The Einstein-Vlasov system in a large data regime

(Based on joint work with P. Mondal)

General Relativity Seminar-Harvard University

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Outline

The Einstein equations of General Relativity

According to the General Theory of Relativity, a spacetime is a 4-dimensional manifold $\mathcal{M}$ equipped with a Lorentzian metric g. The metric satisfies the *Einstein field equations*

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi T_{\mu\nu},$$

where $R_{\mu\nu}$ is the Ricci curvature tensor of $(\mathcal{M}, g)$, R is the scalar curvature of $(\mathcal{M}, g)$ and $T_{\mu\nu}$ denotes the stress-energy-momentum tensor of the matter field under study.

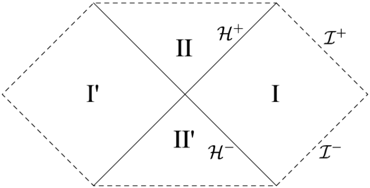
The Einstein equations of General Relativity II - Explicit Solutions and the Schwarzschild spacetime

In the early years, research followed two broad directions:

- Work towards experimental verification of the theory.
- Work towards identifying explicit solutions to (3).

The first non-trivial solution to the vacuum equations ($T_{\mu\nu} = 0$) was given by Karl Schwarzschild:

$$g_{\mathrm{Schw}} = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}.$$



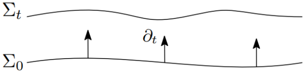
Black holes

It was first realised by Lemaitre that g_{Schw} contains a non-empty region $\mathcal{B}$ with the following features:

- Any observer in $\mathcal{B}$ cannot send a signal to an ideal conformal boundary at infinity, denoted by $\mathcal{I}^+$.
- Every timelike or null geodesic $\gamma(s)$ entering the interior of $\mathcal{B}$ is future incomplete.

Brief detour: The initial value problem

General Relativity is a *dynamical* theory!

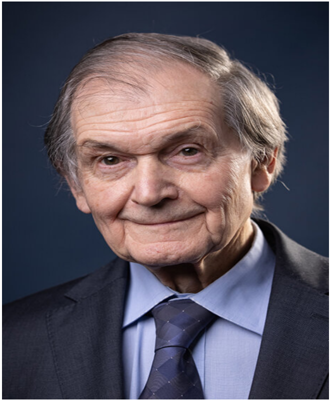


Evolution of initial data on a Cauchy hypersurface.

- A big step towards understanding this fact was made by Choquet-Bruhat (1952) (local existence for sufficiently smooth data).
- Existence of *maximal* solution $(\mathcal{M}, g)$ uniquely determined by initial conditions $(\Sigma_0, \bar{g}, k)$ for the Einstein-equations (Choquet-Bruhat, Geroch '69).
- Characteristic initial value problem (Rendall '90).

Roger Penrose's contribution

In 2020, Sir Roger Penrose was (co)-awarded the Nobel Prize in Physics...



...for the discovery that black hole formation is a robust prediction of the general theory of relativity.

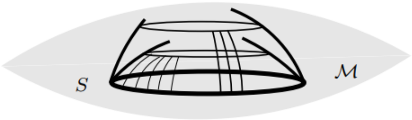
The notion of a trapped surface

Definition

Given a $(3+1)$- dimensional Lorentzian manifold $(\mathcal{M}, g)$, a closed spacelike $2-$surface S is caled **trapped** if the following two fundamental forms χ and $\underline{\chi}$ have everywhere pointwise negative expansions on S:

$$\chi(X, Y) := g(D_X L, Y), \quad \underline{\chi}(X, Y) := g(D_X \underline{L}, Y).$$

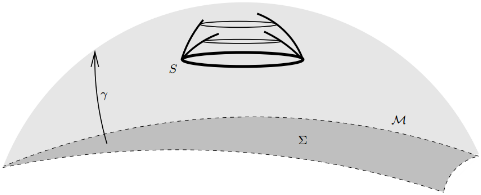
Here D denotes the Levi-Civita connection of g, L and $\underline{L}$ denote a null basis of the 2-dimensional orthogonal complement of $T_p S$ in $T_p \mathcal{M}$, extended as smooth vector fields and X, Y are arbitrary $S-$tangent vector fields.



