particular, the spatial harmonic gauge is not suited for large data problems in the current context without substantial technical machinery.*

In the previous section, we have chosen the CMC gauge as the time gauge. We need to choose a spatial gauge to analyze the evolution system 3.1. We choose CMC transported spatial gauge (previously [40, 16] used this gauge to study the dynamical stability of big-bang singularity formation)

Definition 3 (Transported spatial gauge) *Let (x_0^1, x_0^2, x_0^3) denote the local coordinates of a neighborhood Ω of the initial Cauchy hypersurface $M(T^{'} = T_0)$. The coordinates $(T^{'}, x^1, x^2, x^3)$ are said to be spatially transported coordinates on $[T_0, T] \times \Omega \subset \widetilde{M} = \mathbb{R} \times M$ if $\widehat{\mathbf{n}}(x^i) = 0,\ i = 1, 2, 3$ with the initial condition (x_0^1, x_0^2, x_0^3), where $\widehat{\mathbf{n}}$ is the unit future directed time-like normal field to the CMC hypersurface M.*

This choice of coordinate yields the vanishing of the shift vector field X on $[T_0, T] \times \Omega \subset \widetilde{M} = \mathbb{R} \times M$

$$\widehat{\mathbf{n}}(x^i) = 0 \implies \frac{1}{N}(\partial_{T'} - X^i\partial_i)(x^i) \implies X^i = 0. \tag{3.40}$$

With this choice of spatial gauge, the re-scaled Einstein's equations read as follows

$$\begin{aligned}
\frac{\partial \Sigma_{ij}}{\partial T} &= -(n-1)\Sigma_{ij} - \frac{\varphi}{\tau}N\mathfrak{T}_{ij} + \frac{\varphi}{\tau}\nabla_i\nabla_j(\frac{N}{n} - 1) + \frac{2\varphi}{\tau}N\Sigma_{ik}\Sigma^k_j && (3.41)\\
&\quad -\frac{\varphi}{n\tau}(\frac{N}{n} - 1)g_{ij} - (n-2)(\frac{N}{n} - 1)\Sigma_{ij}, \\
\frac{\partial \mathfrak{T}_{ij}}{\partial T} &= -N\frac{\varphi}{\tau}(\Delta_g\Sigma_{ij} - R^m{}_{jli}\Sigma^l_m - R^m{}_{ilj}\Sigma^l_m) + n\frac{\varphi}{\tau}\nabla^l\nabla_i(\frac{N}{n} - 1)\Sigma_{jl} && (3.42)\\
&\quad + n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n} - 1)\nabla_i\Sigma_{jl} + (2-n)\nabla_i\nabla_j(\frac{N}{n} - 1) + n\frac{\varphi}{\tau}\nabla^l\nabla_j(\frac{N}{n} - 1)\Sigma_{il} \\
&\quad + n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n} - 1)\nabla_j\Sigma_{il} - n\frac{\varphi}{\tau}\Delta_g(\frac{N}{n} - 1)\Sigma_{ij} - 2n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n} - 1)\nabla_l\Sigma_{ij} \\
&\quad - \Delta_g(\frac{N}{n} - 1)g_{ij} + N\frac{\varphi}{\tau}(\mathfrak{T}_{ki}\Sigma^k_j + \mathfrak{T}_{kj}\Sigma^k_i) + \frac{2(n-1)}{n^2}(\frac{N}{n} - 1)g_{ij}
\end{aligned}$$

that are supplemented by the elliptic equation for the lapse function N

$$-\Delta_g(\frac{N}{n} - 1) + (|\Sigma|^2 + \frac{1}{n})(\frac{N}{n} - 1) = -|\Sigma|^2 \tag{3.43}$$

3.1. Integration and norms

Let U be a coordinate patch on (M, g) and ξ_U be the corresponding partition of unity. For a function f, we define its integral over (M, g) as follows

$$\int_M f\mu_g := \sum_U \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\xi_U\sqrt{\det(g_{ij})}dx^1dx^2dx^3. \tag{3.44}$$

Using this definition of the integration, we can proceed to define $L^p,\ 1 \leq p < \infty$ norm of an arbitrary tensor field $\psi \in \Gamma(^L\otimes TM \otimes\ ^KT^*M),\ K + L \geq 1$ as follows

$$||\psi||^p_{L^p(M)} := \int_M \langle\psi, \psi\rangle_g^{\frac{p}{2}}\mu_g, \tag{3.45}$$

where the $g-$inner product is defined as follows

$$\langle\psi,\psi\rangle_g := g^{i_1j_1}g^{i_2j_3}g^{i_3j_3}....g^{i_Lj_L}g_{m_1n_1}g_{m_2n_2}g_{m_3n_3}....g_{m_Kn_K}\psi^{m_1m_2m_3....m_K}_{i_1i_2i_3....i_L}\psi^{n_1n_2n_3....n_K}_{j_1j_2j_3....j_L}. \quad (3.46)$$

When $p=\infty$, we define L^∞ norm as follows

$$||\psi||_{L^\infty(M)} := \sup_{x\in M}\langle\psi,\psi\rangle_g^{\frac{1}{2}}. \quad (3.47)$$

We need to define the norm of the dynamical entities Σ and $\mathfrak{T}$. For $I\leq 4$, define

$$\mathcal{O}_I := ||\nabla^I\Sigma||_{L^2(M)} + ||\nabla^{I-1}\mathfrak{T}||_{L^2(M)} \quad (3.48)$$

and for $I\leq 3$

$$\mathcal{F}_I := ||e^{\gamma T}\nabla^I\Sigma||_{L^2(M)},\ \gamma=1. \quad (3.49)$$

$$\mathcal{N}^\infty := ||e^{\gamma T}\Sigma||_{L^\infty(M)} + ||e^{2\gamma T}(\frac{N}{n}-1)||_{L^\infty(M)}. \quad (3.50)$$

The total norms are defined as follows

$$\mathcal{O} := ||\Sigma||_{L^2(M)} + \sum_{1\leq I\leq 4}\mathcal{O}_I,\ \mathcal{F} := \sum_{0\leq I\leq 3}\mathcal{F}_I. \quad (3.51)$$

We shall see that the optimal value of γ is 1.

3.2. Setting up the bootstrap argument

We will employ boot-strap arguments to derive uniform upper bounds on the norms $\mathcal{O},\mathcal{F}$, and $\mathcal{N}^\infty$ for the twisted Einstein's equations (or Einstein$-\Lambda$ equations). Our initial data verifies the bound

$$\mathcal{O}_{in} + \mathcal{F}_{in} + \mathcal{N}^\infty_{in} \lesssim \mathcal{I}^0 \quad (3.52)$$

for large $\mathcal{I}^0$ i.e., the initial data size is large. Our main goal is to prove that in the spacetime region $\mathcal{B} := \{M\times[T_0,T],\ T_0\leq T\leq T_\infty\}$, the following estimate holds

$$\mathcal{O} + \mathcal{F} + \mathcal{N}^\infty \lesssim \mathcal{I}^0 + 1. \quad (3.53)$$

Once such a uniform (in T) estimate is obtained, the global well-posedness is followed by the standard local well-posedness and continuation condition 3.1. To accomplish the uniform estimate 3.53, we make a boot-strap assumption

$$\mathcal{O}\leq\mathbb{\Gamma},\ \mathcal{F}\leq\mathbb{Y},\ \mathcal{N}^\infty\leq\mathbb{L} \quad (3.54)$$

for large numbers $\mathbb{\Gamma},\mathbb{Y}$, and $\mathbb{L}$ such that

$$(\mathcal{I}^0)^2 + \mathcal{I}^0 + 1 \ll \min(\mathbb{\Gamma},\mathbb{Y},\mathbb{L}) \quad (3.55)$$

but also such that

$$(\mathbb{\Gamma}+\mathbb{Y}+\mathbb{L})^{20} \leq e^{\frac{T_0}{10}}. \quad (3.56)$$

We then prove the global existence result by the standard continuity method. First, we define the set $\mathcal{S} := \{T|T\in[T_0,T_\infty]$ *and bootstrap assumption* 3.54 *holds*$\}$. Our goal is to first improve the boot-strap assumption on the entire interval $[T_0, T_\infty]$ uniformly and then prove that $\mathcal{S}$ is both open and closed. Then since it is connected it must be equal to $[T, T_\infty]$. First, by the local well-posedness theory, there exists an $\epsilon > 0$ such that given data on T_0, there exists a solution to the Einstein-Λ equation in $[T_0, T_0 + \epsilon]$ and there holds

$$\mathcal{O} \lesssim 2\mathcal{I}^0,\ \mathcal{F} \lesssim 2\mathcal{I}^0,\ \mathcal{N}^\infty \lesssim 2\mathcal{I}^0. \tag{3.57}$$

Therefore, $\mathcal{S}$ is not empty since $[T_0, T_0 + \epsilon] \subset \mathcal{S}$. Closedness follows from uniform estimates and continuity of solutions in T, which does not pose a challenge. The next part is the hardest and constitutes the main part of the article. This entails proving a hierarchy of estimates. In particular, the hierarchy is $\mathcal{N}^\infty \to \mathcal{F} \to \mathcal{O}$. $\mathcal{N}^\infty$ is controlled by means of elliptic estimates and Sobolev inequality, $\mathcal{F}$ is estimated by means of integrating the transport equation for Σ, and finally $\mathcal{O}$ is controlled by means of energy estimates. In particular, the estimates that are obtained are as follows

$$\mathcal{N}^\infty \lesssim \mathcal{O} + \mathcal{F}^2 + \mathcal{F}, \tag{3.58}$$
$$\mathcal{F} \lesssim \mathcal{O} + \mathcal{I}^0 + 1, \tag{3.59}$$
$$\mathcal{O} \lesssim \mathcal{I}^0 + 1 \tag{3.60}$$

uniformly in $T \in [T_0, T_\infty]$ which improves the upper bound on the boot-strap assumption 3.54. Therefore, by the local well-posedness theory, the solution to the Einsten-Λ equation can be extended a bit towards a larger value of T implying the openness of $\mathcal{S}$. Since $\mathcal{S}$ is both open and closed, being connected it must be the full interval $[T_0, T_\infty]$. The estimates obtained are in particular independent of T_∞ which can be taken to ∞ implying global existence.

3.3. Estimates for the metric components

Proposition 3.1 *Under the bootstrap assumption 3.54 and the assumption of the main theorem 1.1, we have for the dynamical metric $g(T')$ on $[T_0, T] \times M$ verifying the Einstein's evolution and constraint equations*

$$C_1 \leq \det g_{ij} \leq C_2, \tag{3.61}$$

were C_1 and C_2 are constants depending only on the initial data.

