

Einstein-Yang-Mills equations in the double null framework

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Abstract

We prove a semi-global gauge-invariant estimate for the solutions of the characteristic initial value problem associated with the coupled Einstein-Yang-Mills equations. In particular, we prove the existence of a future development of regular initial data on a pair of incoming and outgoing null hypersurfaces emanating from a spacelike topological 2-sphere. This semi-global existence result is to be used in a potential future proof of trapped surface formation in the context of coupled Einstein-Yang-Mills dynamics.

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1 Introduction and motivation

One of the important problems of modern general relativity is the dynamical formation of spacetime singularities and their stability properties. According to Penrose's *weak cosmic censorship conjecture*, [1], the singularities in a general relativistic system can not be accessed by a future observer. If the singularities were to occur, they had to be hidden behind a horizon and therefore are not accessible to an observer located in the domain of outer communication. In the original singularity theorem of Penrose, the formation of a

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future singularity was understood in terms of the null geodesic incompleteness and such an incompleteness required the formation of a *trapped* surface in a spacetime with certain topological properties (such as the spacetime admits a non-compact Cauchy hypersurface) [2]. Following Penrose's analysis, trapped surface formation implies geodesic incompleteness and therefore at a formal level formation of a trapped surface corresponds to the formation of a black hole. However, the major challenge in the fully general relativistic setting (and possibly without symmetry) is the precise condition under which a trapped surface may form. A few years after Penrose's incompleteness theorem was published, Schoen and Yau [3] proved that for an asymptotically flat initial data set with mass density large on a large region, there is a closed trapped surface in the initial data. The future evolution of such data would then generate a geodesically incomplete spacetime according to Penrose's theorem. Much Later, [4] proved the formation of trapped surface in an evolutionary manner. More specifically, he showed that regular dispersed initial data that contains no trapped surface *can* lead to the formation of a trapped surface under the Einsteinian evolution of vacuum spacetime. Later [5] presented a simplified proof of the formation of a trapped surface in vacuum gravity and enlarged the admissible set of initial data.

Moving one step further, one would like to understand the formation of black holes (trapped surfaces) including suitable sources. This is of course motivated by the fact that our universe contains the structure and such structure is expected to arise due to matter (or radiation)-gravity interaction. Therefore, it is important to couple Einstein's equations with suitable sources and subsequently study the coupled dynamics. There has been progress in studying the trapped surface formation in the context of source coupled Einstein dynamics over the past few years. [9] proved the dynamical formation of a trapped surface by coupling electromagnetic field to Einstein's gravity without symmetry assumption. [10] established a trapped surface formation criterion for the Einstein-Maxwell-charged scalar field system under the assumption of spherical symmetry. There are several other studies including Vlasov matter source ([11]), perfect fluid source [12], and null dust source [13] and studies in the context of trapped surface formation by focusing incoming gravitational radiation from the past null infinity [14] as well.

Apart from the formation of singularities that are hidden behind a horizon and therefore inaccessible to the observers located in the domain of outer communication, naked singularities are of significant importance in general relativity. As we have mentioned in beginning, the existence of this type of singularity is ruled out by Penrose's weak cosmic censorship conjecture. In other words, the existence of such a singularity that is accessible by an observer at future null infinity would indicate a pathological breakdown of Einstein's theory. Christodoulou [15] showed a possible formation of a naked singularity in the context of Einstein-scalar field dynamics right before the collapse to a black hole. The genericity of such singularity is known to be violated i.e., a perturbation seems to destroy such singularity [16, 17] and as such they appear to be rather an artifact of high symmetry of the spacetimes. Apart from the study of [15], recently [18, 23] introduced a new type of geometric twisting phenomenon that contributes to the formation of a 'naked' singularity in a self-similar vacuum setting. However, the genericity of such solutions remains to be studied. Recently, Yau, Chen, and Du [24] constructed a remarkable family of spherically symmetric solutions of the Einstein-Yang-Mills equations that possess the property of being regular at the center of symmetry. However, the spacetime Riemann curvature (suitable invariant) is shown to blow up at the apparent horizon. Robust numerical studies suggest C^0 -stability of such solutions in the class of spherical symmetry. Contrary to the Einstein-Maxwell system or Einstein-scalar field system, Einstein-Yang-Mills equations are tremendously rich even in spherical symmetry and exhibit non-trivial dynamics. The numerical result of Bartnik [25] first showed the existence of a countable family of soliton-type solutions that are globally regular. Later Yau, Wasserman, and Smoller [26] rigorously proved the existence of such soliton-like solutions. However, such solutions were proven to be unstable against perturbations [31]. Later [32] also proved the existence of an infinite family of black hole solutions with a regular event horizon. The existence of these nontrivial solutions essentially unfolds the rich characteristics of the Einstein-Yang-Mills system. Due to the non-linear characteristics of the Yang-Mills fields, the fully coupled Einstein-Yang-Mills system is dynamically flexible i.e., both the possibility of the existence of regular solutions and the formation of singularities are open. This is precisely due to the fact that the non-linearity of Yang-Mills

fields can counterbalance the non-linearity of gravity and the formation of singularity or regularity of the solutions is essentially dictated by the dominating one which in turn depends on several additional conditions. Returning back to the EYM solution [24] containing a naked singular horizon, one is compelled to ask the following question: can these solutions arise in an evolutionary manner? In other words, one would want to study an initial value problem where the initial data is assumed to be sufficiently regular and possesses a degree of genericity and investigate whether such data can yield these naked singular solutions in finite time. This is motivated by the weak cosmic censorship conjecture [1] that rules out the possibility of the existence of evolutionary naked singularity (arising from regular and generic data). In addition, one would also like to provide analytical arguments supporting the stability (instability) of these solutions.

Motivated by these fundamental problems, we initiate the study of the dynamics of the Einstein-Yang-Mills system in the setting of a characteristic initial value formulation. In particular, we want to explore the nonlinear interaction of gravity and the Yang-Mills field and study two problems in the potential future: deducing the criteria to form trapped surfaces and naked singularities. The first step towards proving a trapped surface formation result is to establish a semi-global existence property of the coupled system i.e., one needs to ensure that the spacetime exists for long enough to form a trapped surface. Since the null hypersurfaces *are* the carrier of the gravitational and Yang-Mills radiation (both have the same characteristics), it is most natural to work in this *double null* framework. In addition, the naked singular solution of [24] does not arise at the origin but rather on a sphere of finite radius and therefore the question of stability of such solutions translates to an exterior stability problem. In other words, one would like to understand if one perturbs these solutions, can the energy of the perturbations escape through the outgoing null cones or can they potentially focus to form a trapped surface thereby destroying the naked singularities. In order to address such a question, the double null framework once again seems to be the most natural one to adapt.

The study of characteristic initial value problem for vacuum Einstein equation was initiated by Rendall [22]. In particular, [22] proved the existence of a solution to the characteristic initial value problem in a small enough neighborhood of the intersection of an outgoing and an incoming null hypersurface. This construction is not very useful in the context of studying trapped surface formation since in the latter one ought to evolve the initial data long enough along one of the null directions. Later Luk [8] improved the time of existence along one of the null directions in the context of vacuum gravity in a fairly straightforward way. However, it turns out that coupling to Yang-Mills source (or Maxwell for that matter) complicates the analysis and as such the analysis of [8] does not apply due to obstruction of closing the regularity argument. Roughly, the complication arises due to the presence of Yang-Mills source terms in the null Bianchi equations for the Weyl curvature. The appearance is such that one requires the Yang-Mills curvature components to have a regularity level one order higher than that of Weyl curvature components. In other words, if we work with K ($K \geq 3$) angular derivatives of Weyl curvature in $L^2(H, \bar{H})$ (H and $\bar{H}$ denote the outgoing and incoming null hypersurfaces to be defined later), then from the null Bianchi equations for the Weyl curvature, one would need to control $K + 1$ angular derivatives of the Yang-Mills curvature. This in turn would require control on the $K + 1$ angular derivatives of the connection coefficients from the null Yang-Mills equations. However, this seems to be incompatible with the analysis of [8] since the latter is compatible with controlling K angular derivatives of the connection coefficients on the topological 2-spheres. Therefore, we need to work with the optimal regularity level of [5] (or a higher-order regularity level consistent with the optimal regularity in a relative sense). In particular, we work with only 1 angular derivative of the Weyl curvature bounded in $L^2(H, \bar{H})$. This in turn requires control of 2 angular derivatives of the Yang-Mills curvature and the space-time connection coefficients. This regularity argument can be closed by means of the elliptic estimates and trace estimates. Of course, one can propagate these estimates to successive higher orders yielding estimates for a classical solution. In addition to the subtlety associated with the regularity level, the Yang-Mills theory is a gauge theory and one ought to make a choice of gauge. However, this gauge issue can be avoided since the Yang-Mills equations are manifestly hyperbolic in the double null framework if one works with the fully gauge covariant derivative instead of splitting it into the spacetime covariant derivative part and the pure gauge part (a similar strategy was used in the global

existence proof of Yang-Mills fields on the Minkowski background through the construction of a parametrix for the wave equation by Klainerman and Rodnianski [20]). Therefore, in the double null framework, we work with a manifestly hyperbolic system of coupled Einstein-Yang-Mills Bianchi equations supplemented by the constraints (of elliptic nature) and transport equations. In addition, our analysis does not require smallness assumption on the initial data.

The structure of the article is as follows. Starting from the null structure equations and a bootstrap assumption on the connection coefficients, we derive the necessary estimates for the connection coefficients which allows us to estimate the sectional curvature of the topological 2-spheres throughout the spacetime slab of interest using the null Hamiltonian constraint. This in turn allows us to utilize the null Codazzi equations to obtain elliptic estimates. Utilizing these estimates, we then use a direct integration by parts argument and the null evolution equations for the Weyl curvature and the Yang-Mills curvature to obtain the energy estimates in terms of the initial data thereby closing the bootstrap argument. A few conclusions are drawn based on our result and we make a conjecture about the nonlinear exterior stability of the Minkowski space under Einstein-Yang-Mills perturbations.

2 Notations and facts

Let the two null hypersurfaces H_0 and $\bar{H}_0$ (to be defined later) intersect at a topological 2-sphere $S_{0,0}$ in a globally hyperbolic spacetime M equipped with a Lorentzian metric g . The null hypersurfaces H and $\bar{H}$ are described by the level sets of the optical functions u and $\bar{u}$, respectively. In other words u and $\bar{u}$ satisfy the Eikonal equations

$$g^{\mu\nu}\partial_\mu u\partial_\nu u = 0, \quad g^{\mu\nu}\partial_\mu \bar{u}\partial_\nu \bar{u} = 0. \quad (1)$$

Through the variation of u and $\bar{u}$, we can foliate a spacetime slab $\mathcal{D}_{u,\bar{u}}$ by these two family of null hypersurfaces. The geodesic generators of the double null foliation are the vector fields L and $\bar{L}$ given by

$$L := -g^{\mu\rho}\partial_\rho u\partial_\mu, \quad \bar{L} := -g^{\mu\rho}\partial_\rho \bar{u}\partial_\mu \quad (2)$$

and manifestly they satisfy

$$\nabla_L L = 0 = \nabla_{\bar{L}} \bar{L}. \quad (3)$$

Whenever, we say H (resp. $\bar{H}$) we will always mean H_u (resp. $\bar{H}_{\bar{u}}$). In this notation, H_0 and $\bar{H}_0$ are the two initial null hypersurfaces corresponding to $u = 0$ and $\bar{u} = 0$, respectively on which the data is to be provided. Intersection of H and $\bar{H}$ is a topological 2-sphere $S_{u,\bar{u}}$. Evidently, the slab $\mathcal{D}_{u,\bar{u}}$ (see the picture) is the causal future of $S_{0,0}$ extended up to $u = \epsilon$ and $\bar{u} = J$. The spacetime metric in the double null gauge may be written as (in a local chart $(u, \bar{u}, \theta^1, \theta^2)$)

$$g := -2\Omega^2(du \otimes d\bar{u} + d\bar{u} \otimes du) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du), \quad (4)$$

where Ω is the null lapse function and $b := b^A \frac{\partial}{\partial \theta^A}$ is the null shift vector field. $\{\theta^A\}_{A=1}^2$ are the coordinates on the topological sphere $S_{u,\bar{u}}$. The induced metric on $S_{u,\bar{u}}$ is γ_{AB} . We can identify a normalized null frame (e_4, e_3, e_1, e_2) such that $g(e_4, e_4) = g(e_3, e_3) = 0 = g(e_4, e_3) = g(e_4, e_1) = g(e_4, e_2) = 0$ and $g(e_1, e_1) = g(e_2, e_2) = -2$, where (e_1, e_2) is an arbitrary frame on $S_{u,\bar{u}}$. We may identify e_4 and e_3 as follows

$$e_4 = \Omega^{-1} \frac{\partial}{\partial \bar{u}}, \quad e_3 = \Omega^{-1} \left(\frac{\partial}{\partial u} + b^A \frac{\partial}{\partial \theta^A} \right). \quad (5)$$

For the coordinate system, we may first define a coordinate chart $\mathcal{A}$ on $S_{0,0}$ then drag it by the flow of the vector field e_4 along H_0 and then drag it by the flow of e_3 to fill out the entire slab $\mathcal{D}_{u,\bar{u}}$. For more detailed

information about the double null foliation of a spacetime, see [21, 5, 8]

To formulate the Yang-Mills equations over the spacetime slab $\mathcal{D}_{u,\bar{u}}$, one first chooses a semi-simple Lie group G (the gauge group) which, for physical applications, is normally required to be compact and considers connections defined on a principle G -bundle. If a section of this bundle is chosen and the connection pulled back to the base manifold, then it yields a one-form field on the base which takes values in the Lie algebra $\mathfrak{g}$ of the chosen gauge group. The curvature of this connection (a two-form field with values in $\mathfrak{g}$) is defined in the usual way and the Yang-Mills equations (in the absence of sources) correspond to setting the natural (spacetime and gauge) covariant divergence of this curvature two-form to zero. By virtue of its definition in terms of the connection, this curvature also satisfies the Bianchi identity that asserts the vanishing of its gauge covariant exterior derivative. Taken together these equations provide a geometrically natural nonlinear generalization of Maxwell's equations and play a fundamental role in modern elementary particle physics. If nontrivial bundles are considered or nontrivial spacetime topologies are involved, then the foregoing so-called 'local trivializations' of the bundles in question must be patched together to give global descriptions but, by virtue of the covariance of the formalism, there is a natural way of carrying out this patching procedure at least over those regions of spacetime where the connections are well-defined.

For our purposes however, it is more convenient to consider a slight reformulation of the basic equations which will link up more naturally with the Cartan formalism that we have already used for general relativity. To this end we choose a vector space V and a matrix representation for the action of G on V . For simplicity, let us confine our attention to real representations though in fact this restriction is inessential. We now consider vector bundles over spacetime (so-called 'associated' bundles) with standard fiber $\simeq V$. Cross sections of such bundles would represent, in physical terms, multiplets of Higgs fields. To formulate field equations for such Higgs fields that are naturally covariant with respect to automorphisms of the associated vector bundle (i.e., with respect to gauge transformations acting on the Higgs fields) one needs a covariant derivative operator or connection defined on this bundle. Such an object is naturally induced from a 'fundamental' connection on the principal G -bundle described above and in turn induces (when expressed relative to a local trivialization) a one-form on (some local chart for) the base manifold with values in the chosen matrix representation for the Lie algebra $\mathfrak{g}$. Let us consider the dimension of the group G to be $\dim_G$ and since $\mathfrak{g} := T_e G$, it has a natural vector space structure. Assume that the vector space $\mathfrak{g}$ has a basis $\{\chi_A\}_{A=1}^{\dim_G}$ given by a set of $k \times k$ real valued matrices (k being the dimension of the representation V of the Lie algebra $\mathfrak{g}$). The connection 1-form field is then defined to be

$$A := A_\mu^A \chi_A dx^\mu = A_\mu^A (\chi_A)_Q^P dx^\mu = A^P{}_{Q\mu} dx^\mu, \quad P, Q = 1, 2, 3, \dots, k. \quad (6)$$

From now on by the connection 1-form field A_μ , we will always mean $A^P{}_{Q\mu}$. In the current setting $A \in \Omega^1(\mathcal{D}_{u,\bar{u}}; \text{End}(V))$, where $\text{End}(V)$ denotes the space of endomorphisms of the vector space V . The curvature of this connection is defined to be the Yang-Mills field $F \in \Omega^2(\mathcal{D}_{u,\bar{u}}; \text{End}(V))$

$$F^P{}_{Q\mu\nu} := \partial_\mu A^P{}_{Q\nu} - \partial_\nu A^P{}_{Q\mu} + [A, A]^P{}_{Q\mu\nu}, \quad (7)$$

where the bracket is defined on the Lie algebra $\mathfrak{g}$ and given by the commutator of matrices under multiplication. The Yang-Mills coupling constant is set to unity. Since G is compact, it admits a positive definite adjoint invariant metric on $\mathfrak{g}$. We choose a basis of $\mathfrak{g}$ such that this adjoint invariant metric takes the Cartesian form δ_{AB} and work with representations for which the bases satisfy

$$-tr(\chi_A \chi_B) = (\chi_A)_Q^P (\chi_B)_P^Q = \delta_{AB}. \quad (8)$$

Under a gauge transformation by $\mathcal{U}$, the $\mathfrak{g}$ valued 1-form field A transforms as

$$A_\mu \mapsto \mathcal{U} A_\mu \mathcal{U} + \mathcal{U} \partial_\mu \mathcal{U}^{-1} \quad (9)$$

and therefore A_μ is not a tensor in the sense that it is *not* a $(1, 1)$ section of the associated V -bundle over $\mathcal{D}_{u,\bar{u}}$. For any $\mathfrak{g}$ valued section $\mathcal{K}^P_{Q\mu_1\mu_2\mu_3\dots\mu_k}$ of a vector bundle over $\mathcal{D}_{u,\bar{u}}$ that transforms as a tensor under the gauge transformation, the gauge covariant derivative is defined to be

$$\begin{aligned} & \hat{D}_\alpha \mathcal{K}^P_{Q\mu_1\mu_2\mu_3\dots\mu_k} \\ := & D_\alpha \mathcal{K}^P_{Q\mu_1\mu_2\mu_3\dots\mu_k} + A^P_{R\alpha} \mathcal{K}^R_{Q\mu_1\mu_2\mu_3\dots\mu_k} - A^R_{Q\alpha} \mathcal{K}^P_{R\mu_1\mu_2\mu_3\dots\mu_k} \\ = & D_\alpha \mathcal{K}^P_{Q\mu_1\mu_2\mu_3\dots\mu_k} + [A, \mathcal{K}]^P_{Q\mu_1\mu_2\mu_3\dots\mu_k}, \end{aligned} \quad (10)$$

where D_α is the ordinary spacetime covariant derivative with respect to a Lorentzian metric on $\mathcal{D}_{u,\bar{u}}$. Even though connection of the gauge bundle appears in the definition of the gauge covariant derivative, we will never make explicit use of it in the current context rather work with the fully gauge covariant derivative $\hat{D}$. More specifically, in our analysis, we will encounter the commutator of the fully gauge covariant derivative which yields Riemann curvature and Yang-Mills curvature components. In other words, using the fully gauge covariant derivative, we do not encounter the connection term allowing us to obtain estimates in a gauge-invariant way. The commutator of the fully gauge covariant derivative while acting on a $\mathfrak{g}$ valued section of a vector bundle $\mathcal{K}^P_{Q\mu_1\mu_2\mu_3\dots\mu_k}$ (or a section of the mixed bundle) yields

$$[\hat{D}_\alpha, \hat{D}_\beta] \mathcal{K}^P_{Q\mu_1\mu_2\mu_3\dots\mu_k} = F^P_{R\alpha\beta} \mathcal{K}^R_{Q\mu_1\mu_2\mu_3\dots\mu_k} - F^R_{Q\alpha\beta} \mathcal{K}^P_{R\mu_1\mu_2\mu_3\dots\mu_k} - \sum_i R^\gamma_{\mu_i\alpha\beta} \mathcal{K}^P_{Q\mu_1\mu_2\mu_3\dots\hat{\gamma}\dots\mu_k}, \quad (11)$$

where $\hat{\gamma}$ indicates the removal of the index μ_i and replacing by γ . Note that $\hat{D}$ is compatible with both the metrics and therefore the commutator produces curvature of the mixed bundle (Yang-Mills curvature and spacetime curvature). Action of $\hat{D}$ is only well defined on *sections* of the mixed bundle.

Now we define the norms that are adapted to the double null framework. We need to define the norms within the bulk spacetime $\mathcal{D}_{u,\bar{u}}$, on the null hypersurfaces (H and $\bar{H}$), and on the topological 2-sphere $S_{u,\bar{u}}$. First, we need an integration measure. On $\mathcal{D}_{u,\bar{u}}$, we can use the canonical volume measure corresponding to the spacetime metric. On the null hypersurfaces, the metric is degenerate and therefore no canonical choice of volume form is available. For a function f , we define its integration over $\mathcal{D}_{u,\bar{u}}$, H , $\bar{H}$, and $S_{u,\bar{u}}$ as follows

$$\int_{S_{u,\bar{u}}} f := \int_{S_{u,\bar{u}}} f \mu_\gamma, \quad \int_{\mathcal{D}_{u,\bar{u}}} f := \int_0^u \int_0^{\bar{u}} \int_S \Omega^2 f \mu_\gamma du d\bar{u}, \quad (12)$$

$$\int_H f := \int_0^{\bar{u}} \int_S f \Omega \mu_\gamma d\bar{u}, \quad \int_{\bar{H}} f := \int_0^u \int_S f \Omega \mu_\gamma du, \quad (13)$$

where μ_γ is the volume form associated with the metric γ on $S_{u,\bar{u}}$. Now we define the norm of a $\mathfrak{g}$ -valued section associated with $S_{u,\bar{u}}$, H , and $\bar{H}$. Let $\mathcal{G}$ be a $\mathfrak{g}$ -valued section of a vector bundle over $S_{u,\bar{u}}$ (we will also call $\mathcal{G}$ as a $\mathfrak{g}$ -valued horizontal tensor field). Its L^p norms ($1 \leq p < \infty$) are defined as follows

$$\|\mathcal{G}\|_{L^p(S_{u,\bar{u}})}^p := \int_{S_{u,\bar{u}}} (|\mathcal{G}|_{\gamma,\delta}^2)^{\frac{p}{2}} \mu_\gamma, \quad (14)$$

$$\|\mathcal{G}\|_{L^p(H)}^p := \int_0^{\bar{u}} \int_S (|\mathcal{G}|_{\gamma,\delta}^2)^{\frac{p}{2}} \Omega \mu_\gamma d\bar{u}, \quad \|\mathcal{G}\|_{L^p(\bar{H})}^p := \int_0^u \int_S (|\mathcal{G}|_{\gamma,\delta}^2)^{\frac{p}{2}} \Omega \mu_\gamma du, \quad (15)$$

where $|\mathcal{G}|_{\gamma,\delta}^2$ is defined as follows

$$|\mathcal{G}|_{\gamma,\delta}^2 := \mathcal{G}^P_{QA_1A_2A_3\dots A_N} \mathcal{G}^Q_{PB_1B_2B_3\dots B_N} \gamma^{A_1B_1} \gamma^{A_2B_2} \gamma^{A_3B_3} \dots \gamma^{A_NB_N}. \quad (16)$$

L^∞ norm over $S_{u,\bar{u}}$ is defined as $\|\mathcal{G}\|_{L^\infty(S_{u,\bar{u}})} := \sup_{\theta^1, \theta^2 \in S_{u,\bar{u}}} \sqrt{|\mathcal{G}|_{\gamma,\delta}^2(\theta^1, \theta^2)}$.

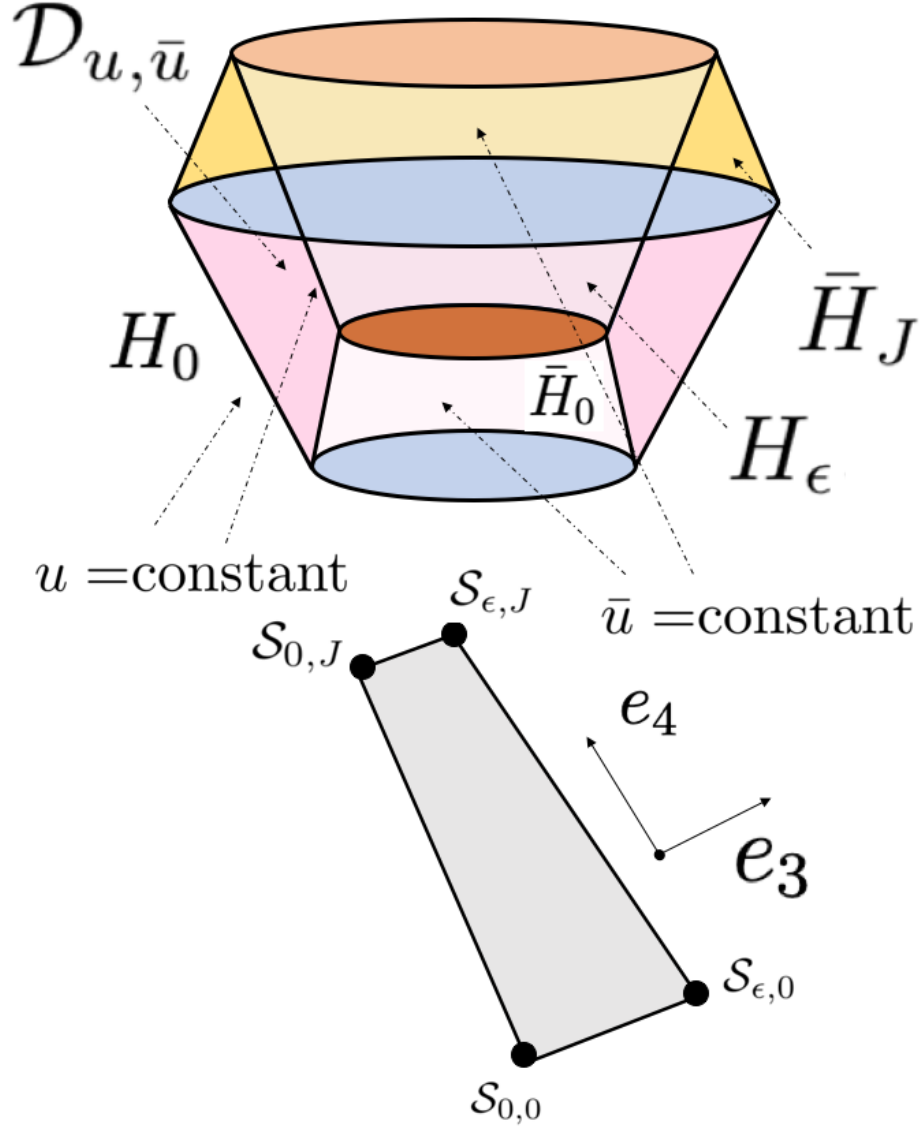


Figure 1: Figure depicting the double null foliation adapted to the current framework. The bottom picture is constructed after taking a formal quotient of the top figure by the topological 2-spheres i.e., each point of the bottom figure denotes a topological 2-sphere. The theorem of [22] deals with the existence of a solution to the characteristic initial value problem in an ϵ neighborhood of $S_{0,0}$. Note $\mathcal{D}_{u,\bar{u}}$ denotes the bulk spacetime region that is causal future of $S_{0,0}$ up to $u = \epsilon, \bar{u} = J$.

3 Field equations in the double null framework

In this section, we explicitly define all the entities associated with the double null foliation and write down structure equations. The spacetime covariant derivative D admits the following usual decomposition in terms of its components parallel and orthogonal to the topological 2-sphere $S_{u,\bar{u}}$

$$D_{e_a} e_b = \nabla_{e_a} e_b - \frac{1}{2} \langle D_{e_a} e_b, e_3 \rangle e_4 - \frac{1}{2} \langle D_{e_a} e_b, e_4 \rangle e_3, \quad (17)$$

where ∇ is the $S_{u,\bar{u}}$ -parallel covariant derivative. Similar to the spacetime covariant derivative, the space-time gauge covariant derivative $\hat{D}$ admits the following decomposition

$$\begin{aligned} \hat{D}_{e_a} e_b &= \hat{\nabla}_{e_a} e_b - \frac{1}{2} \langle \hat{D}_{e_a} e_b, e_3 \rangle e_4 - \frac{1}{2} \langle \hat{D}_{e_a} e_b, e_4 \rangle e_3 \\ &= \nabla_{e_a} e_b - \frac{1}{2} \langle D_{e_a} e_b, e_3 \rangle e_4 - \frac{1}{2} \langle D_{e_a} e_b, e_4 \rangle e_3 \end{aligned} \quad (18)$$

simply because the basis $(e_3, e_4, e_a)_{a=1}^2$ are not sections of the gauge bundle and therefore gauge covariant derivative acts as ordinary covariant derivative. Now recalling the following definitions of the outgoing and incoming null second fundamental form of $S_{u,v}$

$$\chi_{ab} := \langle D_{e_a} e_4, e_b \rangle, \quad \bar{\chi}_{ab} := \langle D_{e_a} e_3, e_b \rangle, \quad (19)$$

the spacetime covariant (also gauge covariant) derivative satisfies

$$D_{e_a} e_b = \nabla_{e_a} e_b + \frac{1}{2} \bar{\chi}_{ab} e_4 + \frac{1}{2} \chi_{ab} e_3. \quad (20)$$

Now let $K \in \text{sections}\{TS_{u,\bar{u}}\}$ and Z be a $\mathfrak{g}$ -valued frame vector field on $S_{u,\bar{u}}$ that transforms as a tensor under gauge transformation, then using the definition of the gauge covariant derivative (10), the following holds for the gauge covariant derivative

$$\hat{D}_K Z = \hat{\nabla}_K Z + \frac{1}{2} \bar{\chi}(K, Z) e_4 + \frac{1}{2} \chi(K, Z) e_3, \quad (21)$$

where $\chi(K, Z) := \chi_{ab} K_a Z^P{}_{Qb}$ and $\bar{\chi}(K, Z) := \bar{\chi}_{ab} K_a Z^P{}_{Qb}$. Now we recall the definitions of the remaining connection coefficients adapted to the double null framework

$$\begin{aligned} \eta_a &:= -\frac{1}{2} \langle D_{e_3} e_a, e_4 \rangle, \quad \omega := -\frac{1}{4} \langle D_{e_4} e_3, e_4 \rangle = -\frac{1}{2} D_{e_4} \ln \Omega, \\ \bar{\eta}_a &:= -\frac{1}{2} \langle D_{e_4} e_a, e_3 \rangle, \quad \bar{\omega} := -\frac{1}{4} \langle D_{e_3} e_4, e_3 \rangle = -\frac{1}{2} D_{e_3} \ln \Omega \end{aligned} \quad (22)$$

and the torsion $\zeta_a := \frac{1}{2} \langle D_{e_a} e_4, e_3 \rangle = \frac{1}{2} (\eta - \bar{\eta})$. Utilizing these definitions, let us write down the kinematical set of structural equations

$$D_{e_a} e_3 = \bar{\chi}_{ab} e_b + \zeta_a e_3, \quad D_{e_3} e_a = \nabla_{e_3} e_a + \eta_a e_3, \quad (23)$$

$$D_{e_4} e_a = \nabla_{e_4} e_a + \bar{\eta}_a e_4, \quad \nabla_{e_3} e_3 = -2\bar{\omega} e_3, \quad (24)$$

$$D_{e_4} e_4 = -2\omega e_4, \quad \nabla_{e_4} e_3 = 2\omega e_3 + 2\bar{\eta}_a e_a, \quad (25)$$

$$D_{e_3} e_4 = 2\bar{\omega} e_4 + 2\eta_a e_a, \quad D_{e_4} e_a = \nabla_{e_4} e_a + \bar{\eta}_a e_4, \quad (26)$$

$$D_{e_a} e_4 = \chi_{ab} e_b - \zeta_a e_4. \quad (27)$$

Also note $\eta_a = \zeta_a + \nabla_{e_a} \ln \Omega$, $\bar{\eta}_a = -\zeta_a + \nabla_{e_a} \ln \Omega$. Utilizing these kinematical structure equations, we obtain the dynamical set of structural equations suitable trace of which are nothing but the Einstein's

equations sourced by Yang-Mills stress energy tensor and expressed in the double null framework. Before writing down such equations, let us recall the following decomposition of the spacetime Riemann curvature tensor

$$R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}) + \frac{1}{6}R(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}), \quad (28)$$

where W is the Weyl curvature tensor describing pure gravitational degrees of freedom. W is trace-free and enjoys the same algebraic symmetry of the Riemann curvature. Since the Ricci and the scalar curvatures are fixed by the Einstein's field equations, we ought to write down the equation for the null components of the Weyl curvature. These equations will constitute the null Bianchi equations. The components of the Weyl curvature are defined as follows

$$\alpha_{ab} := W(e_a, e_4, e_b, e_4), \quad \bar{\alpha}_{ab} := W(e_a, e_3, e_b, e_3), \quad (29)$$

$$2\beta_a := W(e_4, e_3, e_4, e_a), \quad 2\bar{\beta}_a := W(e_3, e_4, e_3, e_a), \quad (30)$$

$$\rho := \frac{1}{4}W(e_4, e_3, e_4, e_3), \quad \sigma := \frac{1}{4}{}^*W(e_4, e_3, e_4, e_3), \quad (31)$$

where *W is the left Hodge dual of W defined as follows

$${}^*W_{\alpha\beta\gamma\delta} := \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta}, \quad (32)$$

where $\epsilon_{\alpha\beta\mu\nu}$ is the volume form on the spacetime M . In addition to the Weyl field, we also have the Yang-Mills curvature $F := \frac{1}{2}F^P{}_{Q\mu\nu}dx^\mu \wedge dx^\nu \in \Omega^2(M; \text{end}(V))$, $P, Q = 1, 2, 3, \dots, \dim(V)$. The components of the Yang-Mills curvature F are defined as follows

$$\begin{aligned} \alpha_a^F &:= F^P{}_{Q}(e_a, e_4), \quad \bar{\alpha}_a^F := F^P{}_{Q}(e_a, e_3), \quad \rho^F := \frac{1}{2}F^P{}_{Q}(e_3, e_4), \\ \sigma^F &= \frac{1}{2}{}^*F^P{}_{Q}(e_3, e_4) = F^P{}_{Q}(e_1, e_2). \end{aligned} \quad (33)$$

Also decompose the null second fundamental forms $\chi, \bar{\chi}$ into their trace and trace-free components

$$\chi = \hat{\chi} + \frac{1}{2}\text{tr}\chi\gamma, \quad \bar{\chi} = \hat{\bar{\chi}} + \frac{1}{2}\text{tr}\bar{\chi}\gamma. \quad (34)$$

The Einstein's equations (with the choice of unit $8\pi G = c = 1$)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \mathfrak{T}_{\mu\nu} \quad (35)$$

in the double null framework reads

$$\nabla_4 \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 = -|\hat{\chi}|_\gamma^2 - 2\omega \text{tr} \chi - \mathfrak{T}_{44} \quad (36)$$

$$\nabla_4 \hat{\chi} + \text{tr} \chi \hat{\chi} = -2\omega \hat{\chi} - \alpha \quad (37)$$

$$\nabla_3 \text{tr} \bar{\chi} + \frac{1}{2} (\text{tr} \bar{\chi})^2 = -|\hat{\bar{\chi}}|_\gamma^2 - 2\bar{\omega} \text{tr} \bar{\chi} - \mathfrak{T}_{33} \quad (38)$$

$$\nabla_3 \hat{\bar{\chi}} + \text{tr} \bar{\chi} \hat{\bar{\chi}} = -2\bar{\omega} \hat{\bar{\chi}} - \bar{\alpha} \quad (39)$$

$$\nabla_4 \eta_a = -\chi \cdot (\eta - \bar{\eta}) - \beta - \frac{1}{2} \mathfrak{T}_{a4} \quad (40)$$

$$\nabla_3 \bar{\eta}_a = -\bar{\chi} \cdot (\bar{\eta} - \eta) + \bar{\beta} + \frac{1}{2} \mathfrak{T}_{a3} \quad (41)$$

$$\begin{aligned} \nabla_4 \bar{\omega} &= 2\omega \bar{\omega} + \frac{3}{4} |\eta - \bar{\eta}|^2 - \frac{1}{4} (\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) - \frac{1}{8} |\eta + \bar{\eta}|^2 + \frac{1}{2} \rho + \frac{1}{4} \mathfrak{T}_{43} \\ \nabla_3 \omega &= 2\omega \bar{\omega} + \frac{3}{4} |\eta - \bar{\eta}|^2 + \frac{1}{4} (\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) - \frac{1}{8} |\eta + \bar{\eta}|^2 + \frac{1}{2} \rho + \frac{1}{4} \mathfrak{T}_{43} \\ \nabla_4 \text{tr} \bar{\chi} + \frac{1}{2} \text{tr} \chi \text{tr} \bar{\chi} &= 2\omega \text{tr} \bar{\chi} + 2 \text{div} \bar{\eta} + 2|\bar{\eta}|_\gamma^2 + 2\rho - \hat{\chi} \cdot \hat{\bar{\chi}} \end{aligned} \quad (42)$$

$$\nabla_3 \text{tr} \chi + \frac{1}{2} \text{tr} \bar{\chi} \text{tr} \chi = 2\bar{\omega} \text{tr} \chi + 2 \text{div} \eta + 2|\eta|^2 + 2\rho - \hat{\chi} \cdot \hat{\bar{\chi}} \quad (43)$$

$$\nabla_4 \hat{\bar{\chi}} + \frac{1}{2} \text{tr} \chi \hat{\bar{\chi}} = \nabla \hat{\otimes} \bar{\eta} + 2\omega \hat{\bar{\chi}} - \frac{1}{2} \text{tr} \bar{\chi} \hat{\bar{\chi}} + \bar{\eta} \hat{\otimes} \bar{\eta} + \hat{\mathfrak{T}}_{ab} \quad (44)$$

$$\nabla_3 \hat{\chi} + \frac{1}{2} \text{tr} \bar{\chi} \hat{\chi} = \nabla \hat{\otimes} \eta + 2\bar{\omega} \hat{\chi} - \frac{1}{2} \text{tr} \chi \hat{\chi} + \eta \hat{\otimes} \eta + \hat{\mathfrak{T}}_{ab} \quad (45)$$

$$\text{div} \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \frac{1}{2} (\eta - \bar{\eta}) \cdot (\hat{\chi} - \frac{1}{2} \text{tr} \chi \gamma_{ab}) - \beta + \frac{1}{2} \mathfrak{T}(e_4, \cdot) \quad (46)$$

$$\text{div} \hat{\bar{\chi}} = \frac{1}{2} \nabla \text{tr} \bar{\chi} - \frac{1}{2} (\bar{\eta} - \eta) \cdot (\hat{\bar{\chi}} - \frac{1}{2} \text{tr} \bar{\chi} \gamma_{ab}) - \bar{\beta} + \frac{1}{2} \mathfrak{T}(e_3, \cdot) \quad (47)$$

$$\text{curl} \eta = \hat{\chi} \wedge \hat{\bar{\chi}} + \sigma \epsilon = -\text{curl} \bar{\eta} \quad (48)$$

$$K - \frac{1}{2} \hat{\chi} \cdot \hat{\bar{\chi}} + \frac{1}{4} \text{tr} \chi \text{tr} \bar{\chi} = -\rho + \frac{1}{4} \mathfrak{T}_{43}, \quad (49)$$

where $\mathfrak{T}_{\mu\nu} := \frac{1}{2} (F^P{}_{Q\mu\alpha} F^Q{}_{P\nu}{}^\alpha + {}^* F^P{}_{Q\mu\alpha} {}^* F^Q{}_{P\nu}{}^\alpha)$ is the Yang-Mills stress-energy tensor and K is the sectional curvature of the topological 2-sphere $S_{u,\bar{u}}$ (or a constant multiple of Gauss curvature). Here equations (46-49) are the null-constraint equations. Now recall the Bianchi identities

$$D_\mu R_{\alpha\beta\nu\lambda} + D_\nu R_{\alpha\beta\lambda\mu} + D_\lambda R_{\alpha\beta\mu\nu} = 0 \quad (50)$$

which together with the decomposition (28) and the Einstein's equations yields the following Yang-Mills type equations for the Weyl curvature

$$D^\alpha W_{\alpha\beta\gamma\delta} = J[\mathfrak{T}]_{\beta\gamma\delta}, \quad D_{[\mu} W_{\gamma\delta]\alpha\beta} = \frac{1}{3} \epsilon_{\nu\mu\gamma\delta} J[\mathfrak{T}]^{\nu}{}_{\alpha\beta}. \quad (51)$$

Here $J[\mathfrak{T}]$ is the source term determined fully by the Yang-Mills stress energy tensor $\mathfrak{T}$. After elementary algebraic manipulations, the differential equations for the Weyl curvature may be cast into the following

double-null form ¹

$$\nabla_3 \alpha_{ab} + \frac{1}{2} \text{tr} \bar{\chi} \alpha_{ab} = (\nabla \hat{\otimes} \beta)_{ab} + 4\bar{\omega} \alpha_{ab} - 3(\hat{\chi} \rho + {}^* \hat{\chi} \sigma)_{ab} + ((\zeta + 4\eta) \hat{\otimes} \beta)_{ab} + \frac{1}{2} (D_3 R_{44} - D_4 R_{43}) \gamma_{ab} \quad (52)$$

$$\nabla_4 \beta_a + 2 \text{tr} \chi \beta_a = (\text{div} \alpha)_a - 2\omega \beta_a + (\eta \cdot \alpha)_a - \frac{1}{2} (D_a R_{44} - D_4 R_{4a}) \quad (53)$$

$$\nabla_3 \beta_a + \text{tr} \bar{\chi} \beta_a = \nabla_a \rho + {}^* \nabla_a \sigma + 2\bar{\omega} \beta_a + 2(\hat{\chi} \cdot \bar{\beta})_a + 3(\eta \rho + {}^* \eta \sigma)_a + \frac{1}{2} (D_a R_{34} - D_4 R_{3a})$$

$$\nabla_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma = -\text{div} {}^* \beta + \frac{1}{2} \hat{\chi} \cdot {}^* \alpha - \zeta \cdot {}^* \beta - 2\bar{\eta} \cdot {}^* \beta - \frac{1}{4} (D_\mu R_{4\nu} - D_\nu R_{4\mu}) \epsilon^{\mu\nu}{}_{34}$$

$$\nabla_3 \sigma + \frac{3}{2} \text{tr} \bar{\chi} \sigma = -\text{div} {}^* \bar{\beta} + \frac{1}{2} \hat{\chi} \cdot {}^* \bar{\alpha} - \zeta \cdot {}^* \bar{\beta} - 2\eta \cdot {}^* \bar{\beta} + \frac{1}{4} (D_\mu R_{3\nu} - D_\nu R_{3\mu}) \epsilon^{\mu\nu}{}_{34}$$

$$\nabla_4 \rho + \frac{3}{2} \text{tr} \chi \rho = \text{div} \beta - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2\bar{\eta} \cdot \beta - \frac{1}{4} (D_3 R_{44} - D_4 R_{34}) \quad (54)$$

$$\nabla_3 \rho + \frac{3}{2} \text{tr} \bar{\chi} \rho = -\text{div} \bar{\beta} - \frac{1}{2} \hat{\chi} \cdot \bar{\alpha} + \zeta \cdot \bar{\beta} - 2\eta \cdot \bar{\beta} + \frac{1}{4} (D_3 R_{34} - D_4 R_{33}) \quad (55)$$

$$\nabla_4 \bar{\beta}_a + \text{tr} \chi \bar{\beta}_a = -\nabla_a \rho + {}^* \nabla_a \sigma + 2\omega \bar{\beta}_a + 2(\hat{\chi} \cdot \beta)_a - 3(\bar{\eta} \rho - {}^* \bar{\eta} \sigma)_a - \frac{1}{2} (D_a R_{43} - D_3 R_{4a})$$

$$\nabla_3 \bar{\beta}_a + 2 \text{tr} \bar{\chi} \bar{\beta}_a = -(\text{div} \bar{\alpha})_a - 2\bar{\omega} \bar{\beta}_a + (\bar{\eta} \cdot \bar{\alpha})_a + \frac{1}{2} (D_a R_{33} - D_3 R_{3a}) \quad (56)$$

$$\nabla_4 \bar{\alpha}_{ab} + \frac{1}{2} \text{tr} \chi \bar{\alpha} = -(\nabla \hat{\otimes} \bar{\beta})_{ab} + 4\omega \bar{\alpha}_{ab} - 3(\hat{\chi}_{ab} \rho - {}^* \hat{\chi}_{ab} \sigma) + ((\zeta - 4\bar{\eta}) \hat{\otimes} \bar{\beta})_{ab} + \frac{1}{2} (D_4 R_{33} - D_3 R_{34}) \gamma_{ab}, \quad (57)$$

where $R_{\mu\nu} = \mathfrak{T}_{\mu\nu}$ due to the trace-free property of the Yang-Mills stress energy tensor. The Yang-Mills equations

$$\hat{D}_\mu F^P{}_{Q\nu\lambda} + \hat{D}_\nu F^P{}_{Q\lambda\mu} + \hat{D}_\lambda F^P{}_{Q\mu\nu} = 0, \quad g^{\alpha\beta} \hat{D}_\alpha F^P{}_{Q\beta\mu} = 0 \quad (58)$$

imply the following double null Yang-Mills equations

$$\hat{\nabla}_4 \bar{\alpha}^F + \frac{1}{2} \text{tr} \chi \bar{\alpha}^F = -\hat{\nabla} \rho^F - {}^* \hat{\nabla} \sigma^F - 2 {}^* \bar{\eta} \sigma^F - 2 \bar{\eta} \rho^F + 2\omega \bar{\alpha}^F - \hat{\chi} \cdot \alpha^F \quad (59)$$

$$\hat{\nabla}_3 \alpha^F + \frac{1}{2} \text{tr} \bar{\chi} \alpha^F = -\hat{\nabla} \rho^F + {}^* \hat{\nabla} \sigma^F - 2 {}^* \eta \sigma^F + 2 \eta \rho^F + 2\bar{\omega} \alpha^F - \hat{\chi} \cdot \bar{\alpha}^F \quad (60)$$

$$\hat{\nabla}_4 \rho^F = -\hat{\text{div}} \alpha^F - \text{tr} \chi \rho^F - (\eta - \bar{\eta}) \cdot \alpha^F \quad (61)$$

$$\hat{\nabla}_4 \sigma^F = -\hat{\text{curl}} \alpha^F - \text{tr} \chi \sigma^F + (\eta - \bar{\eta}) \cdot {}^* \alpha^F \quad (62)$$

$$\hat{\nabla}_3 \rho^F = -\hat{\text{div}} \bar{\alpha}^F + \text{tr} \bar{\chi} \rho^F + (\eta - \bar{\eta}) \cdot \bar{\alpha}^F \quad (63)$$

$$\hat{\nabla}_3 \sigma^F = -\hat{\text{curl}} \bar{\alpha}^F - \text{tr} \bar{\chi} \sigma^F + (\eta - \bar{\eta}) \cdot {}^* \bar{\alpha}^F, \quad (64)$$

where $\hat{\nabla}$ is the horizontal (tangential to $S_{u,\bar{u}}$) component of the spacetime gauge covariant derivative $\hat{D}$. We have the following lemma regarding the properties of the null Bianchi and null Yang-Mills equations.