Penrose's incompleteness theorem

Theorem (Penrose '65)

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime with a non-compact Cauchy hypersurface. Let $\mathcal{M}$ satisfy $R(V, V) \geq 0$ for all null vectors V. If $\mathcal{M}$ contains a closed trapped surface, then $\mathcal{M}$ is future causally geodesically incomplete.



The dynamical formation of trapped surfaces

- The first existence result is due to Schoen -Yau (conditions for a trapped surface to exist on the initial slice).
- A series of papers by Christodoulou for the gravitational collapse of a scalar field ('91, '94, '99 et al).
- The breakthrough by Christodoulou ('08) in the absence of symmetry for the vacuum equations ($\delta-$*short pulse method*).
- The work of Klainerman-Rodnianski ('09) on trapped surface formation using a different scaling.
- The work of An-Luk ('14) on a scale-critical trapped surface formation criterion in a finite region.
- The work of An ('19) on a scale-critical formation criterion for the vacuum equations and An-A. ('20) on the Einstein-Maxwell system.

The dynamical formation of trapped surfaces

- The work A.-Mondal-Yau on the Einstein-Yang-Mills system.
- The work of Chen and Klainerman ('24) on trapped surface formation under a geodesic foliation.
- The works of Luk-Rodnianski ('12)-('13)

The three tests of reality

- **Rigidity:** Can we characterize all vacuum black holes?
- **Stability:** Are black hole solutions stable under perturbations of initial data giving rise to them?
- **Formation:** Can black holes form dynamically (from a "time" when they did not exist)?

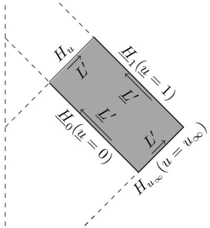
First-ever real-life picture of a black hole at M87 (EHT).

The Einstein-Vlasov system: Important works

Contrary to the Einstein-vacuum or Einstein-Maxwell systems, the Einstein-Vlasov system exhibits non-trivial dynamics, even in spherical symmetry.

- A pivotal result was given by Martin Taylor ('17), namely the Stability of Minkowski space as a solution to the Einstein-massless Vlasov system- also important and relevant to our work.
- Taylor and Lindblad ('20) obtain the same result but for the massive Vlasov system, using a harmonic coordinate gauge. Initially, f is assumed to have compact support (in space and momentum variables).
- Fajman, Joudioux and Smulevici ('21) extend this result and allow for non-compact support in the momentum variables initially for f.
- Further improved (for the massless case) by Bigorgne, Fajman, Joudioux, Smulevici and Thaller ('21): No compact support assumptions on the initial data of the Vlasov field in space or the momentum variables.
- Preceded by important works on spherical symmetry (Rein-Rendall ('92) and Dafermos-Rendall ('04)).

The framework of double null foliations



The spacetime $(\mathcal{M}, g)$ that we construct in the present work is covered by a double null foliation given by 3-dimensional incoming null hypersurfaces $\left\{\underline{H}_{\underline{u}'}\right\}_{0\leq\underline{u}'\leq 1}$, outgoing null hypersurfaces $\left\{H_u\right\}_{u_\infty\leq u\leq -a/4}$ and their pairwise intersections $S_{u,\underline{u}}$. We shall be working with normalized vectors e_3, e_4 such that $g(e_3, e_4) = -2$ and the null lapse function Ω. We construct a coordinate system in which the metric takes the form

$$g = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \gamma_{AB}(d\theta^A - d^A du) \otimes (d\theta^B - d^B du).$$

Einstein-Vlasov system on $(\mathcal{M}, g)$

The equations governing the evolution of the Einstein-Vlasov system are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}[f],$$
$$X[f] = 0.$$