Proof. This is a direct consequence of the evolution equation for the metric and the bootstrap assumption 3.54. Recall the evolution equation for the metric g_{ij} in CMC-spatially transported gauge

$$\frac{\partial g_{ij}}{\partial T} = -\frac{2\varphi}{\tau} N\Sigma_{ij} - 2(1 - \frac{N}{n})g_{ij}. \tag{3.62}$$

The evolution of the $\mu_g := \sqrt{\det g_{ij}}$ reads

$$\frac{\partial \mu_g}{\partial T} = \frac{\mu_g}{2}\mathrm{tr}_g \partial_T g = n\mu_g(\frac{N}{n} - 1) \tag{3.63}$$

or

$$\mu_g(T) = \mu_g(T_0)e^{n\int_{T_0}^{T}(\frac{N}{n}-1)dt} \tag{3.64}$$

which upon using the bootstrap assumption $||\frac{N}{n} - 1||_{L^\infty} \leq \mathbb{L}e^{-2\gamma t}$ yields

$$\mu_g(T_0)e^{-n\mathbb{L}\int_{T_0}^{T} e^{-2\gamma t}dt} \leq \mu_g(T) \leq \mu_g(T_0)e^{n\mathbb{L}\int_{T_0}^{T} e^{-2\gamma t}dt} \tag{3.65}$$

or

$$|\mu_g(T) - \mu_g(T_0)| \leq \mu_g(T_0)|e^{\frac{n\mathbb{L}}{2\gamma}e^{-2\gamma T_0}} - 1| \lesssim \mathbb{L}e^{-2\gamma T_0}. \tag{3.66}$$

This completes the proof. □

Corollary 3.1 *Under the bootstrap assumption 3.54, the following volume estimate holds for the dynamical metric for any $T \in [T_0, T_\infty]$*

$$|\, Vol(g(T)) - Vol(g_0)| \lesssim \mathbb{L}e^{-2\gamma T_0} \tag{3.67}$$

Proof. Direct computation as in the previous proposition yields

$$|\mathrm{Vol}(g(T)) - \mathrm{Vol}(g_0))| \lesssim \mathrm{Vol}(g_0) \int_{T_0}^{T} ||\frac{N}{n} - 1||_{L^\infty(M(t'))} dt' \lesssim \mathbb{L}e^{-2\gamma T_0}. \tag{3.68}$$

□

Proposition 3.2 *Let $(x, T) \in M \times [T_0, T_\infty]$ denote the CMC-transported coordinates on the spacetime $M \times \mathbb{R}$ and let $\alpha(T)$ and $\beta(T)$ be the largest and smallest eigenvalues of $g(T, x)$ with respect to $g(T_0, x)$. Then the following estimate hold Under the bootstrap assumption 3.54*

$$|\alpha(T) - 1| + |\beta(T) - 1| \lesssim \mathbb{L}e^{-(1+\gamma)T_0} \tag{3.69}$$

Proof. First, define the maximum and minimum eigenvalues of $g(T, x)$ with respect to $g(T_0, x)$ at x

$$\alpha(x, T) = \sup_{0 \neq Y \in T_x M(T_0)} \frac{g(x, T)(Y, Y)}{g(x, 0)(Y, Y)}, \ \beta(x, T) = \inf_{0 \neq Y \in T_x M(T_0)} \frac{g(x, T)(Y, Y)}{g(x, 0)(Y, Y)}. \tag{3.70}$$

Now recall the evolution equation for $g(x, t)$

$$\frac{\partial g_{ij}}{\partial T} = -\frac{2\varphi}{\tau} N \Sigma_{ij} - 2(1 - \frac{N}{n}) g_{ij} \tag{3.71}$$

Compute

$$\partial_T g(Y, Y) = -\frac{2\varphi}{\tau} N \Sigma(Y, Y) - 2(1 - \frac{N}{n}) g(Y, Y) \tag{3.72}$$

First note that $|\Sigma(Y, Y)| \leq \sqrt{g(T_0, x)^{ij} g(T_0, x)^{kl} \Sigma_{ik} \Sigma_{jl}} g(T_0, x)(Y, Y)$. Now integrate 3.72 and use that fact that $|\frac{\varphi(T)}{\tau(T)}| \lesssim e^{-T}$ to obtain

$$\begin{aligned} |g(T, x)(Y, Y) - g(0, x)(Y, Y)| &\lesssim \mathbb{L} e^{\int_{T_0}^{T} ||\frac{N}{n} - 1||_{L^\infty(M)} dt} \int_{T_0}^{T} e^{-\gamma t} e^{-t} e^{\int_{T_0}^{t} ||\frac{N}{n} - 1||_{L^\infty(M)} dt'} g(0, x)(Y, Y) dt \\ &\lesssim \mathbb{L} e^{\mathbb{L} e^{-\gamma T_0}} e^{-(1+\gamma)T_0} g(0, x)(Y, Y) \end{aligned}$$

or

$$|\frac{g(T, x)(Y, Y)}{g(0, x)(Y, Y)} - 1| \lesssim \mathbb{L}e^{-(1+\gamma)T_0} \tag{3.73}$$

which yields

$$|\alpha(T) - 1| + |\beta(T) - 1| \lesssim \mathbb{L}e^{-(1+\gamma)T_0}. \tag{3.74}$$

□

Remark 7 *Propositions 3.1 and 3.2 implies* $g_0 = g(T_0, x) \lesssim g(T, x) \lesssim g(T_0, x) = g_0$ *for all* $T \in [T_0, T_\infty]$ *for a large* $T_0 \geq a$. *This allows us to define Sobolev norms of the tensor fields with respect to the dynamical metric* $g(t, x)$ *and utilize the Sobolev inequalities.*

The following Sobolev inequalities hold true for any smooth tensor field $\psi \in \Gamma({}^K TM \otimes {}^L T^* M)$.

Proposition 3.3 *Let* $\psi \in C^\infty\left(\Gamma({}^K TM \otimes {}^L T^* M)\right)$. *The following inequalities are verified by* ψ *on the closed* 3*− dimensional Riemannian manifold* (M, g)

$$||\psi||_{L^4(M)} \lesssim \sum_{I \leq 1} ||\nabla^I \psi||_{L^2(M)}, \tag{3.75}$$

$$||\psi||_{L^\infty(M)} \lesssim \sum_{I \leq 2} ||\nabla^I \psi||_{L^2(M)} \tag{3.76}$$

Proof. The proof for scalar function $f \in C^\infty(M)$ can be used to obtain the Sobolev type inequality for any arbitrary tensor field ψ. Define the following function $f := \sqrt{|\psi|^2 + \epsilon}$. Now note the following inequality in the light of the metric estimates from 3.1-3.2

$$||f||_{L^4} \lesssim ||f||_{L^2(M)} + ||\nabla f||_{L^2(M)}. \tag{3.77}$$

Now observe

$$\nabla f = \frac{\langle \psi, \nabla \psi \rangle}{\sqrt{|\psi|^2 + \epsilon}} \tag{3.78}$$

by virtue of the metric compatibility of the connection ∇. Therefore

$$|\nabla f| \leq \frac{|\psi||\nabla \psi|}{\sqrt{|\psi|^2 + \epsilon}} \tag{3.79}$$

which yields the desired inequality after letting $\epsilon \to 0$. The other inequality follows in a similar fashion. □

Lastly, we note that on (M, g) we can obtain compactness theorems for Sobolev embeddings. This can be proven by working on local charts using the uniform estimates on the metric $g(t)$ on the interval $[T_0, T_\infty]$ in light of propositions 3.1-3.2 and gluing the charts together in a compatible manner. In particular, on (M, g) for a C^∞ tensor field ψ, the Sobolev norm $H^I(M)$ is defined as

$$||\psi||_{H^I(M)} := \sum_{j \leq I} ||\nabla^j \psi||_{L^2(M)}. \tag{3.80}$$

Proposition 3.4 *The embedding* $H^I(M) \hookrightarrow H^J(M)$ *is compact for* $I > J$.

Theorem 3.1 (Local Well-posedness and continuation) *Let* $C^{-1}\xi_0 \leq g_0 \leq C\xi_0$ *for a smooth background metric* ξ_0 *and* $(\Sigma_0, \mathfrak{T}_0) \in H^I \times H^{I-1}$, $I > \frac{3}{2} + 1$ *be the initial data for the Cauchy problem of the re-scaled evolution equations (3.41-3.42) supplemented by elliptic equation 3.43 in constant mean extrinsic curvature transported spatial gauge satisfying the constraint equations (3.27-3.28). This Cauchy problem is well posed in* $C\left([0, t^*]; H^I \times H^{I-1}\right)$. *In particular, there exists a time* $t^* > 0$ *dependent on* $C, ||\Sigma_0||_{H^I}, ||\mathfrak{T}_0||_{H^{I-1}}$ *such that the solution map* $(\Sigma_0, \mathfrak{T}_0) \mapsto (\Sigma(t), \mathfrak{T}(t), N(t))$ *is continuous as a map*

$$H^I \times H^{I-1} \to H^I \times H^{I-1} \times H^{I+2}.$$

Let $t^{max} \geq t^*$ *be the maximal time of existence of a solution to the Cauchy problem with data* $(\Sigma_0, \mathfrak{T}_0)$, *then either* $t^{max} = \infty$ *or*

$$\lim_{t \to t^{max}} \sup \max\left(\mu_1(t)^{-1}, \mu_2(t)^{-1}, \mu_3(t)^{-1}, ||\nabla\Sigma(t)||_{L^\infty}, ||\mathfrak{T}(t)||_{L^\infty}\right) = \infty, \tag{3.81}$$

where $\{\mu_i(t)\}_{i=1}^3$ *is the set of eigenvalues of* $\xi_0^{-1} g(t)$.

4. Estimates

We need to estimate the spatial Riemann curvature Riem_{ijkl} under the boot-strap assumption in section 3.2. To this end, we use the Ricci decomposition of the Riemann curvature in 3−dimensions

$$\text{Riem}_{ijkl} = \frac{1}{2}(g_{ik}\text{Ric}_{jl} + g_{il}\text{Ric}_{kj} - g_{jk}\text{Ric}_{il} - g_{jl}\text{Ric}_{ik}) + \frac{1}{6}R(g)(g_{il}g_{jk} - g_{ik}g_{jl}), \tag{4.1}$$

where Ric can be written as $\mathfrak{T} - \frac{n-1}{n^2}g$ and the scalar curvature $R(g)$ can be written in terms of the TT variable Σ by the renormalized Hamiltonian constraint equation

$$R(g) + \frac{n-1}{n} = |\Sigma|^2. \tag{4.2}$$

Therefore Riem is completely determined by the dynamical variables $\mathfrak{T}$ and Σ [3].

4.1. Elliptic estimates

Recall the elliptic equation

$$-\Delta_g(\frac{N}{n} - 1) + (|\Sigma|^2 + \frac{1}{n})(\frac{N}{n} - 1) = -|\Sigma|^2. \tag{4.3}$$

We obtain the following estimate for the normalized lapse function $\frac{N}{n} - 1$

Proposition 4.1 *The lapse function N verifies the Sobolev estimate*

$$\sum_{j=0}^{I+2} ||\nabla^j(\frac{N}{n} - 1)||_{L^2(M)} \lesssim (1 + \Gamma)\sum_{j=0}^{I} ||\nabla^j\Sigma||^2_{L^2(M)},\ I = 3, 4 \tag{4.4}$$

assuming the boot-strap condition 3.54 holds.