Lemma 1: *The null Bianchi equations (52-57) and the null Yang-Mills equations (59-265) are manifestly hyperbolic.*

Proof: The proof is a simple consequence of the existence of Bel-Robinson tensor for the Weyl curvature

$$Q_{\alpha\beta\gamma\delta} := W_{\alpha\rho\gamma\sigma} W_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} + {}^* W_{\alpha\rho\gamma\sigma} {}^* W_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} \quad (65)$$

and the canonical stress-energy tensor

$$\mathfrak{T}_{\mu\nu} := \frac{1}{2} (F^P{}_{Q\mu\alpha} F^Q{}_{P\nu}{}^{\alpha} + {}^* F^P{}_{Q\mu\alpha} {}^* F^Q{}_{P\nu}{}^{\alpha}) \quad (66)$$

¹ $A \hat{\otimes} B := (A \otimes B + B \otimes A - A \cdot B \gamma)$, $(A \wedge B) := \epsilon^{ac} \gamma^{bd} A_{ab} B_{cd}$, $(\text{curl} A)_{a_1 \dots a_n} := \epsilon^{cd} \nabla_c A_{da_1 \dots a_n}$

for the Yang-Mills curvature. Indeed one may explicitly obtain energy identities for the Weyl curvature energy as well as Yang-Mills energy. For a future directed unit time-like vector field $n = \frac{1}{2}(e_3 + e_4)$, construct the current

$$\mathfrak{E} := Q(n, n, n, \cdot). \quad (67)$$

Integrating the divergence of $\mathfrak{E}$ over the spacetime domain $\mathcal{D}_{u, \bar{u}}$ yields

$$\int_{\mathcal{D}_{u, \bar{u}}} \mathcal{D}_\mu \mathfrak{E}^\mu = \int_H \mathfrak{E}(e_4) + \int_{\bar{H}_{\bar{u}}} \mathfrak{E}(e_3) - \int_{H_0} \mathfrak{E}(e_4) - \int_{\bar{H}_0} \mathfrak{E}(e_3), \quad (68)$$

where $\mathcal{D}_\mu \mathfrak{E}^\mu$ is algebraic in the Weyl curvature W

$$\begin{aligned} \mathcal{D}_\mu \mathfrak{E}^\mu &= \mathcal{D}^\mu (Q_{\mu\beta\gamma\delta} n^\beta n^\gamma n^\delta) = (W_\beta{}^\mu{}_\delta{}^\nu J(\mathfrak{T})_{\mu\gamma\nu} + W_\beta{}^\mu{}_\gamma{}^\nu J(\mathfrak{T})_{\mu\delta\nu} \\ &+ {}^*W_\beta{}^\mu{}_\delta{}^\nu {}^*J(\mathfrak{T})_{\mu\gamma\nu} + {}^*W_\beta{}^\mu{}_\gamma{}^\nu {}^*J(\mathfrak{T})_{\mu\delta\nu}) + 3Q_{\mu\beta\gamma\delta} \mathcal{D}^\mu n^\beta n^\gamma n^\delta. \end{aligned} \quad (69)$$

Explicit calculations show

$$\mathcal{C}(e_4) = Q(W)(n, n, n, e_4) \approx (|\alpha|_\gamma^2 + |\bar{\beta}|_\gamma^2 + |\beta|_\gamma^2 + \rho^2 + \sigma^2), \quad (70)$$

$$\mathcal{C}(e_3) = Q(W)(n, n, n, e_3) \approx (|\alpha|_\gamma^2 + |\bar{\beta}|_\gamma^2 + |\beta|_\gamma^2 + \rho^2 + \sigma^2), \quad (71)$$

where the involved constants are purely numerical positive constants. This completes the case for the null-Bianchi equations. For the case of the Yang-Mills, construct the energy current

$$\mathfrak{Y} := \mathfrak{T}(n, \cdot) \quad (72)$$

and obtain a similar divergence identity

$$\int_{\mathcal{D}_{u, \bar{u}}} \mathcal{D}_\mu \mathfrak{Y}^\mu \mu_g = \int_H \mathfrak{Y}(e_4) + \int_{\bar{H}_{\bar{u}}} \mathfrak{Y}(e_3) - \int_{H_0} \mathfrak{Y}(e_4) - \int_{\bar{H}_0} \mathfrak{Y}(e_3), \quad (73)$$

where $\mathcal{D}_\mu \mathfrak{Y}^\mu = \mathcal{D}^\mu (\mathfrak{T}_{\mu\nu} n^\nu) = \mathfrak{T}_{\mu\nu} \mathcal{D}^\mu n^\nu$ is purely algebraic in the Yang-Mills curvature F . Explicit calculations show

$$\mathfrak{Y}(e_4) = \mathfrak{T}(n, e_4) \approx (|\alpha^F|_{\gamma, \delta}^2 + \rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F), \quad (74)$$

$$\mathfrak{Y}(e_3) = \mathfrak{T}(n, e_3) \approx (|\bar{\alpha}^F|_{\gamma, \delta}^2 + \rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F), \quad (75)$$

where the involved constants are purely numerical positive constants. These integrals persist at the level of higher order too with the introduction of lower order error terms. This completes the proof. $\square$

We may write down explicitly the null components of the Yang-Mills stress energy tensor

$$\mathfrak{T}_{43} = \rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F, \quad \mathfrak{T}_{4A} = \alpha_A^F \cdot \rho^F + \epsilon_A{}^B \alpha_B^F \cdot \sigma^F, \quad (76)$$

$$\mathfrak{T}_{3A} = -\bar{\alpha}_A^F \cdot \rho^F + \epsilon_A{}^B \bar{\alpha}_B^F \cdot \sigma^F, \quad \mathfrak{T}_{44} = \alpha_A^F \cdot \alpha_B^F \gamma^{AB}, \quad \mathfrak{T}_{33} = \bar{\alpha}_A^F \cdot \bar{\alpha}_B^F \gamma^{AB}, \quad (77)$$

$$\mathfrak{T}_{ab} = \frac{1}{2}(\rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F) \gamma_{AB} - (\alpha_A^F \cdot \bar{\alpha}_B^F + \bar{\alpha}_B^F \cdot \alpha_A^F - \alpha_C^F \cdot \bar{\alpha}_D^F \gamma^{CD} \gamma_{AB}), \quad (78)$$

where $\cdot$ indicates the inner product on the fibres of the gauge bundle. Let us now write down the commutator formulas for $\mathfrak{g}$ -valued sections of vector bundles over $S_{u, \bar{u}}$ that transform as tensors under gauge transformations

$$\begin{aligned} [\hat{\nabla}_4, \hat{\nabla}_B] \mathcal{G}^P{}_{QA_1 A_2 A_3 \dots A_n} &= [\hat{D}_4, \hat{D}_B] \mathcal{G}^P{}_{QA_1 A_2 A_3 \dots A_n} + (\nabla_B \log \Omega) \hat{\nabla}_4 \mathcal{G}^P{}_{QA_1 A_2 A_3 \dots A_n} \\ &\quad - \gamma^{CD} \chi_{BD} \hat{\nabla}_C \mathcal{G}^P{}_{QA_1 A_2 A_3 \dots A_n} - \sum_{i=1}^n \gamma^{CD} \chi_{BD} \bar{\eta}_{A_i} \mathcal{G}^P{}_{QA_1 A_2 A_3 \dots \hat{A}_i C \dots A_n} \\ &\quad + \sum_{i=1}^n \gamma^{CD} \chi_{A_i B} \bar{\eta}_D \mathcal{G}^P{}_{QA_1 A_2 A_3 \dots \hat{A}_i C \dots A_n} \end{aligned}$$

and

$$[\hat{D}_4, \hat{D}_A]\mathcal{G}^P{}_{QA_1A_2\cdots A_n} = (\nabla_A \log \Omega)\hat{\nabla}_4\mathcal{G}^P{}_{QA_1A_2\cdots A_n} - \sum_i R(e_C, e_{A_i}, e_4, e_A)\mathcal{G}^P{}_{QA_1\cdots \hat{A}_i\cdots A_n} \\ + F^P{}_{R4A}\mathcal{G}^R{}_{QA_1A_2\cdots A_n} - F^R{}_{Q4A}\mathcal{G}^P{}_{RA_1A_2\cdots A_n} \quad (79)$$

which through the algebraic Bianchi identity and the definition of the null curvature components may be written in the following schematic forms (indices are suppressed)

$$[\hat{D}_4, \hat{D}_A]\mathcal{G} \sim (\beta + \alpha^F \cdot (\rho^F + \sigma^F))\mathcal{G} + \alpha^F\mathcal{G} + (\eta + \bar{\eta})\hat{\nabla}_4\mathcal{G} \quad (80)$$

and

$$[\hat{\nabla}_4, \hat{\nabla}_B]\mathcal{G} \sim [\hat{D}_4, \hat{D}_B]\mathcal{G} + (\eta + \bar{\eta})\hat{\nabla}_4\mathcal{G} - \chi\hat{\nabla}\mathcal{G} + \chi\bar{\eta}\mathcal{G}.$$

Similar schematic expressions hold for $[\hat{\nabla}_3, \hat{\nabla}_B]\mathcal{G}$

$$[\hat{\nabla}_3, \hat{\nabla}_B]\mathcal{G} \sim [\hat{D}_3, \hat{D}_B]\mathcal{G} + (\eta + \bar{\eta})\hat{\nabla}_3\mathcal{G} - \bar{\chi}\nabla\mathcal{G} + \bar{\chi}\eta\mathcal{G} \quad (81)$$

and

$$[\hat{D}_3, \hat{D}_A]\mathcal{G} \sim (\bar{\beta} + \bar{\alpha}^F \cdot (\rho^F + \sigma^F))\mathcal{G} + \bar{\alpha}^F\mathcal{G} + (\eta + \bar{\eta})\hat{\nabla}_3\mathcal{G}. \quad (82)$$

For a spacetime (and gauge) scalar f the following holds

$$[\hat{\nabla}_4, \hat{\nabla}_A]f \sim (\eta + \bar{\eta})\nabla_4f - \chi\nabla f, \quad [\hat{\nabla}_3, \hat{\nabla}_A]f \sim (\eta + \bar{\eta})\nabla_3f - \bar{\chi}\nabla f. \quad (83)$$

We will use these commutation formulas while deriving higher order energy estimates. Note that we write the schematic form since we would not require the exact form while deriving estimates.

4 Main theorem and idea of the proof

In this section, we describe the main result of the article and sketch the main argument behind the proof. As discussed previously, the Yang-Mills sourced null Bianchi equations (52-57) and null Yang-Mills equations are manifestly hyperbolic contrary to the equations $d\Gamma + \Gamma\Gamma = R$ (36-49) and $dA + [A, A] = F$ which are manifestly non-hyperbolic, where Γ , A , R , and F denote connection coefficients on the frame tangent bundle, connection on the principle G -bundle (or gauge bundle), curvature of frame tangent bundle, and curvature of the principle G -bundle, respectively. Nevertheless, the equations $d\Gamma + \Gamma\Gamma = R$ (36-49) exhibit special structure as we shall observe. In addition we will never use the equations $dA + [A, A] = F$ rather derive estimates with the fully gauge covariant derivatives (since working at the level of connection would require a choice of Yang-Mills gauge). While proving a local existence in the Cauchy problem (this needs to be done at the end utilizing the estimates obtained from analysis in double null gauge) for Einstein-Yang-Mills equations, one ought to work with the connections directly. However, the connections can always be estimated in terms of the gauge invariant norms of the Yang-Mills curvature and a local existence theory can be obtained (e.g., Yang-Mills equations take the form of a symmetric hyperbolic system in temporal gauge and one can estimate the residual spacial connections in terms of the electric field that can be constructed by means of the null components of the Yang-Mills curvature). The primary factor behind the semi-global existence feature in the context of characteristic initial value problem is the remarkable null structure associated with the nonlinearities of the Einstein-Yang-Mills equations while expressed in the double null gauge. Of course, these null structures are more obvious if one writes down the gauge covariant wave equations for the Weyl curvature and the Yang-Mills curvature (see [19, 20]). Essentially this null structure played a crucial role in establishing the non-linear stability of Minkowski space under

vacuum and electromagnetic perturbations [6, 7]. We first write down the theorem and then briefly discuss the idea of the proof and in particular how the remarkable structure of the Einstein-Yang-Mills equations is conducive to proving a semi-global existence result. We omit the description of a characteristic initial value problem since we are only interested in obtaining estimates. For a precise formulation, the reader is referred to the section 2.3 of [8]

Theorem: *Let (M, g) be a $3 + 1$ dimensional Lorentzian manifold and S be a space-like topological 2-sphere. There exists a suitable $\epsilon > 0$ such that a solution to the coupled Einstein-Yang-Mills equations exists in a suitable function space in the future causal domain $\mathcal{D}_{u, \bar{u}}$ of S foliated by the two families of the incoming and outgoing null hypersurfaces H_u and $\bar{H}_{\bar{u}}$ such that $u \in [0, \epsilon]$, $\bar{u} \in [0, J]$. The choice of ϵ depends on the following norm of the initial data*

$$\begin{aligned} \mathcal{O}_0 &:= \sup_{S \subset H_0, S \in \bar{H}_0} \sup_{\varphi \in \{\hat{\chi}, tr\hat{\chi}, \hat{\bar{\chi}}, tr\hat{\bar{\chi}}, \omega, \bar{\omega}, \eta, \bar{\eta}\}} \max \left(\sum_{I=0}^2 \|\nabla^I \varphi\|_{L^2(S)}, \sum_{I=0}^1 \|\nabla^I \varphi\|_{L^4(S)}, \|\varphi\|_{L^\infty(S)} \right) \\ &\quad + \sup_{S \subset H_0, S \in \bar{H}_0} \sup_{\varphi \in (\eta, \bar{\omega})} \|\nabla_3 \varphi\|_{L^2(S)} + \sup_{S \subset H_0, S \in \bar{H}_0} \sup_{\varphi \in (\bar{\eta}, \omega)} \|\nabla_4 \varphi\|_{L^2(S)} \\ &+ \left(\sup_{u, \bar{u}} C |C^{-1} \gamma_{round}(X, X) \leq \gamma(X, X) \leq C \gamma_{round}(X, X), \gamma \text{ is a metric on } S_{u,0} \text{ or } S_{0,\bar{u}}, X \in TS_{u,0} \text{ or } TS_{0,\bar{u}} \right) \\ \mathcal{W}_0 &:= \sup_{\Psi \in \{\alpha, \beta, \bar{\beta}, \rho, \sigma\}} \|\nabla \Psi\|_{L^2(H_0)} + \sup_{\Psi \in \{\bar{\alpha}, \beta, \bar{\beta}, \rho, \sigma\}} \|\nabla \Psi\|_{L^2(\bar{H}_0)} + \max \left(\|\nabla_4 \alpha\|_{L^2(H_0)}, \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H}_0)}, \right. \\ &\quad \left. \|\nabla_4 \beta\|_{L^2(\bar{H}_0)}, \|\nabla_3 \bar{\beta}\|_{L^2(H_0)} \right) + \sup_{S \subset H_0, S \in \bar{H}_0} \sup_{\Psi \in \{\alpha, \bar{\alpha}, \beta, \bar{\beta}, \rho, \sigma\}} \max \left(\sum_{I=0}^1 \|\nabla^I \Psi\|_{L^2(S)}, \|\Psi\|_{L^4(S)} \right), \\ \mathcal{F}_0 &:= \sum_{I=0}^2 \left(\sup_{\Phi \in \{\alpha^F, \rho^F, \sigma^F\}} \|\hat{\nabla}^I \Phi\|_{L^2(H_0)} + \sup_{\Phi \in \{\bar{\alpha}^F, \rho^F, \sigma^F\}} \|\hat{\nabla}^I \Phi\|_{L^2(\bar{H}_0)} + \sup_{\Phi \in \{\rho^F, \sigma^F\}} \|\hat{\nabla}_4^I \Phi\|_{L^2(\bar{H}_0)} \right. \\ &+ \sup_{\Phi \in \{\rho^F, \sigma^F\}} \|\hat{\nabla}_3^I \Phi\|_{L^2(H_0)} + \|\hat{\nabla}_4^I \alpha^F\|_{L^2(H_0)} + \|\hat{\nabla}_3^I \bar{\alpha}^F\|_{L^2(H_0)} + \sup_{\Phi \in (\rho^F, \sigma^F)} \|\hat{\nabla}_4 \hat{\nabla} \Phi\|_{L^2(\bar{H}_0)} + \sup_{\Phi \in (\rho^F, \sigma^F)} \|\hat{\nabla}_3 \hat{\nabla} \Phi\|_{L^2(H_0)} \\ &\quad \left. + \|\hat{\nabla}_4 \hat{\nabla} \alpha^F\|_{L^2(H_0)} + \|\hat{\nabla}_3 \hat{\nabla} \bar{\alpha}^F\|_{L^2(\bar{H}_0)} + \sup_{S \subset H_0, S \subset \bar{H}_0} \sup_{\Phi \in \{\alpha^F, \bar{\alpha}^F, \rho^F, \sigma^F\}} \max \left(\sum_{I=0}^1 \|\hat{\nabla}^I \Phi\|_{L^4(S)} \right) \right), \end{aligned}$$

where γ_{round} is the standard round metric on a 2-sphere. Moreover, the following norm of the Weyl curvature and the Yang-Mills curvature remain bounded in terms of the initial data in the domain of existence i.e.,

$$\begin{aligned} &\sum_{I=0}^1 \left(\sup_u \sup_{\Psi \in \{\alpha, \beta, \bar{\beta}, \rho, \sigma\}} \|\nabla^I \Psi\|_{L^2(H)} + \sup_{\bar{u}} \sup_{\Psi \in \{\bar{\alpha}, \beta, \bar{\beta}, \rho, \sigma\}} \|\nabla^I \Psi\|_{L^2(\bar{H})} \right. \\ &\quad \left. + \sup_u \|\nabla_4 \alpha\|_{L^2(H)} + \sup_{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})} \right), \\ &+ \sum_{I=0}^2 \left(\sup_u \sup_{\Phi \in \{\alpha^F, \rho^F, \sigma^F\}} \|\hat{\nabla}^I \Phi\|_{L^2(H)} + \sup_{\bar{u}} \sup_{\Phi \in \{\bar{\alpha}^F, \rho^F, \sigma^F\}} \|\hat{\nabla}^I \Phi\|_{L^2(\bar{H})} \right) + \sup_u \|\hat{\nabla}_4 \alpha^F\|_{L^2(H)} \\ &+ \sup_{\bar{u}} \|\hat{\nabla}_3 \bar{\alpha}^F\|_{L^2(\bar{H})} + \sup_u \|\hat{\nabla}_4 \hat{\nabla}_4 \alpha^F\|_{L^2(H)} + \sup_{\bar{u}} \|\hat{\nabla}_3 \hat{\nabla}_3 \bar{\alpha}^F\|_{L^2(\bar{H})} + \sup_{\bar{u}} \|\hat{\nabla}_3 \hat{\nabla} \bar{\alpha}^F\|_{L^2(\bar{H})} \\ &\quad + \sup_u \|\hat{\nabla}_4 \hat{\nabla} \alpha^F\|_{L^2(H)} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) \end{aligned}$$

throughout $\mathcal{D}_{u, \bar{u}}$.

Notice that one may continue to prove these estimates to the successive higher orders and therefore one can establish the result for the classical solution. Let us now discuss the rough idea of the proof.

We first assume that the connection coefficients φ enjoy an upper bound (possibly large), namely the bootstrap constant. This upper bound allows us to estimate the ellipticity constant of the metric on the topological 2-spheres in terms of the initial data (independent of the assumed upper bound of the connection coefficients) thereby allowing us to utilize the standard Sobolev inequalities. We start the main estimates with the connection coefficients assuming finiteness of the curvature norms and $L^2(H, \bar{H})$ norm of second angular derivatives of the connection coefficients. The good connection coefficients φ_g satisfy a ∇_3 equation and therefore they gain a smallness factor ϵ through integration. As a consequence, they can be bounded by the initial data alone. On the contrary, the bad connection coefficients satisfying ∇_4 equations are estimated in terms of the curvature norms and the $L^2(H, \bar{H})$ norm of second angular derivatives of φ . The remarkable point to note here is that in the structure equation for these bad connection coefficients φ_b the terms $\varphi_b \varphi_b$ do not appear. This is precisely a consequence of the null structure of the non-linearities present. This allows us to employ Grönwall's inequality to yield the desired estimates. Next, using the available estimates, we show that the undetermined connection norms $\|\nabla^2 \varphi\|_{L^2(H, \bar{H})}$ are determined by the curvature norms through a series of transport-elliptic estimates.

In the next step, we use simple integration by parts arguments to obtain the estimates of the Weyl and Yang-Mills curvatures (contrary to using the Bel-Robinson and Yang-Mills stress-energy tensors). Once again the good curvature components $(\alpha, \beta, \bar{\beta}, \rho, \sigma, \alpha^F, \rho^F, \sigma^F)$ enjoy a gain of a smallness factor ϵ and therefore are innocuous. However, the bad curvature components $(\bar{\alpha}, \bar{\alpha}^F)$ do not gain such a small factor since they are integrated along $\bar{H}$. Therefore, in the energy estimate, they pose a potential obstruction. However, the remarkable null structure of non-linearities once again remedies the situation. In other words, the connection coefficients multiplying the terms $|\nabla \bar{\alpha}|^2$ and $|\nabla \bar{\alpha}^F|^2$ are good connection coefficients (i.e., they satisfy the ∇_3 transport equations) and therefore are estimated completely by means of the initial data. Therefore, we can use a Grönwall inequality to obtain the final estimate. In addition, there are several occasions where the null structure of the Einstein-Yang-Mills equations plays a subtle role. Once we have obtained the final estimates, we may choose the initial bootstrap constant accordingly to close the argument.

5 Important inequalities

Throughout our analysis, we need to employ Sobolev inequalities on the topological sphere S at different stages. However, since the metric on S is dynamical, we need to make appropriate bootstrap assumptions. Technically, one could define the norms and energies with respect to a background metric on S and try to control the additional terms that arise. However, we will start with making a bootstrap assumption on the connection coefficients similar to [8, 5]. Let us assume the following

$$\sup_{u, \bar{u}} \|\varphi\|_{L^\infty(S_{u, \bar{u}})} \leq \Delta, \quad (84)$$

where $\varphi := (tr\bar{\chi}, \hat{\chi}, \bar{\eta}, \omega, tr\chi, \hat{\chi}, \eta, \bar{\omega})$ and Δ is possibly a large constant. Later, we will show through the analysis that one can choose Δ such that the bootstrap assumption (84) is closed. Under this assumption, it is straightforward to prove that the null lapse function Ω , the ellipticity constant of the dynamical metric γ , and the shift vector field $b^A \frac{\partial}{\partial \theta^A}$ are bounded in the spacetime slab $D_{u, \bar{u}}$ ($u \in [0, \epsilon]$, $\bar{u} \in [0, J]$) in terms of the initial data $\mathcal{O}_0$ (see [5, 8]). This of course provides us with estimates on area of $S_{u, \bar{u}}$ ([5, 8]). Once we have the metric components under control in this double null gauge, we may write down the following set of inequalities that will be useful throughout.

1. [5, 8] *Given $\sup_{u, \bar{u}} \|tr\chi, tr\bar{\chi}\|_{L^\infty(S_{u, \bar{u}})} \leq C(\mathcal{O}_0)$, the following inequalities hold throughout $\mathcal{D}_{u, \bar{u}}$ ($u \in$*

$[0, \epsilon], \bar{u} \in [0, J]$) for a sufficiently small ϵ

$$\|\mathcal{G}\|_{L^p(S_{u,\bar{u}})} \leq C(\mathcal{O}_0) \left(\|\mathcal{G}\|_{L^p(S_{u,\bar{u}'})} + \int_{\bar{u}'}^{\bar{u}} \|\hat{\nabla}_4 \mathcal{G}\|_{L^p(S_{u,\bar{u}''})} d\bar{u}'' \right) \quad (85)$$

$$\|\mathcal{G}\|_{L^p(S_{u,\bar{u}})} \leq C \left(\|\mathcal{G}\|_{L^p(S_{u',\bar{u}})} + \int_{u'}^u \|\hat{\nabla}_3 \mathcal{G}\|_{L^p(S_{u'',\bar{u}})} du'' \right) \quad (86)$$

for $1 \leq p \leq \infty$. Here $\mathcal{G}$ can be a section of the mixed bundle and $\hat{\nabla}$ is the gauge covariant connection compatible with the metrics of both fibres.

2. [5, 8] There exists $\epsilon_0 = \epsilon_0(\mathcal{O}_0, \Delta)$ such that the following gauge invariant Sobolev inequalities hold for any horizontal gauge field strength in the spacetime slab $\mathcal{D}_{u,\bar{u}}$ ($u \in [0, \epsilon], \bar{u} \in [0, J]$), $\epsilon \leq \epsilon_0$

$$\|\mathcal{G}\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0) \sum_{I=0}^1 \|\hat{\nabla}^I \mathcal{G}\|_{L^2(S_{u,\bar{u}})}, \quad (87)$$

$$\|\mathcal{G}\|_{L^\infty(S_{u,\bar{u}})} \leq C(\mathcal{O}_0) (\|\mathcal{G}\|_{L^4(S_{u,\bar{u}})} + \|\hat{\nabla} \mathcal{G}\|_{L^4(S_{u,\bar{u}})}), \quad (88)$$

$$\|\mathcal{G}\|_{L^\infty(S_{u,\bar{u}})} \leq C(\mathcal{O}_0) \sum_{I=0}^2 \|\hat{\nabla}^I \mathcal{G}\|_{L^2(S_{u,\bar{u}})}. \quad (89)$$

Proof: Let $f = (\mathcal{G}^P{}_{QA_1A_2\dots A_n} \mathcal{G}^Q{}_{PB_1B_2\dots B_n} \gamma^{A_1B_1} \gamma^{A_2B_2} \dots \gamma^{A_nB_n} + \delta)^{1/2}$ and apply the standard Sobolev embedding for scalars

$$\|f\|_{L^4(S)} \leq C(\mathcal{O}_0) \left(\|f\|_{L^2(S)} + \left\| \frac{\langle \mathcal{G}, \hat{\nabla} \mathcal{G} \rangle}{(|\mathcal{G}|^2 + \delta)^{\frac{1}{2}}} \right\|_{L^2(S)} \right) \leq C \left(\|\mathcal{G}\|_{L^2(S)} + \|\hat{\nabla} \mathcal{G}\|_{L^2(S)} \right)$$

after letting $\delta \rightarrow 0$. Note that $\nabla f^2 = 2\mathcal{G}^P{}_{QA_1A_2\dots A_n} \hat{\nabla} \mathcal{G}^Q{}_{PB_1B_2\dots B_n} \gamma^{A_1B_1} \gamma^{A_2B_2} \dots \gamma^{A_nB_n}$ since f is gauge invariant and $\hat{\nabla}$ is metrics compatible. The second inequality follows in a similar way. The last inequality is a consequence of first two.

3. [5] The following inequalities hold for any horizontal gauge field strength $\mathcal{G}$ throughout $\mathcal{D}_{u,\bar{u}}$ ($u \in [0, \epsilon], \bar{u} \in [0, J]$) under the bootstrap assumption (84)) (which in turn controls the metric on $S_{u,\bar{u}}$)

$$\|\mathcal{G}\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0) \left(\|\mathcal{G}\|_{L^2(S_{u,\bar{u}})}^{\frac{1}{2}} \|\hat{\nabla} \mathcal{G}\|_{L^2(S_{u,\bar{u}})}^{\frac{1}{2}} + \|\mathcal{G}\|_{L^2(S_{u,\bar{u}})} \right), \quad (90)$$

$$\|\mathcal{G}\|_{L^\infty(S_{u,\bar{u}})} \leq C(\mathcal{O}_0) \left(\|\mathcal{G}\|_{L^4(S_{u,\bar{u}})}^{\frac{1}{2}} \|\hat{\nabla} \mathcal{G}\|_{L^4(S_{u,\bar{u}})}^{\frac{1}{2}} + \|\mathcal{G}\|_{L^4(S_{u,\bar{u}})} \right). \quad (91)$$

Note that we will never need to use these inequalities on gauge field strengths.

4. [5, 8] Under the bootstrap assumption (84), following holds for any horizontal gauge field strength $\mathcal{G}$ throughout $\mathcal{D}_{u,\bar{u}}$ ($u \in [0, \epsilon], \bar{u} \in [0, J]$)

$$\|\mathcal{G}\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0) \left(\|\mathcal{G}\|_{L^4(S_{0,\bar{u}})} + \|\mathcal{G}\|_{L^2(\bar{H})}^{1/2} \|\hat{\nabla}_4 \mathcal{G}\|_{L^2(\bar{H})}^{1/4} (\|\mathcal{G}\|_{L^2(\bar{H})} + \|\hat{\nabla} \mathcal{G}\|_{L^2(\bar{H})})^{1/4} \right), \quad (92)$$

$$\|\mathcal{G}\|_{L^4(S_{u,\bar{u}})} \leq C \left(\|\mathcal{G}\|_{L^4(S_{0,\bar{u}})} + \|\mathcal{G}\|_{L^2(\bar{H})}^{1/2} \|\hat{\nabla}_3 \mathcal{G}\|_{L^2(\bar{H})}^{1/4} (\|\mathcal{G}\|_{L^2(\bar{H})} + \|\hat{\nabla} \mathcal{G}\|_{L^2(\bar{H})})^{1/4} \right). \quad (93)$$

All of these inequalities hold true for sections of tangent bundles of $S_{u,\bar{u}}$ (i.e., the non-gauge field strengths) and in such case $\hat{\nabla}$ and ∇ coincide. These will be the main inequalities that we will use throughout.

6 Estimate of the connection and curvature components in terms of the initial data and curvature energy

We divide the connection coefficients into two classes depending on the transport equations they satisfy. Notice that each element of the set $(tr\bar{\chi}, \hat{\chi}, \bar{\eta}, \omega, tr\chi, \hat{\chi})$ satisfies a ∇_3 equation, where as η and $\bar{\omega}$ only satisfy

a ∇_4 equation. We denote each element of $(tr\bar{\chi}, \hat{\bar{\chi}}, \bar{\eta}, \omega, tr\chi, \hat{\chi})$ by φ_g and an element of $(\eta, \bar{\omega})$ by φ_b and element of the whole set $(tr\bar{\chi}, \hat{\bar{\chi}}, \bar{\eta}, \omega, tr\chi, \hat{\chi}, \eta, \bar{\omega})$ by φ . By C , we will always mean an universal constant while a constant that depends on the other entities will be denoted so. We first define the curvature norms (both Weyl and Yang-Mills) that we intend to estimate

$$\mathcal{W}(S) := \sup_{u, \bar{u}} \|\alpha, \bar{\alpha}, \beta, \bar{\beta}, \rho, \sigma\|_{L^4(S_{u, \bar{u}})}, \quad (94)$$

$$\mathcal{F}(S) := \sum_{i=0}^1 \sup_{u, \bar{u}} \|\hat{\nabla}^i(\alpha^F, \bar{\alpha}^F, \rho^F, \sigma^F)\|_{L^4(S)}, \quad (95)$$

$$\begin{aligned} \mathcal{W} := & \sum_{I=0}^1 \left(\sup_u \sup_{\Psi \in \{\alpha, \beta, \bar{\beta}, \rho, \sigma\}} \|\nabla^I \Psi\|_{L^2(H)} + \sup_{\bar{u}} \sup_{\Psi \in \{\bar{\alpha}, \beta, \bar{\beta}, \rho, \sigma\}} \|\nabla^I \Psi\|_{L^2(\bar{H})} \right. \\ & \left. + \sup_u \|\nabla_4 \alpha\|_{L^2(H)} + \sup_{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})} \right), \\ \mathcal{F} := & \sum_{I=0}^2 \left(\sup_u \sup_{\Phi \in \{\alpha^F, \rho^F, \sigma^F\}} \|\hat{\nabla}^I \Phi\|_{L^2(H)} + \sup_{\bar{u}} \sup_{\Phi \in \{\bar{\alpha}^F, \rho^F, \sigma^F\}} \|\hat{\nabla}^I \Phi\|_{L^2(\bar{H})} \right) + \sup_u \|\hat{\nabla}_4 \alpha^F\|_{L^2(H)} \\ & + \sup_{\bar{u}} \|\hat{\nabla}_3 \bar{\alpha}^F\|_{L^2(\bar{H})} + \sup_u \|\hat{\nabla}_4 \hat{\nabla}_4 \alpha^F\|_{L^2(H)} + \sup_{\bar{u}} \|\hat{\nabla}_3 \hat{\nabla}_3 \bar{\alpha}^F\|_{L^2(\bar{H})} + \sup_{\bar{u}} \|\hat{\nabla}_3 \hat{\nabla} \bar{\alpha}^F\|_{L^2(\bar{H})} \\ & + \sup_u \|\hat{\nabla}_4 \hat{\nabla} \alpha^F\|_{L^2(H)}. \end{aligned}$$

Lemma 2: Assume that $\|\nabla^2 \varphi_g\|_{L^2(H, \bar{H})}, \|\nabla^2 \varphi_b\|_{L^2(H, \bar{H})}, \mathcal{W}, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S) < \infty$. Then the connection coefficients satisfy the following point-wise estimates

$$\sup_{u, \bar{u}} \|\varphi_g\|_{L^\infty(S_{u, \bar{u}})} \leq C\mathcal{O}_0, \quad \sup_{u, \bar{u}} \|\varphi_b\|_{L^\infty(S_{u, \bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S), \|\nabla^2 \varphi\|_{L^2(H)}). \quad (96)$$

Proof: The proof relies on the direct transport inequalities

$$\|\varphi\|_{L^\infty(S_{u, \bar{u}})} \leq C(\mathcal{O}_0) \left(\|\varphi\|_{L^\infty(S_{u, \bar{u}'})} + \sup_{\bar{u}, \bar{u}'} \int_{\bar{u}'}^{\bar{u}} |\nabla_4 \varphi| d\bar{u}'' \right) \quad (97)$$

$$\|\varphi\|_{L^\infty(S_{u, \bar{u}})} \leq C \left(\|\varphi\|_{L^\infty(S_{u', \bar{u}})} + \sup_{u, u'} \int_{u'}^u |\nabla_3 \varphi| du'' \right) \quad (98)$$

and delicate trace estimates. First assume the bootstrap assumption of the good connection coefficients φ_g

$$\sup_{u \in [0, \epsilon], \bar{u} \in [0, J]} \|\varphi_g\|_{L^\infty(S_{u, \bar{u}})} \leq 2C\mathcal{O}_0. \quad (99)$$

Using the transport equations and trace-estimates, we will improve this bootstrap estimate thereby completing the proof. First consider the bad connection coefficients $\varphi_g := (\eta, \bar{\omega})^2$

$$\|\eta\|_{L^\infty(S_{u, \bar{u}})} \leq C\|\eta\|_{L^\infty(S_{u, 0})} + C \sup_{\bar{u}, \bar{u}'} \int_0^{\bar{u}} \left| -\chi \cdot (\eta - \bar{\eta}) - \beta - \frac{1}{2}(\alpha^F \rho^F - \alpha^F \sigma^F) \right| d\bar{u}'' \quad (100)$$

which under the boot-strap, Sobolev embedding (89), and the definition of the entity $\mathcal{F}(S)$ yields

$$\|\eta\|_{L^\infty(S_{u, \bar{u}})} \leq C\|\eta\|_{L^\infty(S_{u, 0})} + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\eta\|_{L^\infty(S)} d\bar{u}' + C\mathcal{F}(S) + C \sup_{\bar{u}, \bar{u}'} \left(\int_0^{\bar{u}} |\beta|^2 d\bar{u}' \right)^{\frac{1}{2}}. \quad (101)$$

²note that one may obtain an equation for $\nabla_3 \eta$ by means of the relation $\nabla_3 \eta = -\nabla_3 \bar{\eta} + 2\nabla_3 \nabla \ln \Omega$

Now recall that $\sup_{S_{u,\bar{u}}}(\int_0^{\bar{u}} |\beta|^2 d\bar{u}')^{\frac{1}{2}}$ is nothing but the trace norm $\|\beta\|_{tr(H)}$ defined in ([5]). We may estimate this term by means of the following trace inequality ([5])

$$\|\nabla_4 A\|_{tr(H)} \leq (\|\nabla_4^2 A\|_{L^2(H)} + \|A\|_{L^2(H)})^{1/2} \|\nabla^2 A\|_{L^2(H)}^{1/2} + \|\nabla_4 \nabla A\|_{L^2(H)} + \|\nabla A\|_{L^2(H)} \quad (102)$$

for a section A of $TS_{u,\bar{u}}$. Now in order to estimate $\|\beta\|_{tr(H)}$, we write the following

$$\beta = -\nabla_4 \eta - \underbrace{\chi \cdot (\eta - \bar{\eta}) - \frac{1}{2}(\alpha^F \rho^F - \alpha^F \sigma^F)}_I. \quad (103)$$

The terms I is harmless and its tr norm can be estimated by $C(\mathcal{O}, \mathcal{F}, \mathcal{F}(S))$. Therefore we focus on the $\|\cdot\|_{tr(H)}$ of $\nabla_4 \eta$. The trace inequality (102) yields

$$\|\nabla_4 \eta\|_{tr(H)} \leq (\|\nabla_4^2 \eta\|_{L^2(H)} + \|\nabla_4 \eta\|_{L^2(H)})^{1/2} \|\nabla^2 \eta\|_{L^2(H)}^{1/2} + \|\nabla_4 \nabla \eta\|_{L^2(H)} + \|\nabla \eta\|_{L^2(H)}.$$

Each term on the left hand side may be estimated as follows. Using the equation of motion, $\nabla_4 \eta$ is estimated by the curvature norm

$$\|\nabla_4 \eta\|_{L^2(H)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}(S)). \quad (104)$$

In order to estimate $\|\nabla_4^2 \eta\|_{L^2(H)}$, we act ∇_4 on the transport equation for η

$$\nabla_4^2 \eta = -(\nabla_4 \hat{\chi} + \frac{1}{2} \nabla_4 tr \chi \gamma) \cdot (\eta - \bar{\eta}) - \chi \cdot (\nabla_4 \eta - \underbrace{\nabla_4 \bar{\eta}}_{II}) - \nabla_4 \beta - \frac{1}{2}(\underbrace{\nabla_4 \alpha^F(\rho^F - \sigma^F)}_{III} + \alpha^F(\nabla_4 \rho^F - \nabla_4 \sigma^F)).$$