- The unknowns are a Lorentzian manifold $(\mathcal{M}, g)$ and a particle density/distribution function $f : P \to [0, \infty)$. Here $P \subset T\mathcal{M}$ is a subset of the tangent bundle called the *mass shell*.
- The function $f(x, p)$ describes the density of the matter at $x \in \mathcal{M}$ with velocity $p \in P_x \subset T_x\mathcal{M}$.
- The (x^μ, p^μ) are coordinates on the tangent bundle, with (p^μ) conjugate to (x^μ),so (x^μ, p^μ) denotes the point $p^\mu \partial_{x^\mu}|_x$ in $T\mathcal{M}$.
- The mass shell is given by $\left\{(x, p) \in T\mathcal{M} \mid g_x(p, p) = 0, p \, \text{future directed}\right\}$.
- X is a vector field on $T\mathcal{M}$, i.e $X \in \Gamma(TT\mathcal{M})$, called the *geodesic spray*.
- The Vlasov equation also rewrites as $p^\mu \partial_{x^\mu} f - \Gamma^{\hat{\lambda}}_{\mu\nu} p^\mu p^\nu \partial_{\bar{p}^{\hat{\lambda}}} f = 0$, where $\hat{\lambda} = 1, 2, 3$ and

$$\partial_{\bar{p}^A} = \partial_{p^A} + \frac{\not{g}_{AB} p^B}{2p^3} \partial_{p^4}, \quad \partial_{\bar{p}^3} = \partial_{p^3} - \frac{p^4}{p^3} \partial_{p^4}.$$

Our results

Theorem (A.-Mondal (2024))

Given $\mathcal{I} > 0$, there exists a sufficiently large $a_0 = a_0(\mathcal{I})$ such that the following holds. For any $0 < a_0 < a$ and with smooth initial data $\hat{\chi}_0$ satisfying

- $\sum_{i\leq 10, m\leq 3} a^{-\frac{1}{2}} \|\nabla_4^m (|u_\infty|\nabla)^i \hat{\chi}\|_{L^\infty(S_{u_\infty,\underline{u}})} \leq \mathcal{I}$ *along* $u = u_\infty$,
- *Minkowskian initial data along* $\underline{u} = 0$,

together with a smooth function f_0 satisfying

- $\sup_{u=u_\infty} \sum_{k=0}^{11} \sum_{i_1,\ldots,i_k=1}^{7} |\tilde{F}_{i_1} \ldots \tilde{F}_{i_k} f_0| \leq a$, *along* $u = u_\infty$,
- $f = 0$ *along* $\underline{u} = 0$.

Suppose, moreover, that

$$0 \leq p^3 \leq C_{p^3}, \quad 0 \leq |u|^2 p^4 \leq C_{p^4} p^3, \quad |u^2 p^A| \leq C_{p^A} p^3, \quad \text{for } A = 1, 2,$$

in $\operatorname{supp} f|_{P|_{u=u_\infty}}$, *for some fixed constants $C_{p_1}, \ldots, C_{p^4}$ independent of u_∞. Then the Einstein-Vlasov system admits a unique smooth solution in the region*

$$u_\infty \leq u \leq -a/4, \quad 0 \leq \underline{u} \leq 1.$$

If, in addition, the initial data also satisfy $\int_0^1 |u_\infty|^2 \left(|\hat{\chi}_0|^2 + |T_{44}|_0^2\right)(u_\infty, \underline{u}')\, d\underline{u}' \geq a$, uniformly for every direction along $u = u_\infty$, then the spacetime has a trapped surface at $S_{-a/4,1}$.

