BREAKDOWN/CONTINUATION CRITERIA FOR VACUUM GRAVITY VIA LIGHT CONE ESTIMATES

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Abstract

We provide an improved extension (or breakdown) criteria for a globally hyperbolic vacuum space-time in 1+3 dimensions by showing that the $L_t^\infty L_{\vec{x}}^\infty$ norm of the Riemann curvature remains bounded in the future provided the deformation tensor of the unit timelike vector field normal to the chosen CMC Cauchy hypersurfaces verifies a $L_t^\infty L_{\vec{x}}^\infty$ bound. Specifically, we adopt a gauge-gravity formalism to examine the light cone dynamics. Under the assumption of the deformation bound, Bel-Robinson and higher order energy estimates together with an integral equation for the space-time curvature are employed to control the space-time point-wise norm of the curvature in a bootstrap manner. This provides an alternate proof of the result obtained by [13].

1 Introduction

Given regular initial data for gravity, it is of mathematical and physical interest to obtain analytic criteria that codify whether the gravitational field will evolve to a unique singularity-free (naked) global solution to the Einstein equations. The appeal for mathematicians is obvious as there is a plethora of literature studying the breakdown of solutions to non-linear field equations. A few examples where global existence holds in 1+3 Minkowski space are: the nonlinear wave equation $\Box\varphi = \lambda|\varphi|^{p-1}\varphi$ for $1 \leq p \leq 4$ [1, 3, 4, 5], sine-Gordon equation [2] non-Abelian Yang-Mills-Higgs for a fixed choice of compact gauge group and specific restrictions on the Higgs potential [6, 7]. Yang-Mills fields on a globally hyperbolic background are also known to exhibit a non-blow characteristic [16, 20]. In contrast, some equations that may have finite time blow-ups in 1+3 dimensions include: wave maps (also known as non-linear sigma models) [9], and relativistic perfect fluids [10]. Any physically acceptable classical field is desired to be globally well-posed on any fixed globally hyperbolic space-time.

From the physics perspective, the breakdown/continuation of solutions to Einstein's equations is essential to (dis)validate the deterministic essence of classical general relativity theory. For if the maximal Cauchy development of regular initial data were to not be regular (in a suitable sense), then the future cannot be fully predicted even with perfect knowledge about the present (therefore a loss of information occurs). A worse situation would be if the evolved space-time contains so-called

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naked singularities which in principle should be observable by a timelike observer located in the future and thus pathological. These irregularities such as naked singularities and Cauchy horizons are hypothesized to be absent from nature (or exist as purely mathematical objects which are to be unstable against perturbations) as stated in Penrose's cosmic censorship conjecture [11]. Nevertheless, this problem is rather complicated to tackle if no suitable assumptions are made regarding function space norms and the initial data.

The purpose of this paper is to arrive at a continuation criteria of solutions to the vacuum equations, namely

$$\mathrm{Ric}_{\widehat{g}} = 0 \tag{1}$$

which is assumed to be satisfied everywhere in a time orientable globally hyperbolic space-time endowed with a metric tensor $\widehat{g}$ of Lorentzian signature. Our analysis relies on a gauge-gravity correspondence that makes use of an orthonormal frame $\mathrm{SO}(1,3)$-bundle with Lie algebra-valued Cartan connection 1-form $\widetilde{\omega}$ [17]. These tools are unnecessary in typical treatments of GR, however they are not additional data as one can construct the bundle from tangent structures and the metric. As for the Cartan connection, it is equivalent to a choice of horizontal distribution meeting certain properties.

Previous accounts on breakdown criteria for vacuum solutions used more systematic approaches, at least from a geometric analysis standpoint. Anderson [12] showed that a breakdown occurs when the $L^\infty_t L^\infty_{\vec{x}}$ norm of the Riemannian curvature $R_{\widehat{g}}$ of the space-time $(M, \widehat{g})$ blows up. An improved breakdown criteria requiring one less degree of differentiability in the metric $\widehat{g}$ was later found by Klainerman and Rodnianski [13], namely the $L^\infty_t L^\infty_{\vec{x}}$ blow-up of the deformation tensor $\pi := \mathcal{L}_n \widehat{g}$ of the unit timelike vector field n normal to a CMC foliation of $\mathcal{M}_* \subset M$ (here $\mathcal{L}$ is the Lie derivative). This in fact did not require all derivatives of the metric (that would be non-geometrical) but rather certain components describing the extrinsic geometry of the chosen Cauchy hypersurfaces. The work in [13] was extended by Shao [14] to apply for Einstein-scalar and Einstein-Maxwell space-times. A further improvement due to Wang [22] required the $L^1_t L^\infty_{\vec{x}}$ norm of π to be bounded for continuation of the vacuum CMC foliation. Our treatment attempts to shed light on the advantages of considering gauge-gravity duality by means of the frame bundle/Cartan formalism and a null framework. In particular, these structures combined make a lot of the physics much more transparent, all while condensing the main body of calculations and thereby making this a digestible read. In particular, our approach differs substantially from that of the previous ones.

Einstein's equations are quasi-linear hyperbolic PDEs while expressed in a suitable gauge (e.g., space-time harmonic gauge [23] or constant mean curvature spatial harmonic gauge [33]). In the case of such a non-linear hyperbolic PDE, long-time existence or finite time blow-up is essentially determined by the relative strengths of the non-linearities and the geometric dispersion associated with the wave chracteristics (or energy decay caused by a rapid expansion of the space-time). Oftentimes, the special structure of the non-linearities makes them weak compared to the linear dispersive terms at the level of small data. A large number of studies exist in the literature that deals with this issue of the structure of the non-linearities. Klainerman [26] showed that if the non-linear terms satisfy the so-called *null* condition in $3+1$ dimensions, then the global existence holds for small data limit contrary to a generic non-linearity for which global existence just fails in $3+1$ dimension. The reason for examining energy through the light cones in the current context is that singularities form when there is a causal focusing due to non-linearities (i.e., non-linearities are stronger than the dispersive effect), therefore we ought to determine whether the non-linearities will be strong enough and lead to a finite time blow-up of curvature. We shall observe a null structure present in the non-linear wave equation for the space-time curvature that precisely obstructs a blow-up of a suitable gauge-invariant norm of the space-time curvature. The techniques used here are motivated by the one used on the general Minkowski space Yang-Mills global existence work by Eardley and Moncrief [7], but with the caveat that the field strength is now taken to be space-time curvature, meaning the background geometry is not fixed whereas it would be in non-gravitational field theories. To put it simply, the geometry we are solving for is the exact same geometry we are solving on.

The outline of this presentation is as follows. Sec. 2 will lay out all the underlying structures and motivate the assumptions to be made. Firstly, we consider the spacetime of the topological type $\mathbb{R} \times M$, where M is a closed 3−manifold of negative Yamabe type. Our main aim is to utilize an integral equation for the spacetime curvature over the mantle of a past light cone of an arbitrary point and show that it does not blow up in finite time as a result of Einsteinian evolution of regular Cauchy data. However, in order to make sense of the light cones in the first place, one ought to make sure that the null injectivity radius has a lower bound. In CMC gauge, this is guaranteed by the result of Lefloch and Chen together with the long analysis of [28, 29, 30]. A lower bound on the null injectivity radius guarantees that light rays emanating from the vertex of the cone will not have a common event within the region of space time where the null cone is defined. This property is an invariant concept of space-time and as such there will not be common events in any other coordinate system since events are 'physical'. The next step is to examine the light-cone dynamics. We do so in a geodesic normal coordinate based at the vertex of the cone. Particularly, a boot-strap geodesic bound on the Riemann curvature is going to allow for a well-defined notion of geodesically convex neighborhoods on Lorentzian manifolds thereby allowing one to utilize Moncrief's reduced integral equation [17]. A wave equation for the curvature 2-form will be given as well as an integral equation for its matrix representation which acts as an endomorphism on the fibers of the vector bundle associated with the SO(1, 3) orthonormal frame bundle. Along the way, we make use of an integral equation for the connection 1-form derived by [17, 18]. Sec. 3 sets up a null frame from which light cone estimates stemming from the Bel Robinson tensor and a higher-order stress tensor are used to control the flux of the Riemann curvature and its gradient corresponding to the light cone in question. We explicitly utilize the approximate quasi-local Killing vector fields introduced by Moncrief [17] to construct and control energies. Under the assumption of point-wise bound on the aforementioned deformation tensor, we show the point-wise norm of the Riemann curvature can not blow up in finite time. Smooth extension of the solution then follows by usual elliptic arguments in a CMC spatial harmonic or CMC- spatially transported gauge.

Main Theorem: *Let $(\widehat{M} = M \times \mathbb{R}, \widehat{g})$ be a globally hyperbolic space-time and $M_{t=0}$ be an initial Cauchy hypersurface of negative Yamabe type and on it $(g_0, k_0) \in H^s \times H^{s-1}$, $s \geq 4$ is the initial data for the Cauchy problem of the vacuum Einstein evolution equations in Constant Mean extrinsic Curvature Spatial Harmonic gauge satisfying the constraint equations. This CMCSH Cauchy problem is well posed in $\mathcal{C}\left([0, t^*]; H^s \times H^{s-1}\right)$. In particular, there exists a time $t^* > 0$ dependent on $||g_0||_{H^s}, ||k_0||_{H^{s-1}}$ such that the solution map $(g_0, k_0) \mapsto (g(t), k(t), N(t), X(t))$ is continuous as a map*

$$H^s \times H^{s-1} \to H^s \times H^{s-1} \times H^{s+1} \times H^{s+1}. \tag{2}$$

Let T^* be the maximal time of existence (i.e., $T^* \geq t^*$) of a solution to the CMCSH Cauchy problem with data (g_0, k_0), then either $T^* = \infty$ or

$$\lim_{t \to T^*} \sup ||\pi(t)||_{L^\infty(M_t)} = \infty. \tag{3}$$

Here $\pi := \mathcal{L}_n \widehat{g}$ and n is the CMC Cauchy hypersurface orthogonal unit time-like vector field.

2 A gauge-gravity duality and integral equation for curvature

Let $(M, \widehat{g})$ be a time orientable globally hyperbolic space-time of dimension 1+3. Since it is globally hyperbolic, M can be foliated by a family of Cauchy hypersurfaces as level sets of a time function t. Each level set Σ_{t_1} is homeomorphic to a future one Σ_{t_2} thanks to the flow generated by

$$\partial_t = Nn + X, \tag{4}$$

where N is a lapse function, n is a future unit vector field normal to the level sets, and X is the shift vector field tangent to the Cauchy hypersurfaces. The topology is thus decomposed into $\Sigma \times \mathbb{R}$. Choosing coordinates (t, x^i) gives the ADM metric

$$\widehat{g} = -N^2 dt \otimes dt + g_{ij}\big(dx^i + X^i dt\big) \otimes \big(dx^j + X^j dt\big), \tag{5}$$

here $g_{ij} = g(\partial_i, \partial_j) = \widehat{g}(\partial_i, \partial_j)$ is the induced Riemannian metric on Σ. The second fundamental form of the constant time hypersurfaces are defined as $k_{ij} = (\frac{1}{2} L_n g)_{ij}$. One has a valid Cauchy problem upon choosing an initial level set with suitable regularity conditions (choosing a slicing of the space-time or equivalently choosing a gauge); additionally this foliation allows to define ad-hoc energies and other key quantities on each hypersurface and renders their evolution. In this article we are interested in space-times that are foliated by compact Cauchy hypersurfaces of negative Yamabe type (see [19, 18] for detail about negative Yamabe manifolds)

Over M, we naturally have the structure of an orthonormal frame bundle $\mathrm{F_{SO}}(TM)$ with structure group $\mathrm{SO}(1,3)$ and associated vector bundle $\mathrm{F_{SO}}(TM) \times_\rho \mathbb{R}^{1,3}$ isomorphic to TM where ρ is the fundamental representation of the Lorentz group acting on Minkowski space. Pick orthonormal frame $\{h_a = h_a^\mu \partial_\mu\}_{a,\mu=0}^3$ and co-frame $\{\Theta^a = \Theta^a_\mu dx^\mu\}_{a,\mu=0}^3$ at a point $p \in M$ that diagonalize the metric $\widehat{g}_{\mu\nu} = \Theta^a_\mu \Theta^b_\nu \eta_{ab}$ where η is the Minkowski metric. We may parallel transport the frame and co-frame along the radial geodesics in the normal chart $\mathcal{N}_p \subset M$ without destroying the duality $\widehat{g}(h_a, \Theta^b) = \delta_a^b$ and orthonormality $\widehat{g}(h_a, h_b) = \eta_{ab}$ since the parallel propagator is an isometry, thus $\{h_a\}$ and $\{\Theta^a\}$ are well defined fields on all of $\mathcal{N}_p$. In turn, this induces a local section $\sigma : \mathcal{N}_p \longrightarrow \mathrm{F_{SO}}(TM)$ that sends a point in the normal chart to the respective frame $(h_1, \dots, h_4)$. Now pullback the Cartan connection 1-form along σ to get a local base spin connection denoted as $\omega := \sigma^* \tilde{\omega}$, then use the frame fields as the basis for a matrix representation of the action of ω on the fibres of $\mathrm{F_{SO}}(TM) \times_\rho \mathbb{R}^{1,3}$ (which happens to be isomorphic to $\mathbb{R}^{1,3}$, i.e. the representation space of ρ). The coordinate components of the matrix entries are given by

$$\omega^a{}_{b\mu} = \Theta^a_\nu h_b^\alpha \Gamma^\nu_{\alpha\mu} + \Theta^a_\nu \partial_\mu h_b^\nu, \tag{6}$$

where $\Gamma^\nu_{\alpha\mu}$ are the Christoffel symbols associated to the Levi-Civita connection ∇ of $(M, \widehat{g})$.