Now notice the terms II and III do not have explicit expressions in terms of the transport equations. However, term III can be estimated by $\mathcal{F}$. Using the evolution equations for $\hat{\chi}$, $tr \chi$, and β we obtain

$$\begin{aligned} \nabla_4^2 \eta_a = & - \left((-tr \chi \hat{\chi} - 2\omega \hat{\chi} - \alpha + \frac{1}{2}(-\frac{1}{2}(tr \chi)^2 - |\hat{\chi}|_\gamma^2 - 2\omega tr \chi - \mathfrak{T}_{44})\gamma) \cdot (\eta - \bar{\eta}) \right)_a \\ & - \underbrace{(\chi \cdot (\nabla \omega + \frac{1}{2} \chi \cdot (\bar{\eta} - \zeta) + \omega(\bar{\eta} + \zeta) - \frac{1}{2} \beta - \frac{1}{2} {}^* \sigma^F \cdot \alpha^F - \frac{1}{2} \rho^F \cdot \alpha^F))_a}_V \\ & - (-2tr \chi \beta_a + (div \alpha)_a - 2\omega \beta_a + (\eta \cdot \alpha)_a - \frac{1}{2}(\underbrace{\mathcal{D}_a R_{44} - \mathcal{D}_4 R_{4a}}_{IV}) - \frac{1}{2}(\underbrace{\nabla_4 \alpha_a^F(\rho^F - \sigma^F)}_{VI})) \\ & + \alpha_a^F(-\hat{div} \alpha^F - tr \chi \rho^F - (\eta - \bar{\eta}) \cdot \alpha^F - (-\hat{curl} \alpha^F - tr \chi \sigma^F + (\eta - \bar{\eta}) \cdot {}^* \alpha^F)). \end{aligned} \quad (105)$$

Explicit computation using the Einstein's equation with Yang-Mills source i.e., $R_{\mu\nu} = \mathfrak{T}_{\mu\nu}$ yields

$$IV = \langle \alpha^F, \hat{\nabla}_b \alpha^F \rangle - \chi_{bc} \mathfrak{T}_{c4} + \eta_b \mathfrak{T}_{44} - 2\omega \mathfrak{T}_{4b} - \hat{\nabla}_4(\alpha_b^F \cdot \rho^F - \alpha_b^F \cdot \sigma^F) \quad (106)$$

and therefore one more use of the evolution equation confirms that $\|IV\|_{L^2(H)}$ is controlled by $\mathcal{F}$ and $\mathcal{O}_0$. Similarly $\|V\|_{L^2(H)}$ and $\|VI\|_{L^2(H)}$ are controlled by $\mathcal{W}$, $\mathcal{F}$, and $\mathcal{O}_0$ under the bootstrap assumption (99). Luckily we have a ∇_4 equation for $\eta - \bar{\eta}$ (the term V arises due to ∇_4 acting on $\eta - \bar{\eta}$). Also note that $\|\Phi^F\|_{L^\infty(S)} := \|(\alpha^F, \bar{\alpha}^F, \rho^F, \sigma^F)\|_{L^\infty(S)}$ can be controlled by $\|\hat{\nabla} \Phi^F\|_{L^4(S)}$ or equivalently $\mathcal{F}(S)$. Collecting all the terms and applying Sobolev embedding, we have

$$(\|\nabla_4^2 \eta\|_{L^2(H)} + \|\nabla_4 \eta\|_{L^2(H)})^{1/2} \|\nabla^2 \eta\|_{L^2(H)}^{1/2} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2 \varphi\|_{L^2(H)}), \quad (107)$$

where φ can be any element of the set $(tr\bar{\chi}, \hat{\bar{\chi}}, \bar{\eta}, \omega, tr\chi, \hat{\chi}, \eta, \bar{\omega})$. The remaining is to estimate $\|\nabla_4 \nabla \eta\|_{L^2(H)}$. We can do this by means of the evolution equation for η . Commuting the evolution equation for η with the Horizontal derivative ∇ yields (schematically)

$$\begin{aligned} \nabla_4 \nabla \eta = & -\nabla \chi \cdot (\eta - \bar{\eta}) - \chi \cdot (\nabla \eta - \nabla \bar{\eta}) - \nabla \beta - \frac{1}{2} \hat{\nabla} \alpha^F (\rho^F - \sigma^F) - \frac{1}{2} \alpha^F (\hat{\nabla} \rho^F - \hat{\nabla} \sigma^F) \\ & + (\beta + \alpha^F (\rho^F - \sigma^F) + \alpha^F) \eta + (\eta + \bar{\eta}) (-\chi(\eta - \bar{\eta}) - \beta - \frac{1}{2} \alpha^F (\rho^F - \sigma^F)) - \chi \nabla \eta + \chi \bar{\eta} \eta \end{aligned} \quad (108)$$

which utilizing Sobolev embedding on $S_{u,\bar{u}}$ (to handle the term $\|\eta\|_{L^4(S_{u,v})}$) can be estimated as

$$\|\nabla_4 \nabla \eta\|_{L^2(H)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \|\nabla^2 \varphi\|_{L^2(H)}). \quad (109)$$

Collecting all the terms together, we obtain

$$\|\eta\|_{L^\infty(S_{u,\bar{u}})} \leq C\|\eta\|_{L^\infty(S_{u,0})} + C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S)) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\eta\|_{L^\infty(S)} d\bar{u}' \quad (110)$$

which upon using Grönwall yields

$$\|\eta\|_{L^\infty(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2 \eta\|_{L^2(H)}). \quad (111)$$

Later, we shall see that $\mathcal{F}(S)$ is actually dominated by $\mathcal{F}$ and $\|\nabla^2 \eta\|_{L^2(H)}$ is dominated by $\mathcal{W}$. Now we want to estimate $\sup_{u,\bar{u}} \|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}})}$. Once again, use of transport inequality yields

$$\|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}})} \leq C \left(\|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}'})} + \sup_{S_{u,\bar{u}}} \int_0^{\bar{u}} |2\omega\bar{\omega} + \frac{3}{4}|\eta - \bar{\eta}|^2 - \frac{1}{4}(\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) - \frac{1}{8}|\eta + \bar{\eta}|^2 + \frac{1}{2}\rho + \frac{1}{4}\mathfrak{T}_{43}|d\bar{u}'' \right).$$

Once again, we observe that all the terms except ρ enjoy estimates that are controlled by $\mathcal{O}_0, \mathcal{W}, \mathcal{F}$, and $\mathcal{F}(S)$. Therefore, we focus on the term ρ . The previous inequality reduces to

$$\|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}})} \leq C\|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}'})} + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}})} d\bar{u}' + \mathcal{C}(\mathcal{O}_0, \mathcal{F}(S)) + C \sup_{S_{u,\bar{u}}} \left(\int_0^{\bar{u}} \rho^2 d\bar{u}' \right)^{\frac{1}{2}}. \quad (112)$$

Now notice $\sup_{S_{u,\bar{u}}} \left(\int_0^{\bar{u}} \rho^2 d\bar{u}' \right)^{\frac{1}{2}} = \|\rho\|_{tr(H)}$. Now we follow the same procedure as before i.e., utilize the trace inequality (102). Since ω does not satisfy a ∇_4 equation, we want to get rid of ω . Write ρ as follows after the re-scaling $\bar{\omega} = \Omega \tilde{\omega}$

$$\rho = \Omega \nabla_4 \tilde{\omega} - \underbrace{\frac{3}{4}|\eta + \bar{\eta}|^2 + \frac{1}{4}(\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) + \frac{1}{8}|\eta + \bar{\eta}|^2 - \frac{1}{4}((\rho^F)^2 + (\sigma^F)^2)}_V.$$

Here the term V is harmless following the bootstrap (99) and the previous estimate (111). Therefore, we only focus on the term $\Omega \nabla_4 \tilde{\omega}$ and noting $\|\Omega\|_{L^\infty(S_{u,\bar{u}})} \leq C(\mathcal{O}_0)$ under bootstrap, estimating $\|\nabla_4 \tilde{\omega}\|_{tr(H)}$ suffices. Now we use the trace inequality to estimate $\|\nabla_4 \tilde{\omega}\|_{tr(H)}$

$$\|\nabla_4 \tilde{\omega}\|_{tr(H)} \leq \left(\|\nabla_4^2 \tilde{\omega}\|_{L^2(H)} + \|\nabla_4 \tilde{\omega}\|_{L^2(H)} \right)^{1/2} \|\nabla^2 \tilde{\omega}\|_{L^2(H)}^{1/2} + \|\nabla_4 \nabla \tilde{\omega}\|_{L^2(H)} + \|\nabla \tilde{\omega}\|_{L^2(H)}.$$

Following the same procedure as before and controlling $\|\nabla_4 \bar{\eta}\|_{L^2(H)}$ by $\|\nabla^2 \bar{\eta}\|_{L^2(H)}$ and $\mathcal{W}$. Collecting all the terms together, one obtains

$$\|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}})} \leq C\|\bar{\omega}\|_{L^\infty(S_{u,0})} + \mathcal{C}(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2 \varphi\|_{L^2(H)}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}})} d\bar{u}' \quad (113)$$

which upon utilizing Grönwall yields

$$\|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}})} \leq \mathcal{C}(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2 \varphi\|_{L^2(H)}). \quad (114)$$

Now we estimate the good connection coefficients φ_g i.e., the ones satisfying ∇_3 equations. We use the transport inequality. First consider $tr\bar{\chi}$ and use that fact that $\|\hat{\nabla}\bar{\alpha}^F\|_{L^4(S)}$ is controlled by $\mathcal{F}$ together with Sobolev embedding (89)

$$\begin{aligned} \|tr\bar{\chi}\|_{L^\infty(S_{u,\bar{u}})} &\leq \frac{C}{2} \left(\|tr\bar{\chi}\|_{L^\infty(S_{0,\bar{u}})} + \int_0^u \left\| -\frac{1}{2}(tr\bar{\chi})^2 - |\hat{\chi}|_\gamma^2 - 2\bar{\omega}tr\bar{\chi} - \mathfrak{T}_{33} \right\|_{L^\infty(S_{u,\bar{u}})} du' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon C(\mathcal{W}, \mathcal{F}, \mathcal{F}(S)) \leq C\mathcal{O}_0 \end{aligned} \quad (115)$$

if we choose $\epsilon > 0$ sufficiently small. Now consider $\hat{\chi}$

$$\begin{aligned} \|\hat{\chi}\|_{L^\infty(S_{u,\bar{u}})} &\leq \frac{C}{2} \left(\|\hat{\chi}\|_{L^\infty(S_{0,\bar{u}})} + \sup_{S_{u,\bar{u}}} \int_0^u |\nabla_3 \hat{\chi}| du' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \frac{C}{2} \int_0^u \left\| -tr\bar{\chi}\hat{\chi} - 2\bar{\omega}\hat{\chi} \right\|_{L^\infty(S)} du' + \frac{C}{2} \epsilon^{\frac{1}{2}} \sup_{S_{u,\bar{u}}} \left(\int_0^{\bar{u}} \bar{\alpha}^2 d\bar{u}' \right)^{\frac{1}{2}}. \end{aligned} \quad (116)$$

Now we need to estimate $\sup_{S_{u,\bar{u}}} \left(\int_0^{\bar{u}} \bar{\alpha}^2 d\bar{u}' \right)^{\frac{1}{2}} = \|\bar{\alpha}\|_{tr(\bar{H})}$. Proceeding exactly the similar way as before i.e., write $\bar{\alpha}$ as follows

$$\bar{\alpha} = -\Omega \nabla_3 \hat{\tilde{\chi}} - tr\bar{\chi}\hat{\tilde{\chi}} \quad (117)$$

where $\hat{\tilde{\chi}} = \frac{1}{\Omega}\hat{\chi}$ since we do not have a ∇_3 equation for $\bar{\omega}$ and $\|\Omega\|_{L^\infty(S)} \leq C(\mathcal{O}_0)$ under the bootstrap assumption 84. Proceeding exactly the same way and collecting all the terms we obtain

$$\begin{aligned} \|\hat{\tilde{\chi}}\|_{L^\infty(S_{u,\bar{u}})} &\leq \frac{C\mathcal{O}_0}{2} + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2(\bar{\eta}, \eta)\|_{L^2(H)}) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2(\bar{\eta}, \eta)\|_{L^2(H)}) \\ &\leq C\mathcal{O}_0 \end{aligned}$$

after choosing sufficiently small ϵ . Since in the trace estimate we need $\|\nabla_3^2 \hat{\tilde{\chi}}\|_{L^2(\bar{H})}$, we will need $\|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}$ and therefore we include $\|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}$ in the definition of $\mathcal{W}$ from the beginning. Exact similar procedure yields

$$\|\bar{\eta}\|_{L^\infty(S)} \leq C\mathcal{O}_0. \quad (118)$$

Now for $tr\chi$ and $\hat{\chi}$, we will encounter terms that can be estimated by $\|\nabla^2(\eta, \bar{\eta})\|_{L^2(H, \bar{H})}$, $\mathcal{W}$, $\mathcal{F}$, and $\mathcal{F}(S)$. The transport inequality yields

$$\begin{aligned} \|tr\chi\|_{L^\infty(S_{u,\bar{u}})} &\leq C\|tr\chi\|_{L^\infty(S_{0,\bar{u}})} + C \int_0^u \left\| -\frac{1}{2}tr\bar{\chi}tr\chi + 2\bar{\omega}tr\chi + 2|\eta|^2 - \hat{\chi} \cdot \hat{\chi} \right\|_{L^\infty(S)} du' + C\epsilon^{\frac{1}{2}} \sup_{S_{u,\bar{u}}} \left(\int_0^u (\nabla\eta)^2 du' \right)^{\frac{1}{2}} \\ &\quad + C\epsilon^{\frac{1}{2}} \sup_{S_{u,\bar{u}}} \left(\int_0^u \rho^2 du' \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore notice that we need to estimate two terms $\sup_{S_{u,\bar{u}}} \left(\int_0^u (\nabla\eta)^2 du' \right)^{\frac{1}{2}} = \|\nabla\eta\|_{tr(\bar{H})}$ and $\sup_{S_{u,\bar{u}}} \left(\int_0^u \rho^2 du' \right)^{\frac{1}{2}} := \|\rho\|_{tr(\bar{H})}$. We have the following estimate

$$\|tr\chi\|_{L^\infty(S_{u,\bar{u}})} \leq \frac{C\mathcal{O}_0}{2} + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C\epsilon^{\frac{1}{2}} \|\nabla\eta\|_{tr(\bar{H})} + C\epsilon^{\frac{1}{2}} \|\rho\|_{tr(\bar{H})} \leq C\mathcal{O}_0. \quad (119)$$

Similarly, for $\hat{\chi}$, we obtain through the use of the transport inequality

$$\|\hat{\chi}\|_{L^\infty(S_{u,\bar{u}})} \leq \frac{C\mathcal{O}_0}{2} + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C\epsilon^{\frac{1}{2}} \|\nabla\eta\|_{tr(\bar{H})} \leq C\mathcal{O}_0, \quad (120)$$

where we can estimate $\|\rho\|_{tr(\bar{H})}$ and $\|\nabla\eta\|_{tr(\bar{H})}$ in terms of $\|\nabla^2\varphi\|_{L^2(\bar{H})}$ and $(\mathcal{W}(S), \mathcal{F}(S), \mathcal{W}, \mathcal{F})$ in the similar way as the other entities before. The most important point to note here is the presence of a factor of $\epsilon^{\frac{1}{2}}$ with $tr(\bar{H})$ norm of ρ and $\nabla\eta$. This concludes the L^∞ estimates for the connection coefficients. $\square$

Lemma 3: Assume $\mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H,\bar{H})} < \infty$. Then $\|\varphi_g\|_{L^4(S)} \leq C\mathcal{O}_0$ and $\|\varphi_b\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H,\bar{H})})$, where $\varphi_g := (\hat{\chi}, \bar{\chi}, tr\chi, tr\bar{\chi}, \bar{\eta}, \omega)$ and $\varphi_b := (\eta, \bar{\omega})$

Proof: We prove it under the assumption

$$\|\varphi_g\|_{L^4(S)} \leq C\mathcal{O}_0. \quad (121)$$

and later try to improve it. We start with η since it satisfies a ∇_4 equation. A direct application of the transport inequality yields

$$\begin{aligned} \|\eta\|_{L^4(S_{u,\bar{u}})} &\leq C \left(\|\eta\|_{L^4(S_{u,0})} + \int_0^{\bar{u}} \|\nabla_4\eta\|_{L^4(S)} d\bar{u}' \right) \\ &= C \left(\|\eta\|_{L^4(S_{u,0})} + \int_0^{\bar{u}} \left\| -\chi \cdot (\eta - \bar{\eta}) - \beta - \frac{1}{2}\alpha^F \cdot (\rho^F - \sigma^F) \right\|_{L^4(S)} d\bar{u}' \right) \\ &\leq C(\mathcal{O}_0, \mathcal{W}(S), \mathcal{F}(S)) + C(\mathcal{O}_0) \int_0^u \|\eta\|_{L^4(S)} d\bar{u}'. \end{aligned} \quad (122)$$

An application of Grönwall's inequality leads to the desired estimate for η

$$\|\eta\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}(S), \mathcal{F}(S)) e^{C(\mathcal{O}_0)\bar{u}} \leq C(\mathcal{O}_0, \mathcal{W}(S), \mathcal{F}(S)) \quad (123)$$

since $\bar{u} \leq J$. Now we repeat the same procedure for $\bar{\omega}$ which also satisfies a ∇_4 transport equation

$$\begin{aligned} \|\bar{\omega}\|_{L^4(S_{u,\bar{u}})} &\leq C \left(\|\bar{\omega}\|_{L^4(S_{u,0})} + \int_0^{\bar{u}} \|\nabla_4\bar{\omega}\|_{L^4(S)} d\bar{u}' \right) \\ &= C \left(\|\bar{\omega}\|_{L^4(S_{u,0})} + \int_0^{\bar{u}} \left\| 2\omega\bar{\omega} + \frac{3}{4}|\eta - \bar{\eta}|^2 - \frac{1}{4}(\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) - \frac{1}{8}|\eta + \bar{\eta}|^2 + \frac{1}{2}\rho + \rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F \right\|_{L^4(S)} d\bar{u}' \right) \\ &\leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\bar{\omega}\|_{L^4(S)} d\bar{u}' \\ &\leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}) e^{C(\mathcal{O}_0)J} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}), \end{aligned} \quad (124)$$

where the last step follows from an application of the Grönwall's inequality and $\bar{u} \leq J$. Notice an extremely important fact: $\|\eta\|_{L^4(S)}$ only depends on $\mathcal{O}_0, \mathcal{W}(S)$, and $\mathcal{F}(S)$, whereas $\|\bar{\omega}\|_{L^4(S_{u,\bar{u}})}$ depends on $\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}$. This will be vital when we show $\|\nabla^2\varphi\|_{L^2(H/\bar{H})} < \infty$ given $\mathcal{W}, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S) < \infty$. Now we move on to estimating φ_g i.e., the connection coefficients that satisfy ∇_3 equation. We start with $\bar{\eta}$. Using the assumption (121) together with previous estimate (lemma 2) and estimates for $\|\eta\|_{L^4(S)}$, we obtain through the transport inequality

$$\begin{aligned} \|\bar{\eta}\|_{L^4(S)} &\leq \frac{C}{2} \left(\|\bar{\eta}\|_{L^4(S_{0,\bar{u}})} + \int_0^u \|\nabla_3\bar{\eta}\|_{L^4(S)} du' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon C(\mathcal{O}_0, \mathcal{W}(S), \mathcal{F}(S)) \leq C\mathcal{O}_0 \end{aligned} \quad (125)$$

for sufficiently small ϵ . Estimate for $\|tr\chi\|$ (and similar others involving $\nabla\eta/\bar{\eta}$) follows in a similar way but now we need to use $\|\nabla\varphi\|_{L^4(S)} \leq \|\nabla^2\varphi\|_{L^2(S)}$ Sobolev embedding to close the estimate. Transport

inequality yields

$$\begin{aligned}
\|tr\chi\|_{L^4(S)} &\leq \frac{C}{2} \left(\|tr\chi\|_{L^4(S_0, \bar{u})} + \int_0^u \|\nabla_3 tr\chi\|_{L^4(S)} du' \right) \\
&= \frac{C}{2} \left(\|tr\chi\|_{L^4(S_0, \bar{u})} + \int_0^u \left\| -\frac{1}{2} tr\bar{\chi} tr\chi + 2\bar{\omega} tr\chi + 2div\eta + 2|\eta|^2 + 2\rho - \hat{\chi} \cdot \hat{\chi} \right\|_{L^4(S)} du' \right) \\
&\leq \frac{C\mathcal{O}_0}{2} + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}) + \epsilon^{\frac{1}{2}} C(\|\nabla^2\eta\|_{L^2(\bar{H})}) \leq C\mathcal{O}_0
\end{aligned} \tag{126}$$

for sufficiently small ϵ . Through similar argument, we obtain the improved estimates for the remaining connection coefficients that is we prove

$$\|\varphi_g\|_{L^4(S)} \leq C\mathcal{O}_0. \tag{127}$$

This concludes the proof of the lemma. $\square$

These estimates will be extremely crucial in proving the following lemma as well as estimating $\|\nabla^2\varphi\|_{L^2(H/\bar{H})}$ in terms of $\mathcal{W}, \mathcal{F}, \mathcal{W}(S)$, and $\mathcal{F}(S)$ in near future. Notice another important point: since $\|\eta\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$, we have $\|\bar{\omega}\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$ i.e., $L^2(S)$ of $\bar{\omega}$ does not depend on $\|\nabla^2\varphi\|_{L^2(H)}$.

Lemma 4: Assume $\mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H, \bar{H})} < \infty$. Then $\|\nabla\varphi_g\|_{L^2(S)} \leq C\mathcal{O}_0$ and $\|\nabla\varphi_b\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H, \bar{H})})$, where $\varphi_g := (\hat{\chi}, \bar{\chi}, tr\chi, tr\bar{\chi}, \bar{\eta}, \omega)$ and $\varphi_b := (\eta, \bar{\omega})$.

Proof: Similar to the previous estimates, we will prove this under the assumption

$$\|\nabla\varphi_g\|_{L^2(S)} \leq 2C\mathcal{O}_0. \tag{128}$$

We will obtain a better estimate therefore closing the argument. As usual, we first start with η since it satisfies a ∇_4 equation. We commute ∇ with the transport equation (40) satisfied by η to obtain (we write it in a schematic way)

$$\begin{aligned}
\nabla_4 \nabla \eta &= -\nabla \chi (\eta - \bar{\eta}) - \chi (\nabla \eta - \nabla \bar{\eta}) - \nabla \beta - \frac{1}{2} \nabla \alpha^F \cdot (\rho^F - \sigma^F) - \frac{1}{2} \alpha^F (\nabla \rho^F - \nabla \sigma^F) \\
&\quad + (\beta + \alpha^F \cdot (\rho^F - \sigma^F)) \eta + (\eta + \bar{\eta}) (-\chi (\eta - \bar{\eta}) - \beta - \frac{1}{2} \alpha^F (\rho^F - \sigma^F)).
\end{aligned} \tag{129}$$

The transport inequality for $\nabla \eta$ reads

$$\|\nabla \eta\|_{L^2(S_{u, \bar{u}})} \leq C \left(\|\nabla \eta\|_{L^2(S_{u, 0})} + \int_0^{\bar{u}} \|\nabla_4 \nabla \eta\|_{L^2(S)} d\bar{u}' \right). \tag{130}$$

Under the assumption (128) and the previous estimates (lemma) we obtain

$$\|\nabla_4 \nabla \eta\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla \eta\|_{L^2(S)} d\bar{u}' \tag{131}$$

which after using Grönwall's inequality yields

$$\|\nabla \eta\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}) e^{C(\mathcal{O}_0 \bar{u})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}) \tag{132}$$

since $\bar{u} \leq J$. A similar argument for $\bar{\omega}$ yields

$$\|\nabla \bar{\omega}\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}). \tag{133}$$

Now we want to estimate the good connection coefficients i.e., the ones that satisfy ∇_3 equation. Let us start with $\bar{\eta}$. Commuting the transport equation of $\bar{\eta}$ with ∇ yields

$$\begin{aligned}
\nabla_3 \nabla \bar{\eta} &= -\nabla \bar{\chi} (\bar{\eta} - \eta) - \bar{\chi} (\nabla \eta - \nabla \bar{\eta}) + \nabla \bar{\beta} + \frac{1}{2} \hat{\nabla} \bar{\alpha}^F \cdot (\rho^F - \sigma^F) + \frac{1}{2} \bar{\alpha}^F \cdot (\nabla \rho^F - \nabla \sigma^F) \\
&\quad + (\bar{\beta} + \bar{\alpha}^F \cdot (\rho^F - \sigma^F)) \bar{\eta} + (\eta + \bar{\eta}) (-\bar{\chi} (\bar{\eta} - \eta) + \bar{\beta} + \frac{1}{2} \bar{\alpha}^F \cdot (\rho^F - \sigma^F)).
\end{aligned} \tag{134}$$

Similarly, we may apply the transport inequality for $\|\nabla\bar{\eta}\|_{L^2(S)}$ to obtain

$$\begin{aligned} \|\nabla\bar{\eta}\|_{L^2(S)} &\leq \frac{C}{2} \left(\|\nabla\bar{\eta}\|_{L^2(S_{0,u})} + \int_0^u \|\nabla_3 \nabla\bar{\eta}\|_{L^2(S)} du' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon C(\mathcal{O}_0, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}) + \epsilon^{\frac{1}{2}} C(\mathcal{W}) \end{aligned} \quad (135)$$

which can be made to satisfy the following estimate after choosing sufficiently small ϵ yields

$$\|\nabla\bar{\eta}\|_{L^2(S)} \leq C\mathcal{O}_0. \quad (136)$$

Now we estimate $tr\chi$. Commuting ∇ with the transport equation of $tr\chi$ yields the following equation (schematic)

$$\nabla_3 \nabla tr\chi + \frac{1}{2} \nabla tr\bar{\chi} tr\chi + \frac{1}{2} tr\bar{\chi} \nabla tr\chi = 2\nabla\bar{\omega} tr\chi + 2\bar{\omega} \nabla tr\chi + 2\nabla^2\eta + 2\eta \nabla\eta + 2\nabla\rho - \nabla\hat{\chi}\hat{\chi} - \hat{\chi}\nabla\hat{\chi}. \quad (137)$$

The transport inequality provides the following estimate for $\nabla tr\chi$ under the assumption (128) together with the previous estimates for η and $\bar{\omega}$

$$\begin{aligned} \|\nabla tr\chi\|_{L^2(S)} &\leq \frac{C}{2} \left(\|\nabla tr\chi\|_{L^2(S_{0,\bar{u}})} + \int_0^u \|2\nabla\bar{\omega} tr\chi + 2\bar{\omega} \nabla tr\chi + 2\nabla^2\eta + 2\eta \nabla\eta + 2\nabla\rho - \nabla\hat{\chi}\hat{\chi} - \hat{\chi}\nabla\hat{\chi}\|_{L^2(S)} du' \right. \\ &\quad \left. + \int_0^u \|\frac{1}{2} \nabla tr\bar{\chi} tr\chi + \frac{1}{2} tr\bar{\chi} \nabla tr\chi\|_{L^2(S)} du' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon C(\mathcal{O}_0, \mathcal{F}, \mathcal{W}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(H)}) + \epsilon^{\frac{1}{2}} C(\|\nabla^2\varphi\|_{L^2(\bar{H})}, \mathcal{W}) \leq C\mathcal{O}_0 \end{aligned}$$

for sufficiently small ϵ . Notice that here we needed $\|\nabla^2\varphi\|_{L^2(\bar{H})}$ or more precisely $\|\nabla^2\eta\|_{L^2(\bar{H})}$. The remaining connection coefficients are estimated in an exact similar way. We sketch the estimates below

$$\begin{aligned} \|\nabla\hat{\chi}\|_{L^2(S_{u,\bar{u}})} &\leq \frac{C}{2} \left(\|\nabla\hat{\chi}\|_{L^2(S_{u',\bar{u}})} + \int_{u'}^u \|\nabla_3 \nabla\hat{\chi}\|_{L^2(S_{u'',\bar{u}})} du'' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{W}, \mathcal{W}(S)\mathcal{F}, \mathcal{F}(S)) \leq C\mathcal{O}_0, \end{aligned} \quad (138)$$

$$\begin{aligned} \|\nabla\bar{\eta}\|_{L^2(S_{u,\bar{u}})} &\leq C \left(\|\nabla\bar{\eta}\|_{L^2(S_{u',\bar{u}})} + \int_{u'}^u \|\nabla_3 \nabla\bar{\eta}\|_{L^2(S_{u'',\bar{u}})} du'' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{W}, \mathcal{W}(S)\mathcal{F}, \mathcal{F}(S)) \leq C\mathcal{O}_0, \end{aligned} \quad (139)$$

$$\begin{aligned} \|\nabla\omega\|_{L^2(S_{u,\bar{u}})} &\leq C \left(\|\nabla\omega\|_{L^2(S_{u',\bar{u}})} + \int_{u'}^u \|\nabla_3 \nabla\omega\|_{L^2(S_{u'',\bar{u}})} du'' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{W}, \mathcal{W}(S)\mathcal{F}, \mathcal{F}(S)) \leq C\mathcal{O}_0, \end{aligned} \quad (140)$$

$$\begin{aligned} \|\nabla tr\chi\|_{L^2(S_{u,\bar{u}})} &\leq C \left(\|\nabla tr\chi\|_{L^2(S_{u',\bar{u}})} + \int_{u'}^u \|\nabla_3 \nabla tr\chi\|_{L^2(S_{u'',\bar{u}})} du'' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{W}, \mathcal{W}(S)\mathcal{F}, \mathcal{F}(S), \|\nabla^2\eta\|_{L^2(\bar{H})}) \leq C\mathcal{O}_0, \end{aligned}$$

$$\begin{aligned} \|\nabla\hat{\chi}\|_{L^2(S_{u,\bar{u}})} &\leq C \left(\|\nabla\hat{\chi}\|_{L^2(S_{u',\bar{u}})} + \int_{u'}^u \|\nabla_3 \nabla\hat{\chi}\|_{L^2(S_{u'',\bar{u}})} du'' \right) \\ &\leq \frac{C\mathcal{O}_0}{2} + \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{W}, \mathcal{W}(S)\mathcal{F}, \mathcal{F}(S), \|\nabla^2\eta\|_{L^2(\bar{H})}) \leq C\mathcal{O}_0 \end{aligned}$$

after choosing ϵ sufficiently small. In this case, we will hit $\mathfrak{T}_{ab} \sim \rho^F \rho^F + \alpha^F \bar{\alpha}^F + \sigma^F \sigma^F$ by $\hat{\nabla}_3$ derivative and therefore we must incorporate $\|\hat{\nabla}_3 \bar{\alpha}^F\|_{L^2(\bar{H})}$ in the $\mathcal{F}$ norm. This completes the proof of the lemma. $\square$

Corollary 1: The Gauss curvature K of the topological 2-sphere $S_{u,\bar{u}}$ satisfies

$$\|K\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}(S), \mathcal{F}(S)), \quad (141)$$

$$\|\nabla K\|_{L^2(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{F}(S)) + \|\nabla \rho\|_{L^2(S_{u,\bar{u}})}. \quad (142)$$

Proof: A direct consequence of the null Hamiltonian constraint (49), lemma (2), (3), (4), and the definitions of $\mathcal{W}(S)$ and $\mathcal{F}(S)$.

Lemma 5: Let $\varphi_g := (tr\bar{\chi}, \hat{\chi}, \bar{\eta}, \omega, tr\chi, \hat{\chi})$ and $\varphi_b := (\eta, \bar{\omega})$ and $\mathcal{W}, \mathcal{F}, \mathcal{F}(S) < \infty$. Then $\|\nabla^2 \varphi_g\|_{L^2(H, \bar{H})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$ and $\|\nabla^2 \varphi_b\|_{L^2(H, \bar{H})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$.

Proof: Following [5], we prove this lemma by means of constructing a transport-Hodge system. The basic idea is the following. We construct a set of new entities $\Xi := \{\nabla tr\chi, \mu, \bar{\xi}\}$ and $\bar{\Xi} := \{\nabla tr\bar{\chi}, \bar{\mu}, \xi\}$. We first obtain their transport equations. We proceed exactly the same way as [5], only keep track of the additional Yang-Mills curvature terms. Yang-Mills curvature terms are harmless in this context since they have one order higher regularity than the Weyl curvature. We define $\mu, \bar{\mu}$ and $\xi, \bar{\xi}$ as follows

$$\mu := -div\eta - \rho, \quad \bar{\mu} := -div\bar{\eta} - \rho, \quad \xi := \nabla\omega + {}^*\nabla\omega^\dagger - \frac{1}{2}\beta, \quad (143)$$

$$\bar{\xi} := -\nabla\bar{\omega} + {}^*\nabla\bar{\omega}^\dagger - \frac{1}{2}\bar{\beta}, \quad (144)$$

where $\omega^\dagger$ and $\bar{\omega}^\dagger$ are the auxiliary variables that satisfy the following boundary valued equations

$$\nabla_3\omega^\dagger = \frac{1}{2}\sigma, \quad \omega^\dagger = 0 \text{ on } H_0, \quad \nabla_4\bar{\omega}^\dagger = \frac{1}{2}\sigma, \quad \bar{\omega}^\dagger = 0 \text{ on } \bar{H}_0. \quad (145)$$

By definition, we have the Hodge system

$${}^*\mathcal{D}_1\langle\bar{\omega}\rangle = \bar{\xi} + \frac{1}{2}\bar{\beta}, \quad {}^*\mathcal{D}_1\langle\omega\rangle = \xi + \frac{1}{2}\beta, \quad (146)$$

$$div\eta = -\mu - \rho, \quad curl\eta = \hat{\chi} \wedge \hat{\chi} + \sigma, \quad (147)$$

$$div\hat{\chi} = \frac{1}{2}\nabla tr\chi - \frac{1}{2}(\eta - \bar{\eta}) \cdot (\hat{\chi} - \frac{1}{2}tr\chi\delta_{ab}) - \beta + \frac{1}{2}\mathfrak{T}(e_4, \cdot) \quad (148)$$

$$div\hat{\bar{\chi}} = \frac{1}{2}\nabla tr\bar{\chi} - \frac{1}{2}(\bar{\eta} - \eta) \cdot (\hat{\bar{\chi}} - \frac{1}{2}tr\bar{\chi}\delta_{ab}) - \bar{\beta} + \frac{1}{2}\mathfrak{T}(e_3, \cdot), \quad (149)$$

where ${}^*\mathcal{D}_1\langle\omega\rangle := \nabla\omega + {}^*\nabla\omega^\dagger$ and ${}^*\mathcal{D}_1\langle\bar{\omega}\rangle := -\nabla\bar{\omega} + {}^*\nabla\bar{\omega}^\dagger$. $\Xi = \mu$ satisfies the following type of transport equation (schematically)

$$\begin{aligned} \nabla_4\mu = & \underbrace{-div(-\chi \cdot (\eta - \bar{\eta}))}_A - \underbrace{div\beta}_{-ID} - \underbrace{div(\frac{1}{2}\alpha^F(\rho^F - \sigma^F))}_{ID} - \underbrace{(-div\beta - \frac{3}{2}tr\chi\rho - \frac{1}{2}\hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2\bar{\eta} \cdot \beta)}_{IA} \\ & - \underbrace{\frac{1}{4}(\alpha^F \cdot \hat{\nabla}_3\alpha^F - \bar{\omega}|\alpha^F|^2 + 4\eta\alpha^F \cdot (\rho^F - \sigma^F) - \rho^F \cdot \hat{\nabla}_4\rho^F + \sigma^F \cdot \hat{\nabla}_4\sigma^F - 2\bar{\eta}\alpha^F \cdot (\rho^F + \sigma^F))}_{IB} \\ & + \underbrace{(\beta + (\alpha^F \cdot (\rho^F - \sigma^F) + \alpha^F))\eta + (\eta + \bar{\eta})(-\chi \cdot (\eta - \bar{\eta}) - \beta - \frac{1}{2}\alpha^F \cdot (\rho^F - \sigma^F)) - \chi\nabla\eta + \chi\bar{\eta}\eta}_{IC} \end{aligned}$$

where all the terms involving Yang-Mills curvature can be controlled by means of null Yang-Mills equations. The most important point is to note that $div\beta$ terms cancel each other in a point-wise way. This is the purpose of constructing the new functions μ (and similar others) so that the derivative of the Weyl

curvature does not appear since that would obstruct closing the regularity argument. Now observe an extremely important point: term A would contain $\text{div}(\chi)(\eta - \bar{\eta})$ and we can need to estimate this term in L^2S in order to estimate $\|\mu\|_{L^2(S)}$ using the transport inequality. Notice the following calculations

$$\begin{aligned} \|A\|_{L^2(S)} &= \| -\text{div}(-\chi \cdot (\eta - \bar{\eta})) \|_{L^2(S)} \leq \|\chi\|_{L^\infty(S)} (\|\nabla\eta\|_{L^2(S)} + \|\nabla\bar{\eta}\|_{L^2(S)}) \\ &\quad + \|\nabla\chi\|_{L^4} (\|\eta\|_{L^4(S)} + \|\bar{\eta}\|_{L^4(S)}) \end{aligned} \quad (150)$$

Now since $\hat{\chi}$ and $\text{tr}\chi$ both satisfy ∇_3 equation, we may estimate $\|\nabla\chi\|_{L^4(S)}$ solely by means of the initial data $\mathcal{O}_0$ using codimension-1 trace inequality

$$\|\nabla\varphi\|_{L^4(S_{u,\bar{u}})} \leq C \left(\|\nabla\varphi\|_{L^4(S_{0,\bar{u}})} + \|\nabla\varphi\|_{L^2(\bar{H})}^{1/2} \|\hat{\nabla}_3 \nabla\varphi\|_{L^2(\bar{H})}^{1/4} (\|\nabla\varphi\|_{L^2(\bar{H})} + \|\nabla\nabla\varphi\|_{L^2(\bar{H})})^{1/4} \right) \quad (151)$$

given $\mathcal{W}, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S), \|\nabla^2\varphi\|_{L^2(\bar{H})} < \infty$. This is simply because, we gain $\epsilon^{1/4}$ from the term $\|\nabla\varphi\|_{L^2(\bar{H})}^{1/2}$. From lemma (3), we have the estimates for $\|\eta\|_{L^4(S)}$ in terms of $\mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)$ and for $\|\bar{\eta}\|_{L^4(S)}$ in terms of $\mathcal{O}_0$. Therefore after choosing ϵ sufficiently small, we have

$$\|A\|_{L^2(S)} \leq C(\mathcal{O}_0) \|\nabla\eta\|_{L^2(S)} + C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \quad (152)$$

Now an application of transport inequality

$$\|\mu\|_{L^2(S_{u,\bar{u}})} \leq C(\mathcal{O}_0) \left(\|\mu\|_{L^2(S_{u,0})} + \int_0^{\bar{u}} \|\nabla_4\mu\|_{L^2(S)} d\bar{u}'' \right) \quad (153)$$

yields

$$\|\mu\|_{L^2(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla\eta\|_{L^2(S)} d\bar{u}' + C \int_0^{\bar{u}} (\|IA, IB, IC\|_{L^2(S)}) d\bar{u}'$$

and from the Hodge system (147) (can also be obtained from)

$$\|\nabla\eta\|_{L^2(S)} \leq C(\mathcal{O}_0) (\|K\|_{L^2(S)} + \|\rho\|_{L^2(S)} + \|\sigma\|_{L^2(S)}) + \|\mu\|_{L^2(S)}. \quad (154)$$

Therefore we are left to estimate $\|IA\|_{L^2(S)}, \|IB\|_{L^2(S)}$, and $\|IC\|_{L^2(S)}$. Using the null Yang-Mills equations and elementary inequality such as Holder's inequality, we obtain

$$\|IA\|_{L^2(S)} + \|IB\|_{L^2(S)} + \|IC\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}(S)) + C(\mathcal{O}_0) (\|\alpha\|_{L^2(S)} + \|\beta\|_{L^2(S)} + \|\rho\|_{L^2(S)})$$

substitution of which in (154), an application of Grönwall, and integration in $\bar{u}$ yield

$$\|\mu\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S)) \quad (155)$$

which in turn yields

$$\|\nabla\eta\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S)) \quad (156)$$

since $\|\alpha\|_{L^2(S)}, \|\beta\|_{L^2(S)}, \|\rho\|_{L^2(S)} \leq C(\mathcal{W}(S))$. In this process, we obtain $\|\nabla\eta\|_{L^2(S)}$ independent of $\|\nabla^2\varphi\|_{L^2(H,\bar{H})}$ and therefore improve (132).