The Einstein-Vlasov system in a double null foliation

We work with the system

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \mathfrak{T}_{\mu\nu},$$

$$\mathfrak{T}_{\mu\nu} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f\, p_\mu\, p_\nu \frac{\sqrt{\det \not g}}{p^3} \mathrm{d}p^1 \mathrm{d}p^2 \mathrm{d}p^3.$$

We introduce null tetrads $\{e_a, e_b, e_3, e_4\}$ where $a, b = 1, 2$ and require

$$g(e_a, e_b) = \delta_{ab}, \quad g(e_3, e_4) = -2, \quad g(e_a, e_3) = 0, \quad g(e_a, e_4) = 0.$$

For the Weyl curvature $W_{\mu\nu\lambda\tau}$, we define the null Weyl curvature components:

$$\begin{aligned}
\alpha_{ab} &= W(e_a, e_4, e_b, e_4), & \underline{\alpha}_{ab} &= W(e_a, e_3, e_b, e_3),\\
\beta_a &= \frac{1}{2} W(e_a, e_4, e_3, e_4), & \underline{\beta}_a &= \frac{1}{2} W(e_a e_3, e_3, e_4),\\
\rho &= \frac{1}{4} W(e_4, e_3, e_4, e_3), & \sigma &= \frac{1}{4} {}^*W(e_4, e_3, e_4, e_3).
\end{aligned}$$

In the same frame, we have the Ricci coefficients

$$\begin{aligned}
\chi_{AB} &= g(D_A e_4, e_B), & \underline{\chi}_{AB} &= g(D_A e_3, e_B),\\
\eta_A &= -\frac{1}{2} g(D_3 e_A, e_4), & \underline{\eta}_A &= -\frac{1}{2} g(D_4 e_A, e_3),\\
\omega &= -\frac{1}{4} g(D_4 e_3, e_4), & \underline{\omega} &= -\frac{1}{4} g(D_3 e_4, e_3), \quad \zeta_A = \frac{1}{2} g(D_A e_4, e_3).
\end{aligned}$$

The transport and constraint equations

We note that the mass shell relation relates p^4 to p^A, p^3 through $\not{g}_{AB}p^Ap^B - 4p^3p^4 = 0$. The structure equations read as follows:

$$\nabla_4 \mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2_{\gamma} - 2\omega \mathrm{tr}\chi - \mathfrak{T}_{44}$$

$$\nabla_4 \hat{\chi} + \mathrm{tr}\chi\hat{\chi} = -2\omega\hat{\chi} - \alpha$$

$$\nabla_3 tr\underline{\chi} + \frac{1}{2}(tr\underline{\chi})^2 = -|\underline{\hat{\chi}}|^2_{\gamma} - 2\underline{\omega}\, tr\underline{\chi} - \mathfrak{T}_{33}$$

$$\nabla_3 \underline{\hat{\chi}} + tr\underline{\chi}\underline{\hat{\chi}} = -2\underline{\omega}\underline{\hat{\chi}} - \underline{\alpha}$$

$$\nabla_4 \eta_a = -\chi \cdot (\eta - \underline{\eta}) - \beta - \frac{1}{2}\mathfrak{T}_{a4}$$

$$\nabla_3 \underline{\eta}_a = -\underline{\chi} \cdot (\underline{\eta} - \eta) + \underline{\beta} + \frac{1}{2}\mathfrak{T}_{a3}$$

$$\nabla_4 \underline{\omega} = 2\omega\underline{\omega} + \frac{3}{4}|\eta - \underline{\eta}|^2 - \frac{1}{4}(\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) - \frac{1}{8}|\eta + \underline{\eta}|^2 + \frac{1}{2}\rho + \frac{1}{4}\mathfrak{T}_{43}$$

$$\nabla_3 \omega = 2\omega\underline{\omega} + \frac{3}{4}|\eta - \underline{\eta}|^2 + \frac{1}{4}(\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) - \frac{1}{8}|\eta + \underline{\eta}|^2 + \frac{1}{2}\rho + \frac{1}{4}\mathfrak{T}_{43}$$