Remark 8 *Note the loss of decay of $(\frac{N}{n} - 1)$ at the top-most order due to loss of decay of Σ at the top-most order.*

Proof. Under the bootstrap assumption 3.54, first note that Riemann curvature Riem of the metric g verifies the following bound for $0 \leq I \leq 3$

$$||\nabla^I\text{Riem}||_{L^2(M)} \lesssim \Gamma + 1. \tag{4.5}$$

This follows from the Ricci decomposition of the Riemann curvature and vanishing of Weyl curvature in $n = 3$

$$R_{ijkl} = \frac{1}{2}(g_{ik}R_{jl} + g_{il}R_{kj} - g_{jk}R_{il} - g_{jl}R_{ik}) + \frac{1}{6}R(g)(g_{il}g_{jk} - g_{ik}g_{jl}) \tag{4.6}$$

and the constraint equation

$$R(g) = -\frac{n-1}{n} + |\Sigma|^2. \tag{4.7}$$

In light of the propositions 3.1-3.2, this can be obtained schematically as follows

$$|\nabla\text{Riem}| \lesssim |\nabla\mathfrak{T}| + |\nabla R(g)| \lesssim |\nabla\mathfrak{T}| + |\Sigma||\nabla\Sigma|, \tag{4.8}$$

[3] In $n \geq 4$ one needs to estimate the additional Weyl curvature as well

$$|\nabla^2\mathrm{Riem}| \lesssim |\nabla^2\mathfrak{T}| + |\nabla\Sigma|^2 + |\Sigma||\nabla^2\Sigma|, \tag{4.9}$$
$$|\nabla^3\mathrm{Riem}| \lesssim |\nabla^3\mathfrak{T}| + |\nabla\Sigma||\nabla^2\Sigma| + |\Sigma||\nabla^3\Sigma| \tag{4.10}$$

which implies by the Minkowski inequality and Sobolev embedding

$$||\nabla\mathrm{Riem}||_{L^2(M)} \lesssim ||\nabla\mathfrak{T}||_{L^2(M)} + ||\nabla\Sigma||_{L^2(M)}||\nabla^2\Sigma||_{L^2(M)} \lesssim \Gamma + e^{-2\gamma T}\mathbb{Y}^2 \lesssim \Gamma + 1, \tag{4.11}$$
$$||\nabla^2\mathrm{Riem}||_{L^2(M)} \lesssim ||\nabla^2\mathfrak{T}||_{L^2(M)} + ||\nabla^2\Sigma||^2_{L^2(M)} \lesssim \Gamma + e^{-2\gamma T}\mathbb{Y}^2 \lesssim \Gamma + 1, \tag{4.12}$$
$$||\nabla^3\mathrm{Riem}||_{L^2(M)} \lesssim \Gamma + 1 \tag{4.13}$$

and similarly $||\mathrm{Riem}||_{L^2(M)} \lesssim 1 + \Gamma$. Now we are ready to obtain the estimates for $\frac{N}{n} - 1$. Let us recall the elliptic equation

$$-\Delta_g(\frac{N}{n} - 1) + (|\Sigma|^2 + \frac{1}{n})(\frac{N}{n} - 1) = -|\Sigma|^2. \tag{4.14}$$

Let us denote the operator $-\Delta_g + (|\Sigma|^2 + \frac{1}{n})$ by $\mathscr{L}$. We claim that an estimate of the following type holds for $u \in C^\infty(M)$

$$\sum_{j=0}^{I+2} ||\nabla^j u||_{L^2(M)} \lesssim \sum_{j=0}^{I} ||\nabla^j(\mathscr{L}u)||_{L^2(M)} \tag{4.15}$$

with the regularity $\nabla^{I-1}\mathrm{Riem} \in L^2(M)$. To do this we commute the lapse equation with ∇^I for $0 \leq I \leq 4$

$$\begin{aligned}\mathscr{L}\nabla^I(\frac{N}{n} - 1) &= -\Delta_g\nabla^I(\frac{N}{n} - 1) + (|\Sigma|^2 + \frac{1}{n})\nabla^I(\frac{N}{n} - 1) = \nabla^I\mathscr{L}(\frac{N}{n} - 1)\\
&+ \sum_{J_1+J_2+J_3=I-1} \nabla^{J_1+1}\Sigma\nabla^{J_2}\Sigma\nabla^{J_3}(\frac{N}{n} - 1) + \sum_{m=0}^{I-1}\sum_{J_1+J_2=m} \nabla^{J_1}\mathrm{Ric}\nabla^{J_2+I-m}(\frac{N}{n} - 1)\\
&+ \sum_{m=0}^{I-1}\sum_{J_1+J_2=m} \nabla^{J_1}\mathrm{Riem}\nabla^{J_2+I-m}(\frac{N}{n} - 1) + \sum_{m=0}^{I-2}\sum_{J_1+J_2=m} \nabla^{J_1+1}\mathrm{Riem}\nabla^{J_2+I-m-1}(\frac{N}{n} - 1).\end{aligned}$$

The elliptic regularity implies

$$\begin{aligned}||\nabla^{I+2}(\frac{N}{n} - 1)||_{L^2(M)} &\lesssim ||\nabla^I\mathscr{L}(\frac{N}{n} - 1)||_{L^2(M)}\\
&+||(|\Sigma|^2 + \frac{1}{n})\nabla^I(\frac{N}{n} - 1)||_{L^2(M)} + ||\sum_{J_1+J_2+J_3=I-1} \nabla^{J_1+1}\Sigma\nabla^{J_2}\Sigma\nabla^{J_3}(\frac{N}{n} - 1)||_{L^2(M)}\\
&+||\sum_{m=0}^{I-1}\sum_{J_1+J_2=m} \nabla^{J_1}\mathrm{Ric}\nabla^{J_2+I-m}(\frac{N}{n} - 1)||_{L^2(M)} + ||\sum_{m=}^{I-1}\sum_{J_1+J_2=m} \nabla^{J_1}\mathrm{Riem}\nabla^{J_2+I-m}(\frac{N}{n} - 1)||_{L^2(M)}\\
&+||\sum_{m=0}^{I-2}\sum_{J_1+J_2=m} \nabla^{J_1+1}\mathrm{Riem}\nabla^{J_2+I-m-1}(\frac{N}{n} - 1)||_{L^2(M)}.\end{aligned}$$

Under the boot-strap assumption 3.54, we estimate each term separately. First estimate

$$||(|\Sigma|^2 + \frac{1}{n})\nabla^I(\frac{N}{n} - 1)||_{L^2(M)} \lesssim ||\nabla^I(\frac{N}{n} - 1)||_{L^2(M)}(1 + e^{-2\gamma T}\mathbb{Y}^2) \lesssim ||\nabla^I(\frac{N}{n} - 1)||_{L^2(M)}.$$

The next term reads

$$||\sum_{J_1+J_2+J_3=I-1} \nabla^{J_1+1}\Sigma\nabla^{J_2}\Sigma\nabla^{J_3}(\frac{N}{n} - 1)||_{L^2(M)} \lesssim e^{-\gamma T}\mathbb{Y}\Gamma||\nabla^I(\frac{N}{n} - 1)||_{L^2(M)} \tag{4.16}$$

$$\lesssim ||\nabla^I(\frac{N}{n}-1)||_{L^2(M)}.$$

for the following terms we have

$$||\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\nabla^{J_1}\mathrm{Ric}\nabla^{J_2+I-m}(\frac{N}{n}-1)||_{L^2(M)}\lesssim \Gamma||\nabla^I(\frac{N}{n}-1)||_{L^2(M)}, \tag{4.17}$$

$$||\sum_{m=}^{I-1}\sum_{J_1+J_2=m}\nabla^{J_1}\mathrm{Riem}\nabla^{J_2+I-m}(\frac{N}{n}-1)||_{L^2(M)}\lesssim \Gamma||\nabla^I(\frac{N}{n}-1)||_{L^2(M)},$$

$$||\sum_{m=0}^{I-2}\sum_{J_1+J_2=m}\nabla^{J_1+1}\mathrm{Riem}\nabla^{J_2+I-m-1}(\frac{N}{n}-1)||_{L^2(M)}\lesssim \Gamma||\nabla^I(\frac{N}{n}-1)||_{L^2(M)}.$$

Therefore collecting all the terms together, we conclude

$$||(\frac{N}{n}-1)||_{H^{I+2}(M)}\lesssim ||\mathscr{L}(\frac{N}{n}-1)||_{H^I(M)}+(1+\Gamma)||(\frac{N}{n}-1)||_{H^I(M)}. \tag{4.18}$$

Now we prove the desired estimates in an iterative way starting from the first-order estimate. First, recall the elliptic equation

$$-\Delta_g(\frac{N}{n}-1)+(|\Sigma|^2+\frac{1}{n})(\frac{N}{n}-1)=-|\Sigma|^2. \tag{4.19}$$

Now multiply both sides by $\frac{N}{n}-1$ and integrate by parts

$$\int_M|\nabla(\frac{N}{n}-1)|^2\mu_g+\int_M(|\Sigma|^2+\frac{1}{n})(\frac{N}{n}-1)^2\mu_g=-\int_M|\Sigma|^2(\frac{N}{n}-1)\mu_g. \tag{4.20}$$

Now we estimate the term on the right-hand side by Cauchy-Schwartz as follows

$$-\int_M|\Sigma|^2(\frac{N}{n}-1)\mu_g\leq(n+1)\int_M|\Sigma|^4\mu_g+\frac{1}{n+1}\int_M(\frac{N}{n}-1)^2\mu_g. \tag{4.21}$$

Therefore, 4.20 reads

$$\int_M|\nabla(\frac{N}{n}-1)|^2\mu_g+\int_M|\Sigma|^2(\frac{N}{n}-1)^2\mu_g+(\frac{1}{n}-\frac{1}{n+1})\int_M(\frac{N}{n}-1)^2\mu_g\leq(n+1)\int_M|\Sigma|^4\mu_g$$

which by means of the inequality 3.3 yields

$$\begin{aligned}\int_M|\nabla(\frac{N}{n}-1)|^2\mu_g+\int_M|\Sigma|^2(\frac{N}{n}-1)^2\mu_g+\frac{1}{n(n+1)}\int_M(\frac{N}{n}-1)^2\mu_g&\\ \lesssim\left(\int_M|\nabla\Sigma|^2\mu_g+\int_M|\Sigma|^2\mu_g\right)^2&.\end{aligned}$$