The curvature 2-form on the frame bundle is defined in terms of the exterior covariant derivative d_H which itself depends on the choice of horizontal distribution H,

$$\widetilde{R} := d_H \widetilde{\omega}. \tag{7}$$

Applying pullbacks along σ as before, we denote the local versions of the curvature 2-form (Yang-Mills field strength) and exterior covariant derivative as R and d_ω, respectively. Note that the Riemann curvature $R_{\widehat{g}}$ is related to $R \in \Omega^2(\mathcal{N}_p, \mathrm{End}(TM))$ by contraction of the frame and co-frame fields, namely $R^a{}_{b\mu\nu} = R^\alpha{}_{\beta\mu\nu}\Theta^a_\alpha h^\beta_b$. In [17] Moncrief derived curvature-dependent expressions for the connection 1-form and co-frame field in the geodesic normal coordinate system $(\mathcal{N}_p, x)$

$$\omega^a{}_{b\mu}(x) = -\int_0^1 \lambda x^\nu R^a{}_{b\mu\nu}(\lambda x)\, d\lambda, \tag{8}$$

$$\Theta^a_\mu(x) = \Theta^a_\mu(0) + \int_0^1 \omega^a{}_{b\mu}(\lambda x)(\lambda x^\nu \Theta^b_\nu(0))\, d\lambda. \tag{9}$$

Here the point p is identified with the origin $x = 0$. We will make crucial use of these equations while deriving the necessary estimates.

The goal is to arrive at a wave equation for the curvature 2-form and then obtain an integral equation from which standard inequalities can be used to find point-wise bounds. First, recall the second Bianchi identity

$$d_\omega R = 0 \tag{10}$$

Working with coordinates gives a gauge covariant derivative D_μ equivalent to d_ω and the above can be re-expressed as

$$D_\alpha R^a{}_{b\mu\nu} + D_\mu R^a{}_{b\nu\alpha} + D_\nu R^a{}_{b\alpha\mu} = 0, \tag{11}$$

where the action of D_μ depends on the space-time covariant derivative as follows

$$D_\alpha R^a{}_{b\mu\nu} := \nabla_\alpha R^a{}_{b\mu\nu} + \omega^a{}_{c\alpha} R^c{}_{b\mu\nu} - \omega^c_{b\alpha} R^a{}_{c\mu\nu}. \tag{12}$$

Imposition of the vacuum Einstein's equations yields the exact Yang-Mills type equation for the curvature 2−form

$$D^\nu R^a{}_{b\mu\nu} = 0. \tag{13}$$

Using the following commutation relation

$$[D_\alpha, D_\beta] R^a{}_{b\mu\nu} = R^a{}_{c\alpha\beta} R^c{}_{b\mu\nu} - R^c{}_{b\alpha\beta} R^a{}_{c\mu\nu} - R^\gamma{}_{\mu\alpha\beta} R^a{}_{b\gamma\nu} - R^\gamma{}_{\nu\alpha\beta} R^a{}_{b\mu\gamma} \tag{14}$$

as well as the vacuum equation and the first Bianchi identity results in a gauge wave equation (a similar wave equation is satisfied by the Yang-Mills curvature with a different gauge group)

$$D^\alpha D_\alpha R^a{}_{b\mu\nu} = 2R^a{}_{c\mu\beta} R^c{}_{b\nu}{}^\beta - 2R^a{}_{c\nu\beta} R^c{}_{b\mu}{}^\beta - R^\gamma{}_{\beta\mu\nu} R^a{}_{b\gamma}{}^\beta, \tag{15}$$

make use of (12) and drop the gauge indices to transform the above into the desired ∇ wave equation for the curvature 2-form

$$\Box R_{\mu\nu} = [\nabla^\alpha \omega_\alpha, R_{\mu\nu}] + 2[\omega_\alpha, \nabla^\alpha R_{\mu\nu}] + [\omega^\alpha, [\omega_\alpha, R_{\mu\nu}]] + 2[R_{\mu\beta}, R_\nu{}^\beta] - R^\gamma{}_{\beta\mu\nu} R_\gamma{}^\beta \tag{16}$$

where $\Box := \nabla^\alpha \nabla_\alpha$. Friedlander's theory of wave operators [31] applies in this context and hence we may write a local integral equation for valid solutions R to the posed Cauchy problem on geodesically convex causal domains. The existence of these domains rely on a lower bound for the injectivity radius of the geodesic normal chart $\mathcal{N}_p$. In particular we need estimate on both the null injectivity radius and as well as chronological injectivity radius. A result due to Chen and LeFloch [27] provides a lower bound on the null injectivity radius if certain assumptions hold true (such as the ones assumed here).

Theorem 2.1 (Chen and LeFloch together with the result of [28]-[30]). *The null injectivity radius of an observer in an Einstein vacuum spacetime is uniformly controlled solely in terms of the lapse* *function, the second fundamental form of the foliation, and the L^2 curvature and lower volume bounds on some initial hypersurface.*

Goal *We want to show that under the point-wise bound on the CMC hypersurface orthonormal defomation tensor, finite initial hypersurface L^2 bound on the space-time curvature, and lower volume bounds on some initial hypersurface, the point-wise norm of the space-time Riemann curvature can not blow up for any finite interval $[-\mathfrak{L}, 0]$.*

In CMC time gauge, one can use the elliptic equation for N

$$\Delta_g N + |k|^2 N = \frac{\partial \operatorname{tr}_g k}{\partial t},$$

to obtain a point-wise estimate of the lapse function N in terms of the second fundamental form (and therefore the deformation tensor of the unit hypersurface orthogonal time-like vector field). Once we have a lower bound on the null injectivity radius, we can *draw* a light-cone emanating from p and exists throughout the range of the null exponential map. This is the only place in our analysis where we need the point-wise bound on the deformation tensor of the CMC slice orthonomal vector field n. In other words, in order to even consider a light cone, we ought to apply the theorem of Chen or Lefloch (together with the prior study of Klainerman and Rodnianski [28]-[30] on causal geometry of vacuum spacetime). In other words, the null injectivity radius of an observer in an Einstein vacuum spacetime is uniformly controlled solely in terms of the lapse function, the second fundamental form associated to the foliation, the L^2 norm of curvature, and lower bounds on the volume of some initial hypersurface. Once one establishes the existence of a light cone emanating from point p and extending up to a length depending on the point-wise bound on the deformation tensor (this is a very non-trivial fact), our approach would yield a global existence result. For the later part, to study the light-cone dynamics, we work in a geodsic normal coordinate based at the vertex of the light cone (recall that the light cones are invariant features of spacetime i.e., if two light rays do not have a common event in one coordinate they will not have common event in any other coordinate since *event* is a physical concept of spacetime while coordinates are only used to level them). However, in order to work in the geodesic normal coordinate, we need an additional point-wise bound on the space-time curvature (a boot-strap assumption on the curvature must be made). Under such a bound, another theorem of Lefloch yields the existence of a chronological geodesic normal neighbourhood. However, we ought to make sure that the boot-strap assumption is justified. In other words, through our light-cone estimate analysis, we should obtain a better bound for the point-wise norm of the spacetime curvature.

Theorem 2.2 (Chen and LeFloch). *Let (p, T) be an observer in the space-time. Suppose the domain of the exponential map $\exp_p$ contains a ball $B_T(0, r) \subset T_pM$ and the Riemann curvature satisfies*

$$\sup_{\gamma} \left|R_{\widehat{g}}\right|_{T_\gamma} \leq \frac{1}{r^2}, \tag{17}$$

where supremum is taken over every $\widehat{g}-$geodesic $\gamma \subset \mathcal{N}_p$ initiating from a vector lying in $B_T(0, r)$, then there exists a uniform constant $C \in (0, 1)$ such that the following is satisfied by the injectivity radius

$$\frac{\operatorname{inj}_{\widehat{g}}(M, p, T)}{r} \geq C \frac{\operatorname{vol}_e(\mathcal{B}_T(p, Cr))}{r^4}. \tag{18}$$

Here $\mathcal{B}_T(p, r) = \exp_p(B_T(0, r))$ and the norm $\left|R_{\widehat{g}}\right|^2_{T_\gamma} = \sup_{T_\gamma} \langle R, R\rangle_{T_\gamma}$ is given by a positive definite moving frame metric $e = \delta^{ab} h_a \otimes h_b$ with $h_0 = T_\gamma$ the $\widehat{g}$-parallel propagation of T along γ through the geodesics. This frame metric permits one to define L^p and Sobolev H^k norms for tensor fields [27] which will be particularly useful for the energy estimates. We make the following bootstrap assumption on the curvature

Boot-strap assumption: $\sup_\gamma \left|R_{\widehat{g}}\right|_{T_\gamma} \leq \frac{1}{\delta^2}$ and $\mathrm{vol}_e(\mathcal{B}_T(p, C\delta)) \geq \frac{\delta^4}{C}$.

Under this assumption, we have a lower bound on the injectivity radius $\mathrm{inj}_{\widehat{g}}(M, p, T) \geq \delta$. This enables us to utilize the light cones emanating from point P and extend at least up to δ Euclidean distance. We are only interested in extending the backward light cone from a point p (let's say 0 in a local chart) back to a Cauchy hypersurface of time $t = -\delta$. The equivalence principle of relativity theory stipulates the existence of exponential maps at all points in M, this is so one can construct normal coordinates at every point which are to be interpreted as inertial coordinate systems. Therefore, the theorem almost comes for free except one has to assume the $R_{\widehat{g}}$ geodesic bound.