Similarly, analysing the pair $(\bar{\xi}, \langle\bar{\omega}\rangle)$, we can estimate $\|\nabla\bar{\omega}\|_{L^2(S)}$ by means of $C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$ utilizing the estimate for $\|\nabla\eta\|_{L^2(S)}$ which is now independent of $\|\nabla^2\varphi\|_{L^2(H,\bar{H})}$. This is the whole point of obtaining estimates in a hierarchical way i.e., start with η and estimate $\bar{\omega}$ by using the estimate for η and then continue to do so for the remaining connection coefficients. An extremely important point is that $\bar{\alpha}$ does not appear due to the special structure of the null-Bianchi equations (recall $\bar{\alpha}$ can not be controlled on H). Now we want to estimate $\|\nabla^2\eta\|_{L^2(H)}$. We will do one case and use the $\varphi = (\varphi_g, \varphi_b)$ for the rest of

the cases. Recall, we have constructed the entities $\mu, \bar{\mu}, \xi, \bar{\xi}$ which adding $\nabla \text{tr} \chi$ and $\nabla \text{tr} \bar{\chi}$ constructs the set $\Xi := \{\nabla \text{tr} \chi, \mu, \bar{\xi}\}$ and $\bar{\Xi} := \{\nabla \text{tr} \bar{\chi}, \bar{\mu}, \xi\}$. We obtain a set of following transport equations presented in a schematic way for $\nabla \Xi$ and $\nabla \bar{\Xi}$

$$\begin{aligned} \nabla_4 \nabla \Xi &= \varphi \nabla^2 \varphi + \nabla \varphi \nabla \varphi + \varphi \nabla \Psi + \Psi \nabla \varphi + \Phi^F \cdot \hat{\nabla}^2 \Phi^F + \hat{\nabla} \Phi^F \cdot \hat{\nabla} \Phi^F + \varphi \Phi^F \cdot \hat{\nabla} \Phi^F + \Phi^F \cdot \Phi^F \nabla \varphi \\ &\quad + \varphi \nabla \varphi + \varphi_g \varphi \nabla \varphi + \varphi \varphi \nabla \varphi_g + \varphi_g \nabla \Xi + \varphi \nabla_4 \Xi + \varphi \varphi_g \Xi := \mathcal{F}_1 \end{aligned} \quad (157)$$

where Ψ can consist of any of the ‘good’ Weyl curvature components i.e., $(\alpha, \beta, \bar{\beta}, \rho, \sigma)$ or more precisely it does not contain α . Similarly, Φ^F can contain all the Yang-Mills curvature components $(\alpha^F, \bar{\alpha}^F, \rho^F, \sigma^F)$. However, there is no term involving $\hat{\nabla}^2 \bar{\alpha}^F$ i.e., the topmost derivative operator does not act on $\bar{\alpha}^F$. This is once again a consequence of the special structure of the Yang-Mills equations. Of course Yang-Mills equations do satisfy a null condition. In addition, a good φ (φ_g) always appears multiplied with the top derivative of φ . Similarly, we obtain the following transport equation for $\bar{\Xi}$

$$\begin{aligned} \nabla_3 \nabla \bar{\Xi} &= \varphi \nabla^2 \varphi + \nabla \varphi \nabla \varphi + \varphi \nabla \Psi + \Psi \nabla \varphi + \Phi^F \cdot \hat{\nabla}^2 \Phi^F + \hat{\nabla} \Phi^F \cdot \hat{\nabla} \Phi^F + \varphi \Phi^F \cdot \hat{\nabla} \Phi^F + \varphi \nabla \varphi \\ &\quad + \Phi^F \cdot \Phi^F \nabla \varphi + \varphi_g \varphi \nabla \varphi + \varphi \varphi \nabla \varphi_g + \varphi_g \nabla \bar{\Xi} + \varphi \nabla_3 \bar{\Xi} + \varphi \varphi_g \bar{\Xi} := \mathcal{F}_2. \end{aligned} \quad (158)$$

Similarly, Ψ represents the Weyl curvature components belonging to the set $(\bar{\alpha}, \beta, \bar{\beta}, \rho, \sigma)$. Φ^F can contain all the Yang-Mills curvature components $(\alpha^F, \bar{\alpha}^F, \rho^F, \sigma^F)$. However, we do not have $\hat{\nabla}^2 \alpha^F$ term. This is favourable to us since we do not have a control of $\hat{\nabla}^2 \alpha^F$ on $\bar{H}$. Now we may apply the direct transport inequalities to obtain estimates for $\nabla \Xi$ and $\nabla \bar{\Xi}$. Also notice $\nabla_4 \Xi$ satisfies equation of the following type

$$\nabla_4 \Xi = \varphi \nabla \varphi + \varphi \Psi + \Phi^F \hat{\nabla} \Phi^F + \varphi \Phi^F \Phi^F + \varphi_g \varphi \varphi + \varphi \varphi, \quad (159)$$

where Ψ contains the Weyl curvature components $(\alpha, \beta, \bar{\beta}, \rho, \sigma)$ i.e., the ones that can be controlled over H . Similarly, $\nabla_3 \bar{\Xi}$ satisfies equation of the following type

$$\nabla_3 \bar{\Xi} = \varphi \nabla \varphi + \nabla \varphi_g \varphi + \varphi \Psi + \Phi^F \hat{\nabla} \Phi^F + \varphi \Phi^F \Phi^F + \varphi_g \varphi \varphi + \varphi \varphi. \quad (160)$$

Once again Ψ appearing in $\nabla_3 \bar{\Xi}$ equation contains the Weyl curvature components that can be controlled over $\bar{H}$. Remarkably note that equations for $\nabla_4 \Xi$ and $\nabla_3 \bar{\Xi}$ do not contain derivatives of the Weyl curvature. As mentioned previously, this is vital to close the regularity. Also $L^2(S)$ and $\dot{H}^1(S)$ of φ_g can be controlled only by means of the initial data $\mathcal{O}_0$. We utilize the transport inequalities to estimate $\nabla \Xi$ and $\nabla \bar{\Xi}$

$$\|\nabla \Xi\|_{L^2(S_{u, \bar{u}})} \leq C(\mathcal{O}_0) \left(\|\nabla \Xi\|_{L^2(S_{u, 0})} + \int_0^{\bar{u}} \|\nabla_4 \nabla \Xi\|_{L^2(S)} d\bar{u}'' \right), \quad (161)$$

$$\|\nabla \bar{\Xi}\|_{L^2(S_{u, \bar{u}})} \leq C \left(\|\nabla \bar{\Xi}\|_{L^2(S_{u, 0})} + \int_0^{\bar{u}} \|\nabla_3 \nabla \bar{\Xi}\|_{L^2(S)} d\bar{u}'' \right). \quad (162)$$

Therefore we will need to estimate $\mathcal{F}_1$ and $\mathcal{F}_2$ in $L^2(S)$. We first estimate different elements of $\mathcal{F}_1$ and $\mathcal{F}_2$ (157-158)

$\mathcal{F}_1, \mathcal{F}_2$:

$$\begin{aligned}
& \|\varphi \nabla^2 \varphi\|_{L^2(S)} \leq \|\varphi\|_{L^\infty(S)} \|\nabla^2 \varphi\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \|\nabla^2 \varphi\|_{L^2(S)}, \\
& \|\nabla \varphi \nabla \varphi\|_{L^2(S)} \leq \|\nabla \varphi\|_{L^4(S)}^2 \lesssim (\|\nabla \varphi\|_{L^2(S)}^{1/2} \|\nabla^2 \varphi\|_{L^2(S)}^{1/2} + \|\nabla \varphi\|_{L^2(S)}^2) \lesssim \|\nabla^2 \varphi\|_{L^2(S)}, \\
& \|\varphi \nabla \Psi\|_{L^2(S)} \leq \|\varphi\|_{L^\infty(S)} \|\nabla \Psi\|_{L^2(S)} \lesssim \|\nabla \varphi\|_{L^4(S)} \|\nabla \Psi\|_{L^2(S)}, \\
& \lesssim (\|\nabla \varphi\|_{L^2(S)}^{1/2} \|\nabla^2 \varphi\|_{L^2(S)}^{1/2} + \|\nabla \varphi\|_{L^2(S)}) \|\nabla \Psi\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) (\|\nabla^2 \varphi\|_{L^2(S)} + \|\nabla \Psi\|_{L^2(S)}^2), \\
& \|\Psi \nabla \varphi\|_{L^2(S)} \leq \|\Psi\|_{L^4(S)} \|\nabla \varphi\|_{L^4(S)} \lesssim \|\Psi\|_{L^4(S)} (\|\nabla \varphi\|_{L^2(S)}^{1/2} \|\nabla^2 \varphi\|_{L^2(S)}^{1/2} + \|\nabla \varphi\|_{L^2(S)}), \\
& \lesssim C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \|\Psi\|_{L^4(S)} \|\nabla^2 \varphi\|_{L^2(S)}^{1/2}, \|\Phi^F \cdot \hat{\nabla}^2 \Phi^F\|_{L^2(S)} \leq \|\Phi^F\|_{L^\infty(S)} \|\hat{\nabla}^2 \Phi^F\|_{L^2(S)} \leq C(\mathcal{F}(S)) \|\hat{\nabla}^2 \Phi^F\|_{L^2(S)}, \\
& \|\hat{\nabla} \Phi^F \cdot \hat{\nabla} \Phi^F\|_{L^2(S)} \leq \|\hat{\nabla} \Phi^F\|_{L^4(S)}^2 \leq C(\mathcal{F}(S)), \|\varphi \nabla \varphi\|_{L^2(S)} \leq \|\varphi\|_{L^4(S)} \|\nabla \varphi\|_{L^4(S)} \lesssim \|\nabla \varphi\|_{L^2(S)} \|\nabla^2 \varphi\|_{L^2(S)} \\
& \lesssim \|\nabla^2 \varphi\|_{L^2(S)}, \|\varphi \Phi^F \cdot \hat{\nabla} \Phi^F\|_{L^2(S)} \lesssim \|\varphi\|_{L^4(S)} \|\Phi^F\|_{L^\infty(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \leq C(\mathcal{F}(S)) \\
& (\|\varphi\|_{L^2(S)}^{1/2} \|\nabla \varphi\|_{L^2(S)}^{1/2} + \|\varphi\|_{L^2(S)}), \|\varphi \nabla \varphi\|_{L^2(S)} \lesssim \|\varphi\|_{L^4(S)} \|\nabla \varphi\|_{L^4(S)} \lesssim (\|\nabla \varphi\|_{L^2(S)}^{1/2} \|\nabla^2 \varphi\|_{L^2(S)}^{1/2} + \|\nabla \varphi\|_{L^2(S)}) \\
& \lesssim C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \|\nabla^2 \varphi\|_{L^2(S)}, \|\varphi_g \varphi \nabla \varphi\|_{L^2(S)} \leq \|\varphi_g\|_{L^\infty(S)} \|\varphi\|_{L^4(S)} \|\nabla \varphi\|_{L^4(S)} \lesssim \|\nabla^2 \varphi\|_{L^2(S)}, \\
& \|\varphi \varphi \nabla \varphi_g\|_{L^2(S)} \lesssim \|\varphi\|_{L^\infty(S)}^2 \|\nabla \varphi_g\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S)), \|\varphi \nabla \Xi\|_{L^2(S)} = \|\nabla^2 \varphi + \nabla \Psi\|_{L^2(S)} \\
& \leq \|\nabla^2 \varphi\|_{L^2(S)} + \|\nabla \Psi\|_{L^2(S)}, \|\varphi_g \nabla \Xi\|_{L^2(S)} \leq \|\varphi_g\|_{L^\infty(S)} \|\nabla \Xi\|_{L^2(S)} \lesssim \|\nabla^2 \varphi\|_{L^2(S)} + \|\nabla \Psi\|_{L^2(S)}, \\
& \|\varphi \nabla_4 \Xi\|_{L^2(S)} = \|\varphi(\varphi \nabla \varphi + \varphi \Psi + \Phi^F \hat{\nabla} \Phi^F + \varphi \Phi^F \Phi^F + \varphi_g \varphi \varphi + \varphi \varphi)\|_{L^2(S)} \lesssim \|\varphi\|_{L^\infty} \|\varphi\|_{L^4(S)} \|\nabla \varphi\|_{L^4(S)} \\
& + \|\varphi\|_{L^\infty(S)} \|\varphi\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} + \|\varphi\|_{L^4(S)}^2 \|\Phi^F\|_{L^\infty(S)}^2 + \|\varphi_g\|_{L^\infty(S)} \|\varphi\|_{L^4(S)}^2 \|\varphi\|_{L^\infty(S)} + \|\varphi\|_{L^\infty} \|\varphi\|_{L^4(S)} \|\Psi\|_{L^4(S)} \\
& + \|\varphi\|_{L^\infty} \|\varphi\|_{L^4(S)}^2 \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S)) + C(\mathcal{O}_0, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S)) \|\nabla^2 \varphi\|_{L^2(S)}, \\
& \|\varphi \nabla_3 \Xi\|_{L^2(S)} \lesssim C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S)) + C(\mathcal{O}_0, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S)) \|\nabla^2 \varphi\|_{L^2(S)}, \\
& \|\varphi \varphi_g \Xi\|_{L^2(S)} \lesssim C \|\nabla^2 \varphi\|_{L^2(S)}, \|\varphi \varphi_g \Xi\|_{L^2(S)} \lesssim C \|\nabla^2 \varphi\|_{L^2(S)},
\end{aligned}$$

where we extensively used the inequalities (87-91). Notice that we once again proceed in a hierarchical fashion i.e., start (μ, η) and then use that result to obtain estimates for $(\bar{\xi}, \bar{\omega})$ and so on. This is one of the the main reasons why $\mathcal{F}_1$ and $\mathcal{F}_2$ are estimated by means of $C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$ and $\|\nabla^2 \varphi\|_{L^2(S)}$ (linearly in the latter). Collecting all the terms together, we obtain the following two inequalities satisfied by $\nabla \Xi$ and $\nabla \bar{\Xi}$

$$\|\nabla \Xi\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) (1 + \int_0^{\bar{u}} \|\nabla^2 \varphi\|_{L^2(S)} d\bar{u}'), \quad (163)$$

$$\|\nabla \bar{\Xi}\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) (1 + \int_0^u \|\nabla^2 \varphi\|_{L^2(S)} du') \quad (164)$$

Now we use the elliptic estimates resuting from the Hodge system. After an application of $\mathcal{D}^*$, the Hodge system (146-149) reduces to the following second order elliptic equation

$$\mathcal{D}^* \mathcal{D} \varphi = K \varphi + \nabla(\Xi/\bar{\Xi}) + \nabla \Psi + \varphi \nabla \varphi + \nabla \varphi + \Phi^F \cdot \hat{\nabla} \Phi^F \quad (165)$$

which yields an estimate of type

$$\begin{aligned}
\|\nabla^2 \varphi\|_{L^2(S)} & \lesssim \|K \varphi\|_{L^2(S)} + \|\nabla(\Xi/\bar{\Xi})\|_{L^2(S)} + \|\nabla \Psi\|_{L^2(S)} + \|\varphi \nabla \varphi\|_{L^2(S)} + \|\nabla \varphi\|_{L^2(S)} + \|\Phi^F \cdot \hat{\nabla} \Phi^F\|_{L^2(S)} \\
& \lesssim C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + \|\nabla(\Xi/\bar{\Xi})\|_{L^2(S)} + \|\nabla \Psi\|_{L^2(S)}
\end{aligned} \quad (166)$$

substitution of which in the previous inequalities (163-164) and an application of Grönwall's inequality yields

$$\|\nabla \Xi\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \|\nabla \bar{\Xi}\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (167)$$

Substitution of (167) into the elliptic estimate (166) yields

$$\|\nabla^2 \varphi\|_{L^2(S)} \lesssim C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + \|\nabla \Psi\|_{L^2(S)}, \quad (168)$$

for $\varphi \in (\hat{\chi}, \bar{\hat{\chi}}, tr\bar{\chi}, tr\bar{\hat{\chi}}, \eta, \bar{\eta}, \omega, \bar{\omega})$ and the most important point is that Ψ in the elliptic estimate (168) does not contain $\alpha, \bar{\alpha}$. Therefore integrating over H and $\bar{H}$, we obtain

$$\|\nabla^2 \varphi\|_{L^2(H)}, \|\nabla^2 \varphi\|_{L^2(\bar{H})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (169)$$

This concludes the proof of the lemma. $\square$

Corollary 2: $\|\nabla \varphi_g\|_{L^4(S)} \leq C(\mathcal{O}_0)$, $\|\nabla \varphi_b\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$.

Proof: The proof relies on the co-dimension 1 trace inequalities

$$\|\nabla \varphi\|_{L^4(S_{u, \bar{u}})} \leq C \left(\|\nabla \varphi\|_{L^4(S_{u, 0})} + \|\nabla \varphi\|_{L^2(H)}^{1/2} \|\nabla_4 \nabla \varphi\|_{L^2(H)}^{1/4} (\|\nabla \varphi\|_{L^2(H)} + \|\nabla \nabla \varphi\|_{L^2(H)})^{1/4} \right), \quad (170)$$

$$\|\nabla \varphi\|_{L^4(S_{0, \bar{u}})} \leq C \left(\|\nabla \varphi\|_{L^4(S_{0, \bar{u}})} + \|\nabla \varphi\|_{L^2(\bar{H})}^{1/2} \|\hat{\nabla}_3 \nabla \varphi\|_{L^2(\bar{H})}^{1/4} (\|\nabla \varphi\|_{L^2(\bar{H})} + \|\nabla \nabla \varphi\|_{L^2(\bar{H})})^{1/4} \right), \quad (171)$$

the null evolution equations and lemma (4) and (5). For the good connection coefficients φ_g i.e., the ones satisfying ∇_3 equations, we easily observe the following using lemma (4)

$$\|\nabla \varphi_g\|_{L^2(\bar{H})}^2 = \int_0^u \|\nabla \varphi_g\|_{L^2(S)}^2 du' \leq \epsilon C(\mathcal{O}_0). \quad (172)$$

In addition using the commuted null transport equation, $\|\hat{\nabla}_3 \nabla \varphi_g\|_{L^2(\bar{H})}$ is controlled by $\|\nabla^2 \varphi_g\|_{L^2(\bar{H})}$ which in turn is controlled by $\mathcal{C}(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$ by lemma (5). For example, if we look at the commuted ∇_3 equation for $\hat{\chi}$

$$\begin{aligned} \nabla_3 \nabla \hat{\chi} \sim & -\frac{1}{2} \nabla(tr\bar{\chi})\hat{\chi} - \frac{1}{2} tr\bar{\chi} \nabla \hat{\chi} + \nabla^2 \eta + 2\nabla(\bar{\omega}\hat{\chi}) - \frac{1}{2} \nabla(tr\chi\hat{\chi}) + \eta \nabla \eta + \nabla(\rho^F \cdot \rho^F + \alpha^F \cdot \bar{\alpha}^F + \sigma^F \cdot \sigma^F) \\ & + (\bar{\beta} + \bar{\alpha}^F \cdot (\rho^F + \sigma^F))\hat{\chi} + (\eta + \bar{\eta})\nabla_3 \hat{\chi} - \bar{\chi} \nabla \hat{\chi} + \bar{\chi} \eta \hat{\chi}, \end{aligned}$$

we observe every term at the right hand side can be estimated in $L^2(\bar{H})$ by $C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$. Similar results hold for other good connection coefficients. Therefore we gain an overall factor of $\epsilon^{\frac{1}{4}}$ arising from the integration over $\bar{H}$ i.e.,

$$\|\nabla \varphi_g\|_{L^4(S_{u, \bar{u}})} \leq C(\mathcal{O}_0) + \epsilon^{\frac{1}{4}} \mathcal{C}(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \quad (173)$$

which for sufficiently small ϵ yields

$$\|\nabla \varphi_g\|_{L^4(S_{u, \bar{u}})} \leq C(\mathcal{O}_0). \quad (174)$$

Now for the bad connection coefficients φ_b that is the ones satisfying ∇_4 equations, we would not have $\|\nabla \varphi_b\|_{L^4(S)}$ determined solely in terms of the initial data rather by $C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$. This is because we do not gain a factor ϵ from the integral over H . Therefore putting everything together, using lemma (4) and (5) along with the ∇ commuted ∇_4 equations for φ_g , we obtain

$$\|\nabla \varphi_b\|_{L^4(S_{u, \bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (175)$$

This completes the proof of the lemma. $\square$

In the spirit of the previous lemma, we will prove a similar estimate for mixed derivatives of connection coefficients that can not be estimated directly using their evolution equations. We do so in the following lemma.

Lemma 6: Let $\varphi_g := (tr\bar{\chi}, \hat{\chi}, \bar{\eta}, \omega, tr\chi, \hat{\chi})$ and $\varphi_b := (\eta, \bar{\omega})$ and $\mathcal{W}, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S) < \infty$. Then $\|\nabla \nabla_3 \eta\|_{L^2(H)}$,

$$\|\nabla\nabla_3\bar{\omega}\|_{L^2(H)}, \|\nabla\nabla_4\bar{\eta}\|_{L^2(\bar{H})}, \|\nabla\nabla_4\omega\|_{L^2(\bar{H})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))$$

Proof: The proof follows in the similar fashion as that of lemma 5. Given the estimates obtained in the previous lemmas, we will only prove it for one connection coefficient η . The remaining estimates can be obtained in an exact similar fashion. Let us recall the Hodge transport system for the pair (η, μ) (schematically)

$$\operatorname{div}\eta = -\mu - \rho, \operatorname{curl}\eta = \hat{\chi} \wedge \hat{\chi} + \sigma \quad (176)$$

$$\begin{aligned} \nabla_4\mu &= -\operatorname{div}(-\chi \cdot (\eta - \bar{\eta})) - \operatorname{div}(\frac{1}{2}\alpha^F(\rho^F - \sigma^F)) - (-\frac{3}{2}\operatorname{tr}\chi\rho - \frac{1}{2}\hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2\bar{\eta} \cdot \beta \\ &\quad - \frac{1}{2}(\alpha^F \cdot \hat{\nabla}_3\alpha^F - \bar{\omega}|\alpha^F|^2 + 4\eta\alpha^F \cdot (\rho^F - \sigma^F) - \rho^F \cdot \hat{\nabla}_4\rho^F + \sigma^F \cdot \hat{\nabla}_4\sigma^F - 2\bar{\eta}\alpha^F \cdot (\rho^F + \sigma^F)) \\ &\quad + ((\alpha^F \cdot (\rho^F - \sigma^F) + \alpha^F))\eta + (\eta + \bar{\eta})(-\chi \cdot (\eta - \bar{\eta}) - \frac{1}{2}\alpha^F \cdot (\rho^F - \sigma^F)) - \bar{\eta}\beta - \chi\nabla\eta + \chi\bar{\eta}\eta. \end{aligned}$$

If we commute the transport equation for μ with ∇_3 we obtain an equation of following type (we keep the potentially dangerous terms in exact form and write the harmless terms in an schematic way)

$$\begin{aligned} \nabla_4\nabla_3\mu &\sim \chi\nabla\nabla_3\eta + \underbrace{\nabla\chi\nabla_3\eta}_{IIA} + \operatorname{div}(\hat{\nabla}_3\alpha^F) \cdot (\rho^F - \sigma^F) - \underbrace{\frac{1}{2}\alpha^F \cdot (\hat{\nabla}\hat{\nabla}_3\rho^F - {}^*\hat{\nabla}\hat{\nabla}_3\sigma^F)}_{IIC1} \quad (177) \\ &\quad + \hat{\operatorname{div}}\alpha^F \cdot \hat{\nabla}_3(\rho^F - \sigma^F) + \hat{\nabla}_3\alpha^F \cdot \hat{\operatorname{div}}(\rho^F - \sigma^F) + \operatorname{tr}\chi\nabla_3\rho + \rho\nabla_3\operatorname{tr}\chi + \nabla_3\hat{\chi}\alpha + \hat{\chi}\nabla_3\alpha \\ &\quad + \underbrace{\beta\nabla_3\zeta + \zeta\nabla_3\beta + \nabla_3\bar{\eta}\beta + \bar{\eta}\nabla_3\beta}_{IIB} + \hat{\nabla}_3\alpha^F \cdot \hat{\nabla}_3\alpha^F - \underbrace{\frac{1}{2}\alpha^F \cdot \hat{\nabla}_3^2\alpha^F}_{IIC2} + \underbrace{\nabla_3\bar{\omega}|\alpha^F|^2}_{IIC2} + \bar{\omega}\alpha^F \cdot \hat{\nabla}_3\alpha^F \\ &\quad + \underbrace{\nabla_3\eta\alpha^F \cdot (\rho^F - \sigma^F) + \eta\hat{\nabla}_3(\alpha^F) \cdot (\rho^F - \sigma^F) + \eta\alpha^F \cdot \hat{\nabla}_3(\rho^F - \sigma^F)}_{IID} + \hat{\nabla}_3\rho^F \cdot \hat{\nabla}_4\rho^F + \underbrace{\rho^F \cdot \hat{\nabla}_3\hat{\nabla}_4\rho^F}_{IIE} \\ &\quad + \hat{\nabla}_3\sigma^F \cdot \hat{\nabla}_4\sigma^F + \underbrace{\sigma^F \cdot \hat{\nabla}_3\hat{\nabla}_4\sigma^F}_{IIF} + \nabla_3\bar{\eta}\alpha^F \cdot (\rho^F + \sigma^F) + \bar{\eta}\hat{\nabla}_3(\alpha^F) \cdot (\rho^F + \sigma^F) + \bar{\eta}\alpha^F \cdot \hat{\nabla}_3(\rho^F + \sigma^F) \\ &\quad + \underbrace{\chi\eta\nabla_3\eta + \chi\bar{\eta}\nabla_3\bar{\eta} + (|\eta|^2 + |\bar{\eta}|^2)\nabla_3\chi + \eta\bar{\eta}\nabla_3\chi + \chi\eta\nabla_3\bar{\eta} + \chi\bar{\eta}\nabla_3\eta}_{IIG} + \underbrace{\nabla_3\chi\nabla\eta}_{IIH} + \underbrace{\nabla_3\chi\nabla\eta}_{IIK} \\ &\quad + \omega\nabla_3\mu + \bar{\omega}\nabla_4\mu + (\eta - \bar{\eta})\nabla\mu. \end{aligned}$$

Let us identify the terms that do need care. The term $IIC1$ is extremely dangerous since $\hat{\nabla}\hat{\nabla}_3\rho^F$ and $\hat{\nabla}\hat{\nabla}_3\sigma^F$ contain terms of the type $\hat{\nabla}^2\bar{\alpha}^F$ which can not be controlled on H . Now, the previous equation is a ∇_4 transport equation for $\nabla_3\mu$ and therefore after using the transport inequality, $|\hat{\nabla}^2\bar{\alpha}^F|^2$ gets integrated over H which is absolutely not under control. However, we also do have the term $IIC2$ which contains terms that cancel the dangerous terms of $IIC1$ in a point-wise manner thereby allowing us to close the argument. This is an extremely important point to note about the special structure of the Einstein-Yang-Mills equations. Without this cancellation, we would not have an obstruction to a potential blow up of bad norms (that are not under control) in finite time. We first show this cancellation. Write down the expression for $\hat{\nabla}_3^2\alpha^F$ using the $\hat{\nabla}_3$ transport equation for α^F (once again we write down the most important term exactly and the remaining terms are written in an schematic way) $\hat{\nabla}_3^2\alpha^F$

$$\begin{aligned} \hat{\nabla}_3^2\alpha^F &\sim \nabla_3\operatorname{tr}\bar{\chi}\alpha^F + \operatorname{tr}\bar{\chi}\hat{\nabla}_3\alpha^F - \hat{\nabla}(-\hat{\operatorname{div}}\bar{\alpha}^F + \operatorname{tr}\bar{\chi}\rho^F + (\eta - \bar{\eta}) \cdot \bar{\alpha}^F) + {}^*\hat{\nabla}(-\hat{\operatorname{curl}}\bar{\alpha}^F - \operatorname{tr}\bar{\chi}\sigma^F + (\eta - \bar{\eta}) \cdot {}^*\bar{\alpha}^F) \\ &\quad + \nabla_3\eta\sigma^F + \eta\nabla_3\sigma^F + \nabla_3\eta\rho^F + \eta\hat{\nabla}_3\rho^F + \nabla_3\bar{\omega}\alpha^F + \bar{\omega}\hat{\nabla}_3\alpha^F - (\nabla_3\hat{\chi})\bar{\alpha}^F + \hat{\chi}\hat{\nabla}_3\bar{\alpha}^F + (\bar{\beta} + \bar{\alpha}^F + \bar{\alpha}(\rho^F - \sigma^F)) \\ &\quad (\rho^F + \sigma^F) + (\eta + \bar{\eta})\hat{\nabla}_3(\rho^F + \sigma^F) + \chi\hat{\nabla}\rho^F + \bar{\chi}\eta(\rho^F + \sigma^F). \end{aligned}$$

$$\begin{aligned}
IIC1 + IIC2 &= -\frac{1}{2}\alpha^F \cdot (\hat{\nabla}\hat{\nabla}_3\rho^F - {}^*\hat{\nabla}\hat{\nabla}_3\sigma^F) - \frac{1}{2}\alpha^F \cdot \hat{\nabla}_3^2\alpha^F \quad (178) \\
&= -\frac{1}{2}\alpha^F \cdot \hat{\nabla}(-\hat{div}\bar{\alpha}^F) + \frac{1}{2}\alpha^F \cdot \hat{\nabla}(-\hat{curl}\bar{\alpha}^F) - \frac{1}{2}\alpha^F \cdot \hat{\nabla}(\hat{div}\bar{\alpha}^F) + \frac{1}{2}\alpha^F \cdot {}^*\hat{\nabla}\hat{curl}\bar{\alpha}^F + \hat{\chi}\alpha^F \cdot \hat{\nabla}_3\bar{\alpha}^F \\
&\quad + \nabla_3\bar{\omega}\alpha^F \cdot \alpha^F + III = \hat{\chi}\alpha^F \cdot \hat{\nabla}_3\bar{\alpha}^F + \nabla_3\bar{\omega}\alpha^F \cdot \alpha^F + III,
\end{aligned}$$

where III denotes the collection of terms $L^2(S)$ norm of which can be easily controlled by the available estimates. $\nabla_3\bar{\omega}$ can be estimated by commuting ∇_4 transport equation of $\bar{\omega}$ with $\hat{\nabla}_3$ and the direct use of transport inequalities. Using the $\hat{\nabla}_3$ evolution equation for ρ^F , we can easily estimate the terms $\eta\hat{\nabla}_3\rho^F$ and therefore both $\eta\hat{\nabla}_3\rho^F$ and $\nabla_3\bar{\omega}\alpha^F$ are under control. Similar to $\nabla_3\bar{\omega}$, there are other entities arise which can not be reduced by means of transport equations. This of course includes $\nabla_3\eta$. Therefore let us obtain estimates for $\nabla_3\eta$ and the estimates for the similar entities such as $\nabla_3\bar{\omega}$ will follow in the exact same way. First commute the transport equation for η with ∇_3 to yield (schematically)

$$\begin{aligned}
\nabla_4\nabla_3\eta &\sim \nabla_3\chi(\eta - \bar{\eta}) + \chi(\nabla_3\eta - \nabla_3\bar{\eta}) + \nabla_3\beta + (\hat{\nabla}_3\alpha^F) \cdot (\rho^F - \sigma^F) + \alpha^F \cdot (\hat{\nabla}_3\rho^F - \hat{\nabla}_3\sigma^F) + \omega\nabla_3\eta \\
&\quad + \bar{\omega}\nabla_4\eta + (\eta - \bar{\eta})\nabla\eta + \sigma\eta + \Phi^F \cdot \Phi^F\eta.
\end{aligned} \quad (179)$$

Once again, we note that $\nabla_4\eta$ terms may be eliminated by means of the ∇_4 transport equation for η . Similarly we have

$$\begin{aligned}
\nabla_3\beta &\sim tr\bar{\chi}\beta + \nabla\rho + {}^*\nabla\sigma + 2\bar{\omega}\beta + \hat{\chi} \cdot \bar{\beta} + (\eta\rho + {}^*\eta\sigma) + \frac{1}{2}(\hat{\nabla}(\rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F) \\
&\quad + \bar{\chi}\alpha^F \cdot (\rho^F - \sigma^F) - \hat{\nabla}_4(\bar{\alpha}^F \cdot \rho^F - \bar{\alpha}^F \cdot \sigma^F) + \omega\bar{\alpha}^F(\sigma^F - \rho^F) + 2\bar{\eta}_a(\rho^F \cdot \rho^F + \alpha^F \cdot \bar{\alpha}^F + \sigma^F \cdot \sigma^F) \\
&\quad + \bar{\eta}(\rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F)),
\end{aligned}$$

$$\hat{\nabla}_3\alpha^F \sim tr\bar{\chi}\alpha^F - \hat{\nabla}\rho^F + {}^*\hat{\nabla}\sigma^F - {}^*\eta\sigma^F + \eta\rho^F + \bar{\omega}\alpha^F - \hat{\chi} \cdot \bar{\alpha}^F, \quad (180)$$

$$\hat{\nabla}_4\bar{\alpha}^F \sim tr\bar{\chi}\bar{\alpha}^F + \hat{\nabla}\rho^F - {}^*\hat{\nabla}\sigma^F - {}^*\bar{\eta}\sigma^F - \bar{\eta}\rho^F + \omega\bar{\alpha}^F - \hat{\chi} \cdot \alpha^F, \quad (181)$$

$$\hat{\nabla}_3\rho^F \sim \hat{div}\bar{\alpha}^F + tr\bar{\chi}\rho^F + (\eta - \bar{\eta}) \cdot \bar{\alpha}^F, \quad (182)$$

$$\hat{\nabla}_3\sigma^F \sim \hat{curl}\bar{\alpha}^F - tr\bar{\chi}\sigma^F + (\eta - \bar{\eta}) \cdot {}^*\bar{\alpha}^F, \quad (183)$$

Utilizing these evolution equations, the definitions of $\mathcal{W}, \mathcal{F}$ and $\mathcal{F}(S)$, the L^∞ estimate of the connection coefficients, and lemma (2), we note the following

$$\int_0^{\bar{u}} \|\nabla_3\beta\|_{L^2(S)} d\bar{u}' \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (184)$$

Going back to (179) and utilizing available transport equations, we observe

$$\int_0^{\bar{u}} \|\nabla_4\nabla_3\eta\|_{L^2(S)} d\bar{u}' \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3\eta\|_{L^2(S)} d\bar{u}'. \quad (185)$$

The transport inequality applied to (179) yields

$$\|\nabla_3\eta\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3\eta\|_{L^2(S)} d\bar{u}' \quad (186)$$

which through Grönwall's inequality leads to

$$\|\nabla_3\eta\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))e^{C(\mathcal{O}_0)\bar{u}} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (187)$$

This inequality of course yields the estimate

$$\|\nabla_3 \eta\|_{L^2(H)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (188)$$

This later estimate (188) will be used in proving the next lemma. In an exact similar way, we obtain

$$\|\nabla_3 \bar{\omega}\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \quad (189)$$

$$\|\nabla_4 \bar{\eta}\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \quad (190)$$

$$\|\nabla_4 \omega\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (191)$$

These estimates will be extremely useful in the future. Now we may estimate $\hat{\nabla}_3 \bar{\alpha}^F$ through the codimension-1 trace inequality (243)

$$\|\hat{\nabla}_3 \bar{\alpha}^F\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{F}_0, \mathcal{F}). \quad (192)$$

Now we estimate $L^2(S)$ norm of the underlined entities in the expression of $\nabla_4 \nabla_3 \mu$ as follows

$$\|IIA\|_{L^2(S)} = \|\nabla \chi \nabla_3 \eta\|_{L^2(S)} \leq \|\nabla \chi\|_{L^4(S)} \|\nabla_3 \eta\|_{L^4(S)} \leq C(\mathcal{O}_0) \|\nabla \nabla_3 \eta\|_{L^2(S)}, \quad (193)$$

$$\begin{aligned} \|IIB\|_{L^2(S)} &= \|\beta \nabla_3 \zeta\|_{L^2(S)} = \|\beta\|_{L^4(S)} \|\nabla \bar{\omega} - \frac{1}{2} \bar{\chi} \cdot (\eta + \zeta) + \bar{\omega}(\zeta - \eta) - \frac{1}{2} \bar{\beta} + \frac{1}{2} {}^* \sigma^F \bar{\alpha}^F\|_{L^4(S)} \\ &\leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \end{aligned} \quad (194)$$

$$\begin{aligned} \|IID\|_{L^2(S)} &= \|\nabla_3 \eta \alpha^F \cdot (\rho^F - \sigma^F)\|_{L^2(S)} \leq \|\nabla_3 \eta\|_{L^2(S)} \|\alpha^F\|_{L^\infty(S)} (\|\rho^F\|_{L^\infty(S)} + \|\sigma^F\|_{L^\infty(S)}) \\ &\leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \end{aligned} \quad (195)$$

$$\|IIE\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) (\|\hat{\nabla}^2 \rho^F\|_{L^2(S)} + \|\hat{\nabla}^2 \sigma^F\|_{L^2(S)}) + C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \quad (196)$$

$$\|IIF\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) (\|\hat{\nabla}^2 \rho^F\|_{L^2(S)} + \|\hat{\nabla}^2 \sigma^F\|_{L^2(S)}) + C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \quad (197)$$

$$\|IIG, IIH\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \quad (198)$$

$$\|IIK\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \|\nabla \eta\|_{L^4(S)}^2 \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)).$$

Putting together all the estimates and an use of the transport inequality

$$\|\nabla_3 \mu\|_{L^2(S)} \leq C \left(\|\nabla_3 \mu\|_{L^2(S_{u,0})} + \int_0^{\bar{u}} \|\nabla_4 \nabla_3 \mu\|_{L^2(S)} d\bar{u}' \right) \quad (199)$$

yields

$$\|\nabla_3 \mu\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \int_0^{\bar{u}} \|\nabla \nabla_3 \eta\|_{L^2(S)} d\bar{u}'. \quad (200)$$

Now we go back to the definition of Hodge system (146-149) to obtain

$$\text{div}(\nabla_3 \eta) = -\nabla_3 \mu - \nabla_3 \rho + [\text{div}, \nabla_3] \eta, \quad \text{curl}(\nabla_3 \eta) = \nabla_3 \hat{\chi} \wedge \hat{\chi} + \hat{\chi} \wedge \bar{\chi} + \nabla_3 \sigma. \quad (201)$$

This Hodge system yields the following elliptic estimate since K is under control

$$\|\nabla \nabla_3 \eta\|_{L^2(S)} \leq C(\|\nabla_3 \mu\|_{L^2(S)} + \|\nabla \Psi\|_{L^2(S)} + C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S))) \quad (202)$$

where we have used the ∇_3 transport equations for the entities present in the right hand side of the Hodge system (201). Substitution of (202) into the estimate (200) yields

$$\begin{aligned} \|\nabla_3 \mu\|_{L^2(S)} &\leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3 \mu\|_{L^2(S)} d\bar{u}' \\ &\quad + C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \int_0^{\bar{u}} \|\nabla \Psi\|_{L^2(S)} d\bar{u}' \end{aligned} \quad (203)$$

and since Ψ is not $\bar{\alpha}$ or α , we can write

$$\|\nabla_3 \mu\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3 \mu\|_{L^2(S)} d\bar{u}' \quad (204)$$

and therefore an application of Grönwall's inequality yields

$$\|\nabla_3 \mu\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) e^{C(\mathcal{O}_0 \bar{u})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \quad (205)$$

since $\bar{u} \leq J$. Therefore after integrating the elliptic estimate (201), we have

$$\|\nabla \nabla_3 \eta\|_{L^2(H)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \quad (206)$$

Proceeding in an exact similar way we obtain

$$\|\nabla \nabla_3 \bar{\omega}\|_{L^2(H)}, \|\nabla \nabla_4 \bar{\eta}\|_{L^2(\bar{H})}, \|\nabla \nabla_4 \omega\|_{L^2(\bar{H})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (207)$$

The estimates for the rest of the connection coefficients can be estimated directly through their transport equations since the remaining connection coefficients satisfy both ∇_4 and ∇_3 transport equations. In the following lemma we prove the $L^4(S)$ estimates of the difficult derivatives of the connection coefficients.

Lemma 7: *Let $\mathcal{W}, \mathcal{FW}(S), \mathcal{F}(S) < \infty$, then the following estimate holds*

$$\|\nabla_3 \eta\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{FW}(S), \mathcal{F}(S)), \quad \|\nabla_3 \bar{\omega}\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{FW}(S), \mathcal{F}(S)), \quad (208)$$

$$\|\nabla_4 \bar{\eta}\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{FW}(S), \mathcal{F}(S)), \quad \|\nabla_4 \omega\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{FW}(S), \mathcal{F}(S)) \quad (209)$$

Proof: In order to prove these estimates, we use the co-dimension 1 trace inequalities for any field φ (be it a section of the gauge bundle or tangent bundle or mixed)

$$\begin{aligned} \|\varphi\|_{L^4(S)} &\leq C \left(\|\varphi\|_{L^4(\bar{S}')} + \|\varphi\|_{L^2(\bar{H})}^{1/2} \|\hat{\nabla}_3 \varphi\|_{L^2(\bar{H})}^{1/4} (\|\varphi\|_{L^2(\bar{H})} + \|\hat{\nabla} \varphi\|_{L^2(\bar{H})})^{1/4} \right), \\ \|\varphi\|_{L^4(S)} &\leq C \left(\|\varphi\|_{L^4(\bar{S}')} + \|\varphi\|_{L^2(H)}^{1/2} \|\hat{\nabla}_4 \varphi\|_{L^2(H)}^{1/4} (\|\varphi\|_{L^2(H)} + \|\hat{\nabla} \varphi\|_{L^2(H)})^{1/4} \right). \end{aligned}$$

We prove for one of the connection coefficients. Rest of the connection coefficients can be handled exactly similar way. Let's consider $\varphi = \nabla_3 \eta$ and write

$$\|\nabla_3 \eta\|_{L^4(S)} \leq C \left(\|\nabla_3 \eta\|_{L^4(\bar{S}')} + \|\nabla_3 \eta\|_{L^2(H)}^{1/2} \|\nabla_4 \nabla_3 \eta\|_{L^2(H)}^{1/4} (\|\nabla_3 \eta\|_{L^2(H)} + \|\hat{\nabla} \nabla_3 \eta\|_{L^2(H)})^{1/4} \right). \quad (210)$$

Note that every term except $\nabla_4 \nabla_3 \eta$ on the right hand side is estimated. In order to estimate this term we can differentiate the ∇_4 transport equation for η with respect to e_3 . Such an operation yields (schematically)

$$\begin{aligned} \nabla_4 \nabla_3 \eta &= -\nabla_3 \chi(\eta - \bar{\eta}) - \chi(\nabla_3 \eta - \nabla_3 \bar{\eta}) - \nabla_3 \beta - (\hat{\nabla}_3 \alpha^F)(\rho^F + \sigma^F) - \alpha^F(\hat{\nabla}_3 \rho^F + \hat{\nabla}_3 \sigma^F) \\ &\quad + \omega \nabla_3 \eta + \bar{\omega} \nabla_4 \eta + (\eta - \bar{\eta}) \nabla \eta + \sigma \eta + (\rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F) \eta. \end{aligned} \quad (211)$$

Using the estimates from previous lemma (2-6) and the transport equations we obtain

$$\|\nabla_4 \nabla_3 \eta\|_{L^2(S)} \leq \mathcal{C}(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) + \|\nabla \Psi\|_{L^2(S)}. \quad (212)$$

Here Ψ does not contain $\bar{\alpha}$ and the Yang-Mills curvature components are estimated by means of $\mathcal{F}$ and $\mathcal{F}(S)$ since they enjoy one order higher regularity than the Weyl curvature. Therefore, we obtain

$$\|\nabla_4 \nabla_3 \eta\|_{L^2(H)} \leq \mathcal{C}(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (213)$$

Now if we plug this estimate into (210), we obtain

$$\|\nabla_3 \eta\|_{L^4(S)} \leq \mathcal{C}(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)). \quad (214)$$

Proceeding exact similar way, we obtain the remaining estimates

$$\begin{aligned} \|\nabla_3 \bar{\omega}\|_{L^4(S)} &\leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{FW}(S), \mathcal{F}(S)), \|\nabla_4 \bar{\eta}\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{FW}(S), \mathcal{F}(S)), \\ \|\nabla_4 \omega\|_{L^4(S)} &\leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{FW}(S), \mathcal{F}(S)). \end{aligned} \quad (215)$$

This concludes the proof of this lemma. $\square$

Lemma 8: *Let φ be any connection coefficients belonging to the set $(tr\chi, tr\bar{\chi}, \hat{\chi}, \hat{\bar{\chi}}, \eta, \bar{\eta}, \omega, \bar{\omega})$, then the following estimates hold*

$$\|\nabla_3 \varphi\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)), \|\nabla_4 \varphi\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \quad (216)$$

Proof: The proof is a straightforward consequence of the previous lemma (7) and the null evolution equations. The connection coefficients that satisfy the ∇_4 and ∇_3 null transport equations, we can directly estimate their $\|\nabla_4 \varphi\|_{L^4(S)}$ since the terms on the right hand side of such equations satisfy $L^4(S)$ estimate. The connection coefficients $(\eta, \bar{\omega}, \bar{\eta}, \omega)$ that do satisfy only one of the ∇_4, ∇_3 transport equations, the previous lemma yields the result.