More concretely, we have the two lowest-order estimates

$$||\frac{N}{n}-1||_{L^2(M)}\lesssim||\Sigma||^2_{H^1(M)}, \tag{4.22}$$

$$||\frac{N}{n}-1||_{H^1(M)}\lesssim||\Sigma||^2_{H^1(M)}. \tag{4.23}$$

Now iterate this using 4.18 to obtain

$$||\frac{N}{n}-1||_{H^2(M)}\lesssim|||\Sigma|^2||_{L^2(M)}+(1+\Gamma)||\frac{N}{n}-1)||_{L^2(M)}\lesssim|||\Sigma|^2||_{L^2(M)}+(1+\Gamma)||\Sigma||^2_{H^1(M)},$$

$$||\frac{N}{n}-1||_{H^3(M)} \lesssim |||\Sigma|^2||_{H^1(M)} + (1+\Gamma)||\frac{N}{n}-1)||_{H^1(M)} \lesssim |||\Sigma|^2||_{H^1(M)} + (1+\Gamma)||\Sigma||^2_{H^1(M)},$$
$$||\frac{N}{n}-1||_{H^4(M)} \lesssim |||\Sigma|^2||_{H^2(M)} + (1+\Gamma)||\frac{N}{n}-1)||_{H^2(M)} \lesssim |||\Sigma|^2||_{H^2(M)} + (1+\Gamma)||\Sigma||^2_{H^1(M)}$$

and so on. ow use the embedding $H^{i_1}(M) \hookrightarrow H^{i_2}(M)$ for $i_1 > i_2$, and the algebra property of the Sobolev spaces $H^s(M)$ for $s > \frac{n}{2}$, we conclude the desired estimate

$$\sum_{j=0}^{I+2} ||\nabla^j(\frac{N}{n}-1)||_{L^2(M)} \lesssim (1+\Gamma)\sum_{j=0}^{I} ||\nabla^j\Sigma||^2_{L^2(M)},\ I > 2. \tag{4.24}$$

Now let us provide a rationale behind the existence of such an estimate. Note that we have implicitly utilized the special property of the operator $\mathscr{L}$ that it is injective between appropriate Sobolev spaces. This in fact allowed us to effectively replace the lower order term $||\frac{N}{n}-1||_{H^I(M)}$ in the estimate 4.18 by the source term $|\Sigma|^2$ using the integration by parts and Cauchy-Schwartz trick. This can be understood in terms of the compact embedding $H^{I+2}(M) \hookrightarrow H^I(M)$ for M compact together with the injectivity of $\mathscr{L}$ between $H^{I+2}(M)$ and $H^I(M)$ and ignore the lower order term. For this, we need to show that the kernel

$$\ker\mathscr{L} := \{\zeta \in H^{I+2}(M)|\mathscr{L}\zeta = 0\} = \{\emptyset\}. \tag{4.25}$$

This is trivial since by means of the uniform ellipticity of $g(t)$ due to proposition 3.1-3.2, an elementary calculation by Stokes yields $\forall\zeta \in \ker\mathscr{L}$

$$\int_M |\nabla\zeta|^2\mu_g + n\int_M (|\Sigma|^2 + \frac{1}{n})\zeta^2\mu_g = 0 \tag{4.26}$$

implying $\zeta = 0$ or $\ker\mathscr{L} = \{\emptyset\}$. Now we prove the improved claim by contradiction. Assume that an estimate of the type $||u||_{H^{I+2}(M)} \lesssim ||\mathscr{L}u||_{H^I(M)}$ does not hold. Then there exists a sequence $\{u_i\}_{i=1}^{\infty}$ with $||u_i||_{H^{s+1}} = 1$ and $||\mathscr{L}u_i||_{H^{s-1}} \to 0$ as $i \to \infty$. Now M is compact and therefore $H^{I+2}(M)$ is compactly embedded into $H^I(M)$. This yields a sub-sequence $\{u_{i_j}\}$ converging to $u^* \in H^{I+2}$ strongly in H^I, which by construction satisfies $\mathscr{L}u^* = 0$. This contradicts the fact that the operator $\mathscr{L}$ is injective. Therefore an estimate of the type $||u||_{H^{I+2}(M)} \lesssim ||\mathscr{L}u||_{H^I(M)}$ holds for $I \geq 0$.
The elliptic estimate yields together with the algebra property of Sobolev spaces for $I > 2$

$$||\frac{N}{n}-1||_{H^{I+2}(M)} \lesssim ||\Sigma||^2_{H^I(M)} \tag{4.27}$$

□

The next part entails controlling the lower order norm $\mathcal{F}$ of Σ in terms of the top order norm $\mathcal{O}$ and the initial data norm $\mathcal{I}^0$.

4.2. Estimating the lower order norm of TT tensor Σ via transport equation for Σ
In this section, we prove that the weighted lower order norm of the TT tensor Σ is controlled by the top order norm of $(\mathfrak{T}, \Sigma)$ and the initial data. Recall the equation for Σ

$$\begin{aligned}\frac{\partial\Sigma_{ij}}{\partial T} &= -(n-1)\Sigma_{ij} - \frac{\varphi}{\tau}N\mathfrak{T}_{ij} + \frac{\varphi}{\tau}\nabla_i\nabla_j(\frac{N}{n}-1) + \frac{2\varphi}{\tau}N\Sigma_{ik}\Sigma^k_j \\ &\quad -\frac{\varphi}{n\tau}(\frac{N}{n}-1)g_{ij} - (n-2)(\frac{N}{n}-1)\Sigma_{ij}.\end{aligned} \tag{4.28}$$

To estimate the lower order norm $\mathcal{F}$, we will treat the equation for Σ as a transport equation.

Definition 4

$$\mathscr{E}^{low} := \frac{1}{2}\sum_{I\leq 3}\int_M |\nabla^I\Sigma|^2\mu_g \tag{4.29}$$

First note that $\mathcal{F}^2 \approx \mathscr{E}^{low}$. We have the following theorem estimating $\mathcal{F}^2$.

Theorem 4.1 *The lower order norm the entropy $\mathcal{E}^{low}$ verifies the estimate*

$$\mathcal{F}^2 \approx e^{2\gamma T}\mathscr{E}^{low} \lesssim 1 + (\mathcal{I}^0)^2 + \mathcal{O}^2,\ \gamma = 1 \tag{4.30}$$

Proof. Recall definition 4

$$\mathscr{E}^{low}(T) := \frac{1}{2}\sum_{I\leq 3}\int_{M(T)}\langle\nabla^I\Sigma, \nabla^I\Sigma\rangle\mu_g. \tag{4.31}$$

Explicit application of the time evolution operator $\frac{d}{dT}$ together with the transport theorem [27] yields

$$\frac{d\mathscr{E}^{low}}{dT} = \sum_{I\leq 3}\int_M \langle\nabla^I\partial_T\Sigma, \nabla^I\Sigma\rangle\mu_g + \sum_{I\leq 3}\int_M \langle[\partial_T, \nabla^I]\Sigma, \nabla^I\Sigma\rangle\mu_g + \mathscr{E}\mathscr{R}_2, \tag{4.32}$$

where the error terms $\mathscr{E}\mathscr{R}_2$ reads schematically

$$\mathscr{E}\mathscr{R}_2 \sim \sum_{I\leq 3}\int_{M(T)}\partial_T(g^{-1}....g^{-1}g.....g)\nabla^I\Sigma\nabla^I\Sigma\mu_g + \sum_{I\leq 3}\int_{M(T)}|\nabla^I\Sigma|^2\partial_T\mu_g. \tag{4.33}$$

Using the equation for $\partial_T\Sigma$, $\frac{d\mathscr{E}^{low}}{dT}$ reads

$$\begin{aligned}
\frac{d\mathscr{E}^{low}}{dT} &= -2(n-1)\mathscr{E} + \sum_{I\leq 3}\frac{\varphi}{\tau}\int_{M(T)} N\nabla^I\mathfrak{T}\nabla^I\Sigma\mu_g \\
&+ \sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{M(T)}\frac{\varphi}{\tau}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\Sigma\nabla^I\Sigma\mu_g \\
&+ \sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2+J_3=m}\int_{M(T)}\frac{\varphi}{\tau}\nabla^{J_1}N\nabla^{J_2+1}\Sigma\nabla^{J_3+I-m-1}\Sigma\nabla^I\Sigma\mu_g \\
&+ \sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\Sigma\nabla^I\Sigma\mu_g \\
&+ \frac{\varphi}{\tau}\sum_{I\leq 3}\sum_{J_1+J_2+J_3=I}\int_{M(T)}\nabla^{J_1}N\nabla^{J_2}\Sigma\nabla^{J_3}\Sigma\nabla^I\Sigma\mu_g \\
&+ \sum_{I\leq 3}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{I+2}(\frac{N}{n}-1)\nabla^I\Sigma\mu_g \\
&+ \sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{M(T)}\nabla^{J_1}(\frac{N}{n}-1)\nabla^{J_2}\Sigma\nabla^I\Sigma\mu_g \\
&+ \sum_{I\leq 3}\frac{\varphi}{\tau}\int_M\sum_{J_1+J_2=I-1}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2}\mathfrak{T}\nabla^I\Sigma\mu_g + \mathscr{E}\mathscr{R}_2.
\end{aligned} \tag{4.34}$$

Now we estimate each term separately. First note that theorem 4.2 implies boundedness of $\sum_{I\leq 3}||\nabla^I\mathfrak{T}||_{L^2(M)}$ by $\mathcal{I}^0$. Therefore

$$|\sum_{I\leq 3}\frac{\varphi}{\tau}\int_{M(T)} N\nabla^I\mathfrak{T}\nabla^I\Sigma\mu_g| \lesssim e^{-T}\mathcal{O}^2. \tag{4.35}$$

The next term is estimated as follows

$$|\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{M(T)}\frac{\varphi}{\tau}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\Sigma\nabla^I\Sigma\mu_g| \lesssim e^{-(1+4\gamma)T}\mathbb{Y}^4, \tag{4.36}$$

$$|\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2+J_3=m}\int_{M(T)}\frac{\varphi}{\tau}\nabla^{J_1}N\nabla^{J_2+1}\Sigma\nabla^{J_3+I-m-1}\Sigma\nabla^I\Sigma\mu_g| \lesssim e^{-(1+3\gamma)T}\mathbb{Y}^3, \tag{4.37}$$

$$|\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\Sigma\nabla^I\Sigma\mu_g| \lesssim e^{-4\gamma T}\mathbb{Y}^4, \tag{4.38}$$

$$|\frac{\varphi}{\tau}\sum_{I\leq 3}\sum_{J_1+J_2+J_3=I}\int_{M(T)}\nabla^{J_1}N\nabla^{J_2}\Sigma\nabla^{J_3}\Sigma\nabla^I\Sigma\mu_g| \lesssim e^{-(1+3\gamma)T}\mathbb{Y}^3, \tag{4.39}$$

$$|\sum_{I\leq 3}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{I+2}(\frac{N}{n}-1)\nabla^I\Sigma\mu_g| \lesssim e^{-(1+3\gamma)T}\mathbb{Y}^3, \tag{4.40}$$

$$|\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{M(T)}\nabla^{J_1}(\frac{N}{n}-1)\nabla^{J_2}\Sigma\nabla^I\Sigma\mu_g| \lesssim e^{-4\gamma T}\mathbb{Y}^4, \tag{4.41}$$

$$|\sum_{I\leq 3}\frac{\varphi}{\tau}\int_M\sum_{J_1+J_2=I-1}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2}\mathfrak{T}\nabla^I\Sigma\mu_g| \lesssim e^{-(1+3\gamma)T}\Gamma\mathbb{Y}^3. \tag{4.42}$$