Equipped with a lower bound for the injectivity radius, [17, 18] used Friedlander's methods to get the following integral equation for the frame components $R^a{}_{bmn} = R^a{}_{b\mu\nu} h^\mu_m h^\nu_n$ of the curvature 2-form[1]

$$\begin{aligned}
R^a{}_{bmn}(x) = \frac{1}{2\pi}\int_{C_p \subset \mathcal{G}_p} & \mu_\Gamma(x')\Big\{\Big[-\omega^p{}_{m\sigma}(x')D^\sigma(k(x,x')R^a{}_{bpn}(x')) \\
& - \omega^p{}_{n\sigma}(x')D^\sigma(k(x,x')R^a{}_{bmp}(x')) - \omega^c{}_{b\sigma}(x')D^\sigma(k(x,x')R^a{}_{cmn}(x')) \\
& + \omega^a{}_{c\sigma}(x')D^\sigma(k(x,x')R^c{}_{bmn}(x'))\Big] + k(x,x')\Big[-2R^a{}_{cmp}(x')R^c{}_{bn}{}^p(x') \\
& + 2R^a{}_{cnp}(x')R^c{}_{bm}{}^p(x') + R^a{}_{bpq}(x')R_{mn}{}^{pq}(x')\Big] \\
& + R^a{}_{bmn}(x')\Box k(x,x') + 2\nabla^\sigma k(x,x')\Big[\omega^p{}_{m\sigma}(x')R^a{}_{bpn}(x') \\
& + \omega^p{}_{n\sigma}(x')R^a{}_{bmp}(x') + \omega^c{}_{b\sigma}(x')R^a{}_{cmn}(x') - \omega^a{}_{c\sigma}(x')R^c{}_{bmn}(x')\Big] \\
& + k(x,x')\Big[R^a{}_{bnp}(x')R^p{}_m(x') - R^a{}_{bnp}(x')R^p_n(x')\Big]\Big\} \\
+ \frac{1}{2\pi}\int_{\sigma_p} & d\sigma_p\Big\{2k(x,x')\xi^\sigma(x')D_\sigma R^a{}_{bmn}(x') \\
& + k(x,x')\Phi(x')R^a{}_{bmn}(x') + k(x,x')\xi^\sigma(x')\Big[R^a{}_{bpn}(x')\omega^p{}_{m\sigma}(x') \\
& + R^a{}_{bmp}(x')\omega^p{}_{n\sigma}(x') + R^a{}_{cmn}(x')\omega^c{}_{b\sigma}(x') - R^c{}_{bmn}(x')\omega^a{}_{c\sigma}(x')\Big]\Big\}.
\end{aligned} \tag{19}$$

Here the point p with coordinates x has geodesically convex neighborhood $\mathcal{G}_p$ and C_p is the mantle of its past light cone, μ_Γ is a Leray form corresponding to the optical function $\Gamma(x, x')$, σ_p is the intersection of C_p with the initial Cauchy hypersurface in the past of p, $k(x, x')$ is the transport bi-scalar that reads as $k(0, x') = \mu_{\widehat{g}}(0)/\mu_{\widehat{g}}(x')$ in the geodesic normal coordinates about $x(p) = 0$ with $\mu_{\widehat{g}}(x') := \sqrt{-\det(\widehat{g}_{\mu\nu})}$ the canonical volume form. Φ is interpreted as the dialation of $d\sigma_p$ along the outgoing null hypersurface $\bar{C}_p$ of σ_p, in a sense it behaves as the trace of the null second fundamental form associated to the null geodesic generator of $\bar{C}_p$ since the latter controls the evolution of the surface area of σ_p along the outgoing direction. ξ is tangent to $\bar{C}_p$ and satisfies $\widehat{g}(\xi, \nabla\Gamma) = -1$ ($\xi, \nabla\Gamma$, and two orthonormal σ_p-tangent frame fields constitute a usual double null frame on σ_p).

[1] Note that this equation is substantially different from the ones used by [13, 22].

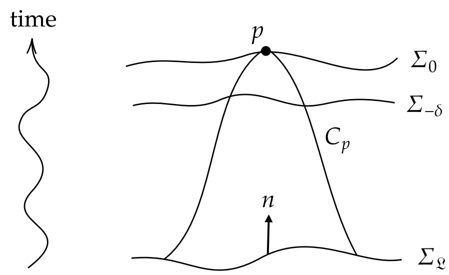

Figure 1: The first step is to use the L^∞ bound on the deformation tensor of CMC hypersurface orthogonal unit vector field n to yield a lower bound on the null injectivity radius to draw the light cone. Now we want to study the dynamics of this light cone in a normal coordinate based at its vertex. To this end we make the boot-strap assumption on the space-time curvature.

Remark: *Notice that in Friedlander's analysis, the choice of the Cauchy hyper surface is not fixed. This was the intuition behind Moncrief's reduction of the Freidlander's equation.*

For further details on how to arrive at such integral as well as concrete definitions, reader is refered to [17, 18]. An important feature of the equation is that all connection terms do not involve derivatives thereof, this means we can substitute in (8) to essentially have everything in terms of curvature. The most vital property of the integral equation (19) is that it does not contain divergence of the spin connection $\omega^a{}_{b\mu}$. Using equation (8), one may not evaluate the space-time covariant divergence of the connections since the curvature is evaluated at a different point (λx) than the connection (evaluated at x). Equations (19) and (8) have exact similarity with those of the Yang-Mills system in Crönstorm gauge [7].

3 Null structure and energy estimates

Motivated by [17], we work with the frame Euclidean metric $e := \delta_{ab}\Theta^a \otimes \Theta^b$. We also define an extension of the Euclidean inner product via e to apply on tensors

$$R \cdot R := R^{abef} R^{cdgh} e_{ca} e_{db} e_{ge} e_{hf}. \tag{20}$$

We now wish to control the following gauge-invariant point-wise L^∞ norm of the Riemann curvature over a ball $B \subset M$

$$\left\| R_{\widehat{g}} \right\|_{L^\infty(B)} := \sup_{x \in B} \sqrt{R \cdot R}. \tag{21}$$

Controlling the integral equation (19) is not an easy job since it exhibits non-linear terms and it is never pleasant to deal with them mathematically. Furthermore, non-trivial phenomena is often in disguise on these kind of terms. For example, the strong interactions posses a non-linear term in the SU(3)-symmetric Lagrangian which permits gluons to communicate amongst each other as opposed to the linear U(1) electromagnetic theory where the photons simply bypass one another (these non-linearities would manifest themselves in the quantum theory where it makes sense to talk about fundamental particles and moreover these non-linear self interactions make the quantization of pure Yang-Mills theory tremendously difficult and as such not known to exist currently). Gravity is different. Since an exponential map exists at every point, geodesics cannot intersect within the normal neighborhoods and the convex causal domains. Hence, any coordinate invariant finite time blow up within the causal domain $\mathcal{G}_p$ is then likely due to incompleteness of the space-time. In other words, if the geodesics were to terminate it will be due to a concentration of energy at the vertex of the light cone which prevents them from escaping towards infinity (this exact non-escaping of the null geodesics in the strong gravity regime leads to he formation of trapped surfaces and with a suitable topology of the Cauchy hypersurfaces would hint towards null geodesic incompleteness). All of this is made manifest in the non-linear behavior. In vacuum, the Riemann curvature is equal to the Weyl tensor W whose role is to capture pure gravity as it is projected out of the Einstein equations. One is then pushed to study how the gravitational radiation evolves along and across the light cones as a means to understand the geometric conditions that will trigger irregularities. In the current formalism, we have a gauge-like stress energy tensor given by the Bel-Robinson tensor

$$\begin{aligned} Q_{\alpha\beta\gamma\delta} &:= W_{\alpha\mu\gamma\nu} W_{\beta}{}^{\mu}{}_{\delta}{}^{\nu} + \star W_{\alpha\mu\gamma\nu} \star W_{\beta}{}^{\mu}{}_{\delta}{}^{\nu} \\ &= R_{\alpha\mu\gamma\nu} R_{\beta}{}^{\mu}{}_{\delta}{}^{\nu} + \star R_{\alpha\mu\gamma\nu} \star R_{\beta}{}^{\mu}{}_{\delta}{}^{\nu}, \end{aligned} \tag{22}$$

where $\star W_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu} W^{\mu\nu}{}_{\gamma\delta}$ is the Hodge dual of W. The Bel-Robinson tensor is traceless in general and divergence-free in vacuum

$$\begin{aligned} \mathrm{tr}_{\widehat{g}}\, Q &= 0, \\ \mathrm{div}_{\nabla}\, Q &= 0. \end{aligned} \tag{23}$$

Clearly, Q satisfies certain requirements to be a sound definition for local energy of the gravitational field (note that an universal definition of local energy density is not allowed in general relativistic settings due to the equivalence principle, nevertheless, there is a good definition of energy in a quasi-local sense [32]). Yet, a generic space-time need not have a timelike Killing vector field meaning the energy is not guaranteed to be a conserved quantity. In our adapted geodesic normal coordinate, we will utilize the time-like frame fields to construct energy densities.

3.1 Approximate Killing vector fields and energy estimates

We choose the following pair of time-like frame and co-frame fields introduced by Moncrief [17]

$$h_0 := h_0^{\mu}\partial_{\mu},\ \Theta^0 := \Theta^0{}_{\mu} dx^{\mu}. \tag{24}$$

By the definition of the frame-fields (and co-frame fields), we confirm the following time-like property of h_0 (and Θ^0)

$$\hat{g}(h_0, h_0) = h_0^{\mu} h_0^{\nu} \hat{g}_{\mu\nu} = \eta_{00} = -1,\ \hat{g}^{-1}(\Theta^0, \Theta^0) = \Theta^0_{\mu}\Theta^0_{\nu}\hat{g}^{\mu\nu} = \eta^{00} = -1. \tag{25}$$

Moreover they satisfy the following component-wise and point-wise bound uniformly in δ

$$|\Theta^0| = O(1),\ |h_0| = O(1) \tag{26}$$

as a consequence of the parallel-propagated formula of [17]

$$\omega^a{}_{b\mu}(x) = -\int_0^1 \lambda x^{\nu} R^a{}_{b\mu\nu}(\lambda x)\, d\lambda, \tag{27}$$

$$\Theta^a_{\mu}(x) = \Theta^a_{\mu}(0) + \int_0^1 \omega^a{}_{b\mu}(\lambda x)(\lambda x^{\nu}\Theta^b_{\nu}(0))\, d\lambda. \tag{28}$$

and the bootstrap assumption the curvature $||R_{\widehat{g}}||_{L^{\infty}} \leq \frac{1}{\delta^2}$. Using the following facts

$$D_{\mu}\Theta^a_{\nu} = 0,\ \nabla_{\mu}\Theta^a_{\nu} + \omega^a{}_{b\mu}\Theta^b{}_{\nu} = 0, \tag{29}$$

one can calculate the strain tensors for Θ^0 and h_0 to yield

$$^{\Theta^0}\pi_{\mu\nu} = \nabla_\mu \Theta_\nu + \nabla_\nu \Theta_\mu = -\omega^0{}_{b\mu}\Theta^b{}_\nu - \omega^0{}_{b\nu}\Theta^b{}_\mu, \tag{30}$$

$$^{h_0}\pi^{\mu\nu} = \nabla^\mu h^\nu + \nabla^\nu h^\mu = \hat{g}^{\mu\alpha}\omega^b{}_{0\alpha}h^\nu{}_b + \hat{g}^{\nu\alpha}\omega^b{}_{0\alpha}h^\mu{}_b \tag{31}$$

Note that as one approaches the point p along a radial geodesic, the strain tensors approach zero via (8) under the bootstrap condition on the Riemann curvature. Thus, h_0 and Θ^0 can be seen as approximate/quasi-local Killing fields and yield almost conserved Bel-Robinson energies. More precisely, under the boot-strap assumption of $|R_{\widehat{g}}| \leq \frac{1}{\delta^2}$, we obtain a the following point-wise estimates

$$|\omega| \leq \frac{C}{\delta},\ |\Theta| = O(1),\ |h| = O(1),\ |^{h_0}\pi| \leq \frac{C}{\delta},\ |^{\Theta^0}\pi| \leq \frac{C}{\delta} \tag{32}$$

throughout the light-cone of height $\approx \delta$ and therefore

$$\int_{t-\delta}^{t} ||\omega||_{L^\infty(B_{t'})}dt' \lesssim 1, \qquad \int_{t-\delta}^{t} ||\ ^{h_0,\Theta^0}\pi||_{L^\infty(B_{t'})}dt' \lesssim 1. \tag{33}$$

This is one of the most crucial parts of our argument. To be more precise we summarize the result as the following proposition.