The transport, constraint and Bianchi equations

$$\nabla_4 tr\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi tr\underline{\chi} = 2\omega tr\underline{\chi} + 2\mathrm{div}\underline{\eta} + 2|\underline{\eta}|^2_\gamma + 2\rho - \hat{\chi}\cdot\hat{\underline{\chi}}$$

$$\nabla_3 \mathrm{tr}\chi + \frac{1}{2}tr\underline{\chi}\mathrm{tr}\chi = 2\underline{\omega}\mathrm{tr}\chi + 2\mathrm{div}\eta + 2|\eta|^2 + 2\rho - \hat{\chi}\cdot\hat{\underline{\chi}}$$

$$\nabla_4\hat{\underline{\chi}} + \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} = \nabla\hat{\otimes}\underline{\eta} + 2\omega\hat{\underline{\chi}} - \frac{1}{2}tr\underline{\chi}\hat{\chi} + \underline{\eta}\hat{\otimes}\underline{\eta} + \hat{\mathfrak{T}}_{ab}$$

$$\nabla_3\hat{\chi} + \frac{1}{2}tr\underline{\chi}\hat{\chi} = \nabla\hat{\otimes}\eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} + \eta\hat{\otimes}\eta + \hat{\mathfrak{T}}_{ab},$$

as well as the constraint equations

$$\mathrm{div}\hat{\chi} = \frac{1}{2}\nabla\mathrm{tr}\chi - \frac{1}{2}(\eta - \underline{\eta})\cdot(\hat{\chi} - \frac{1}{2}\mathrm{tr}\chi\gamma) - \beta + \frac{1}{2}\mathfrak{T}(e_4,\cdot)$$

$$\mathrm{div}\hat{\underline{\chi}} = \frac{1}{2}\nabla tr\underline{\chi} - \frac{1}{2}(\underline{\eta} - \eta)\cdot(\hat{\underline{\chi}} - \frac{1}{2}tr\underline{\chi}\gamma) - \underline{\beta} + \frac{1}{2}\mathfrak{T}(e_3,\cdot)$$

$$\mathrm{curl}\eta = \hat{\underline{\chi}}\wedge\hat{\chi} + \sigma\epsilon = -\mathrm{curl}\underline{\eta}$$

$$K - \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} + \frac{1}{4}\mathrm{tr}\chi tr\underline{\chi} = -\rho + \frac{1}{4}\mathfrak{T}_{43}.$$

The Bianchi equations, expressed in a double null gauge, take the following form:

$$\begin{aligned}\nabla_3\alpha + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha =& \nabla\hat{\otimes}\beta + 4\underline{\omega}\alpha - 3\left(\hat{\chi}\rho + {}^*\hat{\chi}\sigma\right) + (\zeta + 4\eta)\hat{\otimes}\beta \\ &- D_4 R_{AB} + \frac{1}{2}\left(D_B R_{4A} + D_A R_{4B}\right) + \frac{1}{2}\left(D_4 R_{43} - D_3 R_{44}\right)\not{g}_{AB}\end{aligned}$$

The Bianchi equations

$$\nabla_4\beta + 2\mathrm{tr}\chi\beta = \mathrm{div}\alpha - 2\omega\beta + (\eta - 2\zeta)\cdot\alpha + \frac{1}{2}\left(D_4 R_{4A} - D_A R_{44}\right)$$

$$\nabla_3\beta + \mathrm{tr}\underline{\chi}\beta = \nabla\rho + {}^*\nabla\sigma + 2\underline{\omega}\beta + 2\hat{\chi}\cdot\underline{\beta} + 3\left(\eta\rho + {}^*\eta\sigma\right) + \frac{1}{2}\left(D_A R_{43} - D_4 R_{3A}\right)$$

$$\nabla_4\sigma + \frac{3}{2}\mathrm{tr}\chi\sigma = -\mathrm{div}\,{}^*\beta + \frac{1}{2}\,\underline{\hat{\chi}}\cdot{}^*\alpha - (\zeta + 2\underline{\eta})\cdot{}^*\beta - \frac{1}{4}\left(D_\mu R_{4\nu} - D_\nu R_{4\mu}\right)\epsilon^{\mu\nu}{}_{34}$$

$$\nabla_3\sigma + \frac{3}{2}\mathrm{tr}\underline{\chi}\sigma = -\mathrm{div}\,{}^*\underline{\beta} + \frac{1}{2}\,\hat{\chi}\cdot{}^*\underline{\alpha} - (\zeta + 2\eta)\cdot{}^*\underline{\beta} + \frac{1}{4}\left(D_\mu R_{3\nu} - 0D_\nu R_{3\mu}\right)\epsilon^{\mu\nu}{}_{34}$$