Collection of all terms yields

$$\begin{aligned}\frac{d\mathscr{E}}{dT}+2(n-1)\mathscr{E} \lesssim e^{-T}\mathcal{O}^2+e^{-(1+4\gamma)T}\mathbb{Y}^4+e^{-(1+3\gamma)T}\mathbb{Y}^3\\ +e^{-4\gamma T}\mathbb{Y}^4+e^{-(1+3\gamma)T}\Gamma\mathbb{Y}^3\end{aligned} \tag{4.43}$$

integration of which yields

$$\begin{aligned}\mathscr{E}(T) \lesssim e^{-2(n-1)(T-T_0)}\mathscr{E}(T_0)\\ +e^{-2(n-1)T}\left(e^{[2(n-1)-1]T}-e^{[2(n-1)-1]T_0}\right)\mathcal{O}^2\\ +e^{-2(n-1)T}\left(e^{[2(n-1)-(1+4\gamma)]T}-e^{[2(n-1)-(1+4\gamma)]T_0}\right)\mathbb{Y}^4\\ +e^{-2(n-1)T}\left(e^{[2(n-1)-(1+3\gamma)]T}-e^{[2(n-1)-(1+3\gamma)]T_0}\right)\mathbb{Y}^3\\ +e^{-2(n-1)T}\left(e^{[2(n-1)-4\gamma]T}-e^{[2(n-1)-4\gamma]T_0}\right)\mathbb{Y}^4\end{aligned} \tag{4.44}$$

$$+e^{-2(n-1)T}\left(e^{[2(n-1)-(1+3\gamma)]T}-e^{[2(n-1)-(1+3\gamma)]T_0}\right)\Gamma\mathbb{Y}^3$$

or

$$\begin{aligned}
e^{2\gamma T}\mathscr{E}(T) &\leq \frac{e^{2(n-1-\gamma)T_0}}{e^{2(n-1-\gamma)T}}e^{2\gamma T_0}\mathscr{E}(T_0) \\
&+e^{-2(n-1-\gamma)T}\left(e^{[2(n-1)-1]T}-e^{[2(n-1)-1]T_0}\right)\mathcal{O}^2 \\
&+e^{-2(n-1-\gamma)T}\left(e^{[2(n-1)-(1+4\gamma)]T}-e^{[2(n-1)-(1+4\gamma)]T_0}\right)\mathbb{Y}^4 \\
&+e^{-2(n-1-\gamma)T}\left(e^{[2(n-1)-(1+3\gamma)]T}-e^{[2(n-1)-(1+3\gamma)]T_0}\right)\mathbb{Y}^3 \\
&+e^{-2(n-1-\gamma)T}\left(e^{[2(n-1)-4\gamma]T}-e^{[2(n-1)-4\gamma]T_0}\right)\mathbb{Y}^4 \\
&= \frac{e^{2(n-1-\gamma)T_0}}{e^{2(n-1-\gamma)T}}e^{2\gamma T_0}\mathscr{E}(T_0)+e^{-(1-2\gamma)T}\left(1-\frac{e^{[2(n-1)-1]T_0}}{e^{[2(n-1)-1]T}}\right)\mathcal{O}^2 \\
&+e^{-(1+2\gamma)T}\left(1-\frac{e^{[2(n-1)-(1+4\gamma)]T_0}}{e^{[2(n-1)-(1+4\gamma)]T}}\right)\mathbb{Y}^4+e^{-(1+\gamma)T}\left(1-\frac{e^{[2(n-1)-(1+3\gamma)]T_0}}{e^{[2(n-1)-(1+3\gamma)]T}}\right)\mathbb{Y}^3 \\
&+e^{-2\gamma T}\left(1-\frac{e^{[2(n-1)-4\gamma]T_0}}{e^{[2(n-1)-4\gamma]T}}\right)\mathbb{Y}^3
\end{aligned}$$

$$\lesssim e^{2\gamma T_0}\mathscr{E}(T_0)+e^{-(1-2\gamma)T}\mathcal{O}^2+e^{-(1+2\gamma)T_0}\mathbb{Y}^4+e^{-(1+\gamma)T_0}\mathbb{Y}^3+e^{-2\gamma T_0}\mathbb{Y}^3\lesssim e^{-(1-2\gamma)T}\mathcal{O}^2+(\mathcal{I}^0)^2+1.$$

Therefore

$$e^{2\gamma T}\mathscr{E}^{low}(T):=\frac{e^{2\gamma T}}{2}\sum_{I\leq 3}\int_{M(T)}\langle\nabla^I\Sigma,\nabla^I\Sigma\rangle\mu_g\lesssim e^{-(1-2\gamma)T}\mathcal{O}^2+(\mathcal{I}^0)^2+1\ \forall T\in[T_0,T_\infty] \tag{4.45}$$

Now note that $e^{2\gamma T}\mathscr{E}^{low}(T)$ may be estimated by $(\mathcal{I}^0)^2+1$ for $\gamma\leq\frac{1}{2}$ in light of the bootstrap assumption 3.54. However, we will not do so. Instead, we will improve the admissible value of γ to 1 by iteration. We estimate the terms multiplied by $\frac{\varphi}{\tau}\lesssim e^{-T}$ by 4.1 as usual. $\gamma=\frac{1}{2}$ in 4.45 yields

$$e^T\mathcal{E}^{low}(T)\lesssim 1+(\mathcal{I}^0)^2+(\mathcal{O})^2. \tag{4.46}$$

With the decay obtained in this step, write the energy inequality with this improved decay $||\nabla^I\Sigma||_{L^2(M)}\lesssim e^{-\frac{1}{2}T}(\mathcal{O}+(\mathcal{I}^0)^2+1),\ I\leq 3$ i.e., $\gamma=\frac{1}{2}$

$$\begin{aligned}
\frac{d\mathscr{E}^{low}}{dT}+2(n-1)\mathscr{E}^{low}&\lesssim e^{-(1+\frac{1}{2})T}(\mathcal{O})^2+e^{-(1+4\gamma)T}\mathbb{Y}^4+e^{-(1+3\gamma)T}\mathbb{Y}^3 \\
&+e^{-4\gamma T}\mathbb{Y}^4+e^{-(1+3\gamma)T}\mathcal{I}^0\mathbb{Y}^3
\end{aligned} \tag{4.47}$$

integration of which yields

$$e^{(1+\frac{1}{2})T}\mathscr{E}^{low}(T)\lesssim 1+(\mathcal{I}^0)^2+(\mathcal{O})^2. \tag{4.48}$$

Therefore iterating and using the new γ in the next step, we obtain the final decay rate $2\gamma^{optimal}$ for $\mathscr{E}^{low}(T)$

$$2\gamma^{optimal}=1+\frac{1}{2}(1+\frac{1}{2}(1+\cdots\cdot=2\ or\ \gamma^{optimal}=1 \tag{4.49}$$

and therefore

$$e^{2\gamma T}\mathscr{E}^{low}\lesssim 1+(\mathcal{I}^0)^2+\mathcal{O}^2,\ \gamma=1 \tag{4.50}$$

□

As an immediate corollary we obtain

Corollary 4.1 *$\mathcal{F} \lesssim 1 + \mathcal{I}^0 + \mathcal{O}$ and $\mathcal{N}^\infty \lesssim 1 + (\mathcal{I}^0)^2 + \mathcal{O}^2$ by elliptic estimates 4.1.*

4.3. Energy Estimates

With the individual estimates available we are ready to complete the energy estimates for the pair $(\Sigma, \mathfrak{T})$. Let us recall the definition of the top order energy $\mathscr{E}^{top}$

Definition 5

$$\mathscr{E}^{top} := \frac{1}{2}\sum_{I\leq 4}\int_M \langle\nabla^I\Sigma, \nabla^I\Sigma\rangle\mu_g + \frac{1}{2}\sum_{I\leq 4}\int_M \langle\nabla^{I-1}\mathfrak{T}, \nabla^{I-1}\mathfrak{T}\rangle\mu_g. \tag{4.51}$$

First, we recall the following proposition

Proposition 4.2 *Let (Φ, Ψ) be a pair of smooth $\binom{K}{L}$ type tensor field on (M, g) that verifies the wave equation*

$$\partial_T\Phi = -N\Psi + P_\Phi, \tag{4.52}$$
$$\partial_T\Psi = -N\Delta_g\Phi + Q_\Psi \tag{4.53}$$

on $[T_0, T_\infty]$. Denote by $M(T)$ the slice at time t. Then (Φ, Ψ) verify the following integrated energy estimate

$$\int_{M(t=T)}|\nabla\Phi|^2\mu_g + \int_{M(t=T)}|\Psi|^2\mu_g = \int_{M(t=T_0)}|\nabla\Phi|^2\mu_g + \int_{M(t=T_0)}|\Psi|^2\mu_g \tag{4.54}$$
$$+2\int_{T_0}^{T}\int_{M(T)}\left(\langle\nabla P_\Phi, \nabla\Phi\rangle + \langle Q_\Psi, \Psi\rangle\right)\mu_g dt + \int_{T_0}^{T}\mathscr{E}\mathscr{R}(t)dt,$$

where the error term $\mathscr{E}\mathscr{R}(t)$ verifies the following estimate

$$|\mathscr{E}\mathscr{R}(T)| \lesssim \left(||\frac{N}{n} - 1||_{L^\infty(M(T))} + \frac{\varphi}{\tau}||\Sigma||_{L^\infty(M(T))}\right)(||\nabla\Phi||^2_{L^2(M(T))} + |\Psi|^2_{L^2(M(T))}) \tag{4.55}$$
$$+[\frac{\varphi}{\tau}(||\nabla\Sigma||_{L^4(M)} + ||\Sigma||_{L^\infty(M)}||\nabla(\frac{N}{n} - 1)||_{L^4(M)}) + ||\nabla(\frac{N}{n} - 1)||_{L^4(M)}] \tag{4.56}$$
$$||\Phi||_{L^4(M)}||\nabla\Phi||_{L^2(M)}$$

Proof. For a smooth tensor field pair (Φ, Ψ) on compact (M, g), $\int_{M(T)}|\nabla\Phi|^2\mu_g + \int_{M(T)}|\Psi|^2\mu_g$ is finite. Therefore use the transport theorem [27] to obtain

$$\frac{d}{dT}\left(\int_{M(T)}(|\nabla\Phi|^2 + |\Psi|^2)\mu_g\right) = \int_{M(T)}\langle\nabla\partial_T\Phi, \nabla\Phi\rangle\mu_g + \int_{M(T)}\langle\partial_T\Psi, \Psi\rangle\mu_g + \mathscr{E}\mathscr{R}(T) \tag{4.57}$$

where the error term $\mathscr{E}\mathscr{R}(t)$ reads schematically

$$\mathscr{E}\mathscr{R}(T) \sim \int_{M(T)}\langle[\partial_T, \nabla]\Phi, \nabla\Phi\rangle\mu_g + \int_{M(T)}\partial_T(g^{-1}....g^{-1}g.....g)\nabla\Phi\nabla\Phi\mu_g \tag{4.58}$$
$$+\int_{M(T)}\partial_T(g^{-1}....g^{-1}g.....g)\Psi\Psi\mu_g + \int_{M(T)}(|\nabla\Phi|^2 + |\Psi|^2)\partial_T\mu_g$$