Proposition 3.1. *Let the centre of the geodesic normal coordinate be q. Choose $h_a|_q = \delta^\mu_a \partial_\mu$ and parallel propagate it along radial geodesic to extend this in a the full normal neighbourhood. Construct the co-frame fields Θ^a by means of duality. The time components of such frame and co-frame fields are approximately Killing in a sense that their deformation (strain) tensor approaches zero as one approaches the vertex under the bootstrap condition. More precisely, the deformation tensors obey the following estimates uniformly in δ*

$$\int_{t-\delta}^{t} ||\omega||_{L^\infty(B_{t'})}dt' \lesssim 1, \qquad \int_{t-\delta}^{t} ||\ ^{h_0,\Theta^0}\pi||_{L^\infty(B_{t'})}dt' \lesssim 1. \tag{34}$$

As a first step to find the energy estimates, setup a frame $(\widehat{L}, \widehat{\underline{L}}, e_1, e_2)$ where $\widehat{L}$ and $\widehat{\underline{L}}$ are null future-directed and determined by the Eikonal equations in correspondence to an integrable double null foliation of the space-time. $\widehat{L}$ and $\widehat{\underline{L}}$ are thus geodesic generators. Performing a conformal transformation $L = a^2\widehat{L},\ \bar{L} = a^{-2}\widehat{\underline{L}},\ a : M \to \mathbb{R}$, demand the following to be satisfied

$$\widehat{g}(L, e_A) = \widehat{g}(\bar{L}, e_A) = 0, \quad \widehat{g}(L, \bar{L}) = -2, \quad \widehat{g}(e_A, e_B) = \delta_{AB} \tag{35}$$

for A, B running from 1 to 2. Note that L and $\bar{L}$ are tangent to C_p. Moreover, e_1 and e_2 are to be tangential to the topological 2-spheres $\sigma_p(t) := C_p \cap \Sigma_t$ which make up the leafs of the ADM foliation (see Fig. 2). The metric can then be written as

$$\widehat{g} = (L \otimes \bar{L} + \bar{L} \otimes L) + e_1 \otimes e_1 + e_2 \otimes e_2, \tag{36}$$

with component values matching the Minkowski metric in the SO$(1,3)$-frame basis. The timelike unit vector field $\mathfrak{N}$ orthogonal to the space-like level sets has a nice expression in terms of L and $\bar{L}$

$$\mathfrak{N} \approx \frac{1}{2}(\bar{L} + L), \tag{37}$$

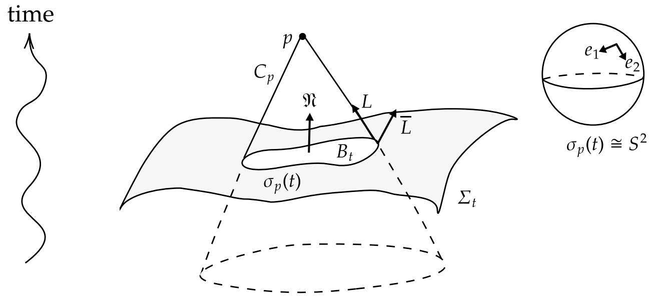

Figure 2: Null frame.

where $\approx$ indicates equality modulo a positive function that is uniformly bounded through the normal neighbourhood by means of the equations (32-33). Let es explicitly write down the different null components of the Riemann curvature

$$\alpha_{AB} := R(e_A, L, e_B, L), \;\; \bar{\alpha}_{AB} := R(e_A, \overline{L}, e_B, \overline{L}), \tag{38}$$

$$2\beta_A := R(L, \overline{L}, L, e_A), \;\; 2\bar{\beta}_A := R(\overline{L}, L, \overline{L}, e_A), \tag{39}$$

$$\rho := \frac{1}{4} R(L, \overline{L}, L, \overline{L}), \;\; e := \frac{1}{4} \, {}^*R(L, \overline{L}, L, \overline{L}). \tag{40}$$

The frame fields may be expressed in terms of the null-frame $h_0 = -\frac{1}{2}\langle h_0, L\rangle \overline{L} - \frac{1}{2}\langle h_0, \overline{L}\rangle L$. Since both L and $\overline{L}$ are future directed, for future directed time-like frame field h_0 satisfies

$$\langle h_0, L\rangle < 0, \qquad \langle h_0, \overline{L}\rangle < 0. \tag{41}$$

Due to the uniform point-wise and component-wise estimate (26) $|\langle h_0, L\rangle| = O(1)$ and $|\langle h_0, \overline{L}\rangle| = O(1)$. Denote $-\frac{1}{2}\langle h_0, L\rangle$ by $\mathfrak{A}_1 > 0$ and $-\frac{1}{2}\langle h_0, \overline{L}\rangle$ by $\mathfrak{A}_2 > 0$. The local energy density associated with the frame field h_0 on the level sets is given by $Q(h_0, h_0, h_0, n)$ which in vacuum reads

$$Q(h_0, h_0, h_0, \mathfrak{N}) = Q(\mathfrak{A}_1 L + \mathfrak{A}_2 \overline{L}, \mathfrak{A}_1 L + \mathfrak{A}_2 \overline{L}, \mathfrak{A}_1 L + \mathfrak{A}_2 \overline{L}, \frac{1}{2}(L + \overline{L})) \tag{42}$$
$$\sim \mathfrak{A}_1^3(|\alpha|^2 + |\beta|^2) + \mathfrak{A}_1^2\mathfrak{A}_2(|\beta|^2 + \rho^2 + e^2) + \mathfrak{A}_1\mathfrak{A}_2^2(|\underline{\beta}|^2 + \rho^2 + e^2) + \mathfrak{A}_2^3(|\underline{\alpha}|^2 + |\underline{\beta}|^2).$$

Due to $\mathfrak{A}_1, \mathfrak{A}_2 > 0$ and $O(1)$ uniformly throughout the light-cone, we obtain

$$C^{-1}(|\alpha|^2 + \underline{\alpha}^2 + |\beta|^2 + \underline{\beta}|^2 + \rho^2 + e^2) \leq Q(h_0, h_0, h_0, \mathfrak{N}) \leq C(|\alpha|^2 + \underline{\alpha}^2 + |\beta|^2 + \underline{\beta}|^2 + \rho^2 + e^2) \tag{43}$$

for a uniform constant C (i.e., independent of δ)

where $|\cdot|^2$ is the Euclidean norm squared. Since each term is positive definite, one can then easily see that the energy $E_B(t) := \int_{\Sigma_t} Q(h_0, h_0, h_0, \mathfrak{N}) \, \mu_g$ controls the $L^2(\Sigma_t)$ norm of the Riemann tensor. This yields the precise relation between radiation and space-time curvature that we wanted. Now focus on light cone energy by computing the following flux density

$$Q(h_0, h_0, h_0, L) \sim \mathfrak{A}_1^3|\alpha|^2 + \mathfrak{A}_1^2\mathfrak{A}_2|\beta|^2 + \mathfrak{A}_1\mathfrak{A}_2^2(\rho^2 + e^2) + \mathfrak{A}_2^3|\underline{\beta}|^2. \tag{44}$$

Notice $Q(h_0, h_0, h_0, L) \geq 0$, more importantly the term $\underline{\alpha} = R_{\widehat{g}}(\bar{L}, e_A, \bar{L}, e_B)$ does not appear in the flux density indicating that $\mathcal{F}_{C_p} := \int_{C_p} Q(h_0, h_0, h_0, L) \, \mu_{\widehat{g}}\big|_{C_p}$ only controls energy flowing across the light cone but not along it.

Proposition 3.2. *Let $t_1 = t - \delta$ and $t_2 = t$. Also let C_p be the mantle of the past light cone of a point p such that it extends to a family of Cauchy hypersurfaces $\{\Sigma_t\}_{t\in[t_1,t_2]}$ and the intersection $\sigma_p(t) := C_p \cap \Sigma_t$ is a topological 2-sphere for any $t \in [t_1, t_2]$. Express $C_p^{t_1,t_2}$ as the portion of C_p that is truncated by e_{t_1} and e_{t_2} and denote B_t as the topological ball bounded by $\sigma_p(t)$. Then the following energy estimates hold*

$$E_B(t_2) \leq CE_B(t_1), \quad \mathcal{F}_{C_p^{t_1,t_2}} \leq \mathcal{C}E_B(t_1) \tag{45}$$

where C and $\mathcal{C}$ are constants independent of δ.

Proof. Verify the divergence identity

$$\nabla_\mu Q_{\alpha\beta\gamma}{}^{\mu} h_0^\alpha h_0^\beta h_0^\gamma = \cancelto{0}{(\nabla_\mu Q_{\alpha\beta\gamma}{}^{\mu})} h_0^\alpha h_0^\beta h_0^\gamma + Q_{\alpha\beta\gamma}{}^{\mu}\nabla_\mu(h_0^\alpha h_0^\beta h_0^\gamma) = \frac{3}{2}Q^{\alpha}{}_{\beta\gamma}{}^{\delta}\, {}^{A}\pi_{\alpha\delta} h_0^\beta h_0^\gamma, \tag{46}$$

where ${}^{A}\pi$ is the deformation tensor associated with the frame field h_0. Through explicit calculation in the normal coordinate using the bootstrap assumption on curvature yields $\|{}^{A}\pi\|_{L^\infty(B_{t'})} \lesssim \frac{1}{\delta}$ in the region $t' \in [t-\delta, t]$. If h_0 is Killing, the above is equal to zero and the claim is automatically true by conservation of energy. This is not the case in general so we integrate over the interior of the truncated cone, which we call $D_p^{t_1,t_2}$, and apply Stokes' theorem as a means to obtain energies on the topological balls. Here we assume the domain is Stokes regular, i.e. has compact support and so forth

$$\begin{aligned}\int_{D_p^{t_1,t_2}} \nabla_\mu Q_{\alpha\beta\gamma}{}^{\mu} h_0^\alpha h_0^\beta h_0^\gamma &= -\int_{C_p^{t_1,t_2}} Q(h_0,h_0,h_0,L)\,\mu_{\widehat{g}} + \int_{B_{t_1}} Q(h_0,h_0,h_0,-\mathfrak{N}) + \int_{B_{t_2}} Q(h_0,h_0,h_0,\mathfrak{N}) \\ \implies E_B(t_2) + \mathcal{F}_{C_p^{t_1,t_2}} &= E_B(t_1) + \int_{D_p^{t_1,t_2}} \frac{3}{2} Q^{\alpha}{}_{\beta\gamma}{}^{\delta}\, {}^{h_0}\pi_{\alpha\delta} h_0^\beta h_0^\gamma\, \mu_{\widehat{g}}.\end{aligned} \tag{47}$$

We can now estimate the integral containing ${}^{h_0}\pi_{\alpha\delta}$. Since the domain is $D_p^{t_1,t_2}$ we can first integrate over the ball $B_{t'}$ and then over the interval $[t_1, t_2]$. The fact that $B_{t'}$ is spacelike allows for the use of the volume form induced by the Riemannian metric g on the Cauchy hypersurfaces, but this comes at the cost of an extra factor of N in the integrand due to relation $\mu_{\widehat{g}} = N\mu_g$. The estimate is then

$$\begin{aligned}\left|\frac{3}{2}\int_{D_p^{t_1,t_2}} Q^{\alpha}{}_{\beta\gamma}{}^{\delta}\, {}^{h_0}\pi_{\alpha\delta} h_0^\beta h_0^\gamma\, \mu_{\widehat{g}}\right| &= \left|\frac{3}{2}\int_{t_1}^{t_2}\int_{B_{t'}} Q^{\alpha}{}_{\beta\gamma}{}^{\delta}\, {}^{h_0}\pi_{\alpha\delta} h_0^\beta h_0^\gamma N\,\mu_g dt'\right| \\ &\leq c\int_{t_1}^{t_2} \left\|N\ {}^{h_0}\pi\right\|_{L^\infty(B_{t'})} E_B(t')\,dt'\end{aligned} \tag{48}$$

for a constant c independent of δ. The lapse function verifies $-N^2 = \widehat{g}_{00} - \widehat{g}_{0i}\widehat{g}_{0j}g^{ij} = \Theta_0^a\Theta_0^b\eta_{ab} - (\Theta_0^a\Theta_i^b\eta_{ab})(\Theta_0^c\Theta_j^d\eta_{cd})g^{ij}$ on $\mathcal{N}_p$, hence the $L^\infty(B_{t'})$ norm of N is controlled by means of the spin connection (8) and co-frame field (9) integral equations. Thus, N is uniformly bounded under the bootstrap assumption on the curvature from Theorem 1.1 since $|x\frac{1}{\delta}| \lesssim 1$. One can now leverage the signs of $Q(h_0,h_0,h_0,\mathfrak{N})$ and $Q(h_0,h_0,h_0,L)$ to obtain to the inequality below

$$E_B(t_2) \leq E_B(t_1) + c\int_{t_1}^{t_2} \left\|\ {}^{h_0}\pi\right\|_{L^\infty(B_{t'})} E_B(t')\,dt', \tag{49}$$

finally apply the Grönwall inequality to obtain the following bound for the energy at a later time

$$E_B(t_2) \leq E_B(t_1) e^{c \int_{t_1}^{t_2} \| \, {}^{h_0}\pi \|_{L^\infty(B_{t'})} \, dt'}. \tag{50}$$

Now substitute back this estimate into the flux equation (47) and apply another iteration of Grönwall to yield

$$\begin{aligned} \mathcal{F}_{C_p^{t_1,t_2}} &\leq E_B(t_1) - E_B(t_2) + c \int_{t_1}^{t_2} \left\| \, {}^{h_0}\pi \right\|_{L^\infty(B_{t'})} E_B(t_1) e^{c \int_{t_1}^{t'} \| {}^{h_0}\pi \|_{L^\infty(B_{t''})} \, dt''} \, dt' \\ &\leq E_B(t_1) \left(1 + e^{c \int_{t_1}^{t_2} \| \, {}^{h_0}\pi \|_{L^\infty(B_{t'})} \, dt'} + c \int_{t_1}^{t_2} \left\| \, {}^{h_0}\pi \right\|_{L^\infty(B_{t'})} e^{c \int_{t_1}^{t'} \| \, {}^{h_0}\pi \|_{L^\infty(B_{t''})} \, dt''} \, dt' \right), \end{aligned} \tag{51}$$

where we used the fact that the exponential function is monotonically increasing in t_2. The uniform integrability condition on ${}^{h_0}\pi$ (34) yields the estimate for $E_B(t)$. Non-negativity of $E_B(t_2)$ yields the estimate for the flux. ■