$$\nabla_4\rho + \frac{3}{2}\mathrm{tr}\chi\rho = \mathrm{div}\beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha + (\zeta + 2\underline{\eta})\cdot\beta - \frac{1}{4}\left(D_3 R_{44} - D_4 R_{43}\right)$$

$$\nabla_3\rho + \frac{3}{2}\mathrm{tr}\underline{\chi}\rho = -\mathrm{div}\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} + (\zeta - 2\eta)\cdot\underline{\beta} + \frac{1}{4}\left(D_3 R_{34} - D_4 R_{33}\right),$$

$$\nabla_4\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = -\nabla\rho + {}^*\nabla\sigma + 2\omega\underline{\beta} + 2\underline{\hat{\chi}}\cdot\beta - 3\left(\underline{\eta}\rho - {}^*\underline{\eta}\sigma\right) - \frac{1}{2}\left(D_A R_{43} - D_3 R_{4A}\right),$$

$$\nabla_3\underline{\beta} + 2\mathrm{tr}\underline{\chi}\underline{\beta} = -\mathrm{div}\underline{\alpha} - 2\underline{\omega}\underline{\beta} + \underline{\eta}\cdot\underline{\alpha} + \frac{1}{2}\left(D_A R_{33} - D_3 R_{3A}\right),$$

$$\nabla_4\underline{\alpha} + \frac{1}{2}\mathrm{tr}\chi\,\underline{\alpha} = -\nabla\hat{\otimes}\underline{\beta} + 4\omega\,\underline{\alpha} - 3\left(\underline{\hat{\chi}}\rho - {}^*\underline{\hat{\chi}}\sigma\right) + \left(\zeta - 4\underline{\eta}\right)\hat{\otimes}\underline{\beta}$$

$$- D_3 R_{AB} + \frac{1}{2}\left(D_A R_{3B} + D_B R_{3A}\right) + \frac{1}{2}\left(D_3 R_{34} - D_4 R_{43}\right)\not{g}_{AB}.$$

Scale-invariant norms

Definition

For $\phi \in \left\{\alpha, \tilde{\beta}, \rho, \sigma, \underline{\tilde{\beta}}, \underline{\alpha}, \chi, \underline{\chi}, \zeta, \eta, \underline{\eta}, \omega\right\}$ we assign a number s_2 called the *signature for decay rates* to it:

$$s_2(\phi) := 0 \cdot N_4(\phi) + \frac{1}{2} N_A(\phi) + 1 \cdot N_3(\phi) - 1.$$

Here $N_j(\phi)$ is the number of times e_j appears in the definition of ϕ. For such ϕ, we define

$$\|\phi\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} := a^{-s_2(\phi)} |u|^{2s_2(\phi)+1} \|\phi\|_{L^\infty(S_{u,\underline{u}})},$$

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} := a^{-s_2(\phi)} |u|^{2s_2(\phi)} \|\phi\|_{L^2(S_{u,\underline{u}})},$$

$$\|\phi\|_{\mathcal{L}^1_{(sc)}(S_{u,\underline{u}})} := a^{-s_2(\phi)} |u|^{2s_2(\phi)-1} \|\phi\|_{L^1(S_{u,\underline{u}})},$$

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} := \int_0^{\underline{u}} \|\phi\|^2_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}'})} \, d\underline{u}',$$

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u_\infty,u)})} := \int_{u_\infty}^{u} \frac{a}{|u'|^2} \|\phi\|^2_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})} \, du'$$

For the Vlasov components $\mathcal{T}_{44}, \mathcal{T}_{43}, \mathcal{T}_{33}, \mathcal{T}_4, \mathcal{T}_3, \mathcal{T}$, we assign the signature so that there is conservation of signature throughout the structure and Bianchi equations.

Why these norms?