First estimate the error terms. Recall the evolution equation for g and g^{-1}

$$\frac{\partial g_{ij}}{\partial T} = -\frac{2\varphi}{\tau}N\Sigma_{ij} - 2(1 - \frac{N}{n})g_{ij}, \tag{4.59}$$

$$\frac{\partial g^{ij}}{\partial T} = \frac{2\varphi}{\tau} N\Sigma^{ij} + 2(1 - \frac{N}{n}) g^{ij} \tag{4.60}$$

and similarly $\partial_T \mu_g = \frac{1}{2}\mu_g \mathrm{tr}_g \partial_T g = -n(1 - \frac{N}{n})\mu_g$. Moreover,

$$[\partial_T, \nabla]\Phi \sim \partial_T \Gamma[g]\Phi \sim \frac{\varphi}{\tau}\nabla(N\Sigma) + \nabla(\frac{N}{n} - 1) \tag{4.61}$$

which yields by means of the proposition 3.1-3.2

$$|[\partial_T, \nabla]\Phi| \lesssim \frac{\varphi}{\tau}(|\nabla\Sigma| + |\Sigma||\nabla(\frac{N}{n} - 1)|) + |\nabla(\frac{N}{n} - 1)|. \tag{4.62}$$

This leads to the following by Hölder

$$|\int_{M(T)} \langle [\partial_T, \nabla]\Phi, \nabla\Phi\rangle \mu_g| \lesssim \frac{\varphi}{\tau}\left(||\nabla\Sigma||_{L^4(M)} + ||\Sigma||_{L^\infty(M)}||\nabla(\frac{N}{n} - 1||_{L^4(M)}\right) \tag{4.63}$$
$$||\Phi||_{L^4(M)}||\nabla\Phi||_{L^2(M)} + ||\nabla(\frac{N}{n} - 1||_{L^4(M)}||\Phi||_{L^4(M)}||\nabla\Phi||_{L^2(M)}$$

Therefore, the collection of all the terms yields the following estimate for the error term

$$\mathscr{E}\mathscr{R}(T) \lesssim \left(||\frac{N}{n} - 1||_{L^\infty(M(T))} + \frac{\varphi}{\tau}||\Sigma||_{L^\infty(M(T))}\right)(||\nabla\Phi||^2_{L^2(M(T))} + |\Psi|^2_{L^2(M(T))}) \tag{4.64}$$
$$+[\frac{\varphi}{\tau}(||\nabla\Sigma||_{L^4(M)} + ||\Sigma||_{L^\infty(M)}||\nabla(\frac{N}{n} - 1)||_{L^4(M)}) + ||\nabla(\frac{N}{n} - 1)||_{L^4(M)}] \tag{4.65}$$
$$||\Phi||_{L^4(M)}||\nabla\Phi||_{L^2(M)}.$$

Now we study the principal terms

$$\int_{M(T)} \langle \nabla\partial_T\Phi, \nabla\Phi\rangle \mu_g + \int_{M(T)} \langle \partial_T\Psi, \Psi\rangle \mu_g \tag{4.66}$$
$$= \int_{M(T)} \langle \nabla(-N\psi + P_\Phi, \nabla\Phi\rangle \mu_g + \int_{M(T)} \langle -N\Delta_g\Phi + Q_\Psi, \Psi\rangle \mu_g$$
$$= \int_{M(T)} \langle \nabla P_\Phi, \nabla\Phi\rangle \mu_g + \int_{M(T)} \langle Q_\Psi, \Psi\rangle \mu_g$$
$$+ \underbrace{\int_{M(T)} \langle \nabla(-N\Psi), \nabla\Phi\rangle \mu_g + \int_{M(T)} \langle -N\Delta_g\Phi, \Psi\rangle \mu_g}_{I}$$

and observe that the $I = 0$ by integration by parts and Stokes theorem on closed (M, g). This cancellation is a vital property of hyperbolic equation since it yields the preservation of regularity (note that I contains terms that require control of L^2 of derivative of Ψ which is not provided at the initial data-therefore would cause a problem if such terms causing loss of regularity are not cancelled). Therefore integration of the equation 4.57 in the time interval $[T_0, T]$ yields the desired result. □

Remark 9 *The terms such as $||\Phi||_{L^4(M)}$ can further be controlled by $\sum_{I\leq 1}||\nabla^I\Phi||_{L^2(M)}$ by means of proposition 3.3.*

Theorem 4.2 *Under the assumption of the main theorem 1.1, the wave pair $(\Sigma, \mathfrak{T})$ verifies the following energy estimate*

$$\mathscr{E}^{top} \lesssim 1 + \mathcal{I}^0 \tag{4.67}$$

Proof. The proof of the energy estimates for the top-order term is based on the proposition 4.2. First, recall the definition of the top-order energy

$$\mathscr{E}^{top} := \frac{1}{2}\sum_{I\leq 3}\int_M \langle \nabla^{I+1}\Sigma, \nabla^{I+1}\Sigma\rangle \mu_g + \frac{1}{2}\sum_{I\leq 3}\int_M \langle \nabla^{I}\mathfrak{T}, \nabla^{I}\mathfrak{T}\rangle \mu_g. \tag{4.68}$$

We want to use the proposition 4.2 with $\Phi = \nabla^I\Sigma$ and $\Psi = \nabla^I\mathfrak{T}$ i.e.,

$$\partial_T\nabla^I\Sigma = -N\frac{\varphi}{\tau}\nabla^I\mathfrak{T} + P^I, \tag{4.69}$$

$$\partial_T\nabla^I\mathfrak{T} = -N\frac{\varphi}{\tau}\Delta_g\nabla^I\Sigma + Q^I, \tag{4.70}$$

where the error terms P^I are Q^I are expressed schematically as follows

$$\begin{aligned}
P^I \;&:=\; [\partial_T, \nabla^I]\Sigma + \frac{\varphi}{\tau}[\nabla^I, N]\mathfrak{T} + \nabla^I\Bigg(-(n-1)\Sigma_{ij} + \frac{\varphi}{\tau}\nabla_i\nabla_j(\frac{N}{n}-1) + \frac{2\varphi}{\tau}N\Sigma_{ik}\Sigma^k_j \\
&\quad -\frac{\varphi}{n\tau}(\frac{N}{n}-1)g_{ij} - (n-2)(\frac{N}{n}-1)\Sigma_{ij}\Bigg), \\
Q^I \;&:=\; [\partial_T, \nabla^I]\mathfrak{T} - \frac{\varphi}{\tau}[\nabla^I, N\Delta_g]\Sigma + \nabla^I\Bigg(n\frac{\varphi}{\tau}\nabla^l\nabla_i(\frac{N}{n}-1)\Sigma_{jl} + n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n}-1)\nabla_i\Sigma_{jl} \\
&\quad +(2-n)\nabla_i\nabla_j(\frac{N}{n}-1) + n\frac{\varphi}{\tau}\nabla^l\nabla_j(\frac{N}{n}-1)\Sigma_{il} + n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n}-1)\nabla_j\Sigma_{il} - n\frac{\varphi}{\tau}\Delta_g(\frac{N}{n}-1)\Sigma_{ij} \\
&\quad -2n\frac{\varphi}{\tau}\nabla^l(\frac{N}{n}-1)\nabla_l\Sigma_{ij} - \Delta_g(\frac{N}{n}-1)g_{ij} + N\frac{\varphi}{\tau}(\mathfrak{T}_{ki}\Sigma^k_j + \mathfrak{T}_{kj}\Sigma^k_i) + \frac{2(n-1)}{n^2}(\frac{N}{n}-1)g_{ij}\Bigg).
\end{aligned}$$

A couple of important points to note here. In the expression or P^I, term $\frac{\varphi}{n\tau}\nabla^I(\frac{N}{n}-1)g_{ij}$ is pure trace while Σ is transe-verse traceless. Therefore, this term does not contribute to the energy estimates. Similarly, in the expression of Q^I, the terms $\nabla^{I+2}(\frac{N}{n}-1)g_{ij}$ and $\nabla^I(\frac{N}{n}-1)g_{ij}$ are of pure trace type and therefore in the energy estimate for $\mathfrak{T}$ they contribute to the nonlinear term $|\Sigma|^2$ since $\mathrm{tr}_g\mathfrak{T} = |\Sigma|^2$ by the Hamiltonian constraint 2.7. Now apply proposition 4.2 to the system 4.69-4.70 to obtain

$$\mathscr{E}^{top}(T) = \mathscr{E}^{top}(T_0) + 2\sum_{I\leq 3}\int_{T_0}^{T}\int_{M(T)}\Big(\langle\nabla P^I, \nabla^{I+1}\Sigma\rangle + \langle Q^I, \nabla^I\mathfrak{T}\rangle\Big)\mu_g dt + \int_{T_0}^{T}\mathscr{E}\mathscr{R}(t)dt,$$

where

$$\begin{aligned}
|\mathscr{E}\mathscr{R}(t)| &\lesssim \left(||\frac{N}{n}-1||_{L^\infty(M(T))} + \frac{\varphi}{\tau}||\Sigma||_{L^\infty(M(T))}\right)\sum_{I\leq 3}(||\nabla^{I+1}\Sigma||^2_{L^2(M(T))} + |\nabla^I\mathfrak{T}|^2_{L^2(M(T))}) \\
&\quad +[\frac{\varphi}{\tau}(||\nabla\Sigma||_{L^4(M(T))} + ||\Sigma||_{L^\infty(M(T))}||\nabla(\frac{N}{n}-1)||_{L^4(M(T))}) + ||\nabla(\frac{N}{n}-1)||_{L^4(M(T))}] \\
&\qquad \sum_{I\leq 3}||\nabla^I\Sigma||_{L^4(M(T))}||\nabla^{I+1}\Sigma||_{L^2(M(T))}.
\end{aligned}$$

Our goal is to control the error terms and the spacetime integral terms involving P^I and Q^I. We do so schematically as follows

$$\int_{T_0}^{T}\int_{M(T)}\langle\nabla P^I, \nabla^{I+1}\Sigma\rangle\mu_g dt \tag{4.71}$$

$$
\begin{aligned}
&= \underbrace{-2(n-1)\int_{T_0}^{T}\int_{M(T)}|\nabla^{I+1}\Sigma|^2\mu_g dt}_{I-decay\ term} \\
&+\sum_{I\leq 3}\sum_{m=0}^{I}\sum_{J_1+J_2=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\Sigma\nabla^{I+1}\Sigma\mu_g dt \\
&+\sum_{I\leq 3}\sum_{m=0}^{I}\sum_{J_1+J_2+J_3=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1}N\nabla^{J_2+1}\Sigma\nabla^{J_3+I-m-1}\Sigma\nabla^{I+1}\Sigma\mu_g dt \\
&+\sum_{I\leq 3}\sum_{m=0}^{I}\sum_{J_1+J_2=m}\int_{T_0}^{T}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\Sigma\nabla^{I+1}\Sigma\mu_g dt \\
&+\sum_{I\leq 3}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{I+3}(\frac{N}{n}-1)\nabla^{I+1}\Sigma\mu_g \\
&+\sum_{I\leq 3}\sum_{J_1+J_2+J_3=I+1}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1}N\nabla^{J_2}\Sigma\nabla^{J_3}\Sigma\nabla^{I+1}\Sigma\mu_g \\
&+\sum_{I\leq 3}\sum_{J_1+J_2=I+1}\int_{T_0}^{T}\int_{M(T)}\nabla^{J_1}(\frac{N}{n}-1)\nabla^{J_2}\Sigma\nabla^{I+1}\Sigma\mu_g \\
&+\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2}\mathfrak{T}\nabla^{I+1}\Sigma\mu_g.
\end{aligned}
$$