Remark: *Note that L^2 curvature flux and L^2 curvature energy are bounded by the initial Cauchy data only. In particular these two estimates do not depend on δ.*

Now in order to control the $||D^2R||^2_{L^2(B_{t_1})} + ||DR||^2_{L^2(B_{t_1})}$ norm that is going to show up when we try to estimate the point-wise behaviour of the spacetime curvature using the light-cone integral. In order to control this norm, we need an energy argument. To that end we define ad-hoc higher order stress-energy tensors to control the relevant norms of the first and second derivatives

$$\mathfrak{T}^1_{\mu\nu} := D_\mu R \cdot D_\nu R - \frac{1}{2} \widehat{g}_{\mu\nu} D_\alpha R \cdot D^\alpha R, \tag{52}$$

$$\mathfrak{T}^2_{\mu\nu} := D_\mu DR \cdot D_\nu DR - \frac{1}{2} \widehat{g}_{\mu\nu} D_\alpha DR \cdot D^\alpha DR, \tag{53}$$

As mentioned earlier, this metric is not compatible with the gauge-covariant derivative D and upon application will produce the spin connection ω that verifies an estimate $|\omega| \lesssim \delta^{-1}$. Therefore we need to keep track of these terms involving ω at each step. Let us first compute the divergence of the stress-energy tensors defined above

Proposition 3.3. *The divergence of the higher order stress-energy tensors defined in (52-53) satisfies the following schematic expressions*

$$\mathrm{div}_\nabla \, \mathfrak{T}^1 \sim DR \cdot DR \cdot \omega + DR \cdot R \cdot R, \tag{54}$$

$$\mathrm{div}_\nabla \, \mathfrak{T}^2 \sim D^2R \cdot DR \cdot R + D^2R \cdot D^2R \cdot \omega \tag{55}$$

Proof. An explicit calculations using the integral equation for the curvature and the commutation formula for the curvature yields

$$\begin{aligned} \nabla^\alpha \mathfrak{T}_{\alpha\beta} = \sum_{a,b,e,f} \Big\{ & D_\beta R^a{}_{bef} \widehat{g}^{\alpha\gamma} \Big[-(D_\gamma R^d{}_{bef} \omega^a{}_{d\alpha} + D_\gamma R^a{}_{def} \omega^d{}_{b\alpha} + D_\gamma R^a{}_{bdf} \omega^d{}_{e\alpha} \\ & + D_\gamma R^a{}_{bed} \omega^d{}_{f\alpha} \Big] - D_\gamma R^a{}_{bef} \widehat{g}^{\alpha\nu} [R^a{}_{c\beta\alpha} R^c{}_{bef} - R^c{}_{b\beta\alpha} R^a{}_{cef} - R^c{}_{e\beta\alpha} R^a{}_{bcf} \\ & - R^c{}_{f\beta\alpha} R^a{}_{bec} - D_\alpha R^c{}_{bef} \omega^a{}_{c\beta} + D_\alpha R^a{}_{cef} \omega^c{}_{b\beta} + D_\alpha R^a{}_{bcf} \omega^c{}_{e\beta} + D_\alpha R^a{}_{bec} \omega^c{}_{f\beta} \\ & + D_\beta R^c{}_{bcf} \omega^a{}_{c\alpha} - D_\beta R^a{}_{cef} \omega^c{}_{b\alpha} - D_\beta R^a{}_{bcf} \omega^c{}_{e\alpha} - D_\beta R^a{}_{bec} \omega^c_{f\alpha} - D_\beta R^a{}_{bcf} \omega^c{}_{e\alpha} \\ & - D_\beta R^a{}_{bec} \omega^c{}_{f\alpha}] + D_\beta R^a{}_{bef} [-R_{ef}{}^{cd} R^a{}_{bcd} + 2R^a{}_{ced} R^c{}_{bf}{}^d - 2R^a{}_{cfd} R^c{}_{be}{}^d] \Big\}. \end{aligned} \tag{56}$$

Note the most important point here: due to the incompatibility of the frame metric $\delta_{ab}\Theta^a \otimes \Theta^b$ with the gauge covariant connection D, we do encounter the spin connection ω in the divergence expression. Schematically, we obtain

$$\mathrm{div}_{\nabla}\, \mathfrak{T}^1 \sim DR \cdot DR \cdot \omega + DR \cdot R \cdot R. \tag{57}$$

An exact similar procedure yields

$$\mathrm{div}_{\nabla}\, \mathfrak{T}^2 \sim D^2R \cdot DR \cdot R + D^2R \cdot D^2R \cdot \omega \tag{58}$$

concluding the proof of the lemma. ■

Due to the fact that these ad-hoc stress-energy tensors are not divergence free, we need to explicitly control the error terms. First, we define the energy with respect to the frame vector fields (approximate quasi-local vector field). Compute

$$\mathfrak{T}^1(h_0, \mathfrak{N}) = \mathfrak{T}^1(\mathfrak{A}_1 L + \mathfrak{A}_2 \overline{L}, \frac{1}{2}(L + \overline{L})) \tag{59}$$
$$= \frac{\mathfrak{A}^1}{2}(\mathfrak{T}^1(L, L) + \mathfrak{T}^1(L, \overline{L})) + \frac{\mathfrak{A}^2}{2}(\mathfrak{T}^1(\overline{L}, L) + \mathfrak{T}^1(\overline{L}, \overline{L})).$$

Due to the uniform positivity of $\mathfrak{A}^1$ and $\mathfrak{A}^2$ (i.e., independent of δ), we write

$$C^{-1}\left(|D_L R|_e^2 + |D_{\bar{L}} R|_e^2 + \sum_{A=1,2} |D_A R|_e^2\right) \leq \mathfrak{T}^1(h_{0,n}) \leq C\left(|D_L R|_e^2 + |D_{\bar{L}} R|_e^2 + \sum_{A=1,2} |D_A R|_e^2\right) \tag{60}$$

for a constant that is independent of δ. Similar calculations yield

$$C^{-1}(\left(|D_L R|_e^2 + \sum_{A=1,2} |D_A R|_e^2\right) \leq \mathfrak{T}^1(h_0, L) \leq C(\left(|D_L R|_e^2 + \sum_{A=1,2} |D_A R|_e^2\right), \tag{61}$$

where $|\cdot|_e^2$ is the norm squared with respect to the frame euclidean metric e. Both densities define the higher order energy $E^1(t) := \int_{\Sigma_t} \mathfrak{T}^1(h_0, \mathfrak{N})\, \mu_g$ and flux $\mathcal{F}^1_\chi := \int_\chi \mathfrak{T}^1(h_0, L)\, \mu_{\widehat{g}}\big|_\chi$ over an integral manifold $\chi \subset M$. Similar calculations yield

$$C^{-1}\left(|D_L DR|_e^2 + |D_{\bar{L}} DR|_e^2 + \sum_{A=1,2} |D_A DR|_e^2\right) \leq \mathfrak{T}^2(h_0, \mathfrak{N}) \leq C\left(|D_L DR|_e^2 + |D_{\bar{L}} DR|_e^2 + \sum_{A=1,2} |D_A DR|_e^2\right) \tag{62}$$

for a constant C that is independent of δ. Similar calculations yield

$$C^{-1}(\left(|D_L DR|_e^2 + \sum_{A=1,2} |D_A DR|_e^2\right) \leq \mathfrak{T}^2(h_0, L) \leq C(\left(|D_L DR|_e^2 + \sum_{A=1,2} |D_A DR|_e^2\right). \tag{63}$$

Similarly, one may define the energy and flux associated with $\mathfrak{T}^2$, $E^2(t) := \int_{\Sigma_t} \mathfrak{T}^2(h_0, \mathfrak{N})\, \mu_g$ and flux $\mathcal{F}^2_\chi := \int_\chi \mathfrak{T}^2(h_0, L)\, \mu_{\widehat{g}}\big|_\chi$.

Proposition 3.4. *Let us consider the time interval $[t_1, t_2]$ such that $t_2 = t$ and $t_1 = t - \delta$. The higher order energies satisfy the following estimates*

$$||DR||_{L^2(B_{t_2})} \lesssim (||DR||_{L^2(B_{t_1})} + \int_{t_1}^{t_2} ||R||^2_{L^\infty(B_{t'})} dt')$$
$$\mathcal{F}^1_{C_p^{t_1,t_2}} \leq (||DR||_{L^2(B_{t_1})} + \int_{t_1}^{t_2} ||R||^2_{L^\infty(B_{t'})} dt'), \tag{64}$$
$$||D^2R||_{L^2(B_{t_2})} \lesssim (||D^2R||_{L^2(B_{t_1})} + \int_{t_1}^{t_2} ||DR||^2_{L^2(B_{t'})} ||R||^2_{L^\infty(B_{t'})} dt')$$

Proof. The contraction $\mathfrak{T}^{1,2}(h_0,\cdot)$ is not generally divergence free since there are still nontrivial terms involving the curvature, its gauge derivatives, and the e metric (20). This is to be expected as the high order stress energy tensor is constructed via e which is not compatible with the Levi-Civita connection of $\widehat{g}$, meaning $De \neq 0$. Proceed by using the Stokes integration over interior of truncated cone strategy to extract ball energies and the $C_p^{t_1,t_2}$ flux

$$\mathcal{F}^{1,2}_{C_p^{t_1,t_2}} + E^{1,2}_B(t_2) - E^{1,2}_B(t_1) = -\int_{D_p^{t_1,t_2}} {}^{h_0}\pi^{\mu\nu}\mathfrak{T}^{1,2}_{\mu\nu}\,\mu_{\widehat{g}} - \int_{D_p^{t_1,t_2}} h_0^{\mu}\nabla^{\nu}\mathfrak{T}^{1,2}_{\mu\nu}\,\mu_{\widehat{g}}. \tag{65}$$

Now from (31), we observe ${}^{h_0}\pi \sim \omega$. The integral containing ${}^{h_0}\pi$ can be controlled exactly as how it was done in Proposition 3.1, but now one has to deal with the extra divergence term. Explicit calculations yield

$$||DR||^2_{L^2(B_{t_2})} \lesssim ||DR||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}(\int_{B_{t'}}|DR\cdot R\cdot R|)dt' + \int_{t_1}^{t_2}(\int_{B_{t'}}|DR\cdot DR\cdot\omega|)dt' \tag{66}$$

$$\lesssim ||DR||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}||DR||^2_{L^2(B_{t'})} + \int_{t_1}^{t_2}||R||^2_{L^\infty(B_t)}\left(\int_{B_{t'}} R\cdot R\,\mu_g\right)dt' + \int_{t_1}^{t_2}||\omega||_{L^\infty(B_{t'})}||DR||^2_{L^2(B_{t'})}$$

$$\lesssim ||DR||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}||DR||^2_{L^2(B_{t'})} + \int_{t_1}^{t_2}||\omega||_{L^\infty(B_{t'})}||DR||^2_{L^2(B_{t'})} + \int_{t_1}^{t_2}||R||^2_{L^\infty(B_t)}()dt'.$$

Application of Gronwall's inequality yields

$$||DR||^2_{L^2(B_{t_1})} \lesssim (||DR||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}||R||^2_{L^\infty(B_t)}()dt')e^{\int_{t_1}^{t_2}||\omega||_{L^\infty(B_{t'})}dt'} \tag{67}$$

$$\lesssim ||DR||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}||R||^2_{L^\infty(B_t)}()dt'$$

since $|\omega| \lesssim \frac{1}{\delta}$ and therefore $\int_{t_1}^{t_2}||\omega||_{L^\infty(B_{t'})}dt' \lesssim 1$ for $t_2 - t_1 = \delta$. The flux estimate immediately follows from the positivity of $||DR||^2_{L^2(B_t)}$.
Using the schematic form for $\mathrm{div}_\nabla\mathfrak{T}^2$, the second order estimate for the curvature follows in a similar way

$$||D^2R||^2_{L^2(B_{t_2})} \lesssim ||D^2R||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}(\int_{B_{t'}}|D^2R\cdot DR\cdot R|)dt' + \int_{t_1}^{t_2}(\int_{B_{t'}}|D^2R\cdot D^2R\cdot\omega|)dt' \tag{68}$$

$$\lesssim ||D^2R||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}||D^2R||^2_{L^2(B_{t'})}dt' + \int_{t_1}^{t_2}||DR||^2_{L^2(B_{t'})}||R||^2_{L^\infty(B_{t'})}dt^2 + \int_{t_1}^{t_2}||\omega||_{L^\infty(B_{t'})}||D^2R||^2_{L^2(B_{t'})}dt'.$$

An application of Gronwall yields

$$||D^2R||^2_{L^2(B_{t_1})} \lesssim (||D^2R||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}||DR||^2_{L^2(B_{t'})}||R||^2_{L^\infty(B_t)})e^{\int_{t_1}^{t_2}||\omega||_{L^\infty(B_{t'})}dt'} \tag{69}$$

$$\lesssim (||D^2R||^2_{L^2(B_{t_1})} + \int_{t_1}^{t_2}||DR||^2_{L^2(B_{t'})}||R||^2_{L^\infty(B_t)})dt'$$

∎

We are now in place to control a gauge-invariant $L^\infty_t L^\infty_{\vec{x}}$ norm of curvature 2-form. Recall that the non-linear terms are the ones most susceptible to blow ups, the idea is to use the derived estimates to bound such terms. Normal coordinates will be used to invoke the parallel propagated connection equation (8) and take better care of some non-linearities since (8) contains R in its integrand and all connection terms in (19) are coupled to the curvature. Once the integral equation is controlled, the initial assumed bound for $R_{\widehat{g}}$ will be refined. In turn, this closure of bootstrap leads to the conclusion that the Riemann curvature cannot blow up in finite time. Finally, we sketch an argument for why the gauge-gravity system is well posed in the future.