Lemma 9: $\mathcal{W}(S) \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{W}, \mathcal{F})$, $\mathcal{F}(S) \leq C(\mathcal{O}_0, \mathcal{F}_0, \mathcal{W}, \mathcal{F})$ and also $\|\bar{\alpha}\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F})$, $\|\hat{\nabla} \bar{\alpha}^F\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{F})$.

Proof: Recall the definitions of $\mathcal{W}(S)$ and $\mathcal{F}(S)$

$$\mathcal{W}(S) := \sup_{u, \bar{u}} \|(\alpha, \beta, \bar{\beta}, \rho, \sigma)\|_{L^4(S)}, \quad \mathcal{F}(S) := \sum_{I=0}^1 \sup_{u, \bar{u}} \|\hat{\nabla}^I(\alpha^F, \rho^F, \sigma^F)\|_{L^4(S)}. \quad (217)$$

First we prove the estimate for $\mathcal{W}(S)$. Recall the codimension-1 trace inequalities (remember $\hat{\nabla}$ acts as usual covariant derivative on fields that are not sections of gauge bundle)

$$\|\Psi\|_{L^4(S_{u, \bar{u}})} \leq C \left(\|\Psi\|_{L^4(S_{u, 0})} + \|\Psi\|_{L^2(H)}^{1/2} \|\hat{\nabla}_4 \Psi\|_{L^2(H)}^{1/4} (\|\Psi\|_{L^2(H)} + \|\hat{\nabla} \Psi\|_{L^2(H)})^{1/4} \right), \quad (218)$$

$$\|\Psi\|_{L^4(S_{u, \bar{u}})} \leq C \left(\|\Psi\|_{L^4(S_{0, \bar{u}})} + \|\Psi\|_{L^2(\bar{H})}^{1/2} \|\hat{\nabla}_3 \Psi\|_{L^2(\bar{H})}^{1/4} (\|\Psi\|_{L^2(\bar{H})} + \|\hat{\nabla} \Psi\|_{L^2(\bar{H})})^{1/4} \right), \quad (219)$$

where the constants may depend on the initial data $\mathcal{O}_0$. We shall observe that $\|(\beta, \bar{\beta}, \rho, \sigma)\|_{L^4(S)}$ and $\|\hat{\nabla}^I(\rho^F, \sigma^F)\|_{L^4(S)}$ can be completely determined by the initial data $\mathcal{O}_0$. However, this would not hold true for α and α^F . However, this would not matter since we may use the trace inequalities (218-219). We start with the Weyl curvature components. A direct application of (218-219) applied to α and $\bar{\alpha}$ yields

$$\|\alpha\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{W}), \quad \|\bar{\alpha}\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{W}) \quad (220)$$

since $\|\nabla_4 \alpha\|_{L^2(H)}$ and $\|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}$ are dominated by $\mathcal{W}$. Now since $((\beta, \bar{\beta}, \rho, \sigma))$ satisfy ∇_3 equations, we may write

$$\|\beta\|_{L^2} \leq C \left(\|\beta\|_{L^2(S')} + \int_0^u \|\nabla_3 \beta\|_{L^2} du' \right) \quad (221)$$

$$\begin{aligned} &\sim C \left(\|\beta\|_{L^2(S')} + \int_0^u \left\| -tr\bar{\chi}\beta + D\rho + {}^*D\sigma + 2\bar{\omega}\beta + 2\hat{\chi} \cdot \bar{\beta} \right. \right. \\ &\quad \left. \left. + 3(\eta\rho + {}^*\eta\sigma) + \frac{1}{2}(\hat{\nabla}(|\rho^F|^2 + |\sigma^F|^2) - \bar{\chi}(\alpha^F \cdot \rho^F + \alpha^F \cdot \sigma^F) - \hat{\nabla}_4(\bar{\alpha}^F \cdot \rho^F + \bar{\alpha}^F \cdot \sigma^F) \right. \right. \\ &\quad \left. \left. + \omega(\bar{\alpha}^F \cdot \rho^F + \bar{\alpha}^F \cdot \sigma^F) + 2\bar{\eta}(|\rho^F|^2 + \alpha^F \cdot \bar{\alpha}^F + |\sigma^F|^2) + \bar{\eta}(|\rho^F|^2 + |\sigma^F|^2) \right\|_{L^2(S)} du' \right). \end{aligned} \quad (222)$$

Notice that all of these terms are estimated by $\mathcal{W}(S), \mathcal{W}, \mathcal{F}$ and $\mathcal{F}(S)$ with a factor of $\epsilon^{1/2}$ or ϵ in front. Therefore

$$\|\beta\|_{L^2} \leq C(\mathcal{O}_0, \mathcal{W}_0). \quad (223)$$

after choosing a sufficiently small ϵ . Now we repeat exact similar procedure for the remaining $\bar{\beta}, \rho$, and σ to yield

$$\|\bar{\beta}, \rho, \sigma\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{W}_0). \quad (224)$$

Now we want to use the trace inequality

$$\|\Psi\|_{L^4(S_{u,\bar{u}})} \leq C \left(\|\Psi\|_{L^4(S_{0,\bar{u}})} + \|\Psi\|_{L^2(\bar{H})}^{1/2} \|\nabla_3 \Psi\|_{L^2(\bar{H})}^{1/4} (\|\Psi\|_{L^2(\bar{H})} + \|\nabla \Psi\|_{L^2(\bar{H})})^{1/4} \right) \quad (225)$$

for $\Psi = (\beta, \bar{\beta}, \rho, \sigma)$. Observe the following

$$\|\nabla \Psi\|_{L^2(\bar{H})}^2 = \int_0^u \int_S |\nabla \Psi|^2 \mu_\gamma du \leq C\mathcal{W}^2 \quad (226)$$

and obtain

$$\|\Psi\|_{L^4(S)} \leq C \left(\|\Psi\|_{L^4(S')} + \|\Psi\|_{L^2(\bar{H})}^{1/2} \|\nabla_3 \Psi\|_{L^2(\bar{H})}^{1/4} \mathcal{W}^{1/2} \right). \quad (227)$$

Now

$$\|\nabla_3 \Psi\|_{L^2(\bar{H})} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (228)$$

and

$$\|\Psi\|_{L^2(\bar{H})}^2 = \int_0^u \int_S |\Psi|^2 \mu_\gamma du' \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S)). \quad (229)$$

due to the uniform $L^2(S)$ estimates for $\Psi = (\beta, \bar{\beta}, \rho, \sigma)$. Putting everything together

$$\|(\beta, \bar{\beta}, \rho, \sigma)\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}_0) + \epsilon^{1/4} C(\mathcal{O}_0, \mathcal{W}, \mathcal{W}(S), \mathcal{F}, \mathcal{F}(S)) \leq C(\mathcal{O}_0, \mathcal{W}_0) \quad (230)$$

due to smallness of ϵ .

Now we estimate $\mathcal{F}(S)$ in terms of $\mathcal{F}$. A direct application of the trace inequalities (218-219) yields

$$\|\alpha^F\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{F}_0, \mathcal{F}), \quad \|\bar{\alpha}^F\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{F}_0, \mathcal{F}). \quad (231)$$

Now recall the $\hat{\nabla}_3$ equation satisfied by ρ^F and commute it with $\hat{\nabla}$ to obtain

$$\begin{aligned} \hat{\nabla}_3 \hat{\nabla} \rho^F &= -\hat{\nabla} \hat{div} \bar{\alpha}^F + \nabla(tr \bar{\chi}) \rho^F + tr \bar{\chi} \hat{\nabla} \rho^F + \nabla(\eta - \bar{\eta}) \bar{\alpha}^F + (\eta - \bar{\eta}) \hat{\nabla} \bar{\alpha}^F \\ &\quad + (\bar{\beta} + \bar{\alpha}^F(\rho^F - \sigma^F) + \bar{\alpha}^F) \rho^F + (\eta + \bar{\eta}) \hat{\nabla}_3 \rho^F - \bar{\chi} \hat{\nabla} \rho^F + \bar{\chi} \eta \rho^F. \end{aligned} \quad (232)$$

Now an application of the transport inequality yields

$$\|\hat{\nabla} \rho^F\|_{L^2(S_{u,\bar{u}})} \leq C(\|\hat{\nabla} \rho^F\|_{L^2(S_{0,\bar{u}})} + \int_0^u \|\hat{\nabla}_3 \hat{\nabla} \rho^F\|_{L^2(S)} du'). \quad (233)$$

Now notice each term on the right hand side of (232) can be estimated in L^2 by definition of $\mathcal{W}, \mathcal{F}$ and utilizing the estimates for the connections (lemma)

$$\|\hat{\nabla} \rho^F\|_{L^2(S_{u,\bar{u}})} \leq C\|\hat{\nabla} \rho^F\|_{L^2(S_{0,\bar{u}})} + \epsilon C(\mathcal{O}_0, \mathcal{W}(S), \mathcal{W}, \mathcal{F}, \mathcal{F}(S)) \quad (234)$$

Since $u \leq \epsilon$. We may therefore estimate $\|\hat{\nabla} \rho^F\|_{L^2(S)}$ by means of the initial data i.e.,

$$\|\hat{\nabla} \rho^F\|_{L^2(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{F}_0). \quad (235)$$

We obtain a similar estimate for σ^F since it verifies a $\hat{\nabla}_3$ equation

$$\|\hat{\nabla}\sigma^F\|_{L^2(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{F}_0). \quad (236)$$

Of course using the integration inequality, we also have

$$\|\rho^F, \sigma^F\|_{L^2(S)} \leq C(\mathcal{O}_0, \mathcal{F}_0). \quad (237)$$

Now using the trace inequality we obtain

$$\begin{aligned} \|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^4(S_{u,\bar{u}})} &\leq C \left(\|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^4(S_{0,\bar{u}})} + \|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^2(\bar{H})}^{1/2} \|\hat{\nabla}_3 \hat{\nabla}(\rho^F, \sigma^F)\|_{L^2(\bar{H})}^{1/4} \right. \\ &\quad \left. (\|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^2(\bar{H})} + \|\hat{\nabla}^2(\rho^F, \sigma^F)\|_{L^2(\bar{H})})^{1/4} \right) \end{aligned} \quad (238)$$

where each term on the right hand side are estimated in terms of $\mathcal{F}, \mathcal{F}(S), \mathcal{W}$, and $\mathcal{W}(S)$ and in addition we gain a factor of $\epsilon^{1/2}$ from $\|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^2(\bar{H})}^{1/2}$ due to the estimates (235-236) i.e.,

$$\|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{F}_0) + \epsilon^{1/4} \mathcal{C}(\mathcal{O}_0, \mathcal{F}_0, \mathcal{W}, \mathcal{F}, \mathcal{F}(S), \mathcal{W}(S)) \quad (239)$$

and therefore

$$\|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{F}_0) \quad (240)$$

after choosing sufficiently small ϵ . However, for α^F we can not make use of the $\hat{\nabla}_3$ equation since it is not controlled on $\bar{H}$. Instead, we can make use of the $\hat{\nabla}_4$ trace inequality (218)

$$\|\hat{\nabla}\alpha^F\|_{L^4(S_{u,\bar{u}})} \leq C \left(\|\hat{\nabla}\alpha^F\|_{L^4(S_{u,0})} + \|\hat{\nabla}\alpha^F\|_{L^2(H)}^{1/2} \|\hat{\nabla}_4 \hat{\nabla}\alpha^F\|_{L^2(H)}^{1/4} (\|\hat{\nabla}\alpha^F\|_{L^2(H)} + \|\hat{\nabla}\hat{\nabla}\alpha^F\|_{L^2(H)})^{1/4} \right)$$

where every term in the right hand side is under control

$$\|\hat{\nabla}\alpha^F\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}, \mathcal{F}_0). \quad (241)$$

An exact similar argument for $\bar{\alpha}^F$ yields

$$\|\hat{\nabla}\bar{\alpha}^F\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}, \mathcal{F}_0). \quad (242)$$

This concludes the proof of the lemma. $\square$

Notice that we also have

$$\|\hat{\nabla}_4\alpha^F\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}, \mathcal{F}_0), \quad \|\hat{\nabla}_3\bar{\alpha}^F\|_{L^4(S_{u,\bar{u}})} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}, \mathcal{F}_0) \quad (243)$$

through a direct use of the trace inequalities (218-219).

7 Energy estimates for the Weyl and Yang-Mills curvature components

In this section we estimate the energy associated with the Weyl and Yang-Mills curvature components. This would complete the proof of the main theorem. Once the connection coefficients are estimated, it is straightforward to estimate the curvatures only we need to keep track to bad components ($\bar{\alpha}$ and $\bar{\alpha}^F$) since they controlled only on an incoming null hypersurface $\bar{H}$. However as we have mentioned previously, the connection coefficients that are associated with these bad terms in the curvature estimate are controlled

solely by the initial data $\mathcal{O}_0$. Therefore, we can safely utilize the Grönwall's inequality to close the energy argument. There are several ways to obtain the energy estimate for the curvatures. One of the ways is to utilize the stress-energy tensors and associated divergence identities. Even though we do not have a canonical stress-energy tensor for gravity due to the equivalence principle, we can still use the Bel-Robinson tensor introduced in lemma 1. Yang-Mills field, on the other hand, is equipped its own canonical stress-energy tensor. Despite the fact that the utilization of the stress energy tensor provides us with a direct physical insight into energy propagation (and possible concentration), we will not use it in the current context. Instead, we take a direct approach (the approach that exists in the traditional PDE literature). First, we write down the integration identities that are useful in the current context. For a scalar function $f : M \rightarrow \mathbb{R}$, the following integration by parts identities hold

$$\int_{D_{u,\bar{u}}} \nabla_3 f = \int_{H_0} f - \int_{H_0(0,\bar{u})} f + \int_{\mathcal{D}_{u,\bar{u}}} f(2\bar{\omega} - \frac{1}{2}tr\bar{\chi}), \quad (244)$$

$$\int_{D_{u,\bar{u}}} \nabla_4 f = \int_{\bar{H}_{\bar{u}}(0,u)} f - \int_{\bar{H}_0(0,u)} f + \int_{\mathcal{D}_{u,\bar{u}}} f(2\omega - \frac{1}{2}tr\chi). \quad (245)$$

We omit the proof here since that can be found in [8]. We start the estimates with components of the Weyl curvature. Notice that we need to identify a suitable collections of the curvature components whose energy estimates generate a cancellation of principal terms. This should of course be possible since we have already established the manifestly hyperbolic characteristics of the Yang-Mills sourced Bianchi equations.

Lemma 10: *The null components of the Weyl curvature satisfy the following L^2 energy estimates*

$$\int_H |\alpha|^2 + 2 \int_{\bar{H}_{\bar{u}}} |\beta|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (246)$$

$$\int_{H_u} |\beta|^2 + \int_{\bar{H}_{\bar{u}}} |\rho|^2 + \int_{\bar{H}_{\bar{u}}} |\sigma|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (247)$$

$$\int_{\bar{H}_{\bar{u}}} |\bar{\beta}|^2 + \int_{H_u} |\rho|^2 + \int_{H_u} |\sigma|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (248)$$

$$2 \int_{H_u} |\bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\bar{\alpha}|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\bar{\alpha}\|_{L^2(\bar{H})}^2 \quad (249)$$

Proof: The proof relies on the direct use of the integration by parts inequalities (244-245). First consider (α, β) . Direct use of (244-245) yields

$$\begin{aligned} & 2 \int_{\mathcal{D}_{u,\bar{u}}} (\langle \alpha, \nabla_3 \alpha \rangle + 2 \langle \beta, \nabla_4 \beta \rangle) \\ &= \int_{H_u} |\alpha|^2 + 2 \int_{\bar{H}_{\bar{u}}} |\beta|^2 - \int_{H_0(0,\bar{u})} |\alpha|^2 - 2 \int_{\bar{H}_0(u,0)} |\beta|^2 + \int_{\mathcal{D}_{u,\bar{u}}} |\alpha|^2 (2\bar{\omega} - \frac{1}{2}tr\bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} 2|\beta|^2 (2\omega - \frac{1}{2}tr\chi) \end{aligned}$$

and therefore

$$\begin{aligned} \int_{H_u} |\alpha|^2 + 2 \int_{\bar{H}_{\bar{u}}} |\beta|^2 &= \int_{H_0(0,\bar{u})} |\alpha|^2 + \int_{\bar{H}_0(u,0)} 2|\beta|^2 - \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} |\alpha|^2 (2\bar{\omega} - \frac{1}{2}tr\bar{\chi}) - \int_{\mathcal{D}_{u,\bar{u}}} 2|\beta|^2 (2\omega - \frac{1}{2}tr\chi)}_{ER1} \\ &\quad - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} (\langle \alpha, \nabla_3 \alpha \rangle + \langle \beta, \nabla_4 \beta \rangle)}_{ER2} \end{aligned}$$

We have to estimate the error terms ER_1 and ER_2 . Since α is innocuous in the current context (recall we are trying to prove a semi-global existence result where the size of the domain transverse to H is small),

ER_1 and ER_2 can be easily estimated in terms of the initial data and smallness of u . Nevertheless, we write down the estimates explicitly. Let us recall the definition of integration over the spacetime slab $\mathcal{D}_{u,\bar{u}}$

$$\int_{\mathcal{D}_{u,\bar{u}}} := \int_0^u \int_0^{\bar{u}} \int_{S_{u,\bar{u}}} \Omega^2 \mu_\gamma d\bar{u} du \quad (250)$$

and therefore in the error term ER_1 , we can estimate the connection coefficients in $\sup_{u,\bar{u}} L^\infty(S_{u,\bar{u}})$ and estimate α, β by their L^2 norm on H . In doing so we gain a factor of u ($\leq \epsilon$). Since $\|\Omega\|_{L^\infty(S)} \lesssim 1$, we have

$$\begin{aligned} |ER_1| &\leq \sup_{u,\bar{u}} (\|\bar{\omega}\|_{L^\infty(S_{u,\bar{u}})} + \|tr\bar{\chi}\|_{L^\infty(S_{u,\bar{u}})}) \int_0^u \|\alpha\|_{L^2(H)}^2 du \\ &\quad + \sup_{u,\bar{u}} (\|\omega\|_{L^\infty(S_{u,\bar{u}})} + \|tr\chi\|_{L^\infty(S_{u,\bar{u}})}) \int_0^u \|\beta\|_{L^2(H)}^2 du \\ &\leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \end{aligned} \quad (251)$$

Now we have to estimate the error term ER_2 . We do so using the Yang-Mills sourced Bianchi equations

$$\begin{aligned} ER_2 &= -2 \int_{\mathcal{D}_{u,\bar{u}}} (\langle \alpha, \nabla_3 \alpha \rangle + 2 \langle \beta, \nabla_4 \beta \rangle) \\ &= -2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\underbrace{\langle \alpha, \nabla \hat{\otimes} \beta \rangle}_{PI1} - \frac{1}{2} tr \bar{\chi} \alpha + 4 \bar{\omega} \alpha - 3(\hat{\chi} \rho + {}^* \hat{\chi} \sigma) + (\zeta + 4\eta) \hat{\otimes} \beta \right. \\ &\quad \left. + \frac{1}{2} (D_3 R_{44} - D_4 R_{43}) \gamma \right) + 2 \underbrace{\langle \beta, div \alpha \rangle}_{PI2} - 2 tr \chi \beta - 2 \omega \beta^W + \eta \cdot \alpha - \frac{1}{2} (D_b R_{44} - D_4 R_{4b}) \rangle \\ &\sim \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \alpha, -\frac{1}{2} tr \bar{\chi} \alpha + 4 \bar{\omega} \alpha - 3(\hat{\chi} \rho + {}^* \hat{\chi} \sigma) + (\zeta + 4\eta) \hat{\otimes} \beta \right. \\ &\quad \left. + \langle \beta, -2 tr \chi \beta - 2 \omega \beta + \eta \cdot \alpha - \frac{1}{2} (D_b R_{44} - D_4 R_{4b}) \rangle \right) + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} \langle \alpha, (\eta + \bar{\eta}) \beta \rangle}_{IBP1}, \end{aligned}$$

where we note that the principal terms $PI1$ and $PI2$ cancel each other producing the extra term ($IBP1$) due to the integration by parts procedure. In addition due to γ -traceless property of α , $\langle \alpha, (D_3 R_{44} - D_4 R_{43}) \gamma \rangle = 0$. In order to estimate ER_2 , we need to use the expression for $D_b R_{44} - D_4 R_{4b}$. Since there is no delicate cancellation here, we will write this term schematically

$$D_b R_{44} - D_4 R_{4b} \sim \langle \alpha^F, \hat{\nabla}_b \alpha^F \rangle - \chi_{bc} \mathfrak{T}_{c4} + \eta_b \mathfrak{T}_{44} - 2 \omega \mathfrak{T}_{4b} - (\hat{\nabla}_4 \alpha_b^F \cdot (\rho^F + \sigma^F) - \alpha_b^F \cdot (\hat{\nabla}_4 \rho^F + \hat{\nabla}_4 \sigma^F)).$$

Now note an extremely important fact: sine we do not have a $\hat{\nabla}_4$ equation for α^F , we can only estimate this term using the definition of $\mathcal{F}$ (this is one of the reasons we need to include $\hat{\nabla}_4 \alpha^F$ term in the definition of $\mathcal{F}$). Noting $\mathfrak{T}_{44} \sim |\alpha^F|^2$, $\mathfrak{T}_{4b} \sim \alpha^F(\rho^F + \sigma^F)$ we obtain

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \beta, D_b R_{44} - D_4 R_{4b} \rangle | \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \|\beta\|_{L^2(\mathcal{D})} \|\hat{\nabla}_4 \alpha^F\|_{L^2(\mathcal{D})} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (252)$$

Using the estimates of the connection coefficients, we may estimate the remaining terms of ER_2 . Finally we obtain

$$|ER_2| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (253)$$

The most important point here is that we gain a factor of ϵ from the bulk integral (i.e., the integral over D). Therefore, even though some of the connection coefficients (such as η and $\bar{\omega}$) are dependent on $\mathcal{W}$ and $\mathcal{F}$, they always appear as multiplied by a factor of ϵ and therefore under control. We have the first estimate

$$\int_{H_u} |\alpha|^2 + \int_{\bar{H}_{\bar{u}}} |\beta|^2 = \int_{H_0(0,\bar{u})} |\alpha|^2 + \int_{\bar{H}_0(u,0)} |\beta|^2 + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (254)$$

or

$$\int_{H_u} |\alpha|^2 + \int_{\bar{H}_{\bar{u}}} |\beta|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (255)$$

We proceed in an exact similar manner in order to estimate the remaining curvature components. Next we identify the triple (β, ρ, σ) and write down the integral equations

$$\begin{aligned} 2 \int_{\mathcal{D}_{u,\bar{u}}} (\langle \beta, \nabla_3 \beta \rangle + \langle \sigma, \nabla_4 \sigma \rangle + \langle \rho, \nabla_4 \rho \rangle) &= \int_{H_u} |\beta|^2 + \int_{\bar{H}_{\bar{u}}} |\rho|^2 + \int_{\bar{H}_{\bar{u}}} |\sigma|^2 \\ - \int_{H_0} |\beta|^2 - \int_{\bar{H}_0} |\rho|^2 - \int_{\bar{H}_0} |\sigma|^2 &+ \int_{\mathcal{D}_{u,\bar{u}}} |\beta|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\rho|^2 (2\omega - \frac{1}{2} tr \chi) \\ &+ \int_{\mathcal{D}_{u,\bar{u}}} |\sigma|^2 (2\omega - \frac{1}{2} tr \chi), \end{aligned} \quad (256)$$

that is,

$$\begin{aligned} \int_{H_u} |\beta|^2 + \int_{\bar{H}_{\bar{u}}} |\rho|^2 + \int_{\bar{H}_{\bar{u}}} |\sigma|^2 &= \int_{H_0} |\beta|^2 + \int_{\bar{H}_0} |\rho|^2 + \int_{\bar{H}_0} |\sigma|^2 \\ + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} |\beta|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\rho|^2 (2\omega - \frac{1}{2} tr \chi) + \int_{\mathcal{D}_{u,\bar{u}}} |\sigma|^2 (2\omega - \frac{1}{2} tr \chi)}_{ER3} \\ &\quad - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} (\langle \beta, \nabla_3 \beta \rangle + \langle \sigma, \nabla_4 \sigma \rangle + \langle \rho, \nabla_4 \rho \rangle)}_{ER4}. \end{aligned} \quad (257)$$

Once again, using the estimates of the connection coefficients from lemma (2), we estimate the error term ER_3

$$|ER_3| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (258)$$

In order to estimate ER_4 , we utilize the null Bianchi equations (52-57). Explicit calculations yield

$$\begin{aligned}
ER_4 &= 2 \int_{\mathcal{D}_{u,\bar{u}}} (\langle \beta, \nabla_3 \beta \rangle + \langle \sigma, \nabla_4 \sigma \rangle + \langle \rho, \nabla_4 \rho \rangle) \quad (259) \\
&= 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \beta, -tr\bar{\chi}\beta + \nabla\rho + {}^*\nabla\sigma + 2\bar{\omega}\beta + 2\hat{\chi} \cdot \bar{\beta} + 3(\eta\rho + {}^*\eta\sigma) + \frac{1}{2}(D_b R_{34} - D_4 R_{3b}) \rangle \right) \\
&+ 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \sigma, -\frac{3}{2}tr\chi\sigma - div {}^*\beta + \frac{1}{2}\hat{\chi} \cdot {}^*\alpha - \zeta \cdot {}^*\beta - 2\bar{\eta} \cdot {}^*\beta - \frac{1}{4}(D_\mu R_{4\nu} - D_\nu R_{4\mu})\epsilon^{\mu\nu}{}_{34} \rangle \right) \\
&\quad + 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \rho, -\frac{3}{2}tr\chi\rho + div\beta - \frac{1}{2}\hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2\bar{\eta} \cdot \beta - \frac{1}{4}(D_3 R_{44} - D_4 R_{34}) \rangle \right) \\
&\quad \sim \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \beta, -tr\bar{\chi}\beta + 2\bar{\omega}\beta + 2\hat{\chi} \cdot \bar{\beta} + 3(\eta\rho + {}^*\eta\sigma) + \frac{1}{2}(D_b R_{34} - D_4 R_{3b}) \rangle \right) \\
&+ \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \sigma, -\frac{3}{2}tr\chi\sigma + \frac{1}{2}\hat{\chi} \cdot {}^*\alpha - \zeta \cdot {}^*\beta - 2\bar{\eta} \cdot {}^*\beta - \frac{1}{4}(D_\mu R_{4\nu} - D_\nu R_{4\mu})\epsilon^{\mu\nu}{}_{34} \rangle \right) \\
&\quad + \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \rho, -\frac{3}{2}tr\chi\rho - \frac{1}{2}\hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2\bar{\eta} \cdot \beta - \frac{1}{4}(D_3 R_{44} - D_4 R_{34}) \rangle \right) \\
&\quad + \int_{\mathcal{D}_{u,\bar{u}}} (\langle (\eta + \bar{\eta})\sigma, {}^*\beta \rangle - \langle (\eta + \bar{\eta})\rho, \beta \rangle),
\end{aligned}$$

where the principal terms cancel point-wise through the integration by parts procedure. In order to estimate ER_4 , we need the expressions for the source terms involving Yang-Mills curvature. Using the Einstein's equations $R_{\mu\nu} = \mathfrak{T}_{\mu\nu}$, schematically

$$\begin{aligned}
D_b R_{34} - D_4 R_{3b} &\sim (\langle \hat{\nabla}\rho^F, \rho^F \rangle + \langle \hat{\nabla}\sigma^F, \sigma^F \rangle) - \bar{\chi}\alpha^F(\sigma^F + \rho^F) - \hat{\nabla}_4 \bar{\alpha}^F \cdot (\rho^F + \sigma^F) - \bar{\alpha}^F \cdot (\hat{\nabla}_4 \rho^F + \hat{\nabla}_4 \sigma^F) \\
&\quad + 2\bar{\omega}\bar{\alpha}^F(\rho^F + \sigma^F) + 2\bar{\eta}(\rho^F \cdot \rho^F + \alpha^F \cdot \bar{\alpha}^F + \sigma^F \cdot \sigma^F) + \bar{\eta}(\rho^F \cdot \rho^F + \sigma^F \cdot \sigma^F), \\
(D_\mu R_{4\nu} - D_\nu R_{4\mu})\epsilon^{\mu\nu}{}_{34} &\sim \hat{\nabla}(\alpha^F \cdot \rho^F - \alpha^F \cdot \sigma^F) - \chi(\rho^F \cdot \rho^F + \alpha^F \cdot \bar{\alpha}^F + \sigma^F \cdot \sigma^F) + (\eta - \bar{\eta})\alpha^F \cdot (\rho^F + \sigma^F), \\
D_3 R_{44} - D_4 R_{34} &\sim \langle \alpha^F, \hat{\nabla}_3 \alpha^F \rangle + \bar{\omega}|\alpha^F|^2 + \eta_A(\alpha^F \cdot \sigma^F + \alpha^F \cdot \rho^F) + \langle \rho^F, \hat{\nabla}_4 \rho^F \rangle + \langle \sigma^F, \hat{\nabla}_4 \sigma^F \rangle \\
&\quad + \bar{\eta}(\rho^F \bar{\alpha}^F + \rho^F \cdot \sigma^F)
\end{aligned}$$

where we may use the null evolution equations for the Yang-Mills curvature components

$$\hat{\nabla}_4 \bar{\alpha}^F + \frac{1}{2}tr\chi\bar{\alpha}^F = -\hat{\nabla}\rho^F - {}^*\hat{\nabla}\sigma^F - 2{}^*\bar{\eta}\sigma^F - 2\bar{\eta}\rho^F + 2\bar{\omega}\bar{\alpha}^F - \hat{\chi} \cdot \alpha^F \quad (260)$$

$$\hat{\nabla}_3 \alpha^F + \frac{1}{2}tr\bar{\chi}\alpha^F = -\hat{\nabla}\rho^F + {}^*\hat{\nabla}\sigma^F - 2{}^*\eta\sigma^F + 2\eta\rho^F + 2\bar{\omega}\alpha^F - \hat{\chi} \cdot \bar{\alpha}^F \quad (261)$$

$$\hat{\nabla}_4 \rho^F = -\hat{div}\alpha^F - tr\chi\rho^F - (\eta - \bar{\eta}) \cdot \alpha^F \quad (262)$$

$$\hat{\nabla}_4 \sigma^F = -\hat{curl}\alpha^F - tr\chi\sigma^F + (\eta - \bar{\eta}) \cdot {}^*\alpha^F \quad (263)$$

$$\hat{\nabla}_3 \rho^F = -\hat{div}\bar{\alpha}^F + tr\bar{\chi}\rho^F + (\eta - \bar{\eta}) \cdot \bar{\alpha}^F \quad (264)$$

$$\hat{\nabla}_3 \sigma^F = -\hat{curl}\bar{\alpha}^F - tr\bar{\chi}\sigma^F + (\eta - \bar{\eta}) \cdot {}^*\bar{\alpha}^F, \quad (265)$$

Note that even though $\bar{\alpha}^F$ appears in the source terms, we can estimate its point-wise norm on S by means of $\|\hat{\nabla}\bar{\alpha}^F\|_{L^4(S)}$ through Sobolev embedding. We only write down the estimates involving the potentially problematic source terms since the remaining terms are harmless

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \Psi, \bar{\alpha}^F \hat{\nabla}\Phi^F \rangle \right| \leq \int_0^u \|\Psi\|_{L^2(H)} \|\hat{\nabla}\Phi^F\|_{L^2(H)} \sup_{\bar{u}} \|\nabla\bar{\alpha}^F\|_{L^4(S)} du' \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (266)$$

where $\Psi := (\beta, \rho, \sigma)$ and $\Phi^F := (\alpha^F, \rho^F, \sigma^F)$. Therefore the error term ER_4 is estimated as follows

$$|ER_4| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (267)$$

Putting everything together, we obtain

$$\begin{aligned} \int_{H_u} |\beta|^2 + \int_{\bar{H}_{\bar{u}}} |\rho|^2 + \int_{\bar{H}_{\bar{u}}} |\sigma|^2 &= \int_{H_0} |\beta|^2 + \int_{\bar{H}_0} |\rho|^2 + \int_{\bar{H}_0} |\sigma|^2 + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \\ &\leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \end{aligned} \quad (268)$$

Now we collect the triple $(\bar{\beta}, \rho, \sigma)$ and write down the associated integral equation

$$\begin{aligned} \int_{\bar{H}_{\bar{u}}} |\bar{\beta}|^2 + \int_{H_u} |\rho|^2 + \int_{H_u} |\sigma|^2 &= \int_{\bar{H}_0} |\bar{\beta}|^2 + \int_{H_0} |\rho|^2 + \int_{H_0} |\sigma|^2 \\ + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} |\bar{\beta}|^2 (2\omega - \frac{1}{2} \text{tr} \chi) + \int_{\mathcal{D}_{u,\bar{u}}} |\rho|^2 (2\bar{\omega} - \frac{1}{2} \text{tr} \bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\sigma|^2 (2\bar{\omega} - \frac{1}{2} \text{tr} \bar{\chi})}_{ER5} \\ &\quad - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} (\langle \bar{\beta}, \nabla_4 \bar{\beta} \rangle + \langle \sigma, \nabla_3 \sigma \rangle + \langle \rho, \nabla_3 \rho \rangle)}_{ER6}. \end{aligned} \quad (269)$$

Using the estimates for the connections (lemma), ER_5 is under control

$$|ER_5| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (270)$$

In order to estimate ER_6 , we utilize the null evolution equations

$$\begin{aligned} ER_6 &= 2 \int_{\mathcal{D}_{u,\bar{u}}} (\langle \bar{\beta}, \nabla_4 \bar{\beta} \rangle + \langle \sigma, \nabla_3 \sigma \rangle + \langle \rho, \nabla_3 \rho \rangle) \\ &= 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \bar{\beta}, -\text{tr} \chi \bar{\beta} - \nabla \rho + {}^* \nabla \sigma + 2\omega \bar{\beta} + 2\hat{\chi} \cdot \beta - 3(\bar{\eta} \rho - {}^* \bar{\eta} \sigma) - \frac{1}{2} (D_b R_{43} - D_3 R_{4b}) \rangle \right) \\ &\quad + 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \sigma, -\frac{3}{2} \text{tr} \bar{\chi} \sigma - \text{div} {}^* \bar{\beta} + \frac{1}{2} \hat{\chi} \cdot {}^* \bar{\alpha} - \zeta \cdot {}^* \bar{\beta} - 2\eta \cdot {}^* \bar{\beta} + \frac{1}{4} (D_\mu R_{3\nu} - D_\nu R_{3\mu}) \epsilon^{\mu\nu}{}_{34} \rangle \right) \\ &\quad + 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \rho, -\frac{3}{2} \text{tr} \bar{\chi} \rho - \text{div} \bar{\beta} - \frac{1}{2} \hat{\chi} \cdot \bar{\alpha} + \zeta \cdot \bar{\beta} - 2\eta \cdot \bar{\beta} + \frac{1}{4} (D_3 R_{34} - D_4 R_{33}) \rangle \right) \\ &\quad \sim \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \bar{\beta}, -\text{tr} \chi \bar{\beta} + 2\omega \bar{\beta} + 2\hat{\chi} \cdot \beta - 3(\bar{\eta} \rho - {}^* \bar{\eta} \sigma) - \frac{1}{2} (D_b R_{43} - D_3 R_{4b}) \rangle \right) \\ &\quad + \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \sigma, -\frac{3}{2} \text{tr} \bar{\chi} \sigma + \frac{1}{2} \hat{\chi} \cdot {}^* \bar{\alpha} - \zeta \cdot {}^* \bar{\beta} - 2\eta \cdot {}^* \bar{\beta} + \frac{1}{4} (D_\mu R_{3\nu} - D_\nu R_{3\mu}) \epsilon^{\mu\nu}{}_{34} \rangle \right) \\ &\quad + \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \rho, -\frac{3}{2} \text{tr} \bar{\chi} \rho - \frac{1}{2} \hat{\chi} \cdot \bar{\alpha} + \zeta \cdot \bar{\beta} - 2\eta \cdot \bar{\beta} + \frac{1}{4} (D_3 R_{34} - D_4 R_{33}) \rangle \right) + \int_{\mathcal{D}_{u,\bar{u}}} \langle (\eta + \bar{\eta}) \rho, \bar{\beta} \rangle \\ &\quad + \int_{\mathcal{D}_{u,\bar{u}}} \langle (\eta + \bar{\eta}) \sigma, {}^* \bar{\beta} \rangle. \end{aligned} \quad (271)$$

Once again, we need the source terms explicitly in terms of the Yang-Mills curvature components

$$D_b R_{43} - D_3 R_{4b} \sim \langle \rho^F, \hat{\nabla} \rho^F \rangle + \langle \sigma^F, \hat{\nabla} \sigma^F \rangle - \bar{\chi} \alpha^F (\rho^F + \sigma^F) + \chi \bar{\alpha}^F (\rho^F + \sigma^F), \quad (272)$$

$$\begin{aligned} (D_\mu R_{3\nu} - D_\nu R_{3\mu}) \epsilon^{\mu\nu}{}_{34} &\sim \hat{\nabla} (\bar{\alpha}^F \rho^F - \bar{\alpha}^F \sigma^F) - \bar{\chi} (\rho^F \rho^F + \bar{\alpha}^F \alpha^F + \sigma^F \sigma^F) \\ &\quad + (\eta + \bar{\eta}) \bar{\alpha}^F (\rho^F + \sigma^F), \end{aligned} \quad (273)$$

$$\begin{aligned} D_3 R_{43} - D_4 R_{33} &\sim 2 \langle \rho^F, -\hat{\text{div}} \bar{\alpha}^F + \text{tr} \bar{\chi} \rho^F + (\eta - \bar{\eta}) \bar{\alpha}^F \rangle + 2 \langle \sigma^F, -\hat{\text{curl}} \bar{\alpha}^F - \text{tr} \bar{\chi} \sigma^F + (\eta - \bar{\eta}) {}^* \bar{\alpha}^F \rangle \\ &\quad - 2\eta (\rho^F \bar{\alpha}^F + \rho^F \sigma^F) - \langle \bar{\alpha}^F, -\frac{1}{2} \text{tr} \chi \bar{\alpha}^F - \hat{\nabla} \rho^F - {}^* \hat{\nabla} \sigma^F - 2 {}^* \bar{\eta} \cdot \sigma^F - 2\bar{\eta} \cdot \rho^F + 2\omega \bar{\alpha}^F - \hat{\chi} \cdot \alpha^F \rangle \\ &\quad + 2\omega |\bar{\alpha}^F|^2 - 4\bar{\eta}_a (\bar{\alpha}^F \cdot \rho^F - \bar{\alpha}^F \cdot \sigma^F). \end{aligned} \quad (274)$$

Using these explicit expressions, we will obtain estimates of the following types

$$|\int_{\mathcal{D}_{u,\bar{u}}} \langle \Psi, \hat{\nabla} \bar{\alpha}^F \Phi^F \rangle| \leq |\int_0^u \int_0^{\bar{u}} \|\Psi\|_{L^2(S)} \|\hat{\nabla} \bar{\alpha}^F\|_{L^4(S)} \|\Phi^F\|_{L^4(S)} d\bar{u}' du'| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (275)$$

$$|\int_D \langle \Psi, \bar{\alpha}^F \hat{\nabla} \Phi^F \rangle| \leq |\int_0^u \int_0^{\bar{u}} \|\Psi\|_{L^2(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \|\bar{\alpha}^F\|_{L^4(S)} d\bar{u}' du'| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (276)$$

$$|\int_D \langle \Psi, \bar{\alpha} \varphi \rangle| \leq |\int_0^u \int_0^{\bar{u}} \|\Psi\|_{L^2(S)} \|\bar{\alpha}\|_{L^2(S)} \|\varphi\|_{L^\infty(S)} d\bar{u}' du| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (277)$$

where $\Psi := (\bar{\beta}, \rho, \sigma)$ and $\Phi^F := (\alpha^F, \rho^F, \sigma^F)$ and φ denotes any connection coefficients. Collecting all the terms, we obtain

$$|ER_6| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (278)$$

and therefore

$$\begin{aligned} \int_{\bar{H}_{\bar{u}}} |\bar{\beta}|^2 + \int_{H_u} |\rho|^2 + \int_{H_u} |\sigma|^2 &= \int_{\bar{H}_0} |\bar{\beta}|^2 + \int_{H_0} |\rho|^2 + \int_{H_0} |\sigma|^2 + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \\ &\leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \end{aligned} \quad (279)$$

Now we move onto the last estimate. This one is the most delicate one and crucially depends on the special structure of the Einstein-Yang-Mills equations. Collecting the pair $(\bar{\alpha}, \bar{\beta})$ and applying the integration identities, we obtain

$$\begin{aligned} 2 \int_{H_u} |\bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\bar{\alpha}|^2 &= 2 \int_{H_0} |\bar{\beta}|^2 + \int_{\bar{H}_0} |\bar{\alpha}|^2 + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} 2|\bar{\beta}|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\bar{\alpha}|^2 (2\omega - \frac{1}{2} tr \chi)}_{ER7} \\ &\quad - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} (2\langle \bar{\beta}, \nabla_3 \bar{\beta} \rangle + \langle \bar{\alpha}, \nabla_4 \bar{\alpha} \rangle)}_{ER8}. \end{aligned} \quad (280)$$