We can estimate each term as follows. We use the elliptic estimate whenever necessary. First note that $\frac{\varphi}{\tau}=\frac{e^{-T}}{\sqrt{e^{-2T}+\frac{2n\Lambda}{n-1}}}\leq\sqrt{\frac{n-1}{2n\Lambda}}e^{-T}\lesssim e^{-T}$

$$
\begin{aligned}
&|\sum_{I\leq 3}\sum_{m=0}^{I}\sum_{J_1+J_2=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m}\Sigma\nabla^{I+1}\Sigma|\lesssim\int_{T_0}^{T}e^{-(1+3\gamma)t}\mathbb{Y}^4 dt \\
&\lesssim(e^{-(1+3\gamma)T_0}-e^{-(1+3\gamma)T})\mathbb{Y}^3\Gamma\lesssim e^{-(1+3\gamma)T_0}(1-e^{-(1+3\gamma)(T-T_0)})\mathbb{Y}^3\Gamma\lesssim 1.
\end{aligned}
$$

The maximum derivative on lapse function in the previous estimate is $I+1$ and therefore by elliptic estimate, it is bounded by $||\Sigma||^2_{H^{I-1}(M)}$ which yields $e^{-2\gamma t}$ decay. The next term is estimated as

$$
\begin{aligned}
&|\sum_{I\leq 3}\sum_{m=0}^{I}\sum_{J_1+J_2+J_3=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1}N\nabla^{J_2+1}\Sigma\nabla^{J_3+I-m}\Sigma\nabla^{I+1}\Sigma\mu_g dt|\lesssim\int_{T_0}^{T}e^{-(1+\gamma)t}\mathbb{Y}\Gamma^2 dt \\
&\lesssim e^{-(1+\gamma)T_0}(1-e^{-(1+\gamma)(T-T_0)})\mathbb{Y}\Gamma^2\lesssim 1.
\end{aligned}
$$

The next term is estimated as

$$
\begin{aligned}
&|\sum_{I\leq 3}\sum_{m=0}^{I}\sum_{J_1+J_2=m}\int_{T_0}^{T}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m}\Sigma\nabla^{I+1}\Sigma\mu_g dt|\lesssim\int_{T_0}^{T}e^{-3\gamma t}\Gamma\mathbb{Y}^3 dt \quad (4.72)\\
&\lesssim e^{-3\gamma T_0}(1-e^{-3\gamma(T-T_0)})\Gamma\mathbb{Y}^3\lesssim 1,
\end{aligned}
$$

$$
|\sum_{I\leq 3}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{I+3}(\frac{N}{n}-1)\nabla^{I+1}\Sigma\mu_g|\lesssim\int_{T_0}^{T}e^{-t}\Gamma^3 dt\lesssim e^{-T_0}(1-e^{-(T-T_0)})\Gamma^3\lesssim 1, \quad (4.73)
$$

$$|\sum_{I\leq 3}\sum_{J_1+J_2+J_3=I+1}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1}N\nabla^{J_2}\Sigma\nabla^{J_3}\Sigma\nabla^{I+1}\Sigma\mu_g|\lesssim\int_{T_0}^{T}e^{-(1+\gamma)t}\Gamma^2\mathbb{Y}dt \quad (4.74)$$
$$\lesssim e^{-(1+\gamma)T_0}(1-e^{-(1+\gamma)(T-T_0)})\Gamma^2\mathbb{Y}\lesssim 1.$$

The next term is estimated as follows

$$|\sum_{I\leq 3}\sum_{J_1+J_2=I+1}\int_{T_0}^{T}\int_{M(T)}\nabla^{J_1}(\frac{N}{n}-1)\nabla^{J_2}\Sigma\nabla^{I+1}\Sigma\mu_g|\lesssim\int_{T_0}^{T}e^{-2\gamma t}\Gamma^2\mathbb{Y}^2dt \quad (4.75)$$
$$\lesssim e^{-2\gamma T_0}(1-e^{-2\gamma(T-T_0)})\Gamma^2\mathbb{Y}^2\lesssim 1,$$

where we noted that the top order term in lapse $||\nabla^4(\frac{N}{n}-1)||_{L^2(M)}$ is estimated by $||\Sigma||^2_{H^2(M)}$ which exhibits $e^{-2\gamma t}$ decay. The last term in the expression of P^I is estimated as

$$|\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2}\mathfrak{T}\nabla^{I+1}\Sigma\mu_g|\lesssim\int_{T_0}^{T}e^{-(1+2\gamma)t}\Gamma^2\mathbb{Y}^2dt \quad (4.76)$$
$$\lesssim e^{-(1+2\gamma)T_0}(1-e^{-(1+2\gamma)(T-T_0)})\Gamma^2\mathbb{Y}^2\lesssim 1.$$

Now we estimate the terms in the expressions of Q^I. First, the spacetime integral involving Q^I is explicitly evaluated as follows (written in schematic notation)

$$\int_{T_0}^{T}\int_{M(T)}\langle Q^I,\nabla^I\mathfrak{T}\rangle\mu_g dt \quad (4.77)$$
$$\sim\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\mathfrak{T}\nabla^I\mathfrak{T}\mu_g dt$$
$$+\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2+J_3=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1}N\nabla^{J_2+1}\Sigma\nabla^{J_3+I-m-1}\mathfrak{T}\nabla^I\mathfrak{T}\mu_g dt$$
$$+\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{T_0}^{T}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\mathfrak{T}\nabla^I\mathfrak{T}\mu_g dt$$
$$+\sum_{I\leq 3}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M}\nabla(\frac{N}{n}-1)\nabla^{I+1}\Sigma\nabla^I\mathfrak{T}\mu_g dt$$
$$+\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}N\nabla^{J_1}\mathrm{Riem}\nabla^{J_2+I-m}\Sigma\nabla^I\mathfrak{T}\mu_g dt$$
$$+\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}\mathrm{Riem}\nabla^{J_2+I-m-1}\Sigma,\nabla^I\mathfrak{T}\rangle\mu_g dt$$
$$+\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1}\mathrm{Riem}\nabla^{J_2}\Sigma\nabla^I\mathfrak{T}\mu_g dt$$
$$+\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+2}(\frac{N}{n}-1)\nabla^{J_2}\Sigma,\nabla^I\mathfrak{T}\rangle\mu_g dt$$
$$+\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+1}\Sigma\nabla^I\mathfrak{T}\mu_g dt$$
$$+\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M}\langle\nabla^{J_1}\mathfrak{T}\nabla^{J_2}\Sigma\nabla^I\mathfrak{T}\rangle\mu_g dt$$

$$+\sum_{I\leq 3}\sum_{J_1+J_2=I-1}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+1}\Sigma\nabla^I\mathfrak{T}\mu_g dt.$$

Now we estimate each term separately. The first term is estimated as follows

$$|\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\mathfrak{T}\nabla^I\mathfrak{T}\mu_g dt| \tag{4.78}$$

$$\lesssim\int_{T_0}^{T}e^{-(1+2\gamma)t}\mathbb{Y}^2\Gamma^2 dt\lesssim e^{-(1+2\gamma)T_0}(1-e^{-(1+2\gamma)(T-T_0)})\Gamma^2\mathbb{Y}^2\lesssim 1.$$

Here the elliptic estimate 4.1 for the lapse function is used. The next term is estimated as

$$|\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2+J_3=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1}N\nabla^{J_2+1}\Sigma\nabla^{J_3+I-m-1}\mathfrak{T}\nabla^I\mathfrak{T}\mu_g dt| \tag{4.79}$$

$$\lesssim\int_{T_0}^{T}e^{-(1+\gamma)t}\Gamma^2\mathbb{Y}dt\lesssim e^{-(1+\gamma)T_0}(1-e^{-(1+\gamma)(T-T_0)})\Gamma^2\mathbb{Y}\lesssim 1.$$

Notice that Σ appears as $\nabla^I\Sigma$ at the top most order and therefore is estimated in L^2 yielding extra $e^{-\gamma t}$ decay. The next term is estimated as

$$|\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{T_0}^{T}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+I-m-1}\mathfrak{T}\nabla^I\mathfrak{T}\mu_g dt| \tag{4.80}$$

$$\lesssim\int_{T_0}^{T}e^{-2\gamma t}\mathbb{Y}^2\Gamma^2 dt\lesssim e^{-2\gamma T_0}(1-e^{-2\gamma(T-T_0)})\Gamma^2\mathbb{Y}^2\lesssim 1.$$

The next terms are estimated as

$$|\sum_{I\leq 3}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M}\nabla(\frac{N}{n}-1)\nabla^{I+1}\Sigma\nabla^I\mathfrak{T}\mu_g dt|\lesssim\int_{T_0}^{T}e^{-(1+2\gamma)t}\mathbb{Y}^2\Gamma^2 dt \tag{4.81}$$

$$\lesssim e^{-(1+2\gamma)T_0}(1-e^{-(1+2\gamma)(T-T_0)})\Gamma^2\mathbb{Y}^2\lesssim 1,$$

$$|\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}N\nabla^{J_1}\mathrm{Riem}\nabla^{J_2+I-m}\Sigma\nabla^I\mathfrak{T}\mu_g dt| \tag{4.82}$$

$$\lesssim\int_{T_0}^{T}e^{-(1+\gamma)t}\Gamma^2\mathbb{Y}dt\lesssim e^{-(1+\gamma)T_0}(1-e^{-(1+\gamma)(T-T_0)})\Gamma^2\mathbb{Y}\lesssim 1,$$

$$|\sum_{I\leq 3}\sum_{m=0}^{I-1}\sum_{J_1+J_2=m}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}\mathrm{Riem}\nabla^{J_2+I-m-1}\Sigma,\nabla^I\mathfrak{T}\rangle\mu_g dt| \tag{4.83}$$

$$\lesssim\int_{T_0}^{T}e^{-(1+\gamma)t}\Gamma^2\mathbb{Y}dt\lesssim e^{-(1+\gamma)T_0}(1-e^{-(1+\gamma)(T-T_0)})\Gamma^2\mathbb{Y}\lesssim 1,$$

$$|\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1}\mathrm{Riem}\nabla^{J_2}\Sigma\nabla^I\mathfrak{T}\mu_g dt| \tag{4.84}$$

$$\lesssim \int_{T_0}^{T} e^{-(1+\gamma)t}\Gamma^2\Upsilon dt \lesssim e^{-(1+\gamma)T_0}(1-e^{-(1+\gamma)(T-T_0)})\Gamma^2\Upsilon \lesssim 1,$$

$$|\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+2}(\frac{N}{n}-1)\nabla^{J_2}\Sigma\nabla^I\mathfrak{T}\mu_g dt| \tag{4.85}$$
$$\lesssim \int_{T_0}^{T} e^{-(1+3\gamma)t}\Upsilon^3\Gamma dt \lesssim e^{-(1+3\gamma)T_0}(1-e^{-(1+3\gamma)(T-T_0)})\Gamma\Upsilon^3 \lesssim 1,$$