Proposition 3.5. *Let $t_1 = t - \delta$ correspond to the initial data. Then the gauge-invariant L^∞ norm of $R_{\widehat{g}}$ at $t_2 = t$ has a refined point-wise upper bound dependent the initial B_{t_1} slice energies $E(t_1)$ and $E^1(t_1)$ and $E^2(t_1)$. More precisely*

$$\|R_{\widehat{g}}\|^2_{L^\infty}(t) \;\; \lesssim \Big(||D^2R||^2_{L^2(B_{t_1})} + ||DR||^2_{L^2(B_{t_1})} + \delta^{-1}||R||^2_{L^2(B_{t_1})}\Big). \tag{70}$$

Proof. The main object of study is the integral representation of $R^a{}_{bmn}$, which is decomposed into a light cone C_p term and a $t = t_1$ topological 2-sphere $\sigma_p = \sigma_p(t_1)$ term. The σ_p integral can be dealt with by applying a trace inequality. In particular, using the trace inequality to control the term $\int_{\sigma_p} d\sigma_p \, 2k(x,x^{'})\xi^\sigma(x^{'})D_\sigma R^a{}_{bmn}(x^{'})$ requires gauge and scaled H^2 data of the algebra-valued 2-form R over the initial ball $B_{t=t_1}$. As for the C_p integral, notice that each potentially dangerous term belongs to one out of the following classes

$$\begin{aligned}
I_1 &:= \int_{C_p} k(x,x^{'})\omega^d_{e\sigma}D^\sigma R^a{}_{bdf}\;\mu_\Gamma, \\
I_2 &:= \int_{C_p} k(x,x^{'})(R_{fd}R_e{}^d - R_{ed}R_f{}^d)\;\mu_\Gamma, \\
I_3 &:= \int_{C_p} \nabla^\sigma k(x,x^{'})\omega^d_{e\sigma}R^a{}_{bdf}\;\mu_\Gamma, \\
I_4 &:= \int_{C_p} k(x,x^{'})R^a{}_{bcd}R_{ef}{}^{cd}\;\mu_\Gamma, \\
I_5 &:= \int_{C_p} R^a{}_{bef}\Box k(x,x')\;\mu_\Gamma.
\end{aligned} \tag{71}$$

Each must be controlled to arrive at a L^∞ bound for the curvature 2-form R. Begin with I_1. Notice the integrand contains the spin connection, by working in normal coordinates we can use the parallel propagation condition (8) to get

$$I_1 = -\int_{C_p} k(x,x^{'})\Big(\int_0^1 \lambda x^\nu R^d{}_{e\sigma\nu}(\lambda x)\,d\lambda\Big)D^\sigma R^a{}_{bdf}(x)\;\mu_\Gamma. \tag{72}$$

Along C_p, the Leray form in null-spherical coordinates can be used to find that $\mu_\Gamma \approx \frac{1}{r}\,\mu_{\widehat{g}}\big|_{C_p}$. Going back to normal coordinates, we also have the approximation $x^\mu \approx rNL^\mu$. Using both relations, I_1 is simplified as

$$|I_1| \approx \left|\int_{C_p} k(x,x')N\mathcal{R}^d{}_{e\sigma L}(x)D^\sigma R^a{}_{bdf}(x)\;\mu_{\widehat{g}}\right| \tag{73}$$

where $\mathcal{R}^d{}_{e\sigma L}(x) := \int_0^1 \lambda R^d{}_{e\sigma L}(\lambda x)\,d\lambda$. Now expand $\mathcal{R}^d{}_{e\sigma L}(x)D^\sigma R^a{}_{bdf}(x)$ by way of the null frame metric (36) in order to couple I_1 with the energy flux estimates

$$\mathcal{R}^d{}_{e\sigma L}\widehat{g}^{\sigma\mu}D_\mu R^a{}_{bdf} = \mathcal{R}^d{}_{e\bar{L}L}D_L R^a{}_{bdf} + \sum_{A=1,2}\mathcal{R}^d{}_{eAL}D_A R^a{}_{bdf}. \tag{74}$$

Proceed by switching the order of integration and applying the L^2 version of the Cauchy-Schwartz inequality with respect to the metric e

$$
\begin{aligned}
|I_1| \lesssim \int_0^1 d\lambda\, \lambda \sup_{x,x'} |k(x,x')| \Bigg[\bigg(\int_{C_p} |R^d{}_{e\bar{L}L}(\lambda x)|_e^2\, \mu_{\widehat{g}} \bigg)^{1/2} \bigg(\int_{C_p} |ND_L R^a{}_{bdf}(x)|_e^2\, \mu_{\widehat{g}} \bigg)^{1/2} \\
+ \sum_{A=1,2} \bigg(\int_{C_p} |R^d{}_{eAL}(\lambda x)|_e^2\, \mu_{\widehat{g}} \bigg)^{1/2} \bigg(\int_{C_p} |ND_A R^a{}_{bdf}(x)|_e^2\, \mu_{\widehat{g}} \bigg)^{1/2} \Bigg].
\end{aligned}
\tag{75}
$$

One can show $||R_{\widehat{g}}(\lambda x)||_{L^2(C_p)} = \lambda^{-3/2} ||R_{\widehat{g}}(x)||_{L^2(C_p)}$ through a simple re-scaling argument where the -3 in the exponent of λ is a consequence of the cone C_p being 3-dimensional. We can then make all coordinate-valued functions independent of λ

$$
\begin{aligned}
|I_1| \lesssim \bigg(\int_0^1 \lambda^{-1/2}\, d\lambda \bigg) \bigg(\big\| R^d{}_{e\bar{L}L}(x) \big\|_{L^2(C_p)} \big\| ND_L R(x) \big\|_{L^2(C_p)} \\
+ \sum_{A=1,2} \big\| R^d{}_{eAL}(x) \big\|_{L^2(C_p)} \big\| ND_A R(x) \big\|_{L^2(C_p)} \bigg).
\end{aligned}
\tag{76}
$$

Here we used the fact that $\lim_{x' \to x} k(x,x') = 1$ and $\sup_{x,x'} k(x,x') \lesssim 1$ (see later). Notice that the λ integral is finite and we can bound it. However, for spatial dimensions $n > 3$ the integrand becomes $\lambda \cdot \lambda^{-n/2}$ leading to divergence and failure of our methods. Additionally $|N| \lesssim 1$ following (8-9). We may now apply both the 0th and 1st order flux estimates since they control the L^2 norms. Propositions 3.1 and 3.2 then ensure the final I_1 bound is dependent on c_π, $E_B(t_1)$, and $E_B^1(t_1)$ and $(\int_{t_1}^{t_2} ||R||_{L^\infty}^2 dt)^{\frac{1}{2}}$.

The integrand contains the antisymmetric combination $R_{fd} R_e{}^d - R_{ed} R_f{}^d$ which can be expanded using the null frame metric. The first term reads as

$$
\begin{aligned}
R_{fd} R_{e\mu} \hat{g}^{\mu d} &= R_{f\bar{L}} R_{eL} + R_{fL} R_{e\bar{L}} + \sum_{A=1,2} R_{fA} R_{eA} \\
&= \sum_{A=1,2} R_{\bar{L}A} R_{\bar{L}A} L_e L_f + \text{terms controlled by } \mathcal{F}_{C_p}.
\end{aligned}
\tag{77}
$$

The expansion of $|R(\cdot,\cdot,\bar{L},e_A)|^2$ includes terms such as $|R(\bar{L},e_B,\bar{L},e_A)|^2$ which do not appear in the flux density $Q(h_0,h_0,h_0,L)$. Fortunately, the $(f \leftrightarrow e)$ antisymmetry of the I_2 integrand results in a point-wise cancellation of the problematic $R_{\bar{L}A} R_{\bar{L}A}$ quadratic term. This is a manifestation of the *null structure*, namely the non-linearity responsible for concentration of energy along the null cone (and a potential blow up at the vertex) ends up being too weak. In the end, every time $R_{\bar{L}A}$ appears in the full expansion of $R_{fd} R_e{}^d - R_{ed} R_f{}^d$ it is multiplied by a curvature component that is controlled by the flux $\mathcal{F}_{C_p}$ ($R_{\bar{L}A}$ is then taken out of the integral as sup norm), therefore I_2 is essentially linear and estimated as follows

$$
\begin{aligned}
|I_2| &\lesssim \sup_{x,x'} \big| k(x,x') \big| (\mathcal{F}_{C_p})^{1/2} \bigg(\int_{t_1}^{t_2} ||NR||_{L^\infty(B_{t'})}^2\, dt' \bigg)^{1/2} \\
&\lesssim (\mathcal{F}_{C_p})^{1/2} \bigg(\int_{t_1}^{t_2} ||R||_{L^\infty(B_{t'})}^2\, dt' \bigg)^{1/2}.
\end{aligned}
\tag{78}
$$

To estimate I_3, we work in normal coordinates and use the parallel propagation condition as well as the approximation for x^μ just as in our treatment of I_1. Cauchy-Schwartz then implies

$$\begin{aligned}|I_3| \lesssim &\left(\int_{C_p} |NR^a{}_{bdf}(x)\nabla_L k(x,x')|_e^2\, \mu_{\widehat{g}}\right)^{1/2} \left\|R^d{}_{e\bar{L}L}(x)\right\|_{L^2(C_p)} \\ &+ \sum_{A=1,2}\left(\int_{C_p} |NR^a{}_{bdf}(x)\nabla_A k(x,x')|_e^2\, \mu_{\widehat{g}}\right)^{1/2} \left\|R^d{}_{eAL}(x)\right\|_{L^2(C_p)}.\end{aligned} \tag{79}$$

Both $R^d{}_{e\bar{L}L}$ and $R^d{}_{eAL}$ include terms that are controlled by the light cone flux $\mathcal{F}_{C_p}$, we handle the ∇k terms by taking a sup norm over the topological sphere S^2

$$|I_3| \lesssim (\mathcal{F}_{C_p})^{1/2}\left(\int_{t_1}^{t_2} ||N||_{L^\infty(B_{t'})}||R||^2_{L^\infty(B_{t'})}\, dt'\right)^{1/2}\left((t_1-t_2)^2 \sup\left\{\int_{S^2}\left(|N\nabla_L k|_e^2 + |N\nabla_A k|_e^2\right)\mu_{S^2}\right\}\right)^{1/2}. \tag{80}$$