At first, the definition of such norms might seem unmotivated. However, it achieves two important goals:

- There is a *conservation and balance of signatures* along every transport, constraint, Bianchi equation and for the Vlasov matter:

$$s_2(\phi_1 \cdot \phi_2) = s_2(\phi_1) + s_2(\phi_2).$$

- The Einstein-Vlasov equations are a coupled system of many geometric quantities. The signature for decay rates allows one to give norms such that, in evolution of these large initial data, most of these quantities are bounded by 1 in these norms, with the exception of a few anomalous terms.
- The Hölder inequality, expressed in scale-invariant norms, allows for a smallness factor to appear:

$$\|\phi_1 \cdot \phi_2\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \frac{1}{|u|} \|\phi_1\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \|\phi_2\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})},$$

$$\|\phi_1 \cdot \phi_2\|_{\mathcal{L}^1_{(sc)}(S_{u,\underline{u}})} \lesssim \frac{1}{|u|} \|\phi_1\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \|\phi_2\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

Contrast this to, say, Klainerman-Rodnianski's δ-scaled norms, where

$$\|\phi_1 \cdot \phi_2\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \delta^{\frac{1}{2}} \|\phi_1\|_{\mathcal{L}^\infty_{(sc)}(S_{u,\underline{u}})} \|\phi_2\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})}.$$

This gain in smallness is **crucial** to running the bootstrap arguments required for semi-global existence.

Estimates for $\mathcal{T}$ assuming estimates for f

- Our ultimate goal is to estimate energy-momentum tensor components and their derivatives.
- This is done by controlling f and its derivatives.
- Assume that $\mathcal{D} \in \left\{|u|\nabla\!\!\!\!/_3, \nabla\!\!\!\!/_4, \nabla\!\!\!\!/\right\}$. Consider the corresponding operators $\tilde{\mathcal{D}} \in \left\{|u|e_3, e_4, e_A\right\}$ as sections of $TT\mathcal{M}$. Then there holds

$$\mathcal{D}^k\mathcal{T}[f] = \mathcal{T}[\tilde{\mathcal{D}}^k f] + \text{error terms} \ldots$$

Geometry of null geodesics-Momentum support estimates

The zero-order derivatives for f and by extension $\mathcal{T}$ are handled by first showing that f itself is uniformly bounded on the spacetime. Indeed, since f is preserved by the geodesic flow, then

$$\sup_{\mathcal{P}|_{\{u=u_\infty\}}} f = \sup_{\mathcal{P}} f.$$

The decay of $\mathcal{T}$ as $r \to \infty$ is obtained by the decay of the support of f in $\mathcal{P}_x$.

Let γ denote a null geodesic emanating from $\{u = u_\infty\}$ such that $(\gamma(0), \dot{\gamma}(0)) \in \text{supp}(f)$. At any time s, the tangent vector can be written as

$$\dot{\gamma}(s) = p^A(s)e_A + p^3(s)e_3 + p^4(s)e_4.$$

Proposition

For such a geodesic γ, there holds

$$\frac{1}{2}p^3(0) \leq p^3(s) \leq \frac{3}{2}p^3(0), \quad 0 \leq |u(s)|^2 p^4(s) \leq 2C_{p^4} p^3(0),$$

$$|u(s)^2 p^A(s)| \leq 2C_{p^A} p^3(0), \quad \textit{for} \quad A = 1, 2.$$

Proof.

We use the geodesic equations

$$\dot{p}^4(s) = -\frac{1}{4}\mathrm{tr}\underline{\chi}\,\not{g}_{AB}\,p^A(s)\,p^B(s) - \frac{1}{2}\,\hat{\underline{\chi}}_{AB}\,p^A(s)\,p^B(s) - 2\,\underline{\eta}_A\,p^A(s)p^4(s) - \omega\,(p^4(s))^2.$$