$$|\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+1}\Sigma\nabla^I\mathfrak{T}\mu_g dt| \tag{4.86}$$
$$\lesssim \int_{T_0}^{T} e^{-(1+2\gamma)t}\Upsilon^2\Gamma^2 dt \lesssim e^{-(1+2\gamma)T_0}(1-e^{-(1+2\gamma)(T-T_0)})\Gamma^2\Upsilon^2 \lesssim 1,$$

$$|\sum_{I\leq 3}\sum_{J_1+J_2=I}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M}\nabla^{J_1}\mathfrak{T}\nabla^{J_2}\Sigma\nabla^I\mathfrak{T}\mu_g dt| \lesssim \int_{T_0}^{T} e^{-(1+\gamma)t}\Upsilon\Gamma^2 dt \tag{4.87}$$
$$\lesssim e^{-(1+\gamma)T_0}(1-e^{-(1+\gamma)(T-T_0)})\Gamma^2\Upsilon \lesssim 1,$$

and

$$|\sum_{I\leq 3}\sum_{J_1+J_2=I-1}\int_{T_0}^{T}\frac{\varphi}{\tau}\int_{M(T)}\nabla^{J_1+1}(\frac{N}{n}-1)\nabla^{J_2+1}\Sigma\nabla^I\mathfrak{T}\mu_g dt| \tag{4.88}$$
$$\lesssim \int_{T_0}^{T} e^{-(1+2\gamma)t}\Gamma^2\Upsilon^2 dt \lesssim e^{-(1+2\gamma)T_0}(1-e^{-(1+2\gamma)(T-T_0)})\Gamma^2\Upsilon^2 \lesssim 1.$$

Now note that the first term in the expression involving P^I contains term I which is negative definite. Therefore, the collection of every term yields

$$\mathscr{E}^{top}(T) \lesssim \mathscr{E}^{top}(T_0)+1 \lesssim \mathcal{I}^0+1 \tag{4.89}$$

which completes the proof of the theorem. □

Corollary 4.2 $\mathcal{O} \lesssim 1+\mathcal{I}^0$

Remark 10 *Notice that the estimates are uniform in T, so one can take the limit $T_\infty=\infty$.*

5. Proof of the main results

5.1. Proof of theorem 1.1

In this section, we prove the main theorem with the aid of the uniform estimates obtained in section 4.3. Note that we have already presented the idea of the proof in section 3.2. Here we will use a contradiction argument. First, recall the statement of the main theorem

Theorem (Global existence, $\boldsymbol{\Lambda>0, \sigma(M)\leq 0}$) *Let $(\widehat{M},\widehat{g})$ be a 3+1 dimensional globally hyperbolic spacetime. Suppose the Cauchy slice (M,g) is negative Yamabe type i.e., $\sigma(M)\leq 0$. Given a large $\mathcal{I}^0$, there exists a sufficiently large $a=a(\mathcal{I}^0)>0$. Let the initial* *data set (g_0, Σ_0) for the reduced Einstein-Λ equations 3.24-3.28 in CMC transported spatial gauge verify the constraint equations together with the estimates*

$$C^{-1}\xi_0 \leq g_0 \leq C\xi_0,$$

$$\left(\sum_{0\leq I\leq 4}||\nabla^I\Sigma_0||_{L^2(M)} + \sum_{0\leq I\leq 3}||\nabla^I\mathfrak{T}[g_0]||_{L^2(M)}\right) + \sum_{0\leq I\leq 3}||e^a\nabla^I\Sigma_0||_{L^2(M)} \leq \mathcal{I}^0$$

for a constant $C > 0$, a smooth background Riemannian metric ξ_0. Then a unique solution to the Einstein-Λ equation with the initial data (M, g_0, Σ_0) at CMC Newtonian time $T_0 = a$ exists for all $T \in [T_0, \infty)$ and the following estimates hold

$$\left(\sum_{0\leq I\leq 4}||\nabla^I\Sigma(T)||_{L^2(M)} + \sum_{0\leq I\leq 3}||\nabla^I\mathfrak{T}[g(T)]||_{L^2(M)}\right) + \sum_{0\leq I\leq 3}||e^T\nabla^I\Sigma||_{L^2(M)} \leq C_1(1+\mathcal{I}^0),$$

$$C_2^{-1}g_0 \leq g(T) \leq C_2 g_0 \;\forall T \in [T_0, \infty)$$

for numerical constants C_1, C_2. Moreover, every solution $T \mapsto (g(T), \Sigma(T))$ with initial data solving constraints 3.27-3.28 and verifying the estimates 5.1-5.1 converges to $(\widetilde{g}, 0)$ for a metric $\widetilde{g}$ with point-wise constant negative scalar curvature as $T \to \infty$.

The proof follows by standard continuation argument. First, note that the system of equations 3.41-3.43 is manifestly elliptic hyperbolic. Therefore, a local well-posedness follows by standard theory (for example [11]). In particular, there exists a $T > T_0$ such that given data on T_0, there exists a unique solution to the Einstein-Λ equation in $[T_0, T)$. Assume that $T < \infty$ and the solution can not be continued beyond T. We will use a contradiction argument to prove $T = \infty$.

By uniform estimates and persistence of regularity as proved in section 4.3, each of g, Σ, and $\mathfrak{T}$ has weak limits as $T^{'} \to T$ where $T^{'} \in [T_0, T)$ since the closed balls in L^2 Sobolev spaces used here are weakly compact. Consider a sequence of times $\{T^{'}_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} T^{'}_n = T$. Therefore by uniform boundedness, we have

$$\sum_{I\leq 4}||\nabla^I\Sigma(T)||_{L^2(M)} \leq \lim_{n\to\infty}\inf\sum_{I\leq 4}||\nabla^I\Sigma(T^{'}_n)||_{L^2(M)} \lesssim 1+\mathcal{I}^0, \tag{5.1}$$

$$\sum_{I\leq 3}||\nabla^I\mathfrak{T}(T)||_{L^2(M)} \leq \lim_{n\to\infty}\inf\sum_{I\leq 3}||\nabla^I\mathfrak{T}(T^{'}_n)||_{L^2(M)} \lesssim 1+\mathcal{I}^0. \tag{5.2}$$

Therefore, we can use the $(\Sigma(T), \mathfrak{T}(T))$ as the new initial data for the system 3.41-3.43 and obtain a solution in the interval $[T, T+\epsilon]$ for an appropriate ϵ depending on the data $(\Sigma(T), \mathfrak{T}(T))$. This violates the maximality of T. Therefore T must be ∞. This proves the global existence theorem.

In the second part, we prove that such a unique global solution converges to $(\widetilde{g}, 0)$ in $H^3(M)$. First, integrate the transport equation for the metric to obtain

$$||g(T) - g(\infty)||_{H^3(M)} \leq \int_T^\infty ||\partial_{T'} g dT^{'}||_{H^3(M)} \lesssim \mathcal{I}^0 e^{-(1+\gamma)T} + (\mathcal{I}^0)^2 e^{-2\gamma T} \tag{5.3}$$

and therefore $g(T)$ converges to $g(\infty)$ strongly in $H^3(M)$ norm. set $\widetilde{g} := g(\infty)$. The Hamiltonian constraint equation and the algebra property of Sobolev spaces $H^s(M)$ for $s > \frac{3}{2}$ implies

$$\lim_{T\to\infty}||R(g(T)) + \frac{n-1}{n}||_{H^3(M)} \leq \lim_{T\to\infty}||\Sigma(T)||^2_{H^3(M)} \lesssim \mathcal{I}^0 \lim_{T\to\infty} e^{-2\gamma T} = 0 \tag{5.4}$$

and since $3 > \frac{3}{2}$, $\lim_{T\to\infty} R(g) = -\frac{n-1}{n}$ in a point-wise sense. Therefore $\widetilde{g} := g(\infty) := \lim_{T\to\infty} g(T)$ verifies $R(\widetilde{g}) := R(g(\infty)) = -\frac{n-1}{n}$. This completes the proof of the main theorem.

Remark 11 $\mathfrak{T} \neq 0-$ *due to the fact that M may not admit any Einstein (hyperbolic in the current $n = 3$ case) metric. This can be shown easily by integrating the evolution equation for $\mathfrak{T}$ and utilizing the available regularity estimates. Let us define*

$$S := ||\mathfrak{T}||^2_{L^2(M)} \tag{5.5}$$

and observe

$$\frac{dS}{dT} \geq -C(\mathcal{I}^0)^2 e^{-2T} \tag{5.6}$$

where estimates for the Σ and $\mathfrak{T}$ are used from the previous section 5.1 along with Sobolev inequalities. Direct integration yields

$$S(T) \geq S(T_0) - C(\mathcal{I}^0)^2 e^{-2T_0}(1 - e^{-2(T-T_0)}) \tag{5.7}$$

from which for large T_0 and large $S(T_0)$, one can guarantee that $\lim_{T\to\infty} S(T) \gtrsim 1$. Therefore the limit metric g_∞ is **not** *hyperbolic. In particular, this estimate does not depend on the choice of M i.e., even if M is hyperbolizable, the hyperbolic metric is not attained as a limit in the presence of a positive cosmological constant $\Lambda > 0$.*

5.2. Proof of Corollary 1.2

Recall the following convergence of Riemannian manifolds. The space of Riemannian metrics on a closed 3−manifold M such that

$$|\mathrm{Riem}| \leq C, \mathrm{vol} \geq v, \ \mathrm{diam} \leq D \tag{5.8}$$

is precompact in $C^{1,\alpha}$ and $L^{2,p}$ topologies. In particular, given a closed Riemannian manifold (M, g), choose $\epsilon > 0$ small and define

$$M^\epsilon := \{x \in M | \mathrm{vol}(B_x(1)) \geq \epsilon\}, \quad M_\epsilon := \{x \in M | \mathrm{vol}(B_x(1)) < \epsilon\}. \tag{5.9}$$

M^ϵ and M_ϵ are called the $\epsilon-$thick and $\epsilon-$thin parts of (M, g), respectively. In particular if a sequence of metric g_i on a closed Riemannian manifold M verifies the bound 5.8, then from standard Bishop-Gromov volume comparison theorem [38] $M = M^\epsilon$. The estimates 5.8 are true in the current context by the global well-posedness and convergence theorem 1.1. In fact, a direct estimate of local volume is possible by using the first variation formula

$$|\mathrm{Vol}(B(1))_T - \mathrm{Vol}(B(1))_{T_0})| \lesssim \mathrm{Vol}(B(1))_{T_0}) \int_{T_0}^{T} ||\frac{N}{n} - 1||_{L^\infty(B(t'))} dt' \lesssim e^{-2T_0}(1 - e^{-2(T-T_0)})$$

and for sufficiently large T_0, $\mathrm{Vol}(B(1))_{T_0} \lesssim \mathrm{Vol}(B(1))_T \lesssim \mathrm{Vol}(B(1))_{T_0}$.

6. Funding and/or Conflicts of interests/Competing interests

The authors have no conflicts to disclose.

7. Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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