The norms involving derivatives of the biscalar must be bounded. The claim is that both are $\lesssim 1/\delta^2$ where δ is the lower bound on the injectivity radius. To see this, go to normal coordinates about $x(p) = 0$ and denote $k(x)^2 := k(0,x)^2 = \mu_{\widehat{g}}(0)/\mu_{\widehat{g}}(x)$. Procced by using $x^\beta \approx \delta N L^\beta$

$$|N\nabla_L k(x)|^2 \approx \left|\frac{x^\beta}{\delta}\partial_\beta k(x)\right|^2 = \left|-\frac{1}{4\delta}k(x)\widehat{g}^{\mu\nu}x^\beta\partial_\beta\widehat{g}_{\mu\nu}\right|^2, \tag{81}$$

here one uses the identity $\partial_\beta\mu_{\widehat{g}}(x)/\mu_{\widehat{g}}(x) = 1/2\, \widehat{g}^{\mu\nu}\partial_\beta\widehat{g}_{\mu\nu}$ to obtain the last equality. Recall the diagonalization $\widehat{g}_{\mu\nu} = \Theta^a_\mu\Theta^b_\nu\eta_{ab}$, now use the integral equations for the co-frame field (9) and the spin connection (8) as well as the bootstrap assumption $|R_{\widehat{g}}|_e \leq 1/\delta^2$ to estimate the metric. The connection has a length factor multiplied by the curvature, meaning $|\omega| \lesssim 1/\delta$. The co-frame field has length multiplied by the connection resulting in $|\theta| \lesssim 1$ and thus the metric is estimated as $|\widehat{g}_{\mu\nu}| \lesssim 1$. As for $k(x)$, $\sqrt{\mu_{\widehat{g}}}$ it is nothing but a polynomial of $\widehat{g}_{\mu\nu}$ (due to computation of the determinant), which yields $|k(x)| \lesssim 1$. We are left with the task of estimating $x^\beta\partial_\beta\widehat{g}_{\mu\nu}$, note the following idenity that is satisfied by the metric in the normal coordinates [17]

$$x^\beta\partial_\beta\widehat{g}_{\mu\nu} = \eta_{ab}\left\{\Theta^b_\nu(x)\left(\omega^a_{\mu f}(x)(x^\gamma\theta^f_\gamma(0)) - \int_0^1 \omega^a_{\mu f}(\lambda x)(\lambda x^\gamma\theta^f_\gamma(0))\, d\lambda\right) + (a \leftrightarrow b, \mu \leftrightarrow \nu)\right\} \tag{82}$$

which immediately gives $|N\nabla_L k|^2 \lesssim 1/\delta^2$. Similarly, one finds $|N\nabla_A k|^2 \lesssim 1/\delta^2$ (notice that this is at the same level of $|N\nabla_L k|^2$ by scaling). Noting $t \leq \delta$, we obtain $t^2 \sup\left\{\int_{S^2}\left(|N\nabla_L k|_e^2 + |N\nabla_A k|_e^2\right)\mu_{S^2}\right\} \lesssim 1$. In addition note that $|N| \lesssim 1$.

Continuing with I_4, one can explicitly compute the quadratic curvature factor appearing on the integrand

$$\begin{aligned}R^a{}_{bcd}R_{ef}{}^{cd} = \sum_{A,B=1,2} &R^a{}_{bLB}R_{ef\bar{L}B} + R^a{}_{b\bar{L}L}R_{efL\bar{L}} + R^a{}_{b\bar{L}B}R_{efLB} \\ &+ R^a{}_{bAB}R_{efAB} + R^a{}_{bAL}R_{efA\bar{L}} + R^a{}_{bA\bar{L}}R_{efAL} + R^a{}_{bL\bar{L}}R_{ef\bar{L}L}.\end{aligned} \tag{83}$$

As in the case of I_2, once each term is expanded it will include at least one curvature component controlled by the light cone flux and the other factor needs to be taken out of the integral as a sup norm. This means I_4 shares the same bound as I_2

$$|I_4| \lesssim (\mathcal{F}_{C_p})^{1/2}\left(\int_{t_1}^{t_2} ||R||^2_{L^\infty(B_{t'})}\, dt'\right)^{1/2}. \tag{84}$$

The final integral that must be estimated is I_5, start by applying Cauchy-Schwartz

$$|I_5|^2 \lesssim ||\Box k(x,x')||^2_{L^2(C_p)} \int_{t_1}^{t_2} ||R_{\widehat{g}}||^2_{L^\infty(B_{t'})}\, dt' \tag{85}$$

The bootstrap bound $|R_{\widehat{g}}| \leq \frac{1}{\delta^2}$ yields $||\Box k(x,x')||^2_{L^2(C_p)} \lesssim \frac{1}{\delta}$ by way of a computation in the normal coordinates about $x(p)=0$ (spacetime curvature exhausts the degrees of freedom in the spacetime). The Laplacian of $k(x) := k(0,x)$ is explicitly given by

$$\begin{aligned}
\Box k(x) &= \frac{1}{\mu_{\widehat{g}}(x)}\partial_\alpha\Big(\mu_{\widehat{g}}(x)\widehat{g}^{\alpha\beta}\partial_\beta k(x)\Big)\\
&= -\frac{\mu^{1/2}_{\widehat{g}}(0)}{4\mu^{1/2}_{\widehat{g}}(x)}\widehat{g}^{\alpha\beta}\widehat{g}^{\mu\nu}\partial_\alpha\partial_\beta\widehat{g}_{\mu\nu} - \frac{\mu^{1/2}_{\widehat{g}}(0)}{16\mu^{1/2}_{\widehat{g}}(x)}\widehat{g}^{\alpha\beta}\widehat{g}^{\mu\nu}\widehat{g}^{ab}\partial_\alpha\widehat{g}_{ab}\partial_\beta\widehat{g}_{\mu\nu}\\
&\quad - \frac{\mu^{1/2}_{\widehat{g}}(0)}{4\mu^{1/2}_{\widehat{g}}(x)}\partial_\alpha\widehat{g}^{\alpha\beta}\widehat{g}^{\mu\nu}\partial_\beta\widehat{g}_{\mu\nu} - \frac{\mu^{1/2}_{\widehat{g}}(0)}{4\mu^{1/2}_{\widehat{g}}(x)}\widehat{g}^{\alpha\beta}\partial_\alpha\widehat{g}^{\mu\nu}\partial_\beta\widehat{g}_{\mu\nu}.
\end{aligned} \tag{86}$$

The most dangerous point is the vertex i.e. p, where the first derivatives of the metric vanish in the normal coordinate system

$$\Box k(0) = -\frac{1}{4}\widehat{g}^{\mu\nu}\widehat{g}^{\alpha\beta}\partial_\alpha\partial_\beta\widehat{g}_{\mu\nu}. \tag{87}$$

Another property of normal coordinates is that the second derivatives of the metric capture the space-time curvature, consequently the following index symmetries hold at the origin

$$\begin{aligned}
\partial_c\partial_d\widehat{g}_{ab} &= \partial_a\partial_b\widehat{g}_{cd},\\
\partial_c\partial_d\widehat{g}_{ab} + \partial_d\partial_b\widehat{g}_{ac} &+ \partial_b\partial_c\widehat{g}_{ad} = 0.
\end{aligned} \tag{88}$$

The second derivative at the origin then satisfies

$$\partial_\beta\partial_\nu\widehat{g}_{\mu\alpha}(0) = -\frac{1}{3}(R_{\mu\nu\alpha\beta}(0) + R_{\alpha\nu\mu\beta}(0)), \tag{89}$$

which yields the following simplification to (87) in terms of the scalar curvature associated to the space-time metric

$$\Box k(0) = \frac{1}{6}\mathrm{Scal}_{\widehat{g}}(0) = 0. \tag{90}$$

Here $\mathrm{Scal}_{\widehat{g}}$ vanishes as a result of the vacuum gravity equation. Since the origin is the only possible blow up point and $\Box k$ is controlled by the curvature, $||\Box k(x,x')||_{L^\infty(D)} \lesssim \frac{1}{\delta^2}$ or $||\Box k(x,x')||^2_{L^2(C_p)} \lesssim \frac{1}{\delta}$.

The integral over the topological 2−sphere σ_p can be controlled by means of trace inequality. Note that Θ appearing in this term is the null expansion or the trace of the second fundamental form od the outgoing null hypersurface emanating from σ_p and behaves like $\frac{1}{\delta}$. The estimates read

$$\begin{aligned}
&|\frac{1}{2\pi}\int_{\sigma_p} d\sigma_p\Big\{2k(x,x')\xi^\sigma(x')D_\sigma R^a{}_{bmn}(x')\\
&+k(x,x')\Phi(x')R^a{}_{bmn}(x') + k(x,x')\xi^\sigma(x')\Big[R^a{}_{bpn}(x')\omega^p{}_{me}(x')\\
&+ R^a{}_{bmp}(x')\omega^p{}_{ne}(x') + R^a{}_{cmn}(x')\omega^c{}_{be}(x') - R^c{}_{bmn}(x')\omega^a{}_{ce}(x')\Big]\Big\}|\\
&\lesssim ||\kappa||_{L^2(\sigma_p)}||DR||_{L^2(\sigma_p)} + ||\Theta||_{L^2(\sigma_p)}||R||_{L^2(\sigma_p)} + \|\omega\|_{L^2(\sigma_p)}\|R\|_{L^2(\sigma_p)}\\
&\lesssim \delta||DR||_{L^2(\sigma_p)} + \|R\|_{L^2(\sigma_p)}\\
&\lesssim \delta^{\frac{3}{2}}||D^2R||_{L^2(B_{t_1})} + \delta^{\frac{1}{2}}||DR||_{L^2(B_{t_1})} + \delta^{-\frac{1}{2}}||R||_{L^2(B_{t_1})}
\end{aligned} \tag{91}$$

since $|\kappa| \lesssim 1$, $|\Theta| \lesssim \frac{1}{\delta}$ and $|\omega| \lesssim \frac{1}{\delta}$ and therefore $||\Theta||_{L^2(\sigma_p)} \lesssim 1, \; ||\omega||_{L^2(\sigma_p)} \lesssim 1$. The last inequality follows from the integration by parts and scaling (one needs to make sure that the dimensions match: since curvature has dimension $[length]^{-2}$, $||R||_{L^2(\sigma_p)}$ has dimension $[length]^{-1}$ where as $||R||_{L^2(B_t)}$ has the dimension of $[length]^{-\frac{1}{2}}$ and so one ought to multiply $[length]^{-\frac{1}{2}}$ with $||R||_{L^2(B_t)}$ in order to be dimensionally consistent. The only length scale present here is δ). This presence of $\delta^{-\frac{1}{2}}$ could potentially be problematic. However, we will see this dangerous term appears as multiplied by δ in the later estimates.

Now collect all terms to obtain the estimate for the gauge invariant norm of the Riemann curvature since the right hand side does not depend on the spatial coordinates

$$\begin{aligned}\left\|R_{\widehat{g}}\right\|^2_{L^\infty}(t) &\lesssim \mathcal{F}_{C_p}\mathcal{F}^1_{C_p} + ||\Box k(x,x')||^2_{L^2(C_p)}\int_{t_1}^{t_2}\left\|R_{\widehat{g}}\right\|^2_{L^\infty(B_{t'})}\,dt' + \mathcal{F}_{C_p}\int_{t_1}^{t_2}\left\|R_{\widehat{g}}\right\|^2_{L^\infty(B_{t'})}\,dt' \\ &+\mathcal{F}_{C_p}\left(\int_{t_1}^{t_2}\left\|R_{\widehat{g}}\right\|^2_{L^\infty(B_{t'})}\,dt'\right)\left((t_1-t_2)^2\sup\left\{\int_{S^2}\left(|N\nabla_L k|^2_e + |N\nabla_A k|^2_e\right)\mu_{S^2}\right\}\right) \\ &+\delta^3||D^2R||_{L^2(B_{t_1})} + \delta||DR||_{L^2(B_{t_1})} + \delta^{-1}||R||_{L^2(B_{t_1})}.\end{aligned}$$

Recall the lower bound of the injectivity radius is δ. Introduce the following notation

$$\mathcal{K} := 1 + (t_1-t_2)^2\sup\left\{\int_{S^2}\left(|N\nabla_L k|^2_e + |N\nabla_A k|^2_e\right)\mu_{S^2}\right\} \tag{92}$$

and apply the bounds for $(\mathcal{F}_{C_p}, E_B(t), \mathcal{F}^1_{C_p})$ which are respectively found in (45) and (64)

$$\begin{aligned}\left\|R_{\widehat{g}}\right\|^2_{L^\infty}(t) &\lesssim E_B(t_1)E^1_B(t_1) + ||\Box k(x,x')||^2_{L^2(C_p)}\int_{t_1}^{t_2}\left\|R_{\widehat{g}}\right\|^2_{L^\infty(B_{t'})}\,dt' \\ &+\int_{t_1}^{t_2}E_B(t_1)(E_B(t_1)+\mathcal{K})\left\|R_{\widehat{g}}\right\|^2_{L^\infty(B_{t'})}\,dt' + ||D^2R||^2_{L^2(B_{t_1})} + ||DR||^2_{L^2(B_{t_1})} + \delta^{-1}||R||^2_{L^2(B_{t_1})}\end{aligned} \tag{93}$$