Now notice the term $\int_{\mathcal{D}_{u,\bar{u}}} |\bar{\alpha}|^2 (2\omega - \frac{1}{2} tr \chi)$ in ER_7 . Technically we can estimate $\bar{\alpha}$ in the $L^4(S)$ norm since in this current settings, we do have $\|\bar{\alpha}\|_{L^4(S)}$ under control (lemma 9). However, we want to avoid using such an estimate since, this issue will reappear at the level of higher order regularity as well (and in that case we will not have $\|\nabla \bar{\alpha}\|_{L^4(S)}$ under control). Therefore we want to emphasize the special structure of this term. The connection coefficients ω and $tr \chi$ that are multiplying $|\bar{\alpha}|^2$ in the bulk integral of ER_7 satisfy ∇_3 transport equation and as a consequence we could estimate these solely in terms of the initial data $\mathcal{O}_0$. In other words, we have

$$|\int_{\mathcal{D}_{u,\bar{u}}} |\bar{\alpha}|^2 (2\omega - \frac{1}{2} tr \chi)| \leq C(\mathcal{O}_0) \int_0^u \int_S |\bar{\alpha}|^2 \Omega \mu_\gamma du \leq C(\mathcal{O}_0) \|\bar{\alpha}\|_{L^2(\bar{H})}^2 \quad (281)$$

or

$$|ER_7| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' \quad (282)$$

and therefore we can safely use Grönwall's inequality to estimate $2 \int_{H_u} |\bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\bar{\alpha}|^2$ provided we can

control the bulk term ER_8 . We do so now utilizing the null evolution equations

$$\begin{aligned}
ER_8 &= 2 \int_{\mathcal{D}_{u,\bar{u}}} (2\langle \bar{\beta}, \nabla_3 \bar{\beta} \rangle + \langle \bar{\alpha}, \nabla_4 \bar{\alpha} \rangle) \quad (283) \\
&= 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(2\langle \bar{\beta}, -2tr\bar{\chi}\bar{\beta} - div\bar{\alpha} - 2\bar{\omega}\bar{\omega} + \bar{\eta} \cdot \bar{\alpha} + \frac{1}{2}(D_b R_{33} - D_3 R_{3b}) \rangle \right) \\
&+ 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \bar{\alpha}, -\frac{1}{2}tr\chi\bar{\alpha} - \nabla \hat{\otimes} \bar{\beta} + 4\omega\bar{\alpha} - 3(\hat{\chi}\rho - {}^*\hat{\chi}\sigma) + (\zeta - 4\bar{\eta})\hat{\otimes}\bar{\beta} + \frac{1}{2}(D_4 R_{33} - D_3 R_{34})\gamma \rangle \right) \\
&\sim 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \bar{\beta}, -2tr\bar{\chi}\bar{\beta} - 2\bar{\omega}\bar{\omega} + \bar{\eta} \cdot \bar{\alpha} + \frac{1}{2}(D_b R_{33} - D_3 R_{3b}) \rangle \right) \\
&+ 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\underbrace{\langle \bar{\alpha}, -\frac{1}{2}tr\chi\bar{\alpha} + 4\omega\bar{\alpha} - 3(\hat{\chi}\rho - {}^*\hat{\chi}\sigma) + (\zeta - 4\bar{\eta})\hat{\otimes}\bar{\beta} + \frac{1}{2}(D_4 R_{33} - D_3 R_{34})\gamma \rangle}_A \right) \\
&\quad + 2 \int_D \langle (\eta + \bar{\eta})\bar{\beta}, \bar{\alpha} \rangle
\end{aligned}$$

where notice that the principal terms once again cancel each other in a point-wise manner after integration by parts procedure. An extremely important point to note here is that due to γ -trace-less property of $\bar{\alpha}$, the term $\langle \bar{\alpha}, (D_4 R_{33} - D_3 R_{34})\gamma \rangle$ vanishes. This character persists in the higher order as well. Once again, we need to explicitly evaluate the source term to estimate this bulk integral

$$D_b R_{33} - D_3 R_{3b} \sim \langle \bar{\alpha}^F, \hat{\nabla} \bar{\alpha}^F \rangle - \bar{\chi} \bar{\alpha}^F \cdot (\rho^F + \sigma^F) + \bar{\eta} |\bar{\alpha}^F|^2 - 2\bar{\omega} \bar{\alpha}^F \cdot (\rho^F + \sigma^F) - \hat{\nabla}_3 (\bar{\alpha}^F \cdot \rho^F + \bar{\alpha}^F \cdot \sigma^F).$$

We note that we encounter a term $\hat{\nabla}_3 \bar{\alpha}^F$ which does not arise from a null evolution equation and this is precisely why we needed to add this term in the definition of Yang-Mills curvature energy. Observe that in the most dangerous term A , the connection coefficients $tr\chi$ and ω satisfy ∇_3 evolution equations and therefore using lemma (2), they are controlled by means of initial data $\mathcal{O}_0$ only (even though we could have utilized the $L^4(S)$ bound of $\bar{\alpha}$ at this level, we do not do so since in higher order we would not have the same privilege). The estimates for the bulk integral involves

$$| \int_D \langle \Psi, \varphi \bar{\alpha} \rangle | \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad | \int_D \langle \Psi, \varphi \Psi \rangle | \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (284)$$

$$\begin{aligned}
| \int_D \langle \Psi, \bar{\alpha}^F \hat{\nabla} \bar{\alpha}^F \rangle | &\leq \epsilon C(\mathcal{F}_0, \mathcal{W}, \mathcal{F}), \quad | \int_D \langle \Psi, \bar{\alpha}^F \hat{\nabla} \bar{\alpha}^F \rangle | \leq \epsilon C(\mathcal{F}_0, \mathcal{W}, \mathcal{F}), \\
| \int_D \langle \Psi, \bar{\alpha}^F \hat{\nabla} \Phi^F \rangle | &\leq \epsilon C(\mathcal{F}_0, \mathcal{W}, \mathcal{F}), \quad | \int_D \langle \Psi, \hat{\nabla}_3 \bar{\alpha}^F \rho^F \rangle | \leq \epsilon^{\frac{1}{2}} C(\mathcal{F}_0, \mathcal{W}, \mathcal{F}), \quad (285)
\end{aligned}$$

$$| \int_D \langle \bar{\alpha}, -\frac{1}{2}tr\chi\bar{\alpha} + 4\omega\bar{\alpha} \rangle | \leq C(\mathcal{O}_0) \int_0^{\bar{u}} \|\bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}', \quad (286)$$

where $\Psi := (\bar{\beta}, \rho, \sigma)$, $\Phi^F := (\rho^F, \sigma^F)$ and we have utilized $L^4(S)$ estimate of $\hat{\nabla} \bar{\alpha}^F$ and Φ^F (lemma 9) together with the definitions of $\mathcal{W}$ and $\mathcal{F}$. Putting everything together, we obtain

$$\begin{aligned}
\int_{H_u} |\bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\bar{\alpha}|^2 &= \int_{H_0} |\bar{\beta}|^2 + \int_{\bar{H}_0} |\bar{\alpha}|^2 + C(\mathcal{O}_0) \|\bar{\alpha}\|_{L^2(\bar{H})}^2 + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{F}_0, \mathcal{W}, \mathcal{F}) \\
&\leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}.
\end{aligned}$$

This concludes the proof of the lemma. $\square$

Now we need to obtain estimates for the higher order energy associated with the Weyl curvature. Once we have completed the estimates for the Weyl curvature, we will move on to estimating the Yang-Mills

curvature.

Lemma 11: *The horizontal derivatives of the null components of the Weyl curvature satisfy the following L^2 energy estimates*

$$\int_{H_u} |\nabla \alpha|^2 + \int_{\bar{H}_{\bar{u}}} 2|\nabla \beta|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (287)$$

$$\int_{H_u} |\nabla \beta|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla \rho|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla \sigma|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (288)$$

$$\int_{\bar{H}_{\bar{u}}} |\nabla \bar{\beta}|^2 + \int_{H_u} |\nabla \rho|^2 + \int_{H_u} |\nabla \sigma|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (289)$$

$$\int_{H_u} 2|\nabla \bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla \bar{\alpha}|^2 \leq C(\mathcal{W}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}'. \quad (290)$$

Proof: The proof relies on the fact that the structure of the Yang-Mills sourced null Bianchi equation is preserved after commuting with horizontal derivatives. We proceed in an exact similar manner. First we consider the pair (α, β)

$$\begin{aligned} & \int_{H_u} |\nabla \alpha|^2 + \int_{\bar{H}_{\bar{u}}} 2|\nabla \beta|^2 = \int_{H_0} |\nabla \alpha|^2 + \int_{\bar{H}_0} 2|\nabla \beta|^2 \quad (291) \\ & + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} |\nabla \alpha|^2 (2\bar{\omega} - \frac{1}{2} \text{tr} \bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} 2|\nabla \beta|^2 (2\omega - \frac{1}{2} \text{tr} \chi)}_{ER9} - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} (\langle \nabla \alpha, \nabla_3 \nabla \alpha \rangle + 2\langle \nabla \beta, \nabla_4 \nabla \beta \rangle)}_{ER10}. \end{aligned}$$

Similar to the previous lemma, $ER9$ is controlled by using the estimates of the connection coefficients from lemma (2)

$$|ER9| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (292)$$

In order to estimate $ER10$, we use the commuted null evolution equations

$$\begin{aligned} ER10 &= 2 \int_{\mathcal{D}_{u,\bar{u}}} (\langle \nabla \alpha, \nabla_3 \nabla \alpha \rangle + 2\langle \nabla \beta, \nabla_4 \nabla \beta \rangle) \quad (293) \\ &= 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \alpha, -\frac{1}{2} \nabla (\text{tr} \bar{\chi} \alpha) + \nabla \hat{\otimes} \nabla \beta + 4\nabla (\bar{\omega} \alpha) - 3\nabla (\hat{\chi} \rho + {}^* \hat{\chi} \sigma) + \nabla ((\zeta + 4\eta) \hat{\otimes} \beta) \right. \\ & \quad \left. + \frac{1}{2} \nabla (D_3 R_{44} - D_4 R_{43}) \gamma + [\nabla_3, \nabla] \alpha + [\nabla, \nabla] \hat{\otimes} \beta \right) \\ & \quad + 2 \int_{\mathcal{D}_{u,\bar{u}}} \left(2\langle \nabla \beta, -2\nabla (\text{tr} \chi \beta) + \text{div} \nabla \alpha - 2\nabla (\omega \beta) + \nabla (\eta \cdot \alpha) - \frac{1}{2} \nabla (D_b R_{44} - D_4 R_{4b}) \right. \\ & \quad \left. + [\nabla_4, \nabla] \beta + [\nabla, \text{div}] \alpha \right) \\ & \sim \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \alpha, -\frac{1}{2} \nabla (\text{tr} \bar{\chi} \alpha) + 4\nabla (\bar{\omega} \alpha) - 3\nabla (\hat{\chi} \rho + {}^* \hat{\chi} \sigma) + \nabla ((\zeta + 4\eta) \hat{\otimes} \beta) \right. \\ & \quad \left. + [\nabla_3, \nabla] \alpha + [\nabla, \nabla] \hat{\otimes} \beta \right) \\ & \quad + \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \beta, -2\nabla (\text{tr} \chi \beta) - 2\nabla (\omega \beta) + \nabla (\eta \cdot \alpha) - \frac{1}{2} \nabla (D_b R_{44} - D_4 R_{4b}) \right. \\ & \quad \left. + [\nabla_4, \nabla] \beta + [\nabla, \text{div}] \alpha \right) + \int_D \langle (\eta + \bar{\eta} \nabla \beta, \nabla \alpha) \rangle. \end{aligned}$$

Where once again $\langle \nabla \alpha, \nabla (D_3 R_{44} - D_4 R_{43}) \gamma \rangle$ vanishes due to the γ -trace-less property of α and since γ is

metric. This error term may be estimated by means of the following estimates for the individual terms

$$\begin{aligned}
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, [\nabla_3, \nabla] \alpha \rangle \right| \sim \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, \bar{\beta} \alpha + \bar{\alpha}^F \alpha + \bar{\alpha}^F (\rho^F + \sigma^F) \alpha + (\eta + \bar{\eta}) \nabla_3 \alpha - \bar{\chi} \nabla \alpha + \bar{\chi} \eta \alpha \rangle \right| \\
& \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \\
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, \nabla(tr \bar{\chi} \alpha) \rangle \right| = \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, \nabla(tr \bar{\chi}) \alpha + tr \bar{\chi} \nabla \alpha \rangle \right| \leq \int_0^u \sup_{\bar{u}} \|\nabla tr \bar{\chi}\|_{L^4(S_{u,\bar{u}})} \int_0^{\bar{u}} \|\nabla \alpha\|_{L^2(S_{u,\bar{u}})} \|\alpha\|_{L^4(S_{u,\bar{u}})} \\
& \quad + \int_0^u (\sup_{\bar{u}} \|\nabla tr \bar{\chi}\|_{L^4(S)} \|\alpha\|_{L^4(S)}) \|\nabla \alpha\|_{L^2(H)}^2 du' \leq \epsilon C(\mathcal{O}_0, \mathcal{W}), \\
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, [\nabla, \nabla] \beta \rangle \right| \sim \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, K \beta \rangle \right| \leq \int_0^u \sup_{\bar{u}} \|K\|_{L^4(S_{u,\bar{u}})} \|\beta\|_{L^4(S_{u,\bar{u}})} \int_0^{\bar{u}} \|\nabla \alpha\|_{L^2(S_{u,\bar{u}})} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}),
\end{aligned}$$

where we have estimated the Gauss curvature of S in $L^4(S)$ through the null Hamiltonian constraint $K = \frac{1}{2} \hat{\chi} \hat{\chi} - \frac{1}{4} tr \chi tr \bar{\chi} - \rho + |\rho^F|^2 + |\sigma^F|^2$ and the estimates from corollary (1) i.e., $\|K\|_{L^4(S)} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0, \mathcal{W}, \mathcal{F})$.

Moving on with the other terms

$$\begin{aligned}
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, [\nabla_4, \nabla] \beta \rangle \right| \sim \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \beta \beta + \alpha^F \beta + \alpha^F (\rho^F - \sigma^F) \beta + (\eta + \bar{\eta}) \nabla_4 \beta - \chi \nabla \beta + \chi \bar{\eta} \beta \rangle \right| \quad (294) \\
& \leq \int_0^u \int_0^{\bar{u}} (\|\nabla \beta\|_{L^2(S)} \|\beta\|_{L^4(S)}^2 + \|\nabla \beta\|_{L^2(S)} \|\alpha^F\|_{L^4(S)} \|\beta\|_{L^4(S)} + \|\alpha^F\|_{L^2(S)} \|\rho^F\|_{L^4(S)} \|\beta\|_{L^4(S)} \\
& \quad + \int_0^u (\|\nabla \beta\|_{L^2(H)}^2 \|\chi\|_{L^\infty} + \|\chi\|_{L^\infty} \|\bar{\eta}\|_{L^\infty} \|\beta\|_{L^2} \|\nabla \beta\|_{L^2}) \\
& \quad + \int_0^u \int_0^{\bar{u}} \|\eta + \bar{\eta}\|_{L^\infty} \|\nabla \beta\|_{L^2} \|\nabla_4 \beta\|_{L^2}
\end{aligned}$$

Now use the equation of motion for β

$$\nabla_4 \beta + 2tr \chi \beta = div \alpha - 2\omega \beta + \eta \cdot \alpha - \frac{1}{2} (D_b R_{44} - D_4 R_{4b}) \quad (295)$$

and

$$(D_b R_{44} - D_4 R_{4b}) \sim \langle \alpha^F, \hat{\nabla}_b \alpha^F \rangle - \chi_{bc} \mathfrak{T}_{c4} + \eta_b \mathfrak{T}_{44} - 2\omega \mathfrak{T}_{4b} - \hat{\nabla}_4 (\alpha_b^F \cdot \rho^F + \alpha_b^F \cdot \sigma^F)$$

This is why we need $\hat{\nabla}_4 \alpha^F$ in the derivative estimate. $\hat{\nabla}_4 \rho^F$ and $\hat{\nabla}_4 \sigma^F$ may be controlled by the respective null evolution equations

$$\hat{\nabla}_4 \rho^F = -\hat{div} \alpha^F - tr \chi \rho^F - (\eta - \bar{\eta}) \cdot \alpha^F, \quad (296)$$

$$\hat{\nabla}_4 \sigma^F = -\hat{curl} \alpha^F - tr \chi \sigma^F + (\eta - \bar{\eta}) \cdot {}^* \alpha^F. \quad (297)$$

Now recall $\|\varphi\|_{L^\infty(S)} \leq C \|\nabla \varphi\|_{L^4(S)}$ and therefore

$$\int_0^u \int_0^{\bar{u}} \|\nabla \beta\|_{L^2} \|\hat{\nabla}_4 \alpha^F \rho^F\|_{L^2} \leq C \int_0^u \int_0^{\bar{u}} \|\nabla \beta\|_{L^2} \|\hat{\nabla}_4 \alpha^F\|_{L^2} \|\nabla \rho^F\|_{L^4} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (298)$$

$$\int_0^u \int_0^{\bar{u}} \|\nabla \beta\|_{L^2} \|\alpha^F \hat{\nabla}_4 \alpha^F\|_{L^2} \leq \int_0^u \int_0^{\bar{u}} \|\nabla \beta\|_{L^2} \|\alpha^F\|_{L^4} \|\hat{\nabla}_4 \alpha^F\|_{L^4} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (299)$$

Therefore collecting all the terms together

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, [\nabla_4, \nabla] \beta \rangle \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (300)$$

which indicates that this is a good term since we gain a factor of ϵ . Among other non-trivial terms, we have the following

$$\begin{aligned} & \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \nabla (D_b R_{44} - D_4 R_{4b}) \rangle \\ & \sim \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \nabla (\langle \alpha^F, \hat{\nabla}_b \alpha^F \rangle - \chi_{bc} \mathfrak{T}_{c4} + \eta_b \mathfrak{T}_{44} - 2\omega \mathfrak{T}_{4b} - \hat{\nabla}_4 (\alpha_b^F \cdot \rho^F - \alpha_b^F \cdot \sigma^F)) \rangle. \end{aligned} \quad (301)$$

Here we will encounter several terms of which the potentially problematic terms are estimated as follows

$$\begin{aligned} & \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \alpha^F \hat{\nabla}^2 \alpha^F \rangle \right| \leq \int_{\mathcal{D}_{u,\bar{u}}} \|\nabla \beta\|_{L^2} \|\alpha^F\|_{L^\infty} \|\hat{\nabla}^2 \alpha^F\|_{L^2} \\ & \leq C \int_{\mathcal{D}_{u,\bar{u}}} \|\nabla \beta\|_{L^2} \|\hat{\nabla} \alpha^F\|_{L^4} \|\hat{\nabla}^2 \alpha^F\|_{L^2} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \end{aligned} \quad (302)$$

and

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \hat{\nabla} \alpha^F \hat{\nabla} \alpha^F \rangle \right| \leq \int_{\mathcal{D}_{u,\bar{u}}} \|\nabla \beta\|_{L^2(S)} \|\hat{\nabla} \alpha^F\|_{L^4(S)} \|\hat{\nabla} \alpha^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}).$$

Therefore these are essentially good estimates since we gain a factor of ϵ . Using the $L^4(S)$ estimates of the derivatives of the connection coefficients (corollary 2) and putting everything together, we have

$$|ER10| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (303)$$

and therefore

$$\int_{H_u} |\nabla \alpha|^2 + \int_{\bar{H}_{\bar{u}}} 2|\nabla \beta|^2 = \int_{H_0} |\nabla \alpha|^2 + \int_{\bar{H}_0} 2|\nabla \beta|^2 + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (304)$$

yielding

$$\int_{H_u} |\nabla \alpha|^2 + \int_{\bar{H}_{\bar{u}}} 2|\nabla \beta|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (305)$$

Now consider the triple (β, σ, ρ) and write using (244-245)

$$\begin{aligned} & \int_{H_u} |\nabla \beta|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla \rho|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla \sigma|^2 = \int_{H_0} |\nabla \beta|^2 + \int_{\bar{H}_0} |\nabla \rho|^2 + \int_{\bar{H}_0} |\nabla \sigma|^2 \\ & + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} |\nabla \beta|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\nabla \rho|^2 (2\omega - \frac{1}{2} tr \chi) + \int_{\mathcal{D}_{u,\bar{u}}} |\nabla \sigma|^2 (2\omega - \frac{1}{2} tr \chi)}_{ER11} \\ & - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} (\langle \nabla \beta, \nabla_3 \nabla \beta \rangle + \langle \nabla \sigma, \nabla_4 \nabla \sigma \rangle + \langle \nabla \rho, \nabla_4 \nabla \rho \rangle)}_{ER12}. \end{aligned} \quad (306)$$

Once again, utilizing the estimates for the connection coefficients from lemma (2), we obtain

$$|ER11| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (307)$$

Now we estimate ER_{12} using the null evolution equations

$$\begin{aligned}
-ER_{12} &= 2 \int_{\mathcal{D}_{u,\bar{u}}} (\langle \nabla \beta, \nabla_3 \nabla \beta \rangle + \langle \nabla \sigma, \nabla_4 \nabla \sigma \rangle + \langle \nabla \rho, \nabla_4 \nabla \rho \rangle) \quad (308) \\
&= \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \beta, -\nabla(tr\bar{\chi}\beta) + \nabla \nabla \rho + {}^* \nabla \nabla \sigma + 2\nabla(\bar{\omega}\beta) + 2\nabla(\hat{\chi} \cdot \bar{\beta}) + 3\nabla(\eta\rho + {}^* \eta\sigma) \right. \\
&\quad \left. + \frac{1}{2} \nabla(D_b R_{34} - D_4 R_{3b}) + [\nabla_3, \nabla]\beta + [\nabla, \nabla]\rho + [{}^* \nabla, \nabla]\sigma \right) \\
&\quad + \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \sigma, -\frac{3}{2} \nabla(tr\chi\sigma) - \text{div} {}^* \nabla \beta + \frac{1}{2} \nabla(\hat{\chi} \cdot {}^* \alpha) - \nabla(\zeta \cdot {}^* \beta) - 2\nabla(\bar{\eta} \cdot {}^* \beta) \right. \\
&\quad \left. - \frac{1}{4} \nabla(D_\mu R_{4\nu} - D_\nu R_{4\mu}) \epsilon^{\mu\nu}{}_{34} + [\nabla_4, \nabla]\sigma - [\nabla, \text{div}] {}^* \beta \right) \\
&\quad + \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \rho, -\frac{3}{2} \nabla(tr\chi\rho) + \text{div} \nabla \beta - \frac{1}{2} \nabla(\hat{\chi} \cdot \alpha) + \nabla(\zeta \cdot \beta) + 2\nabla(\bar{\eta} \cdot \beta) \right. \\
&\quad \left. - \frac{1}{4} \nabla(D_3 R_{44} - D_4 R_{34}) + [\nabla_4 \nabla]\rho + [\nabla, \text{div}]\beta \right) \\
&\sim \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \beta, -\nabla(tr\bar{\chi}\beta) + 2\nabla(\bar{\omega}\beta) + 2\nabla(\hat{\chi} \cdot \bar{\beta}) + 3\nabla(\eta\rho + {}^* \eta\sigma) \right. \\
&\quad \left. + \frac{1}{2} \nabla(D_b R_{34} - D_4 R_{3b}) + [\nabla_3, \nabla]\beta + [\nabla, \nabla]\rho + [{}^* \nabla, \nabla]\sigma \right) \\
&\quad + \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \sigma, -\frac{3}{2} \nabla(tr\chi\sigma) + \frac{1}{2} \nabla(\hat{\chi} \cdot {}^* \alpha) - \nabla(\zeta \cdot {}^* \beta) - 2\nabla(\bar{\eta} \cdot {}^* \beta) \right. \\
&\quad \left. - \frac{1}{4} \nabla(D_\mu R_{4\nu} - D_\nu R_{4\mu}) \epsilon^{\mu\nu}{}_{34} + [\nabla_4, \nabla]\sigma - [\nabla, \text{div}] {}^* \beta \right) \\
&\quad + \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla \rho, -\frac{3}{2} \nabla(tr\chi\rho) - \frac{1}{2} \nabla(\hat{\chi} \cdot \alpha) + \nabla(\zeta \cdot \beta) + 2\nabla(\bar{\eta} \cdot \beta) \right. \\
&\quad \left. - \frac{1}{4} \nabla(D_3 R_{44} - D_4 R_{34}) + [\nabla_4 \nabla]\rho + [\nabla, \text{div}]\beta \right) + \int_{\mathcal{D}_{u,\bar{u}}} \langle (\eta + \bar{\eta}) \nabla \rho, \nabla \beta \rangle + \int_{\mathcal{D}_{u,\bar{u}}} \langle (\eta + \bar{\eta}) \nabla \sigma, \nabla \beta \rangle.
\end{aligned}$$

Once again we control the potentially problematic terms that may arise from the coupled sector

$$\begin{aligned}
&\int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \nabla(D_b R_{34} - D_4 R_{3b}) \rangle \quad (309) \\
&\sim \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \hat{\nabla}(\hat{\nabla}_b(|\rho^F|^2 + |\sigma^F|^2) - \bar{\chi}_{ba} \mathfrak{T}_{a4} - \hat{\nabla}_4(\bar{\alpha}^F \cdot \rho^F + \bar{\alpha}^F \cdot \sigma^F) + 2\omega \mathfrak{T}_{3b} \\
&\quad + 2\bar{\eta}_a \mathfrak{T}_{ab} + \bar{\eta}_b \mathfrak{T}_{34}) \rangle
\end{aligned}$$

Collect the possible dangerous terms and estimate them by means of elementary inequalities and the

available lemmas (2-8)

$$\begin{aligned}
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \bar{\alpha}^F \hat{\nabla} \hat{\nabla}_4 \rho^F \rangle \right| \leq \int_{\mathcal{D}_{u,\bar{u}}} \|\nabla \beta\|_{L^2} \|\bar{\alpha}^F\|_{L^\infty} \|\hat{\nabla}^2 \alpha^F\|_{L^2} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \\
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \rho^F \hat{\nabla}^2 \rho^F \rangle \right| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \\
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \hat{\nabla} \bar{\alpha}^F \hat{\nabla}_4 \rho^F \rangle \right| \leq \int_{\mathcal{D}_{u,\bar{u}}} \|\nabla \beta\|_{L^2} \|\hat{\nabla} \bar{\alpha}^F\|_{L^4} \|\hat{\nabla}_4 \rho^F\|_{L^4} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \\
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \beta, \bar{\alpha}^F \hat{\nabla}^2 \alpha^F \rangle \right| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \\
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \sigma, \nabla (D_\mu R_{4\nu} - D_\nu R_{4\mu}) \epsilon^{\mu\nu}{}_{34} \rangle \right| \\
& \sim \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \sigma, \nabla (\hat{\nabla}(\alpha^F \cdot \rho^F - \alpha^F \cdot \sigma^F) - \chi(\rho^F \rho^F + \alpha^F \bar{\alpha}^F + \sigma^F \sigma^F + (\eta - \bar{\eta})\alpha^F(\sigma^F + \rho^F))) \rangle \right| \\
& \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \\
& \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \rho, \nabla (D_3 R_{44} - D_4 R_{34}) \rangle \sim \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \rho, \hat{\nabla}(\alpha^F (tr \bar{\chi} \alpha^F - \hat{\nabla} \rho^F + {}^* \hat{\nabla} \sigma^F - 2 {}^* \eta \sigma^F + 2 \eta \rho^F + 2 \bar{\omega} \alpha^F - \hat{\chi} \bar{\alpha}^F) \\
& \quad - 4 \bar{\omega} |\alpha^F|^2 - 4 \eta \alpha^F (\rho^F - \sigma^F) + \rho^F \hat{\nabla}_4 \rho^F + 2 \sigma^F \hat{\nabla}_4 \sigma^F - 2 \bar{\eta} \alpha^F (\rho^F + \sigma^F)) \rangle \\
& \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F})
\end{aligned}$$

where $\nabla \rho$ and the $\hat{\nabla}^2(\alpha^F, \rho^F, \sigma^F)$ have to be bounded in $L^2(H)$, the remaining algebraic terms can be bounded in $L^\infty(S)$ and then apply Sobolev embedding for the $L^\infty(S)$ factor i.e., $\|\alpha^F, \rho^F, \sigma^F\|_{L^\infty(S)} \leq C(\mathcal{O}_0) \|\hat{\nabla}(\alpha^F, \rho^F, \sigma^F)\|_{L^4(S)}$. There will be term of type $|\int_{\mathcal{D}_{u,\bar{u}}} \nabla \Psi \bar{\alpha}^F \hat{\nabla}^2 \bar{\alpha}^F|$ which may be a cause of concern. However, since we have $L^4(S)$ estimate for $\hat{\nabla} \bar{\alpha}^F$, we can control it in $L^\infty(S)$ through Sobolev embedding. Therefore we obtain

$$\begin{aligned}
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \nabla \Psi \bar{\alpha}^F \hat{\nabla}^2 \bar{\alpha}^F \right| = \left| \int_0^u \int_0^{\bar{u}} \int_S \nabla \Psi \bar{\alpha}^F \hat{\nabla}^2 \bar{\alpha}^F \right| \\
& \leq \sup_{u,\bar{u}} \|\bar{\alpha}^F\|_{L^\infty} \|\nabla \Psi\|_{L^2(\mathcal{D}_{u,\bar{u}})} \|\hat{\nabla}^2 \bar{\alpha}^F\|_{L^2(\mathcal{D}_{u,\bar{u}})} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F})
\end{aligned}$$

where Ψ is the Weyl curvature component belonging to the set $(\alpha, \beta, \rho, \sigma)$. In other words as long as we do not have bad terms like $\nabla \bar{\alpha} \bar{\alpha}^F \hat{\nabla}^2 \bar{\alpha}^F$, we will always have a gain of the factor ϵ . Thus

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \sigma, \nabla (D_\mu R_{3\nu} - D_\nu R_{3\mu}) \epsilon^{\mu\nu}{}_{34} \rangle \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (310)$$

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \rho, \nabla (D_3 R_{44} - D_4 R_{33}) \rangle \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (311)$$

and

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \nabla (D_b R_{43} - D_3 R_{4b}) \rangle \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (312)$$

Putting all the terms together, we obtain

$$|ER12| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0, \mathcal{W}, \mathcal{F}) \quad (313)$$

and therefore

$$\begin{aligned}
& \int_{H_u} |\nabla \beta|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla \rho|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla \sigma|^2 = \int_{H_0} |\nabla \beta|^2 + \int_{\bar{H}_0} |\nabla \rho|^2 + \int_{\bar{H}_0} |\nabla \sigma|^2 \\
& \quad + \epsilon C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0, \mathcal{W}, \mathcal{F}) \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F})
\end{aligned} \quad (314)$$

Proceeding in an exact similar way, we obtain

$$\int_{\bar{H}_u} |\nabla \bar{\beta}|^2 + \int_{H_u} |\nabla \rho|^2 + \int_{H_u} |\nabla \sigma|^2 \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (315)$$

Now we focus on the most dangerous term that is the term arising from the remaining pair $(\bar{\alpha}, \bar{\beta})$

$$\begin{aligned} 2 \int_{H_u} |\nabla \bar{\beta}|^2 + \int_{\bar{H}_u} |\nabla \bar{\alpha}|^2 &= 2 \int_{H_0} |\nabla \bar{\beta}|^2 + \int_{\bar{H}_0} |\nabla \bar{\alpha}|^2 + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} 2|\nabla \bar{\beta}|^2 (2\bar{\omega} - \frac{1}{2}tr\bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\nabla \bar{\alpha}|^2 (2\omega - \frac{1}{2}tr\chi)}_{ER13} \\ &\quad - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} (2\langle \nabla \bar{\beta}, \nabla_3 \nabla \bar{\beta} \rangle + \langle \nabla \bar{\alpha}, \nabla_4 \nabla \bar{\alpha} \rangle)}_{ER14}. \end{aligned}$$

Now note the most important fact about the structure of the Yang-Mills sourced null Bianchi equations. In the term ER_{13} we encounter the term $\int_{\mathcal{D}_{u,\bar{u}}} |\nabla \bar{\alpha}|^2 (2\omega - \frac{1}{2}tr\chi)$ which could potentially be dangerous since we can not touch $|\nabla \bar{\alpha}|^2$ term. But luckily, the connection coefficients $(\omega, tr\chi)$ that multiply $|\nabla \bar{\alpha}|^2$ satisfy ∇_3 evolution equation and therefore are solely estimated by means of the initial data $\mathcal{O}_0$. This crucial fact will allow us to make use of the Grönwall's inequality. ER_{13} is estimated as

$$|ER13| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}'. \quad (316)$$

For the term ER_{14} , we need caution since $\nabla \bar{\alpha}$ is involved. We explicitly evaluate ER_{14} using the commuted evolution equations

$$\begin{aligned} &2 \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \nabla_3 \nabla \bar{\beta} \rangle + \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\alpha}, \nabla_4 \nabla \bar{\alpha} \rangle \quad (317) \\ &= \int_{\mathcal{D}_{u,\bar{u}}} 2\langle \nabla \bar{\beta}, -2\nabla(tr\bar{\chi}\bar{\beta}) - div\nabla \bar{\alpha} - 2\nabla(\bar{\omega}\bar{\beta}) + \nabla(\bar{\eta} \cdot \bar{\alpha}) + \frac{1}{2}\nabla(D_b R_{33} - D_3 R_{3b}) \\ &\quad + [\nabla_3, \nabla]\bar{\beta} - [\nabla, div]\bar{\alpha} \rangle \quad (318) \\ &+ \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\alpha}, -\frac{1}{2}\nabla(tr\chi\bar{\alpha}) - \nabla \hat{\otimes} \nabla \bar{\beta} + 4\nabla(\omega\bar{\alpha}) - 3\nabla(\hat{\chi}\rho - {}^*\hat{\chi}\sigma) + \nabla((\zeta - 4\bar{\eta})\hat{\otimes}\bar{\beta}) \\ &\quad + \frac{1}{2}\nabla(D_4 R_{33} - D_3 R_{34})\gamma + [\nabla_4, \nabla]\bar{\alpha} + [\nabla, \nabla]\hat{\otimes}\bar{\beta} \rangle \\ &\sim \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, -2\nabla(tr\bar{\chi}\bar{\beta}) - 2\nabla(\bar{\omega}\bar{\beta}) + \nabla(\bar{\eta} \cdot \bar{\alpha}) + \frac{1}{2}\nabla(D_b R_{33} - D_3 R_{3b}) \\ &\quad + (\bar{\beta} + \bar{\alpha}^F(\sigma^F + \rho^F))\bar{\beta} + (\eta + \bar{\eta})\nabla_3 \bar{\beta} - \bar{\chi}\nabla \bar{\beta} + \chi\bar{\eta}\bar{\beta} - K\bar{\alpha} \rangle \quad (319) \\ &+ \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\alpha}, -\frac{1}{2}\nabla(tr\chi\bar{\alpha}) + 4\nabla(\omega\bar{\alpha}) - 3\nabla(\hat{\chi}\rho - {}^*\hat{\chi}\sigma) + \nabla((\zeta - 4\bar{\eta})\hat{\otimes}\bar{\beta}) \\ &\quad + \frac{1}{2}\nabla(D_4 R_{33} - D_3 R_{34})\gamma + (\beta + \alpha^F(\rho^F + \sigma^F))\bar{\alpha} + (\eta + \bar{\eta})\nabla_4 \bar{\alpha} - \chi\nabla \bar{\alpha} + \chi\bar{\eta}\bar{\alpha} + K\bar{\beta} \rangle \\ &\quad + \int_{\mathcal{D}_{u,\bar{u}}} \langle (\eta + \bar{\eta})\nabla \bar{\beta}, \nabla \bar{\alpha} \rangle, \end{aligned}$$

where once again the principal terms cancel point-wise after an integration by parts procedure. Apart from the cancellation of the principal terms, note that the γ -trace-less property of $\bar{\alpha}$ removes the term $\langle \nabla \bar{\alpha}, \nabla(D_4 R_{33} - D_3 R_{34})\gamma \rangle$. This essentially indicates the fact that $\bar{\alpha}$ does not interact with the source stress-energy tensor at the level of topmost derivative. In order to estimate the remaining terms, we need

to explicitly write down the Yang-Mills source term

$$\begin{aligned} \nabla(D_b R_{33} - D_3 R_{3b}) &\sim \nabla(\langle \bar{\alpha}^F, \hat{\nabla} \bar{\alpha}^F \rangle - \bar{\chi} \bar{\alpha}^F \cdot (\rho^F + \sigma^F) + \bar{\eta} |\bar{\alpha}^F|^2 - 2\bar{\omega} \bar{\alpha}^F \cdot (\rho^F + \sigma^F) - \hat{\nabla}_3(\bar{\alpha}^F \cdot \rho^F + \bar{\alpha}^F \cdot \sigma^F)) \\ &\sim \langle \hat{\nabla} \bar{\alpha}^F, \hat{\nabla} \bar{\alpha}^F \rangle + \langle \bar{\alpha}^F, \hat{\nabla} \hat{\nabla} \bar{\alpha}^F \rangle - \nabla(\bar{\chi} \bar{\alpha}^F(\rho^F + \sigma^F)) + \bar{\eta} \langle \hat{\nabla} \bar{\alpha}^F, \bar{\alpha}^F \rangle - \nabla(\bar{\omega} \bar{\alpha}^F(\rho^F + \sigma^F)) \\ &\quad - \hat{\nabla} \hat{\nabla}_3 \bar{\alpha}^F(\rho^F + \sigma^F) - \hat{\nabla}_3 \bar{\alpha}^F(\hat{\nabla} \rho^F + \hat{\nabla} \sigma^F), \end{aligned}$$

where we notice that the potentially dangerous terms $\hat{\nabla} \hat{\nabla}_3 \bar{\alpha}^F$ and $\hat{\nabla}_3 \bar{\alpha}^F$ appear. However, these terms can be controlled by $\mathcal{F}$. We have $\|\nabla_3 \bar{\alpha}^F\|_{L^4(S)}$ estimated in terms of $\mathcal{F}$ by means of the co-dimension 1 trace inequality (243) and $\|\hat{\nabla} \hat{\nabla}_3 \bar{\alpha}^F\|_{L^2(\bar{H})}$ is estimated in terms of $\mathcal{F}$. We first write down how to control these two terms

$$\begin{aligned} & \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \hat{\nabla} \hat{\nabla}_3 \bar{\alpha}^F(\rho^F + \sigma^F) \rangle \right| \leq \|\nabla \bar{\beta}\|_{L^2(\mathcal{D}_{u,\bar{u}})} \|\nabla \hat{\nabla}_3 \bar{\alpha}^F(\rho^F + \sigma^F)\|_{L^2(\mathcal{D}_{u,\bar{u}})} \quad (320) \\ & \leq \epsilon^{\frac{1}{2}} \sup_u \|\nabla \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} (\|\hat{\nabla} \rho^F\|_{L^4(S)} + \|\hat{\nabla} \sigma^F\|_{L^4(S)}) \sup_{\bar{u}} \|\hat{\nabla} \hat{\nabla}_3 \bar{\alpha}^F\|_{L^2(\bar{H})} \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \end{aligned}$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \hat{\nabla} \bar{\alpha}^F \hat{\nabla}(\rho^F, \sigma^F) \rangle \right| \leq \epsilon^{\frac{1}{2}} \sup_u \|\nabla \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} \|\hat{\nabla}_3 \bar{\alpha}^F\|_{L^4(S)} \sup_{u,\bar{u}} \|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^4(S)} \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F})$$

where we have utilized the fact that $\nabla \bar{\beta}$ is controlled on H (in addition to $\bar{H}$). In addition we also have terms of the following types

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \hat{\nabla}^2 \bar{\alpha}^F \bar{\alpha}^F \rangle \right| \leq \epsilon^{\frac{1}{2}} \sup_u \|\nabla \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} \|\hat{\nabla} \bar{\alpha}\|_{L^4(S)} \sup_{\bar{u}} \|\hat{\nabla}^2 \bar{\alpha}^F\|_{L^2(\bar{H})} \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (321)$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \hat{\nabla} \bar{\alpha}^F \hat{\nabla} \bar{\alpha}^F \rangle \right| \leq \epsilon \sup_u \|\nabla \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} \|\hat{\nabla} \bar{\alpha}^F\|_{L^4(S)}^2 \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (322)$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \hat{\nabla} \bar{\alpha}^F \varphi \rangle \right| \leq \epsilon \sup_u \|\nabla \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} \|\hat{\nabla} \bar{\alpha}^F\|_{L^4(S)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (323)$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \hat{\nabla}_3 \bar{\alpha}^F \varphi \rangle \right| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (324)$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\alpha}, \alpha^F(\rho^F, \sigma^F) \bar{\alpha} \rangle \right| \leq \epsilon^{\frac{1}{2}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})} \sup_{u,\bar{u}} \|\alpha^F\|_{L^4(S)} \|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^4(S)} \|\bar{\alpha}\|_{L^4(S)} \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (325)$$

Therefore the coupled Yang-Mills-Einstein terms are under control. Now we control the remaining terms using the estimates on the connection coefficients (lemma 2-7) as follows

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \bar{\beta} \nabla \varphi \rangle \right| \leq \epsilon \sup_u \|\nabla \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} \|\bar{\beta}\|_{L^4(S)} \|\nabla \varphi\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (326)$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \nabla \bar{\beta} \varphi \rangle \right| \leq \epsilon \sup_u \|\nabla \bar{\beta}\|_{L^2(H)}^2 \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (327)$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, \nabla \bar{\alpha} \varphi \rangle \right| \leq \epsilon^{\frac{1}{2}} \sup_u \|\nabla \bar{\beta}\|_{L^2(H)} \sup_{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (328)$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\beta}, K \bar{\alpha} \rangle \right| \leq \epsilon \sup_u \|\nabla \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} \|K\|_{L^4(S)} \|\bar{\alpha}\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (329)$$

Now we estimate the most dangerous terms

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\alpha}, \nabla \bar{\alpha} \text{tr} \chi \rangle \right| \leq \sup_{u,\bar{u}} \|\text{tr} \chi\|_{L^\infty(S)} \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' \leq C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}', \quad (330)$$

$$\left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\alpha}, \nabla \bar{\alpha} \omega \rangle \right| \leq \sup_{u,\bar{u}} \|\omega\|_{L^\infty(S)} \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' \leq C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}', \quad (331)$$

where we notice that the connection coefficients that appear multiplied with the top derivative of $\bar{\alpha}$ satisfy ∇_3 equation and therefore solely determined by the initial data $\mathcal{O}_0$. This allows us to use the Grönwall's inequality to complete the energy estimate. The remaining terms are harmless

$$|\int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\alpha}, \nabla \Psi \varphi \rangle| \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad |\int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \bar{\alpha}, \nabla \varphi \Psi \rangle| \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (332)$$

$$|\int_{\mathcal{D}_{u,\bar{u}}} \langle (\eta + \bar{\eta}) \nabla \bar{\beta}, \nabla \bar{\alpha} \rangle| \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (333)$$

where $\Psi \in (\beta, \bar{\beta}, \rho, \sigma, \alpha)$ and φ is any connection coefficients belonging to the set $(\hat{\chi}, \bar{\chi}, tr\chi, tr\bar{\chi}, \bar{\eta}, \omega, \eta, \bar{\omega})$. Collecting all the terms, we may now estimate $ER14$ as follows

$$|ER14| \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}'. \quad (334)$$

Therefore collecting all the terms we obtain

$$\begin{aligned} & 2 \int_{H_u} |\nabla \bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla \bar{\alpha}|^2 \leq 2 \int_{H_0} |\nabla \bar{\beta}|^2 + \int_{\bar{H}_0} |\nabla \bar{\alpha}|^2 \\ & + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' \\ & \leq C(\mathcal{F}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' \end{aligned}$$

This completes the proof of the lemma. $\square$

Now that we have completed the energy estimates for the Weyl curvature, we move on to estimating the energy associated with the Yang-Mills curvature. For this purpose we could use the canonical stress-energy tensor associated with the Yang-Mills theory. However, we will follow the direct integration by parts procedure using the null Yang-Mills equations. In the Yang-Mills case, we will not do the energy estimates separately for each order rather only for the topmost derivatives. If the estimates at the top order closes then so does the lower orders since the hyperbolic structure of the null Yang-Mills equations are preserved after commuting with derivatives.