Make use of the Grönwall inequality for the last time to arrive at the refined L^∞ bound

$$\begin{aligned}\left\|R_{\widehat{g}}\right\|^2_{L^\infty}(t) &\lesssim \left(E^1_B(t_1) + ||D^2R||^2_{L^2(B_{t_1})} + ||DR||^2_{L^2(B_{t_1})} + \delta^{-1}||R||^2_{L^2(B_{t_1})}\right) \\ &\times\exp\left(\int_{t_1}^{t_2}E_B(t_1)(E_B(t_1)+\mathcal{K})\,dt' + \delta||\Box k(x,x')||^2_{L^2(C_p)}\right)\end{aligned} \tag{94}$$

Apply bounds for $\mathcal{K}$ and $\Box k$

$$\left\|R_{\widehat{g}}\right\|^2_{L^\infty}(t) \lesssim \left(||D^2R||^2_{L^2(B_{t_1})} + ||DR||^2_{L^2(B_{t_1})} + \delta^{-1}||R||^2_{L^2(B_{t_1})}\right) \tag{95}$$

In addition, both $\delta||\Box\kappa||^2_{L^2(C_p)} \lesssim 1$ and $\sup_t\mathcal{K}(t) \lesssim 1$ are responsible for making δ appear in the exponential as a linear factor. Now the most important point here to note that $\delta^{-\frac{1}{2}}$ appears with $||R||_{L^2(B_{t_1})}$ which by virtue of the uniform boundedness of $||R||_{L^2(B_{t_1})}$ (uniform in δ) yields a $\delta^{-\frac{1}{2}}$ estimate of for the term $\delta^{-\frac{1}{2}}||R||_{L^2(B_{t_1})}$. Since all the other terms involve a coefficients 1 or a positive exponent of δ, they can be replaced by $\lesssim 1$. ■

Proposition 3.6: *Let $t \in [-\delta, 0]$*

$$||DR||^2_{L^2(B_t)} + ||D^2R||^2_{L^2(B_t)} \lesssim \frac{C}{\delta}, \tag{96}$$

for a C dependent only on the initial hypersurface H^2 norm of the Riemann curvature uniformly δ.

Proof. Now we have to control the norm $(||D^2R||^2_{L^2(B_{t_1})}+||DR||^2_{L^2(B_{t_1})}+\delta^{-1}||R||^2_{L^2(B_{t_1})})$ in order to close the boot-strap. To this end we want to utilize the proposition 3.2. Let us consider $t_2=t$ and $t_1=t-\delta$ and apply proposition 3.2

$$
\begin{aligned}
||DR||^2_{L^2(B_t)} &\lesssim (||DR||_{L^2(B_{t-\delta})}+\sup_{t'\in[t-\delta,t-\frac{\delta}{2}]}(||D^2R||^2_{L^2(B_{t'})}+||DR||^2_{L^2(B_{t'})}+\delta^{-1}||R||^2_{L^2(B_{t'})})\delta \qquad (97)\\
&\lesssim (||DR||_{L^2(B_{t-\delta})}+\sup_{t'\in[t-\delta,t-\frac{\delta}{2}]}(\delta||D^2R||^2_{L^2(B_{t'})}+\delta||DR||^2_{L^2(B_{t'})}+||R||^2_{L^2(B_{t'})}).
\end{aligned}
$$

Using proposition 3.2 for $||D^2R||_{L^2(B_t)}$, and the uniform estimate for $||DR||^2_{L^2(B_t)}$, we obtain

$$
\begin{aligned}
||D^2R||^2_{L^2(B_t)} &\lesssim (||DR||^2_{L^2(B_{t-\delta})}+\sup_{t'\in[t-\delta,t-\frac{\delta}{2}]}(\delta||D^2R||^2_{L^2(B_{t'})}+\delta||DR||^2_{L^2(B_{t'})}+||R||^2_{L^2(B_{t'})}) \qquad (98)\\
&\sup_{t'\in[t-\delta,t-\frac{\delta}{2}]}(\delta||D^2R||^2_{L^2(B_{t'})}+\delta||DR||^2_{L^2(B_{t'})}+||R||^2_{L^2(B_{t'})})+||D^2R||^2_{L^2(B_{t-\delta})}\\
&\lesssim ||DR||^2_{L^2(B_{t-\delta})}+||D^2R||^2_{L^2(B_{t-\delta})}\\
&+\sup_{t'\in[t-\delta,t-\frac{\delta}{2}]}(\delta||D^2R||^2_{L^2(B_{t'})}+\delta||DR||^2_{L^2(B_{t'})}+||R||^2_{L^2(B_{t'})})^2
\end{aligned}
$$

This implies through iteration (since the length of the big interval $[-\mathfrak{L},0]$ is finite; can always cover by finite fixed copies of the δ intervals i.e., $\exists\mathbf{K}$ such that $\mathbf{K}\delta=\mathfrak{L}$), noting that $||R||^2_{L^2(B_t)}$ is uniform in δ, and appearance of δ multiplied by $||D^2R||_{L^2(B_t)}$ and $||DR||_{L^2(B_t)}$, we obtain

$$
||DR||^2_{L^2(B_t)}+||D^2R||^2_{L^2(B_t)}\lesssim \mathbf{K}C+(C\delta)^{\mathbf{K}}\lesssim\frac{C}{\delta}, \qquad (99)
$$

where C is a constant dependent only on the initial data i.e., $C:=C(((||D^2R||^2_{L^2(\Sigma_{t_0})}+||DR||^2_{L^2(\Sigma_{t_0})}+||R||^2_{L^2(\Sigma_{t_0})})),\ t_0=-\mathfrak{L}$. In particular, it does not depend on δ. ■

Now we complete the bootstrap argument.

Theorem. *The gauge-invariant L^∞ norm of $R_{\widehat{g}}$ at $t\in[-\mathfrak{L},0]$ has a refined point-wise upper bound dependent only on the initial data for any finite $\mathfrak{L}$. More precisely*

$$
\left\|R_{\widehat{g}}\right\|^2_{L^\infty}(t)\quad\lesssim C, \qquad (100)
$$

where the constant C depends on the initial data only.

Proof. Proposition (3.5) together with (99) yields for $t\in[-\mathfrak{L},0]$

$$
\left\|R_{\widehat{g}}\right\|^2_{L^\infty}(t)\lesssim C_1+C_2\delta^{-1}=C_1+C_2\frac{\delta^3}{\delta^4}, \qquad (101)
$$

where C_1 and C_2 do not depend on the δ rather they depend only on the initial data and the length of the time interval i.e., $\mathfrak{L}$. $C_1+C_2\frac{\delta^3}{\delta^4}$ can be made to be smaller than $\frac{1}{4\delta^4}$ after choosing a sufficiently small but fixed $\delta>0$ that only depends on the initial data and the time $\mathfrak{L}$ (i.e., $\delta^4\leq\frac{1-4C_2\delta^3}{4C_1}$). Therefore $\left\|R_{\widehat{g}}\right\|^2_{L^\infty}(t)\lesssim C$ for a constant C only dependent on the initial data and $\mathfrak{L}$. ■

Once we have established that a suitably defined gauge-invariant norm of the Riemann curvature can not blow up in any finite time interval $[-\mathfrak{L}, 0]$, the remaining task is to use the necessary elliptic estimates. Using curvature bounds, one obtains necessary bound for the volume $\mathrm{vol}_g(\mathcal{B}_T(p, C\delta))$. In terms of the elliptic estimates, we refer to the appendix of [13, 25] with a slight modification. An application of the local well-posedness theorem of [33] in CMC spatial harmonic coordinates for space-times foliated by compact Cauchy slices of negative Yamabe type together with the arguments presented in [34] regarding recovery of the space-time given curvature bound yields the main theorem.

Remark: *At this point, we want to conjecture a continuation criteria for Einstein equations coupled with sources. This is based on the preliminary analysis of the elliptic equation for the lapse function in the CMC gauge. To satisfy one of the criteria in Lefloch and Chen's theorem, one has to obtain a point-wise bound on the lapse function. Let* $\mathfrak{T}\mathfrak{T} := \mathfrak{T}\mathfrak{T}_{\mu\nu} dx^\mu \otimes dx^\nu$ *be the stress energy tensor of a source in the Einsteinian spacetime. The lapse function* N *verifies the following elliptic equation in the CMC gauge*

$$\Delta_g N + \{|k|^2 + \frac{S}{n-1} + \frac{n-2}{n-1} E\} N = \frac{\partial \operatorname{tr}_g k}{\partial t}, \tag{102}$$

where $E := \mathfrak{T}\mathfrak{T}(n, n)$ *is the energy density and* $S := g^{ij}(\mathfrak{T}\mathfrak{T}(\partial_i, \partial_j))$ *is the trace of the momentum flux density. Maximum principle yields* $\frac{|\frac{\partial \operatorname{tr}_g k}{\partial t}|}{||k||^2_{L^\infty} + ||S||_{L^\infty} + ||E||_{L^\infty}} \lesssim ||N||_{L^\infty} \lesssim |\frac{\partial \operatorname{tr}_g k}{\partial t}|$. *Therefore, the preliminary continuation criterion would be the finiteness of* $|| \, {}^n\pi||_{L^\infty} + ||S||_{L^\infty} + ||E||_{L^\infty}$. *For example,*
(a) in coupled Einstein-Yang-Mills system one would expect that a preliminary continuation criteria would be the boundedness of $|| \, {}^n\pi||_{L^\infty} + ||F||_{L^\infty}$ *(*F *is the Yang-Mills curvature).*
(b) in coupled Einstein-Euler system, the preliminary continuation criteria can be cast as boundedness of $|| \, {}^n\pi||_{L^\infty} + ||(P, \rho, v)||_{L^\infty}$, *where* P, ρ, *and* v *are the pressure, density, and velocity of the fluid. Of course, one ought study the coupled light cone dynamics (sound cone as well for fluid) which is essentially the second important step.*

4 Discussion

The weak cosmic censorship conjecture asserts that all space-time singularities are hidden inside black holes, i.e. the future null infinity is geodesically complete. In a globally hyperbolic vacuum with non-compact Cauchy hypersurfaces, Penrose's singularity theorem [35] states that space-time cannot be future null complete whenever there exists a closed trapped surface. Thus if the weak cosmic censorship hypothesis holds, then it would be impossible for naked singularities to occur. Addressing the question of cosmic censorship is important due to the potential pathologies mentioned in Sec. 1, but it also plays a vital role in other contexts where it is assumed to be true e.g. black hole radiation. The first step to finding a solution to the conjecture is to determine the breakdown criteria for solutions to the Einstein equations. Although global existence is not known to hold for general gravitational fields due to instability issues that can lead to the formation of black holes, a vacuum is the easiest setting for determining the exact conditions that cause a breakdown of solutions. We tackled this problem using physical principles as backbones instead of proceeding with a pure geometric PDE analysis approach as in [13, 22]. Inspiration was taken from [17, 18] to exploit the gauge-gravity structure and dramatically simplify the approach needed to arrive at the continuation criteria, namely the $L^\infty_t L^\infty_{\vec{x}}$ bound for the deformation tensor of the unit timelike vector field normal to the CMC Cauchy hypersurfaces.

Roughly speaking the bound on the deformation tensor is a key obstruction to proving the weak cosmic censorship hypothesis. Even though we only considered space-times that are foliated by the compact Cauchy hypersurfaces, one may obtain the same result for asymptotically flat space-times foliated by maximal slices (the argument behind proving the point-wise bound of the space-time curvature mostly remains unchanged due to its quasi-local nature). However, the deformation tensor bound seems to be reasonable as most realistic physical systems exhibit an almost-timelike symmetry at least in domains of outer communication. By this we mean the past is not so drastically different from the present or the future in accordance to the finiteness of the bound $\Delta < C$ and noting that any finite number is way smaller than infinity, hence the term "almost". Now, of course, on a dynamical space-time, one can never have any a priori information about this deformation tensor and as such, it may blow up in finite time obstructing the continuation of the solution. Leaving aside the *generic* space-times, one may wonder if the large data the global existence result can be proven to be true in certain special space-times. Positive answers to this question are available for Gowdy space-times where a $\mathbb{T}^3$ symmetry is present [37, 36]. A quantum jump in the context of large data global existence result would be that of U(1) problem where a small data result is already established [38, 39]. Another important direction would be to study large data Einstein-Yang-Mills dynamics without symmetry assumption since Yang-Mills repulsion is expected to counterbalance gravity.

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