Lemma 12: *The horizontal derivatives of the null Yang-Mills curvature satisfy the following L^2 energy estimates*

$$\int_{H_u} |\hat{\nabla}^I \alpha^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \rho^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \sigma^F|^2 \leq C(\mathcal{F}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (335)$$

$$\begin{aligned} & \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \bar{\alpha}^F|^2 + \int_{H_u} |\hat{\nabla}^I \rho^F|^2 + \int_{H_u} |\hat{\nabla}^I \sigma^F|^2 \\ & \leq C(\mathcal{W}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla} \bar{\alpha}^F\|_{L^2(\bar{H})}^2 d\bar{u}' \end{aligned} \quad (336)$$

for $0 \leq I \leq 2$, $I \in \mathbb{Z}$.

Proof: In order to prove this estimate, we use the integration identities (244-245) once again. First, we identify the pairs $(\alpha^F, \rho^F, \sigma^F)$ and apply the identities (244-245) to obtain

$$\begin{aligned} & \int_{H_u} |\hat{\nabla}^I \alpha^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \rho^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \sigma^F|^2 = \int_{H_0} |\hat{\nabla}^I \alpha^F|^2 + \int_{\bar{H}_0} |\hat{\nabla}^I \rho^F|^2 + \int_{\bar{H}_0} |\hat{\nabla}^I \sigma^F|^2 \\ & + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} |\hat{\nabla}^I \alpha^F|^2 (2\bar{\omega} - \frac{1}{2} tr\bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\hat{\nabla}^I \rho^F|^2 (2\omega - \frac{1}{2} tr\chi) + \int_{\mathcal{D}_{u,\bar{u}}} |\hat{\nabla}^I \sigma^F|^2 (2\omega - \frac{1}{2} tr\chi)}_{ER15} \\ & - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \hat{\nabla}^I \alpha^F, \hat{\nabla}_3 \hat{\nabla}^I \alpha^F \rangle + \langle \hat{\nabla}^I \rho^F, \hat{\nabla}_4 \hat{\nabla}^I \rho^F \rangle + \langle \hat{\nabla}^I \sigma^F, \hat{\nabla}_4 \hat{\nabla}^I \sigma^F \rangle \right)}_{ER16}. \end{aligned} \quad (337)$$

We can directly estimate $ER15$ by using the estimates of the connection coefficients (lemma 2) and the definition of $\mathcal{F}$ since all the Yang-Mills curvature coefficients can be estimated on H by

$$|ER15| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}). \quad (338)$$

In order to estimate $ER16$, we use the null Yang-Mills equations

$$\begin{aligned} ER16 &= -2 \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \hat{\nabla}^I \alpha^F, \hat{\nabla}_3 \hat{\nabla}^I \alpha^F \rangle + \langle \hat{\nabla}^I \rho^F, \hat{\nabla}_4 \hat{\nabla}^I \rho^F \rangle + \langle \hat{\nabla}^I \sigma^F, \hat{\nabla}_4 \hat{\nabla}^I \sigma^F \rangle \right) \\ &= \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla^I \alpha^F, -\frac{1}{2} \hat{\nabla}^I (tr \bar{\chi} \alpha^F) - \hat{\nabla} \hat{\nabla}^I \rho^F + {}^* \hat{\nabla} \hat{\nabla}^I \sigma^F - 2 \hat{\nabla}^I ({}^* \bar{\eta} \sigma^F) + 2 \hat{\nabla}^I (\bar{\eta} \rho^F) \right. \\ &\quad + 2 \hat{\nabla}^I (\bar{\omega} \alpha^F) - \hat{\nabla}^I (\hat{\chi} \cdot \bar{\alpha}^F) + [\hat{\nabla}_3, \hat{\nabla}^I] \alpha^F + [\hat{\nabla}, \hat{\nabla}^I] \rho^F - [{}^* \hat{\nabla}, \hat{\nabla}^I] \sigma^F \rangle \\ &\quad + \langle \hat{\nabla}^I \rho^F, -\hat{div} \hat{\nabla}^I \alpha^F - \hat{\nabla}^I (tr \chi \rho^F) - \hat{\nabla}^I ((\eta - \bar{\eta}) \cdot \alpha^F) + [\hat{\nabla}_4, \hat{\nabla}^I] \rho^F + [\hat{div}, \hat{\nabla}^I] \alpha^F \rangle \\ &\quad + \langle \hat{\nabla}^I \sigma^F, -\hat{curl} \hat{\nabla}^I \alpha^F - \hat{\nabla}^I (tr \chi \sigma^F) + \hat{\nabla}^I ((\eta - \bar{\eta}) \cdot {}^* \alpha^F) + [\hat{\nabla}_4, \hat{\nabla}^I] \sigma^F - [\hat{curl}, \hat{\nabla}^I] \alpha^F \rangle \Big) \\ &\sim \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla^I \alpha^F, -\frac{1}{2} \hat{\nabla}^I (tr \bar{\chi} \alpha^F) - 2 \hat{\nabla}^I ({}^* \bar{\eta} \sigma^F) + 2 \hat{\nabla}^I (\bar{\eta} \rho^F) \right. \\ &\quad + 2 \hat{\nabla}^I (\bar{\omega} \alpha^F) - \hat{\nabla}^I (\hat{\chi} \cdot \bar{\alpha}^F) + [\hat{\nabla}_3, \hat{\nabla}^I] \alpha^F + [\hat{\nabla}, \hat{\nabla}^I] \rho^F - [{}^* \hat{\nabla}, \hat{\nabla}^I] \sigma^F \rangle \\ &\quad + \langle \hat{\nabla}^I \rho^F, -\hat{\nabla}^I (tr \chi \rho^F) - \hat{\nabla}^I ((\eta - \bar{\eta}) \cdot \alpha^F) + [\hat{\nabla}_4, \hat{\nabla}^I] \rho^F + [\hat{div}, \hat{\nabla}^I] \alpha^F \rangle \\ &\quad + \langle \hat{\nabla}^I \sigma^F, -\hat{\nabla}^I (tr \chi \sigma^F) + \hat{\nabla}^I ((\eta - \bar{\eta}) \cdot {}^* \alpha^F) + [\hat{\nabla}_4, \hat{\nabla}^I] \sigma^F - [\hat{curl}, \hat{\nabla}^I] \alpha^F \rangle \Big) \\ &\quad + \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \alpha^F, ((\eta + \bar{\eta})(\hat{\nabla}^I \rho^F + \hat{\nabla}^I \sigma^F)) \rangle. \end{aligned} \quad (339)$$

We will sketch how we handle each terms. There will be terms of the following type that are estimated easily

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \varphi \hat{\nabla}^I \Phi^F \rangle \leq \int_{\mathcal{D}_{u,\bar{u}}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\varphi\|_{L^\infty(S)} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (340)$$

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \hat{\nabla}^I \varphi \Phi^F \rangle \leq \int_0^u \|\hat{\nabla}^I \Phi^F\|_{L^2(H)} \|\hat{\nabla}^I \varphi\|_{L^2(H)} \|\Phi^F\|_{L^\infty(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (341)$$

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \hat{\nabla} \varphi \hat{\nabla} \Psi \rangle \leq \int_{\mathcal{D}_{u,\bar{u}}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\hat{\nabla} \varphi\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}),$$

where $\Phi^F \in (\alpha^F, \rho^F, \sigma^F)$ and therefore are controllable on H . For the connection coefficients, we use the lemma (2-7). Now we need to control the commutator terms

$$[\hat{\nabla}_3, \hat{\nabla}^2] \alpha^F = [\hat{\nabla}_3, \hat{\nabla}] \hat{\nabla} \alpha^F + \hat{\nabla} [\hat{\nabla}_3, \hat{\nabla}] \alpha^F \quad (342)$$

Now

$$[\hat{\nabla}_3, \hat{\nabla}] \Phi^F \sim \bar{\beta} \Phi^F + \bar{\alpha}^F \Phi^F + \bar{\alpha}^F (\rho^F + \sigma^F) \Phi^F + (\eta + \bar{\eta}) \nabla_3 \Phi^F - \bar{\chi} \nabla \Phi^F + \bar{\chi} \eta \Phi^F,$$

and therefore

$$\begin{aligned} [\hat{\nabla}_3, \hat{\nabla}] \hat{\nabla} \alpha^F &\sim \bar{\beta} \hat{\nabla} \alpha^F + \bar{\alpha}^F \hat{\nabla} \alpha^F + \bar{\alpha}^F (\rho^F - \sigma^F) \hat{\nabla} \alpha^F + (\eta + \bar{\eta}) \hat{\nabla}_3 \hat{\nabla} \alpha^F - \bar{\chi} \nabla \hat{\nabla} \alpha^F + \bar{\chi} \eta \hat{\nabla} \alpha^F \\ &\sim \bar{\beta} \hat{\nabla} \alpha^F + \bar{\alpha}^F \hat{\nabla} \alpha^F + \bar{\alpha}^F (\rho^F - \sigma^F) \hat{\nabla} \alpha^F + (\eta + \bar{\eta}) \hat{\nabla} \hat{\nabla}_3 \alpha^F - \bar{\chi} \nabla \hat{\nabla} \alpha^F + \bar{\chi} \eta \hat{\nabla} \alpha^F + (\eta + \bar{\eta}) [\hat{\nabla}_3, \hat{\nabla}] \alpha^F, \end{aligned} \quad (343)$$

where $\hat{\nabla} \hat{\nabla}_3 \alpha^F$ is evaluated by means of the null Yang-Mills equations. Now

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, [\hat{\nabla}_3, \hat{\nabla}^I] \Phi^F \rangle \quad (344)$$

would contain the following terms

$$|\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \bar{\alpha}^F \hat{\nabla} \Phi^F \rangle| \leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\bar{\alpha}^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{F}), \quad (345)$$

$$\begin{aligned} |\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \bar{\alpha}^F (\rho^F + \sigma^F) \hat{\nabla} \alpha^F \rangle| &\leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\bar{\alpha}^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \|(\rho^F, \sigma^F)\|_{L^\infty(S)} \\ &\leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\bar{\alpha}^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{F}), \end{aligned}$$

$$|\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \varphi \hat{\nabla} \bar{\alpha}^F \rangle| \leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\varphi\|_{L^\infty(S)} \|\varphi\|_{L^4(S)} \|\hat{\nabla} \bar{\alpha}^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}),$$

$$|\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \varphi \hat{\nabla}^I \Phi^F \rangle| \leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)}^2 \|\varphi\|_{L^\infty(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (346)$$

$$|\int_{\mathcal{D}_{u,\bar{u}}} \hat{\nabla}^I \Phi^F \bar{\beta} \hat{\nabla} \Phi^F| \leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\bar{\beta}\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (347)$$

Now note the following commutation relation

$$[\hat{\nabla}_4, \hat{\nabla}^2](\rho^F, \sigma^F) = [\hat{\nabla}_4, \hat{\nabla}] \hat{\nabla}(\rho^F, \sigma^F) + \hat{\nabla}[\hat{\nabla}_4, \hat{\nabla}](\rho^F, \sigma^F) \quad (348)$$

Now

$$\begin{aligned} [\hat{\nabla}_4, \hat{\nabla}](\rho^F, \sigma^F) &\sim \beta(\rho^F, \sigma^F) + \alpha^F(\rho^F, \sigma^F) + \alpha^F(\rho^F + \sigma^F)(\rho^F, \sigma^F) + (\eta + \bar{\eta}) \nabla_4(\rho^F, \sigma^F) - \chi \nabla(\rho^F, \sigma^F) \\ &\quad + \chi \bar{\eta}(\rho^F, \sigma^F), \end{aligned}$$

and therefore

$$\begin{aligned} [\hat{\nabla}_4, \hat{\nabla}] \hat{\nabla}(\rho^F, \sigma^F) &\sim \beta \hat{\nabla}(\rho^F, \sigma^F) + \alpha^F \hat{\nabla}(\rho^F, \sigma^F) + \alpha^F(\rho^F + \sigma^F) \hat{\nabla}(\rho^F, \sigma^F) + (\eta + \bar{\eta}) \hat{\nabla}_4 \hat{\nabla}(\rho^F, \sigma^F) \\ &\quad - \chi \nabla \hat{\nabla}(\rho^F, \sigma^F) + \bar{\chi} \eta \hat{\nabla}(\rho^F, \sigma^F) \\ &\sim \beta \hat{\nabla}(\rho^F, \sigma^F) + \alpha^F \hat{\nabla}(\rho^F, \sigma^F) + \alpha^F(\rho^F - \sigma^F) \hat{\nabla}(\rho^F, \sigma^F) + (\eta + \bar{\eta}) \hat{\nabla} \hat{\nabla}_4(\rho^F, \sigma^F) - \bar{\chi} \nabla \hat{\nabla}(\rho^F, \sigma^F) \\ &\quad + \bar{\chi} \eta \hat{\nabla}(\rho^F, \sigma^F) + (\eta + \bar{\eta}) \left(\beta + \alpha^F(\rho^F + \sigma^F) \right) (\rho^F, \sigma^F) + (\eta + \bar{\eta}) \hat{\nabla}_4(\rho^F, \sigma^F) - \chi \hat{\nabla}(\rho^F, \sigma^F) + \chi \bar{\eta}(\rho^F, \sigma^F), \end{aligned}$$

where $\hat{\nabla} \hat{\nabla}_4(\rho^F, \sigma^F)$ can be further simplified by means of the null Yang-Mills equations. The resulting terms are easy to estimate

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \Phi^F \hat{\nabla} \Phi^F \rangle \leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\Phi^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{F}), \quad (349)$$

$$\begin{aligned} \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \Phi^F (\rho^F + \sigma^F) \hat{\nabla} \Phi^F \rangle &\leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\Phi^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \|(\rho^F, \sigma^F)\|_{L^\infty(S)} \\ &\leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\Phi^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \|\hat{\nabla}(\rho^F, \sigma^F)\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{F}), \end{aligned}$$

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \varphi \hat{\nabla} \Phi^F \rangle \leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \|\varphi\|_{L^\infty(S)} \|\varphi\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}),$$

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \varphi \hat{\nabla}^I \Phi^F \rangle \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (350)$$

where once again $\Phi^F \in (\alpha^F, \rho^F, \sigma^F)$ and φ is any connection coefficients belonging to the set $(\hat{\chi}, \bar{\chi}, tr \bar{\chi}, tr \bar{\chi}, \bar{\eta}, \omega, \eta, \bar{\omega})$. Now note the following

$$[\hat{\nabla}, \hat{\nabla}^2] \Phi^F = \hat{\nabla}[\hat{\nabla}, \hat{\nabla}] \Phi^F + [\hat{\nabla}, \hat{\nabla}] \hat{\nabla} \Phi^F \sim \hat{\nabla}((K + \sigma^F) \Phi^F) + (K + \sigma^F) \hat{\nabla} \Phi^F \quad (351)$$

and therefore

$$\begin{aligned}
& \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, [\hat{\nabla}, \hat{\nabla}^2] \Phi^F \rangle \right| \sim \left| \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \Phi^F, \hat{\nabla}((K + \sigma^F) \Phi^F) + (K + \sigma^F) \hat{\nabla} \Phi^F \rangle \right| \quad (352) \\
& \leq \int_{u,\bar{u}} \|\hat{\nabla}^I \Phi^F\|_{L^2(S)} \left(\|\nabla K\|_{L^2(S)} \|\Phi^F\|_{L^\infty(S)} + \|K\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} + \|\hat{\nabla} \sigma^F\|_{L^4(S)} \|\Phi^F\|_{L^4(S)} \right. \\
& \quad \left. + \|\sigma^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \right) \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F})
\end{aligned}$$

where $\|\hat{\nabla} K\|_{L^2(S)}$, $\|K\|_{L^4(S)}$ are controlled by $\mathcal{W}$ and $\mathcal{F}$ by the virtue of the null Hamiltonian constraint (49, corollary 1). Collecting all the terms together, we obtain

$$|ER16| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (353)$$

and therefore

$$\begin{aligned}
& \int_{H_u} |\hat{\nabla}^I \alpha^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \rho^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \sigma^F|^2 = \int_{H_0} |\hat{\nabla}^I \alpha^F|^2 + \int_{\bar{H}_0} |\hat{\nabla}^I \rho^F|^2 + \int_{\bar{H}_0} |\hat{\nabla}^I \sigma^F|^2 \quad (354) \\
& \quad + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \leq C(\mathcal{W}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}).
\end{aligned}$$

Now in order to prove the second estimate of the lemma, we collect the triple $(\bar{\alpha}, \rho^F, \sigma^F)$ and apply the integration identities (244-245)

$$\begin{aligned}
& \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \bar{\alpha}^F|^2 + \int_{H_u} |\hat{\nabla}^I \rho^F|^2 + \int_{H_u} |\hat{\nabla}^I \sigma^F|^2 = \int_{\bar{H}_0} |\hat{\nabla}^I \bar{\alpha}^F|^2 + \int_{H_0} |\hat{\nabla}^I \rho^F|^2 + \int_{H_0} |\hat{\nabla}^I \sigma^F|^2 \quad (355) \\
& \quad + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} |\hat{\nabla}^I \bar{\alpha}^F|^2 (2\omega - \frac{1}{2} tr \chi) + \int_{\mathcal{D}_{u,\bar{u}}} |\hat{\nabla}^I \rho^F|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\hat{\nabla}^I \sigma^F|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi})}_{ER17} \\
& \quad - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \hat{\nabla}^I \bar{\alpha}^F, \hat{\nabla}_4 \hat{\nabla}^I \bar{\alpha}^F \rangle + \langle \hat{\nabla}^I \rho^F, \hat{\nabla}_3 \hat{\nabla}^I \rho^F \rangle + \langle \hat{\nabla}^I \sigma^F, \hat{\nabla}_3 \hat{\nabla}^I \sigma^F \rangle \right)}_{ER18}.
\end{aligned}$$

Once again notice that the term involving $|\hat{\nabla}^I \bar{\alpha}^F|^2$ contains connection coefficients $(\omega, tr \chi)$ which satisfy ∇_3 evolution equations and therefore are estimated solely by means of the initial data allowing us to use Grönwall's inequality. $ER17$ is estimated as follows

$$|ER17| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla}^I \bar{\alpha}^F\|_{L^2(\bar{H})}^2 d\bar{u}'. \quad (356)$$

Once again $ER18$ is controlled by means of commuted null Yang-Mills equations

$$\begin{aligned}
& \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \hat{\nabla}^I \bar{\alpha}^F, \hat{\nabla}_4 \hat{\nabla}^I \bar{\alpha}^F \rangle + \langle \hat{\nabla}^I \rho^F, \hat{\nabla}_3 \hat{\nabla}^I \rho^F \rangle + \langle \hat{\nabla}^I \sigma^F, \hat{\nabla}_3 \hat{\nabla}^I \sigma^F \rangle \right) \quad (357) \\
& \sim \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \hat{\nabla}^I \bar{\alpha}^F, -\frac{1}{2} \hat{\nabla}^I (tr \chi \bar{\alpha}^F) - 2 \hat{\nabla}^I (*\bar{\eta} \cdot \sigma^F) - 2 \hat{\nabla}^I (\bar{\eta} \cdot \rho^F) + 2 \hat{\nabla}^I (\omega \bar{\alpha}^F) - \hat{\nabla}^I (\hat{\chi} \cdot \alpha^F) \rangle \right. \\
& \quad \left. + \langle \hat{\nabla}^I \rho^F, \hat{\nabla}^I (tr \bar{\chi} \rho^F) + \hat{\nabla}^I ((\eta - \bar{\eta}) \cdot \bar{\alpha}^F) \rangle + \langle \hat{\nabla}^I \sigma^F, -\hat{\nabla}^I (tr \bar{\chi} \sigma^F) + \hat{\nabla}^I ((\eta - \bar{\eta}) \cdot * \bar{\alpha}^F) \rangle \right) \\
& \quad + \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \bar{\alpha}^F, [\hat{\nabla}_4, \hat{\nabla}^I] \bar{\alpha}^F \rangle + \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \bar{\alpha}^F, [\hat{\nabla}, \hat{\nabla}^I] (\rho^F, \sigma^F) \rangle + \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \rho^F, [\hat{\nabla}_3, \hat{\nabla}^I] \rho^F \rangle \quad (358) \\
& \quad + \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \rho^F, [\hat{\nabla}, \hat{\nabla}^I] \bar{\alpha}^F \rangle + \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \sigma^F, [\hat{\nabla}, \hat{\nabla}^I] \bar{\alpha}^F \rangle + \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \alpha^F, ((\eta + \bar{\eta}) (\hat{\nabla}^I \rho^F + \hat{\nabla}^I \sigma^F)) \rangle
\end{aligned}$$

The terms that appear with the top order derivative of $\bar{\alpha}^F$ are $tr\chi, \omega$, and $\hat{\chi}$ which are determined solely by $\mathcal{O}_0$. We use the following commutation relation

$$[\hat{\nabla}_4, \hat{\nabla}]\bar{\alpha}^F \sim \beta\bar{\alpha}^F + \alpha^F\bar{\alpha}^F + \alpha^F(\rho^F - \sigma^F)\bar{\alpha}^F + (\eta + \bar{\eta})\nabla_4\bar{\alpha}^F - \bar{\chi}\nabla\bar{\alpha}^F + \chi\bar{\eta}\bar{\alpha}^F$$

to evaluate $[\hat{\nabla}_4, \hat{\nabla}^2]\bar{\alpha}^F$

$$[\hat{\nabla}_4, \hat{\nabla}^2]\bar{\alpha}^F = [\hat{\nabla}_4, \hat{\nabla}]\hat{\nabla}\bar{\alpha}^F + \hat{\nabla}[\hat{\nabla}_4, \hat{\nabla}]\bar{\alpha}^F. \quad (359)$$

These will contain terms of the type

$$\begin{aligned} \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \bar{\alpha}^F, \hat{\nabla}^I \varphi \bar{\alpha}^F \rangle &\leq \|\hat{\nabla}^I \bar{\alpha}^F\|_{L^2(\mathcal{D}_{u,\bar{u}})} \sup_{u,\bar{u}} \|\hat{\nabla} \bar{\alpha}^F\|_{L^4(S)} \|\hat{\nabla} \varphi\|_{L^2(\mathcal{D}_{u,\bar{u}})} \leq \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{F}), \\ \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \bar{\alpha}^F, \hat{\nabla} \varphi \hat{\nabla} \bar{\alpha}^F \rangle &\leq \int_{\mathcal{D}_{u,\bar{u}}} \|\hat{\nabla}^I \bar{\alpha}^F\|_{L^2(S)} \|\hat{\nabla} \varphi\|_{L^4(S)} \|\hat{\nabla} \bar{\alpha}^F\|_{L^4(S)} \leq \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{F}) \\ \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \alpha^F, \nabla K \bar{\alpha}^F \rangle &\leq \epsilon C(\mathcal{O}_0, \mathcal{F}), \quad \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \alpha^F, K \hat{\nabla} \bar{\alpha}^F \rangle \leq \epsilon C(\mathcal{O}_0, \mathcal{F}) \\ \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \bar{\alpha}^F, K \hat{\nabla} \bar{\alpha}^F \rangle &\leq \epsilon C(\mathcal{O}_0, \mathcal{F}), \quad \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \alpha^F, \nabla K \bar{\alpha}^F \rangle \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{F}), \end{aligned} \quad (360)$$

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \alpha^F, \varphi \hat{\nabla} \bar{\alpha}^F \rangle \leq \epsilon C(\mathcal{O}_0, \mathcal{F}), \quad \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \bar{\alpha}^F, \nabla \varphi \bar{\alpha}^F \rangle \leq \epsilon^{1/2} C(\mathcal{O}_0, \mathcal{F}) \quad (361)$$

$$\int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}^I \bar{\alpha}^F, (\omega, tr\chi) \hat{\nabla}^I \bar{\alpha}^F \rangle \leq C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla}^I \bar{\alpha}^F\|_{L^2(\bar{H})}^2 d\bar{u}', \quad (362)$$

$$(363)$$

Notice that the last term is the Grönwall term and connection coefficients in the last integral do not contain η and $\bar{\omega}$. Therefore, collecting all the terms, we obtain

$$|ER18| \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla} \bar{\alpha}^F\|_{L^2(\bar{H})}^2 d\bar{u}'. \quad (364)$$

Substituting this estimate in the main energy identity (355) we obtain

$$\begin{aligned} \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}^I \bar{\alpha}^F|^2 + \int_{H_u} |\hat{\nabla}^I \rho^F|^2 + \int_{H_u} |\hat{\nabla}^I \sigma^F|^2 &= \int_{\bar{H}_0} |\hat{\nabla}^I \bar{\alpha}^F|^2 + \int_{H_0} |\hat{\nabla}^I \rho^F|^2 + \int_{H_0} |\hat{\nabla}^I \sigma^F|^2 \\ &\quad + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla} \bar{\alpha}^F\|_{L^2(\bar{H})}^2 d\bar{u}' \\ &\leq C(\mathcal{F}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla} \bar{\alpha}^F\|_{L^2(\bar{H})}^2 d\bar{u}' \end{aligned} \quad (365)$$

This completes the proof of the lemma. $\square$

Now we need to estimate the remaining terms of the curvature energy.

Lemma 13: *The null derivatives of the null components of the Weyl curvature satisfy the following L^2 energy estimates*

$$\int_{H_u} |\nabla_4 \alpha|^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (366)$$

$$\int_{\bar{H}_{\bar{u}}} |\nabla_3 \bar{\alpha}|^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' \quad (367)$$

Proof: We proceed exactly the same way as the previous case only we commute the null derivatives with

the evolution equations of the Weyl curvature components. First we identify the set of pairs (α, β) and $(\bar{\alpha}, \bar{\beta})$ and apply the integration identities (244-245)

$$\begin{aligned} & \int_{H_u} |\nabla_4 \alpha|^2 + 2 \int_{\bar{H}_{\bar{u}}} |\nabla_4 \beta|^2 = \int_{H_0} |\nabla_4 \alpha|^2 + 2 \int_{\bar{H}_0} |\nabla_4 \beta|^2 \quad (368) \\ & + \underbrace{\int_{\mathcal{D}_{u, \bar{u}}} |\nabla_4 \alpha|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u, \bar{u}}} 2 |\nabla_4 \beta|^2 (2\omega - \frac{1}{2} tr \chi)}_{ER19} - 2 \underbrace{\int_{\mathcal{D}_{u, \bar{u}}} (\langle \nabla_4 \alpha, \nabla_3 \nabla_4 \alpha \rangle + 2 \langle \nabla_4 \beta, \nabla_4 \nabla_4 \beta \rangle)}_{ER20} \end{aligned}$$

Now notice that $\nabla_4 \beta$ contains the term $\nabla \alpha$ which can not be controlled on $\bar{H}$ and therefore we needed to include $\|\nabla_4 \beta\|_{L^2(\bar{H}_0)}$ term in the initial data for the Weyl curvature. Fortunately, $\nabla_4 \beta$ appears in the error term $ER19$ within the bulk integral multiplied by ω and $tr \chi$ and therefore harmless. We first estimate the error terms

$$\begin{aligned} |ER19| &= \left| \int_{\mathcal{D}_{u, \bar{u}}} |\nabla_4 \alpha|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u, \bar{u}}} 2 |\nabla_4 \beta|^2 (2\omega - \frac{1}{2} tr \chi) \right| \quad (369) \\ &\leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_4 \beta\|_{L^2(\bar{H})}^2 d\bar{u}' \end{aligned}$$

where notice the connection coefficients multiplying $|\nabla_4 \beta|^2$ in the bulk integral is solely determined by the initial data $\mathcal{O}_0$. This allows us to utilize Grönwall estimate. Now for the term $ER20$, we need to use the commuted evolution equations

$$\begin{aligned} ER20 &= -2 \int_{\mathcal{D}_{u, \bar{u}}} (\langle \nabla_4 \alpha, \nabla_3 \nabla_4 \alpha \rangle + 2 \langle \nabla_4 \beta, \nabla_4 \nabla_4 \beta \rangle) \quad (370) \\ &= -2 \int_{\mathcal{D}_{u, \bar{u}}} \left(\langle \nabla_4 \alpha, \nabla \hat{\otimes} \nabla_4 \beta - \frac{1}{2} \nabla_4 (tr \bar{\chi} \alpha) + 4 \nabla_4 (\bar{\omega} \alpha) - 3 \nabla_4 (\hat{\chi} \rho + {}^* \chi \sigma) + \nabla_4 ((\zeta + 4\eta) \hat{\otimes} \beta) \right. \\ &\quad \left. + \frac{1}{2} \nabla_4 (D_3 R_{44} - D_4 R_{33}) \gamma + [\nabla_3, \nabla_4] \alpha + [\nabla_4, \nabla] \hat{\otimes} \beta \rangle + 2 \langle \nabla_4 \beta, div(\nabla_4 \alpha) - 2 \nabla_4 (tr \chi \beta) - 2 \nabla_4 (\omega \beta) \right. \\ &\quad \left. + \nabla_4 (\eta \alpha) - \frac{1}{2} \nabla_4 (D_b R_{44} - D_4 R_{4b}) + [\nabla_4, div] \alpha \rangle \right) \\ &\sim \int_{\mathcal{D}_{u, \bar{u}}} \left(\langle \nabla_4 \alpha, -\frac{1}{2} \nabla_4 (tr \bar{\chi} \alpha) + 4 \nabla_4 (\bar{\omega} \alpha) - 3 \nabla_4 (\hat{\chi} \rho + {}^* \chi \sigma) + \nabla_4 ((\eta + \bar{\eta}) \hat{\otimes} \beta) \right. \\ &\quad \left. + \frac{1}{2} \nabla_4 (D_3 R_{44} - D_4 R_{33}) \gamma + [\nabla_3, \nabla_4] \alpha + [\nabla_4, \nabla] \hat{\otimes} \beta \rangle + \langle \nabla_4 \beta, -2 \nabla_4 (tr \chi \beta) - 2 \nabla_4 (\omega \beta) \right. \\ &\quad \left. + \nabla_4 (\eta \alpha) - \frac{1}{2} \nabla_4 (D_b R_{44} - D_4 R_{4b}) + [\nabla_4, div] \alpha \rangle \right) + \int_{\mathcal{D}_{u, \bar{u}}} \langle (\eta + \bar{\eta}) \nabla_4 \beta, \nabla_4 \alpha \rangle. \end{aligned}$$

Now write down the commutators explicitly

$$\begin{aligned} [\nabla_3, \nabla_4] \alpha &\sim \sigma \alpha + (\rho^F \rho^F + \sigma^F \sigma^F) \alpha + \omega \nabla_3 \alpha + \bar{\omega} \nabla_4 \alpha + (\eta - \bar{\eta}) \nabla \alpha, \\ [\nabla_4, \nabla] \beta &\sim (\beta + \alpha^F (\rho^F + \sigma^F)) \beta + (\eta + \bar{\eta}) \nabla_4 \beta - \chi \nabla \beta + \chi \bar{\eta} \beta, \\ [\nabla_4, \nabla] \alpha &\sim (\beta + \alpha^F (\rho^F + \sigma^F)) \alpha + (\eta + \bar{\eta}) \nabla_4 \alpha - \chi \nabla \beta + \chi \bar{\eta} \alpha \end{aligned} \quad (371)$$

and use the following null evolution equation for β

$$\nabla_4 \beta_a + 2 tr \chi \beta_a = (div \alpha)_a - 2 \omega \beta_a + (\eta \cdot \alpha)_a - \frac{1}{2} (\mathcal{D}_a R_{44} - \mathcal{D}_4 R_{4a}), \quad (372)$$

where the Yang-Mills source term may be evaluated as follows

$$D_b R_{44} - D_4 R_{4b} \sim \langle \alpha^F, \hat{\nabla}_b \alpha^F \rangle - \chi_{bc} \mathfrak{T}_{c4} + \eta_b \mathfrak{T}_{44} - 2 \omega \mathfrak{T}_{4b} - (\hat{\nabla}_4 \alpha_b^F \cdot (\rho^F + \sigma^F) - \alpha_b^F \cdot (\hat{\nabla}_4 \rho^F + \hat{\nabla}_4 \sigma^F)).$$

In addition we will use the null evolution equations for the connection coefficients whenever they are available. In the error term $ER20$, we note that $\nabla_4 \bar{\eta}$ appears. We can use the estimate from lemma (7) to control this term (this was the whole point of proving lemma 5-7). Note $\langle \nabla_4 \alpha, \nabla_4 (D_3 R_{44} - D_4 R_{33}) \gamma \rangle = 0$ (since ∇_4 commutes with γ). Now we may estimate each term separately

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_4 \alpha, \nabla_4 \varphi \alpha \rangle | \leq \epsilon \sup_u \|\nabla_4 \alpha\|_{L^2(H)} \sup_{u,\bar{u}} \|\nabla_4 \varphi\|_{L^4(S)} \|\alpha\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (373)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_4 \alpha, \varphi \nabla_4 \alpha \rangle | \leq \epsilon \sup_u \|\nabla_4 \alpha\|_{L^2(H)}^2 \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \leq \epsilon C(\mathcal{O}, \mathcal{W}, \mathcal{F}) \quad (374)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_4 \alpha \varphi \nabla_4 \Psi \rangle | \leq \epsilon \sup_u \|\nabla_4 \alpha\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \|\nabla_4 \Psi\|_{L^2(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (375)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_4 \alpha \nabla_4 \varphi \Psi \rangle | \leq \epsilon \sup_u \|\nabla_4 \alpha\|_{L^2(H)} \sup_{u,\bar{u}} \|\nabla_4 \varphi\|_{L^4(S)} \|\Psi\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (376)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_4 \alpha \varphi \nabla \Psi \rangle | \leq \epsilon \sup_u \|\nabla_4 \alpha\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \|\nabla \Psi\|_{L^2(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (377)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, \varphi \Phi^F \hat{\nabla}_4 \alpha^F \rangle | \leq \epsilon \sup_u \|\nabla \alpha\|_{L^2(H)} \sup_u \|\hat{\nabla}_4 \alpha^F\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (378)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, \varphi \alpha^F \hat{\nabla}_4 \Phi^F \rangle | \leq \epsilon \sup_u \|\nabla \alpha\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \|\nabla \alpha^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^2(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (379)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla \alpha, \varphi \nabla_4 \alpha \rangle | \leq \epsilon \sup_u \|\nabla \alpha\|_{L^2(H)} \|\nabla_4 \alpha\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}),$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_4 \alpha, \varphi \Phi^F \hat{\nabla}_4 \alpha^F \rangle | \leq \epsilon \sup_u \|\nabla_4 \alpha\|_{L^2(H)} \sup_u \|\hat{\nabla}_4 \alpha^F\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \|\hat{\nabla} \Phi^F\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (380)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_4 \alpha, \varphi \alpha^F \hat{\nabla}_4 \Phi^F \rangle | \leq \epsilon \sup_u \|\nabla_4 \alpha\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \|\nabla \alpha^F\|_{L^4(S)} \|\hat{\nabla} \Phi^F\|_{L^2(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (381)$$

where $\Psi : (\alpha, \beta, \rho, \sigma)$ and $\Phi^F := (\alpha^F, \rho^F, \sigma^F)$. We have also used the $\|\nabla_4 \varphi\|_{L^4(S)}$ estimate from corollary (2). Collecting all the terms, we obtain

$$|ER20| \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (382)$$

and therefore

$$\begin{aligned} & \int_{H_u} |\nabla_4 \alpha|^2 + 2 \int_{\bar{H}_{\bar{u}}} |\nabla_4 \beta|^2 \leq \int_{H_0} |\nabla_4 \alpha|^2 + 2 \int_{\bar{H}_0} |\nabla_4 \beta|^2 \\ & + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_4 \beta\|_{L^2(\bar{H})}^2 d\bar{u}' \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \end{aligned} \quad (383)$$

by means of Grönwall's inequality. Now we need to prove the second part of the lemma. One again we identify the pair $(\bar{\alpha}, \bar{\beta})$ and apply the integration identities (244-245) to yield

$$\begin{aligned} 2 \int_{H_u} |\nabla_3 \bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla_3 \bar{\alpha}|^2 &= 2 \int_{H_0} |\nabla_3 \bar{\beta}|^2 + \int_{\bar{H}_0} |\nabla_3 \bar{\alpha}|^2 + \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} 2|\nabla_3 \bar{\beta}|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u,\bar{u}}} |\nabla_3 \bar{\alpha}|^2 (2\omega - \frac{1}{2} tr \chi)}_{ER21} \\ &\quad - 2 \underbrace{\int_{\mathcal{D}_{u,\bar{u}}} (2\langle \nabla_3 \bar{\beta}, \nabla_3 \nabla_3 \bar{\beta} \rangle + \langle \nabla_3 \bar{\alpha}, \nabla_4 \nabla_3 \bar{\alpha} \rangle)}_{ER22}. \end{aligned}$$

We first control $ER21$. Notice that $\bar{\omega}$ does not satisfy a ∇_3 equation and as a result $\|\bar{\omega}\|_{L^\infty(S)} \leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F})$. Luckily $|\nabla_3 \bar{\beta}|^2$ is controlled on H and therefore the first term of $ER21$ can be estimated as

$\int_0^u C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \|\nabla_3 \bar{\beta}\|_{L^2(H)}^2 du'.$ For the second term, once again $|\nabla_3 \bar{\alpha}|^2$ is multiplied by ω and $tr\chi$ which are solely determined by the norm of the initial data $\mathcal{O}_0$. Therefore, we have

$$\begin{aligned} |ER21| &\leq C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \int_0^u \|\nabla_3 \bar{\beta}\|_{L^2(H)}^2 du' + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' \\ &\leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}^2 + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' \end{aligned} \quad (384)$$

Utilizing the null Bianchi equations, $ER22$ may be written as follows

$$\begin{aligned} ER22 &= -2 \int_{\mathcal{D}_{u, \bar{u}}} (2\langle \nabla_3 \bar{\beta}, \nabla_3 \nabla_3 \bar{\beta} \rangle + \langle \nabla_3 \bar{\alpha}, \nabla_4 \nabla_3 \bar{\alpha} \rangle) \\ &= -2 \int_{\mathcal{D}_{u, \bar{u}}} \left(2\langle \nabla_3 \bar{\beta}, -(\text{div} \nabla_3 \bar{\alpha}) - 2\nabla_3(tr\bar{\chi}\bar{\beta}) - 2\nabla_3(\bar{\omega}\bar{\beta}) + \nabla_3(\bar{\eta} \cdot \bar{\alpha}) + \frac{1}{2}\nabla_3(\mathcal{D}_a R_{33} - \mathcal{D}_3 R_{3a}) + [\text{div}, \nabla_3]\bar{\alpha} \rangle \right. \\ &\quad + \langle \nabla_3 \bar{\alpha}, -(\nabla \hat{\otimes} \nabla_3 \bar{\beta}) - \frac{1}{2}\nabla_3(tr\chi\bar{\alpha}) + 4\nabla_3(\omega\bar{\alpha}) - 3\nabla_3(\hat{\chi}\rho - {}^*\hat{\chi}\sigma) + \nabla_3((\zeta - 4\bar{\eta})\hat{\otimes}\bar{\beta}) \\ &\quad \left. + \frac{1}{2}\nabla_3(\mathcal{D}_4 R_{33} - \mathcal{D}_3 R_{34})\gamma + [\nabla_4, \nabla_3]\bar{\alpha} + [\nabla, \nabla_3]\hat{\otimes}\bar{\beta} \right) \\ &\sim \int_{\mathcal{D}_{u, \bar{u}}} \left(\langle \nabla_3 \bar{\beta}, -2\nabla_3(tr\bar{\chi}\bar{\beta}) - 2\nabla_3(\bar{\omega}\bar{\beta}) + \nabla_3(\bar{\eta} \cdot \bar{\alpha}) + \frac{1}{2}\nabla_3(\mathcal{D}_a R_{33} - \mathcal{D}_3 R_{3a}) + [\text{div}, \nabla_3]\bar{\alpha} \rangle \right. \\ &\quad + \langle \nabla_3 \bar{\alpha}, -\frac{1}{2}\nabla_3(tr\chi\bar{\alpha}) + 4\nabla_3(\omega\bar{\alpha}) - 3\nabla_3(\hat{\chi}\rho - {}^*\hat{\chi}\sigma) + \nabla_3((\zeta - 4\bar{\eta})\hat{\otimes}\bar{\beta}) \\ &\quad \left. + [\nabla_4, \nabla_3]\bar{\alpha} + [\nabla, \nabla_3]\hat{\otimes}\bar{\beta} \right) + \langle (\eta + \bar{\eta})\nabla_3 \bar{\beta}, \nabla_3 \bar{\alpha} \rangle, \end{aligned}$$

where $\langle \nabla_3 \bar{\alpha}, \frac{1}{2}\nabla_3(\mathcal{D}_4 R_{33} - \mathcal{D}_3 R_{44})\gamma \rangle = 0$ due to γ -trace-less property of $\bar{\alpha}$. Now we want to estimate each term separately. First note the commutators property

$$\begin{aligned} [\nabla_3, \nabla_4]\bar{\alpha} &\sim \sigma\bar{\alpha} + (\rho^F \rho^F + \sigma^F \sigma^F)\bar{\alpha} + \omega\nabla_3 \bar{\alpha} + \bar{\omega}\nabla_4 \bar{\alpha} + (\eta - \bar{\eta})\nabla \bar{\alpha}, \\ [\nabla_3, \nabla]\bar{\beta} &\sim (\bar{\beta} + \bar{\alpha}^F(\rho^F + \sigma^F))\bar{\beta} + (\eta + \bar{\eta})\nabla_3 \bar{\beta} - \bar{\chi}\nabla \bar{\beta} + \bar{\chi}\eta\bar{\beta}, \\ [\nabla_3, \nabla]\bar{\alpha} &\sim (\bar{\beta} + \bar{\alpha}^F(\rho^F + \sigma^F))\bar{\alpha} + (\eta + \bar{\eta})\nabla_4 \bar{\alpha} - \bar{\chi}\nabla \bar{\alpha} + \bar{\chi}\eta\bar{\alpha}. \end{aligned} \quad (385)$$

Also recall the null Bianchi equations that we shall make use of

$$\nabla_3 \bar{\beta}_a + 2tr\bar{\chi}\bar{\beta}_a = -(\text{div}\bar{\alpha})_a - 2\bar{\omega}\bar{\beta}_a + (\bar{\eta} \cdot \bar{\alpha})_a + \frac{1}{2}(\mathcal{D}_a R_{33} - \mathcal{D}_3 R_{3a}) \quad (386)$$

$$\nabla_4 \bar{\alpha}_{ab} + \frac{1}{2}tr\chi\bar{\alpha} = -(\nabla \hat{\otimes} \bar{\beta})_{ab} + 4\omega\bar{\alpha}_{ab} - 3(\hat{\chi}_{ab}\rho - {}^*\hat{\chi}_{ab}\sigma) + ((\zeta - 4\bar{\eta})\hat{\otimes}\bar{\beta})_{ab} + \frac{1}{2}(\mathcal{D}_4 R_{33} - \mathcal{D}_3 R_{34})\gamma_{ab}, \quad (387)$$

and

$$D_b R_{33} - D_3 R_{3b} \sim \langle \bar{\alpha}^F, \hat{\nabla} \bar{\alpha}^F \rangle - \bar{\chi}\bar{\alpha}^F \cdot (\rho^F + \sigma^F) + \bar{\eta}|\bar{\alpha}^F|^2 - 2\bar{\omega}\bar{\alpha}^F \cdot (\rho^F + \sigma^F) - \hat{\nabla}_3(\bar{\alpha}^F \cdot \rho^F + \bar{\alpha}^F \cdot \sigma^F).$$

Note an extremely important point that $\|\eta - \bar{\eta}\|_{L^\infty}$ from lemma is completely determined by $C(\mathcal{O}_0)$ since $\eta - \bar{\eta}$ satisfies a ∇_3 equation

$$\nabla_3(\eta - \bar{\eta}) \sim -\nabla\bar{\omega} - \bar{\chi} \cdot (\eta + \zeta) + \bar{\omega}(\zeta - \eta) - \frac{1}{2}\bar{\beta} + \frac{1}{2}\sigma^F \bar{\alpha}^F, \quad (388)$$

where $\zeta = \frac{1}{2}(\eta - \bar{\eta})$. After commuting with ∇ , a direct application of the transport inequality, and lemma (2-4) yields

$$\|\nabla(\eta - \bar{\eta})\|_{L^2(S)} \leq C(\mathcal{O}_0). \quad (389)$$

Application of the trace inequality yields

$$\begin{aligned}
\|\nabla(\eta - \bar{\eta})\|_{L^4(S)} &\leq C \left(\|\nabla(\eta - \bar{\eta})\|_{L^4(S_{0,\bar{u}})} + \|\nabla(\eta - \bar{\eta})\|_{L^2(\bar{H})}^{1/2} \|\hat{\nabla}_3 \nabla(\eta - \bar{\eta})\|_{L^2(\bar{H})}^{1/4} (\nabla\|\eta - \bar{\eta}\|_{L^2(\bar{H})} \right. \\
&\quad \left. + \|\nabla^2(\eta - \bar{\eta})\|_{L^2(\bar{H})})^{1/4} \right) \\
&\leq C(\mathcal{O}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}, \mathcal{W}(S), \mathcal{F}(S)) \\
&\leq C(\mathcal{O}_0)
\end{aligned}$$

after choosing sufficiently small ϵ . Now use the Sobolev inequality (89) to yield³

$$\|\eta - \bar{\eta}\|_{L^\infty(S)} \leq C(\mathcal{O}_0). \quad (390)$$

This is extremely important to handle the term $\int_{D_{u,\bar{u}}} \langle \nabla_3 \bar{\alpha}, (\eta - \bar{\eta}) \nabla \bar{\alpha} \rangle$ that arises as a result of commutation. The remaining terms can be estimated in a similar manner as that of the previous cases. We need only to take care of the terms where $\nabla_3 \bar{\alpha}$ appears in a quadratic way and keep track of the associated coefficients.

³So effectively in this setting $\|\eta\|_{L^\infty(S)} \leq C(\mathcal{O}_0)$ too. We do not really utilize that for clarity.

We estimate each term using lemma (2)-(7) as follows

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta}, (\nabla_3 \varphi) \bar{\beta} \rangle | \leq \epsilon \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}^2 du' + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (391)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta}, \varphi \nabla_3 \bar{\beta} \rangle | \leq \int_0^u \|\nabla_3 \bar{\beta}\|_{L^2(H)}^2 \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}^2, \quad (392)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta}, \nabla_3 \varphi \bar{\alpha} \rangle | \leq \epsilon \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}^2 + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (393)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta}, \varphi \nabla_3 \bar{\alpha} \rangle | \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}, \quad (394)$$

$$\begin{aligned} | \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta} \hat{\nabla}_3^2 \bar{\alpha}^F \Phi^F \rangle | &\leq \|\nabla_3 \bar{\beta}\|_{L^2(\mathcal{D}_{u,\bar{u}})} \|\hat{\nabla}_3^2 \bar{\alpha}^F\|_{L^2(\mathcal{D}_{u,\bar{u}})} \sup_{u,\bar{u}} \|\Phi^F\|_{L^\infty(S)} \\ &\leq \epsilon^{\frac{1}{2}} C(\mathcal{F}_0, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}, \end{aligned} \quad (395)$$

$$\begin{aligned} | \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta} \hat{\nabla}_3 \hat{\nabla}_3^F \bar{\alpha}^F \Phi^F \rangle | &\leq \|\nabla_3 \bar{\beta}\|_{L^2(\mathcal{D}_{u,\bar{u}})} \|\hat{\nabla}_3 \hat{\nabla}_3^F \bar{\alpha}^F\|_{L^2(\mathcal{D}_{u,\bar{u}})} \sup_{u,\bar{u}} \|\Phi^F\|_{L^\infty(S)} \\ &\leq \epsilon^{\frac{1}{2}} C(\mathcal{F}_0, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}, \end{aligned} \quad (396)$$

$$\begin{aligned} | \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta}, \varphi \hat{\nabla}_3 \bar{\alpha}^F \Phi^F \rangle | &\leq \epsilon \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \|\hat{\nabla}_3 \bar{\alpha}^F\|_{L^4(S)} \|\Phi^F\|_{L^4(S)} \\ &\leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)} \end{aligned}$$

$$\begin{aligned} | \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta}, \hat{\nabla}_3 \bar{\alpha}^F \hat{\nabla}_3 \Phi^F \rangle | &\leq \epsilon \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)} \sup_{u,\bar{u}} \|\hat{\nabla}_3 \bar{\alpha}^F\|_{L^4(S)} \|\hat{\nabla}_3 \Phi^F\|_{L^4(S)} \\ &\leq \epsilon C(\mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}, \end{aligned}$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\alpha}, tr \chi \nabla_3 \bar{\alpha} + \omega \nabla_3 \bar{\alpha} \rangle | \leq C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{\bar{H}}^2 d\bar{u}',$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\alpha} \varphi \nabla_3 \Psi \rangle | \leq \epsilon^{\frac{1}{2}} \sup_{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})} \sup_u \|\nabla \Psi\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (397)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\alpha}, \omega \nabla_3 \bar{\alpha} \rangle | \leq C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}', \quad (398)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\alpha}, (\eta - \bar{\eta}) \nabla \bar{\alpha} \rangle | \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) \int_0^{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}', \quad (399)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta}, \varphi \nabla_4 \bar{\alpha} \rangle | \leq \epsilon \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)} \|\nabla_4 \bar{\alpha}\|_{L^2(H)} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}, \quad (400)$$

$$\begin{aligned} | \int_{\mathcal{D}_{u,\bar{u}}} \langle \nabla_3 \bar{\beta}, \varphi \nabla \bar{\alpha} \rangle | &\leq \epsilon^{\frac{1}{2}} \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)} \sup_{\bar{u}} \|\nabla \bar{\alpha}\|_{L^2(\bar{H})} \sup_{u,\bar{u}} \|\varphi\|_{L^\infty(S)} \\ &\leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}, \end{aligned} \quad (401)$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \langle (\eta + \bar{\eta}) \nabla_3 \bar{\beta}, \nabla_3 \bar{\alpha} \rangle | \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}, \quad (402)$$

where we have utilized the available Bianchi equations and null Yang-Mills equations (e.g., we have $\nabla_4 \bar{\alpha}$ which does not contain derivatives of $\bar{\alpha}$ thereby allowing us to control it on H since we have $L^4(S)$ of $\bar{\alpha}$ under control). Once again, we observe that the connection coefficients ω and $tr \chi$ multiplying the most dangerous term $\langle \nabla_3 \bar{\alpha}, \nabla_3 \bar{\alpha} \rangle$ are determined completely by their initial value. Collecting all the terms, we

obtain

$$2 \int_{H_u} |\nabla_3 \bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla_3 \bar{\alpha}|^2 \leq 2 \int_{H_0} |\nabla_3 \bar{\beta}|^2 + \int_{\bar{H}_0} |\nabla_3 \bar{\alpha}|^2 + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)}^2 \\ + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\nabla_3 \bar{\beta}\|_{L^2(H)} + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})}^2 d\bar{u}' + C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) \int_0^{\bar{u}} \|\nabla_3 \bar{\alpha}\|_{L^2(\bar{H})} d\bar{u}'$$

which upon utilizing Grönwall's inequality and smallness of ϵ yields

$$2 \int_{H_u} |\nabla_3 \bar{\beta}|^2 + \int_{\bar{H}_{\bar{u}}} |\nabla_3 \bar{\alpha}|^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0). \quad (403)$$

This concludes the proof of the lemma. $\square$

Now we are left with estimating the similar estimates for the Yang-Mills curvature components. Since the proof goes exactly similar way as the Weyl curvature case, we will only sketch the proof.

Lemma 14: *The null derivatives of the null components of the Weyl curvature satisfy the following L^2 energy estimates*

$$\int_{H_u} |\hat{\nabla}_4^I \alpha^F|^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (404)$$

$$\int_{\bar{H}_{\bar{u}}} |\hat{\nabla}_3^I \bar{\alpha}^F|^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla}_3^I \bar{\alpha}^F\|_{L^2(\bar{H})} d\bar{u}, \quad (405)$$

$$\int_{H_u} |\hat{\nabla}_4 \hat{\nabla} \alpha^F|^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (406)$$

$$\int_{\bar{H}_{\bar{u}}} |\hat{\nabla}_3 \hat{\nabla} \bar{\alpha}^F|^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla}_3 \hat{\nabla} \bar{\alpha}^F\|_{L^2(\bar{H})} d\bar{u} \quad (407)$$

for $1 \leq I \leq 2, I \in \mathbb{Z}$.

Proof: We only sketch the proof for $I = 2$ since that is the most non-trivial case for the Yang-Mills fields. Once again we identify the triple $(\alpha^F, \rho^F, \sigma^F)$ and apply the integration identities (244-245)

$$\begin{aligned} \int_{H_u} |\hat{\nabla}_4^I \alpha^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}_4^I \rho^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}_4^I \sigma^F|^2 &= \int_{H_0} |\hat{\nabla}_4^I \alpha^F|^2 + \int_{\bar{H}_0} |\hat{\nabla}_4^I \rho^F|^2 + \int_{\bar{H}_0} |\hat{\nabla}_4^I \sigma^F|^2 \\ &+ \underbrace{\int_{\mathcal{D}_{u, \bar{u}}} |\hat{\nabla}_4^I \alpha^F|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u, \bar{u}}} |\hat{\nabla}_4^I \rho^F|^2 (2\omega - \frac{1}{2} tr \chi) + \int_{\mathcal{D}_{u, \bar{u}}} |\hat{\nabla}_4^I \sigma^F|^2 (2\omega - \frac{1}{2} tr \chi)}_{ER23} \\ &- 2 \underbrace{\int_{\mathcal{D}_{u, \bar{u}}} \left(\langle \hat{\nabla}_4^I \alpha^F, \hat{\nabla}_3 \hat{\nabla}_4^I \alpha^F \rangle + \langle \hat{\nabla}_4^I \rho^F, \hat{\nabla}_4 \hat{\nabla}_4^I \rho^F \rangle + \langle \hat{\nabla}_4^I \sigma^F, \hat{\nabla}_4 \hat{\nabla}_4^I \sigma^F \rangle \right)}_{ER24}. \end{aligned} \quad (408)$$

Using the estimates on the connection coefficients, the first error term ER_{23} is estimated easily

$$\begin{aligned} |ER23| &= \left| \int_{\mathcal{D}_{u, \bar{u}}} |\hat{\nabla}_4^I \alpha^F|^2 (2\bar{\omega} - \frac{1}{2} tr \bar{\chi}) + \int_{\mathcal{D}_{u, \bar{u}}} |\hat{\nabla}_4^I \rho^F|^2 (2\omega - \frac{1}{2} tr \chi) + \int_{\mathcal{D}_{u, \bar{u}}} |\hat{\nabla}_4^I \sigma^F|^2 (2\omega - \frac{1}{2} tr \chi) \right| \\ &\leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u \|\hat{\nabla}_4^I \alpha^F\|_{L^2(H)}^2 + C(\mathcal{O}_0) \|\hat{\nabla}_4^I \rho^F\|_{L^2(\bar{H})}^2 + C(\mathcal{O}_0) \|\hat{\nabla}_4^I \sigma^F\|_{L^2(\bar{H})}^2 \\ &\leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \|\hat{\nabla}_4^I \rho^F\|_{L^2(\bar{H})}^2 + C(\mathcal{O}_0) \|\hat{\nabla}_4^I \sigma^F\|_{L^2(\bar{H})}^2 \end{aligned} \quad (409)$$

In order to estimate ER_{24} , we utilize the null Yang-Mills equations and proceed in an exact similar way

as the previous case

$$\begin{aligned}
& \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla_4^I \alpha^F, -\frac{1}{2} \hat{\nabla}_4^I (tr \bar{\chi} \alpha^F) - \hat{\nabla} \hat{\nabla}_4^I \rho^F + {}^* \hat{\nabla} \hat{\nabla}_4^I \sigma^F - 2 \hat{\nabla}_4^I ({}^* \bar{\eta} \sigma^F) + 2 \hat{\nabla}_4^I (\bar{\eta} \rho^F) \right. \\
& \quad + 2 \hat{\nabla}_4^I (\bar{\omega} \alpha^F) - \hat{\nabla}_4^I (\hat{\chi} \cdot \bar{\alpha}^F) + [\hat{\nabla}_3, \hat{\nabla}_4^I] \alpha^F + [\hat{\nabla}, \hat{\nabla}_4^I] \rho^F - [{}^* \hat{\nabla}, \hat{\nabla}_4^I] \sigma^F \rangle \\
& \quad + \langle \hat{\nabla}_4^I \rho^F, -\hat{div} \hat{\nabla}_4^I \alpha^F - \hat{\nabla}_4^I (tr \chi \rho^F) - \hat{\nabla}_4^I ((\eta - \bar{\eta}) \cdot \alpha^F) + [\hat{\nabla}_4, \hat{\nabla}_4^I] \rho^F + [\hat{div}, \hat{\nabla}_4^I] \alpha^F \rangle \\
& \quad + \langle \hat{\nabla}_4^I \sigma^F, -\hat{curl} \hat{\nabla}_4^I \alpha^F - \hat{\nabla}_4^I (tr \chi \sigma^F) + \hat{\nabla}_4^I ((\eta - \bar{\eta}) \cdot {}^* \alpha^F) + [\hat{\nabla}_4, \hat{\nabla}_4^I] \sigma^F - [\hat{curl}, \hat{\nabla}_4^I] \alpha^F \rangle \Big) \\
& \quad \sim \int_{\mathcal{D}_{u,\bar{u}}} \left(\langle \nabla_4^I \alpha^F, -\frac{1}{2} \hat{\nabla}_4^I (tr \bar{\chi} \alpha^F) - 2 \hat{\nabla}_4^I ({}^* \bar{\eta} \sigma^F) + 2 \hat{\nabla}_4^I (\bar{\eta} \rho^F) \right. \\
& \quad + 2 \hat{\nabla}_4^I (\bar{\omega} \alpha^F) - \hat{\nabla}_4^I (\hat{\chi} \cdot \bar{\alpha}^F) + [\hat{\nabla}_3, \hat{\nabla}_4^I] \alpha^F + [\hat{\nabla}, \hat{\nabla}_4^I] \rho^F - [{}^* \hat{\nabla}, \hat{\nabla}_4^I] \sigma^F \rangle \\
& \quad + \langle \hat{\nabla}_4^I \rho^F, -\hat{\nabla}_4^I (tr \chi \rho^F) - \hat{\nabla}_4^I ((\eta - \bar{\eta}) \cdot \alpha^F) + [\hat{\nabla}_4, \hat{\nabla}_4^I] \rho^F + [\hat{div}, \hat{\nabla}_4^I] \alpha^F \rangle \\
& \quad + \langle \hat{\nabla}_4^I \sigma^F, -\hat{\nabla}_4^I (tr \chi \sigma^F) + \hat{\nabla}_4^I ((\eta - \bar{\eta}) \cdot {}^* \alpha^F) + [\hat{\nabla}_4, \hat{\nabla}_4^I] \sigma^F - [\hat{curl}, \hat{\nabla}_4^I] \alpha^F \rangle \Big) \\
& \quad + \int_{\mathcal{D}_{u,\bar{u}}} \langle \hat{\nabla}_4^I \alpha^F, ((\eta + \bar{\eta})(\hat{\nabla}_4^I \rho^F + \hat{\nabla}_4^I \sigma^F)) \rangle.
\end{aligned} \tag{410}$$

Now Consider the equation for α^F after commuting $\hat{\nabla}_4$ once

$$\begin{aligned}
\hat{\nabla}_3 \hat{\nabla}_4 \alpha^F &= -\frac{1}{2} (\nabla_4 (tr \bar{\chi}) \alpha^F + tr \bar{\chi} \hat{\nabla}_4 \alpha^F) - \hat{\nabla} \hat{\nabla}_4 \rho^F + {}^* \hat{\nabla} \hat{\nabla}_4 \sigma^F - 2 {}^* \nabla_4 \eta \sigma^F \\
&\quad - 2 \eta \hat{\nabla}_4 \sigma^F + 2 \nabla_4 \eta \rho^F + 2 \eta \hat{\nabla}_4 \rho^F + 2 \hat{\nabla}_4 \bar{\omega} \alpha^F + 2 \bar{\omega} \hat{\nabla}_4 \alpha^F - \hat{\nabla}_4 \hat{\chi} \cdot \bar{\alpha}^F - \hat{\chi} \cdot \hat{\nabla}_4 \bar{\alpha}^F \\
&= -\frac{1}{2} \left(-\frac{1}{2} tr \chi tr \bar{\chi} + 2 \omega tr \bar{\chi} + \underbrace{2 div \bar{\eta}}_I + 2 |\bar{\eta}|_\gamma^2 + 2 \rho - \hat{\chi} \cdot \hat{\chi} \right) \alpha^F - \frac{1}{2} tr \bar{\chi} \hat{\nabla}_4 \alpha^F \\
&\quad - \underbrace{\hat{\nabla} \hat{\nabla}_4 \rho^F + {}^* \hat{\nabla} \hat{\nabla}_4 \sigma^F}_I - 2 {}^* (-\chi \cdot (\eta - \bar{\eta}) - \beta - \frac{1}{2} \mathfrak{T}(\cdot, e_4)) \sigma^F \\
&\quad - 2 \eta (-\hat{curl} \alpha^F - tr \chi \sigma^F + (\eta - \bar{\eta}) \cdot {}^* \alpha^F) + 2 (-\chi \cdot (\eta - \bar{\eta}) - \beta^W - \frac{1}{2} \mathfrak{T}(\cdot, e_4)) \rho^F \\
&\quad + 2 \eta (-\hat{div} \alpha^F - tr \chi \rho^F - (\eta - \bar{\eta}) \cdot \alpha^F) \\
&\quad + 2 (2 \omega \bar{\omega} + \frac{3}{4} |\eta - \bar{\eta}|^2 - \frac{1}{4} (\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) - \frac{1}{8} |\eta + \bar{\eta}|^2 + \frac{1}{2} \rho + \frac{1}{4} \mathfrak{T}_{43}) \alpha^F \\
&\quad + 2 \bar{\omega} \hat{\nabla}_4 \alpha^F - (-tr \chi \hat{\chi} - 2 \omega \hat{\chi} - \alpha) \\
&\quad - \hat{\chi} \cdot (-\frac{1}{2} tr \chi \bar{\alpha}^F - \hat{\nabla} \rho^F - {}^* \hat{\nabla} \sigma^F - 2 {}^* \bar{\eta} \cdot \sigma^F - 2 \bar{\eta} \cdot \rho^F + 2 \omega \bar{\alpha}^F - \hat{\chi} \cdot \alpha^F) \\
&\quad + [\hat{\nabla}_3, \hat{\nabla}_4] \alpha^F - [\hat{\nabla}_4, \hat{\nabla}] \rho^F + [\hat{\nabla}_4, {}^* \hat{\nabla}] \sigma^F.
\end{aligned}$$

Now notice the structure of the previous equation. the principal terms denoted by I are preserved after we take another $\hat{\nabla}_4$ derivative and therefore are cancelled after integrating by parts. Since $\nabla_4^I \alpha^F$ is controlled over H , we can always gain a factor of ϵ . The most problematic terms in the above expression after taking another $\hat{\nabla}_4$ derivative are

$$\nabla_4 \bar{\eta}, \nabla_4 \omega, \nabla_4 \nabla \bar{\eta}. \tag{411}$$

We control $\nabla_4 \bar{\eta}$ and $\nabla_4 \omega$ by their $L^4(S)$ norms. $\nabla_4 \nabla \bar{\eta}$ can be controlled by its $\|\nabla_4 \nabla \bar{\eta}\|_{L^2(\bar{H})}$ norm. Since $\nabla_4 \nabla \bar{\eta}$ is already of top order it appears with innocuous terms that may be estimated as follows

$$| \int_{\mathcal{D}_{u,\bar{u}}} \nabla_4^I \alpha^F \nabla_4 \nabla \bar{\eta} \alpha^F | \leq \epsilon^{\frac{1}{2}} \sup_u \|\nabla_4^I \alpha^F\|_{L^2(H)} \sup_{\bar{u}} \|\nabla_4 \nabla \bar{\eta}\|_{L^2(\bar{H})} \sup_{u,\bar{u}} \|\alpha^F\|_{L^4(S)} \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \tag{412}$$

$$| \int_{\mathcal{D}_{u,\bar{u}}} \nabla_4^I \alpha^F \nabla_4 \varphi \hat{\nabla}_4 \alpha^F | \leq \epsilon \sup_u \|\nabla_4^I \alpha^F\|_{L^2(H)} \sup_{u,\bar{u}} \|\hat{\nabla}_4 \alpha^F\|_{L^4(S)} \|\nabla_4 \varphi\|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \tag{413}$$

where we have utilized the lemma (7) to control the $\nabla_4 \varphi$ in $L^4(S)$. Similar terms arise from the equations for $\hat{\nabla}_4^2(\rho^F, \sigma^F)$ where most problematic terms are of the above type. Since these estimates involve α^F , we will always gain a factor of ϵ . Collecting all the terms, we may obtain

$$\begin{aligned} \int_{H_u} |\hat{\nabla}_4^I \alpha^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}_4^I \rho^F|^2 + \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}_4^I \sigma^F|^2 &\leq \int_{H_0} |\hat{\nabla}_4^I \alpha^F|^2 + \int_{\bar{H}_0} |\hat{\nabla}_4^I \rho^F|^2 + \int_{\bar{H}_0} |\hat{\nabla}_4^I \sigma^F|^2 \\ &+ \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \|\hat{\nabla}_4^I \rho^F\|_{L^2(\bar{H})}^2 + C(\mathcal{O}_0) \|\hat{\nabla}_4^I \sigma^F\|_{L^2(\bar{H})}^2 \end{aligned} \quad (414)$$

which yields

$$\int_{H_u} |\hat{\nabla}_4^I \alpha^F|^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \quad (415)$$

through Grönwall estimate and smallness of ϵ . Exact same calculations but commuting the e.o.m with $\hat{\nabla}$ in the second time yield the estimate for $\int_{H_u} |\hat{\nabla}_4 \hat{\nabla} \alpha^F|^2$. This completes the first part of the lemma. The second part is proved in a similar way. Now we have pay attention to the terms that are associated with the top derivatives of $\bar{\alpha}^F$. Application of the integration identities for the triple $(\bar{\alpha}^F, \rho^F, \sigma^F)$ yields

$$\begin{aligned} \int_{\bar{H}_{\bar{u}}} |\hat{\nabla}_3^I \bar{\alpha}^F|^2 + \int_{H_u} |\hat{\nabla}_3^I \rho^F|^2 + \int_{H_u} |\hat{\nabla}_3^I \sigma^F|^2 &= \int_{\bar{H}_0} |\hat{\nabla}_3^I \bar{\alpha}^F|^2 + \int_{H_0} |\hat{\nabla}_3^I \rho^F|^2 + \int_{H_0} |\hat{\nabla}_3^I \sigma^F|^2 \\ &+ \underbrace{\int_{\mathcal{D}_{u, \bar{u}}} |\hat{\nabla}_3^I \bar{\alpha}^F|^2 (2\omega - \frac{1}{2} \text{tr} \chi) + \int_{\mathcal{D}_{u, \bar{u}}} |\hat{\nabla}_3^I \rho^F|^2 (2\bar{\omega} - \frac{1}{2} \text{tr} \bar{\chi})}_{ER25} \\ &- 2 \underbrace{\int_{\mathcal{D}_{u, \bar{u}}} \left(\langle \hat{\nabla}_3^I \bar{\alpha}^F, \hat{\nabla}_4 \hat{\nabla}_3^I \bar{\alpha}^F \rangle + \langle \hat{\nabla}_3^I \rho^F, \hat{\nabla}_3 \hat{\nabla}_3^I \rho^F \rangle + \langle \hat{\nabla}_3^I \sigma^F, \hat{\nabla}_3 \hat{\nabla}_3^I \sigma^F \rangle \right)}_{ER26}. \end{aligned} \quad (416)$$

As usual notice that the connection coefficients multiplying $|\hat{\nabla}_3^I \bar{\alpha}^F|^2$ satisfy ∇_3 equations and there are completely determined by their initial data. Therefore we obtain

$$|ER25| \leq C(\mathcal{O}_0) \int_0^{\bar{u}} \|\hat{\nabla}_3^I \bar{\alpha}^F\|_{L^2(\bar{H})}^2 + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \sup_u (\|\hat{\nabla}_3^I \rho^F\|_{L^2(H)} + \|\hat{\nabla}_3^I \sigma^F\|_{L^2(H)}). \quad (417)$$

For $ER26$ we utilize the evolution equations

$$\begin{aligned} &\int_{\mathcal{D}_{u, \bar{u}}} \left(\langle \hat{\nabla}_3^I \bar{\alpha}^F, \hat{\nabla}_4 \hat{\nabla}_3^I \bar{\alpha}^F \rangle + \langle \hat{\nabla}_3^I \rho^F, \hat{\nabla}_3 \hat{\nabla}_3^I \rho^F \rangle + \langle \hat{\nabla}_3^I \sigma^F, \hat{\nabla}_3 \hat{\nabla}_3^I \sigma^F \rangle \right) \\ &\sim \int_{\mathcal{D}_{u, \bar{u}}} \left(\langle \hat{\nabla}_3^I \bar{\alpha}^F, -\frac{1}{2} \hat{\nabla}_3^I (\text{tr} \chi \bar{\alpha}^F) - 2 \hat{\nabla}_3^I (*\bar{\eta} \cdot \sigma^F) - 2 \hat{\nabla}_3^I (\bar{\eta} \cdot \rho^F) + 2 \hat{\nabla}_3^I (\omega \bar{\alpha}^F) - \hat{\nabla}_3^I (\hat{\chi} \cdot \alpha^F) \rangle \right. \\ &\quad \left. + \langle \hat{\nabla}_3^I \rho^F, \hat{\nabla}_3^I (\text{tr} \bar{\chi} \rho^F) + \hat{\nabla}_3^I ((\eta - \bar{\eta}) \cdot \bar{\alpha}^F) \rangle + \langle \hat{\nabla}_3^I \sigma^F, -\hat{\nabla}^I (\text{tr} \bar{\chi} \sigma^F) + \hat{\nabla}_3^I ((\eta - \bar{\eta}) \cdot * \bar{\alpha}^F) \rangle \right) \\ &\quad + \int_{\mathcal{D}_{u, \bar{u}}} \langle \hat{\nabla}_3^I \bar{\alpha}^F, [\hat{\nabla}_4, \hat{\nabla}_3^I] \bar{\alpha}^F \rangle + \int_{\mathcal{D}_{u, \bar{u}}} \langle \hat{\nabla}_3^I \bar{\alpha}^F, [\hat{\nabla}, \hat{\nabla}_3^I] (\rho^F, \sigma^F) \rangle + \int_{\mathcal{D}_{u, \bar{u}}} \langle \hat{\nabla}_3^I \rho^F, [\hat{\nabla}_3, \hat{\nabla}_3^I] \rho^F \rangle \\ &+ \int_{\mathcal{D}_{u, \bar{u}}} \langle \hat{\nabla}_3^I \rho^F, [\hat{\nabla}, \hat{\nabla}^I] \bar{\alpha}^F \rangle + \int_{\mathcal{D}_{u, \bar{u}}} \langle \hat{\nabla}_3^I \sigma^F, [\hat{\nabla}, \hat{\nabla}_3^I] \bar{\alpha}^F \rangle + \int_{\mathcal{D}_{u, \bar{u}}} \langle \hat{\nabla}_3^I \alpha^F, ((\eta + \bar{\eta}) (\hat{\nabla}_3^I \rho^F + \hat{\nabla}_3^I \sigma^F)) \rangle. \end{aligned} \quad (418)$$

Now we notice some of the key features of the commuted equations. Consider the $\hat{\nabla}_3$ commuted equation

for $\bar{\alpha}^F$

$$\begin{aligned}
\hat{\nabla}_4 \hat{\nabla}_3 \bar{\alpha}^F &= -\frac{1}{2} \left(-\frac{1}{2} \text{tr} \bar{\chi} \text{tr} \chi + 2\bar{\omega} \text{tr} \chi + 2 \text{div} \eta + 2|\eta|^2 + 2\rho - \hat{\chi} \cdot \hat{\chi} \right) \bar{\alpha}^F \\
&\quad - \frac{1}{2} \text{tr} \chi \nabla_3 \bar{\alpha}^F - \hat{\nabla} \hat{\nabla}_3 \rho^F - {}^* \hat{\nabla} \hat{\nabla}_3 \sigma^F - 2 {}^* \hat{\nabla}_3 \bar{\eta} \cdot \sigma^F - 2 {}^* \bar{\eta} \cdot \hat{\nabla}_3 \sigma^F - 2 \hat{\nabla}_3 \bar{\eta} \cdot \rho^F \\
&\quad - 2 \bar{\eta} \cdot \hat{\nabla}_3 \rho^F + 2 \hat{\nabla}_3 \omega \bar{\alpha}^F + 2 \omega \hat{\nabla}_3 \bar{\alpha}^F - \hat{\nabla}_3 \hat{\chi} \cdot \alpha^F - \hat{\chi} \cdot \hat{\nabla}_3 \alpha^F \\
&= -\frac{1}{2} \left(-\frac{1}{2} \text{tr} \bar{\chi} \text{tr} \chi + 2\bar{\omega} \text{tr} \chi + 2 \text{div} \eta + 2|\eta|^2 + 2\rho - \hat{\chi} \cdot \hat{\chi} \right) \bar{\alpha}^F \\
&\quad - \frac{1}{2} \text{tr} \chi \nabla_3 \bar{\alpha}^F - \hat{\nabla} \hat{\nabla}_3 \rho^F - {}^* \hat{\nabla} \hat{\nabla}_3 \sigma^F - 2 {}^* (-\bar{\chi} \cdot (\bar{\eta} - \eta) + \bar{\beta}^W + \frac{1}{2} \mathfrak{T}(\cdot, e_3)) \cdot \sigma^F \\
&\quad - 2 {}^* \bar{\eta} \cdot (-\hat{c} \text{url} \bar{\alpha}^F - \text{tr} \bar{\chi} \sigma^F + (\eta - \bar{\eta}) \cdot {}^* \bar{\alpha}^F) - 2 (-\bar{\chi} \cdot (\bar{\eta} - \eta) + \bar{\beta}^W + \frac{1}{2} \mathfrak{T}(\cdot, e_3)) \cdot \rho^F \\
&\quad - 2 \bar{\eta} \cdot (-\hat{d} \text{iv} \bar{\alpha}^F + \text{tr} \bar{\chi} \rho^F + (\eta - \bar{\eta}) \cdot \bar{\alpha}^F) \\
&\quad + 2(2\omega \bar{\omega} + \frac{3}{4} |\eta - \bar{\eta}|^2 + \frac{1}{4} (\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) - \frac{1}{8} |\eta + \bar{\eta}|^2 + \frac{1}{2} \rho^W + \frac{1}{4} \mathfrak{T}_{43}) \bar{\alpha}^F \\
&\quad + 2\omega \hat{\nabla}_3 \bar{\alpha}^F - (-\text{tr} \bar{\chi} \hat{\chi} - 2\bar{\omega} \hat{\chi} - \bar{\alpha}^W) \cdot \alpha^F \\
&\quad - \hat{\chi} \cdot (-\frac{1}{2} \text{tr} \bar{\chi} \alpha^F - \hat{\nabla} \rho^F + {}^* \hat{D} \sigma^F - 2 {}^* \eta \sigma^F + 2\eta \rho^F + 2\bar{\omega} \alpha^F - \hat{\chi} \cdot \bar{\alpha}^F) \\
&\quad + [\hat{\nabla}_4, \hat{\nabla}_3] \bar{\alpha}^F + [\hat{\nabla}, \hat{\nabla}_3] \rho^F + [\hat{\nabla}, \hat{\nabla}_3] \sigma^F.
\end{aligned}$$

Similar to the previous case, after another application of ∇_3 produces $\nabla \nabla_3 \eta$. But since this is at the level of top order derivative it contains $\hat{\nabla}^I \bar{\alpha}^F$ in addition to an algebraic term $\bar{\alpha}^F$. Now we may control $\bar{\alpha}^F$ in $L^4(S)$ and $\nabla \nabla_3 \eta$ in $L^2(H)$. This way we gain a factor of ϵ . Similarly, we control $\nabla_3 \eta$ and $\nabla_3 \bar{\omega}$ in $L^4(S)$ using lemma (7). The most dangerous terms are estimated as follows

$$| \int_{\mathcal{D}_{u, \bar{u}}} \langle \hat{\nabla}_3^I \bar{\alpha}^F, (\omega, \text{tr} \chi) \hat{\nabla}_3^I \bar{\alpha}^F \rangle | \leq C(\mathcal{O}_0) \int_0^{\bar{u}} \| \hat{\nabla}_3^I \bar{\alpha}^F \|_{L^2(\bar{H})}^2, \quad (420)$$

$$| \int_{\mathcal{D}_{u, \bar{u}}} \nabla_3^I \bar{\alpha}^F \nabla \nabla_3 \eta \bar{\alpha}^F | \leq \epsilon^{\frac{1}{2}} \sup_u \| \nabla_3^I \bar{\alpha}^F \|_{L^2(\bar{H})} \sup_{\bar{u}} \| \nabla \nabla_3 \eta \|_{L^2(H)} \sup_{u, \bar{u}} \| \bar{\alpha}^F \|_{L^4(S)} \leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (421)$$

$$| \int_{\mathcal{D}_{u, \bar{u}}} \nabla_3^I \bar{\alpha}^F \nabla_3 \varphi \hat{\nabla}_3 \bar{\alpha}^F | \leq \epsilon \sup_u \| \nabla_3^I \bar{\alpha}^F \|_{L^2(\bar{H})} \sup_{u, \bar{u}} \| \hat{\nabla}_3 \bar{\alpha}^F \|_{L^4(S)} \| \nabla_3 \varphi \|_{L^4(S)} \leq \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}), \quad (422)$$

where φ denotes the connection coefficients. Collecting all the terms, we obtain

$$\begin{aligned}
|ER26| &\leq \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \| \hat{\nabla}_3^I \bar{\alpha}^F \|_{L^2(\bar{H})}^2 + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \| \hat{\nabla}_3^I \rho^F \|_{L^2(H)} \\
&\quad + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \| \hat{\nabla}_3^I \sigma^F \|_{L^2(H)}
\end{aligned} \quad (423)$$

and therefore

$$\begin{aligned}
&\int_{\bar{H}_{\bar{u}}} | \hat{\nabla}_3^I \bar{\alpha}^F |^2 + \int_{H_u} | \hat{\nabla}_3^I \rho^F |^2 + \int_{H_u} | \hat{\nabla}_3^I \sigma^F |^2 \leq \int_{\bar{H}_0} | \hat{\nabla}_3^I \bar{\alpha}^F |^2 + \int_{H_0} | \hat{\nabla}_3^I \rho^F |^2 + \int_{H_0} | \hat{\nabla}_3^I \sigma^F |^2 \\
&\leq C(\mathcal{O}_0) \int_0^{\bar{u}} \| \hat{\nabla}_3^I \bar{\alpha}^F \|_{L^2(\bar{H})}^2 d\bar{u}' + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \| \hat{\nabla}_3^I \rho^F \|_{L^2(H)} + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \| \hat{\nabla}_3^I \sigma^F \|_{L^2(H)} \\
&\quad + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \| \hat{\nabla}_3^I \rho^F \|_{L^2(H)} \\
&\quad + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) \| \hat{\nabla}_3^I \sigma^F \|_{L^2(H)}.
\end{aligned} \quad (424)$$

Smallness of ϵ yields

$$\int_{\bar{H}_{\bar{u}}} | \hat{\nabla}_3^I \bar{\alpha}^F |^2 \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0) + \epsilon^{\frac{1}{2}} C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + \epsilon C(\mathcal{O}_0, \mathcal{W}, \mathcal{F}) + C(\mathcal{O}_0) \int_0^{\bar{u}} \| \hat{\nabla}_3^I \bar{\alpha}^F \|_{L^2(\bar{H})} d\bar{u} \quad (425)$$

Exact same procedure but commuting the e.o.m with $\hat{\nabla}$ in the second time yields the estimate for $\int_{\bar{H}_{\bar{u}}} |\hat{\nabla}_3 \hat{\nabla} \bar{\alpha}^F|^2$. This concludes the proof of the lemma and the energy estimates associated to the Weyl and Yang-Mills curvature. $\square$

Corollary: $\mathcal{W} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0)$, $\mathcal{F} \leq C(\mathcal{O}_0, \mathcal{W}_0, \mathcal{F}_0)$.

Proof: A direct consequence of lemma (10)- lemma (14), Grönwall's inequality and the fact that $u \in [0, \epsilon]$ and $\bar{u} \in [0, J]$.

This concludes the proof of the estimates for the Weyl and Yang-Mills curvature throughout $\mathcal{D}_{u, \bar{u}}$ in terms of the initial data. Once these estimates are derived, we can choose the bootstrap constant Δ (84) large enough (but finite) to close the argument. Using these gauge-invariant estimates and standard arguments for the Cauchy problem up to a suitably defined 'time' $\tau = u + \bar{u} = J + \epsilon$, one can prove the existence of a solution to the Einstein-Yang-Mills equations in the slab $\mathcal{D}_{u, \bar{u}}$ through a continuity argument. We omit such detail since they are standard (see section 6 of [8] for example for vacuum case, Yang-Mills equations in temporal gauge are symmetric hyperbolic and so follows in a similar way).

8 Concluding Remarks

Here we have obtained 'semi-global' estimates for the coupled Einstein-Yang-Mills equations in a gauge-invariant way (in the sense of Yang-Mills gauge theory). The null structure of the Einstein-Yang-Mills equations plays a crucial role in achieving this result. This lays the platform to study several problems associated with the coupled Einstein-Yang-Mills dynamics in the immediate future. As we have mentioned, the double null framework is appropriate for the radiation problem associated with gravity or/and Yang-Mills (they both have the same characteristics) equations. Since the Yang-Mills equations are themselves non-linear, they can counterbalance the gravity on occasions (the case of regular soliton-like solutions). As a consequence, all three possibilities (black hole solution, regular solutions, and naked singular solutions) are open in the context of an evolution problem. Therefore it would be interesting to obtain sharp criteria demarcating all three regimes.

Another interesting perspective would be to study the Wang-Yau quasi-local energy [27, 28] contained in the space-like domain bounded by the membrane $S_{u, \bar{u}}$. Ultimately, the formation of singularities is associated with the focusing of energy and since the dynamics of the topological 2- spheres $S_{u, \bar{u}}$ is one of the central parts of the analysis of singularity formation, it is only natural to understand the evolution of the energy contained within it. Therefore, we need a definition of energy in a fully relativistic setting. Luckily as we have mentioned before, [27, 28] constructed a notion of quasi-local energy associated with a topological 2- sphere $S_{u, \bar{u}}$. It was proven in [29] that such quasi-local energy while evolving along the incoming null direction reproduces the Bel-Robinson energy and the matter stress-energy tensor (at different orders of course) at the limit of approaching vertex. Motivated by this result, it is only natural to study the evolution of this quasi-local energy in the outgoing null direction and observe the behavior in the focusing regime (note an expression of the quasi-local energy was obtained in [30] in the presence of a gauge field). We expect an alternate notion of trapped surface formation through the study of this quasi-local energy.

Lastly, we want to mention the fact that our estimates associated with the Yang-Mills curvature components are completely gauge-invariant. In particular, we obtain estimates for the fully gauge covariant angular derivatives. This essentially hints at an apparent similarity with the Maxwell and Yang-Mills theory despite the fact that the latter is a fully non-linear theory because the gauge covariant derivative *hides* the information of the connection and the non-linear coupling shows up only as the commutator of the fully gauge covariant derivatives. This does not cause a problem in the context of obtaining estimates since all the associated inequalities are formulated in terms of the gauge covariant derivatives (and the estimates required for a local existence theory for coupled Yang-Mills equations can be obtained in a gauge-invariant way; note that at the end one ought to choose a gauge and work with the equation for connection. However our proffered choice of gauge is temporal gauge where the Yang-Mills equations

takes the form of a symmetric hyperbolic system and the spatial connection can be determined in terms of the gauge invariant norms of the Yang-Mills curvature). There is of course a physical motivation behind this. Since the double null framework in some sense encodes the information about the *physical* nature of the Yang-Mills fields, the choice of the gauge should not matter and one should expect to obtain gauge-invariant estimates as is done in the current context. In this framework of the gauge-invariant estimates, therefore, the exterior stability of the Minkowski space under coupled gravity-Yang-Mills perturbations is expected to hold. In other words, we conjecture “*Exterior stability of the Minkowski space holds under the coupled gravity-Yang-Mills perturbations*”. Since we have developed the framework in this article, we want to pursue such stability conjecture in near future.

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