$$\dot{p}^3(s) = -\frac{1}{4}\mathrm{tr}\chi\,\not{g}_{AB}\,p^A(s)\,p^B(s) - \frac{1}{2}\,\hat{\chi}_{AB}\,p^A(s)\,p^B(s) - (\eta_A - \underline{\eta}_A)\,p^A(s)p^3(s) + \omega\,p^3(s)\,p^4(s).$$

$$\dot{p}^A(s) = -\not{\Gamma}^A_{BC}p^B\,p^C - \mathrm{tr}\underline{\chi}\,p^A\,p^3 - \mathrm{tr}\chi\,p^A\,p^4 - 2\,\hat{\underline{\chi}}^A_B\,p^B\,p^3 - 2\,\hat{\chi}^A_B\,p^B\,p^4 + (\not{\nabla}_{e_B}b)^A\,p^B\,p^4$$
$$- b^C\,\not{\Gamma}^A_{BC}\,p^B\,p^4 - 2\,p^3\,p^4\,(\eta_A + \underline{\eta}_A),$$

together with control on all the Ricci coefficients, b and $\not{\Gamma}$, we obtain the desired bounds. Using $\frac{du}{ds} = \frac{p^3(s)}{\Omega^2}$, we arrive at equations of the form:

$$\frac{d(|u|^2\,p^4)}{du} = |u|^2\,\Omega^2\left(-(\mathrm{tr}\underline{\chi} + \frac{2}{|u|})\,p^4 - \frac{1}{2}\,\frac{\hat{\underline{\chi}}_{AB}\,p^A\,p^B}{p^3} - \frac{2\,\underline{\eta}_A\,p^A\,p^4}{p^3} - \omega\,\frac{(p^4)^2}{p^3}\right) + 2(\Omega^2 - 1)\,|u|\,p^4.$$

- The appearance of $\mathrm{tr}\underline{\chi} + \frac{2}{|u|}$ in the equations for p^A and p^4 is crucial! It is what makes the RHS integrable.

$\square$

- The decay in the momentum variables gives decay of $\mathcal{T}$, at zero order.

Horizontal and vertical lifts

As f lives on $\mathcal{P} \subset T\mathcal{M}$, to estimate derivatives of f we use vector fields $\in \Gamma(TT\mathcal{M})$, much like the geodesic spray. There is a natural way to "lift" vector fields from $\Gamma(T\mathcal{M})$ to $\Gamma(TT\mathcal{M})$.

<u>Vertical lifts:</u> Given $v \in T_x\mathcal{M}$, its vertical lift at $p \in T_x\mathcal{M}$, $\mathrm{Ver}_{(x,p)}(v)$, is defined to be the vector tangent to the curve $c_{(x,p),V} : (-\varepsilon, \varepsilon) \to T\mathcal{M}$ given by

$$c_{(x,p),v}(s) = (x, p + sv),$$

at $s = 0$.

<u>Horizontal lifts</u> For the horizontal lift, first let $c : (-\varepsilon, \varepsilon) \to \mathcal{M}$ be a curve on $\mathcal{M}$ with $c(0) = x, c'(0) = v$. Extend p to a vector field along c by parallel transport using the Levi-Civita connection of g on $\mathcal{M}$. The horizontal lift of v at (x, p) denoted $\mathrm{Hor}_{(x,p)}(v)$ is then defined to be the tangent vector to the curve $c_{(x,p),H} : (-\varepsilon, \varepsilon) \to T\mathcal{M}$ given by $c_{(x,p),H}(s) = (c(s), p)$ at $s = 0$,

$$\mathrm{Hor}_{(x,p)}(v) = c'_{(x,p),H}(0).$$

Horizontal and vertical lifts

Given the coordinates $p^1, \ldots, p^4$ on $T_x\mathcal{M}$ conjugate to $e_1, \ldots, e_4$, the double null frame on $\mathcal{M}$, one has a frame for $T\mathcal{M}$ given by $(e_1, \ldots, e_4, p^1, \ldots p^4)$. If $v \in T_x\mathcal{M}$ is written as $v = v^\mu e_\mu$, then

$$\mathrm{Ver}_{(x,p)}(v) = v^\mu \partial_{p^\mu},$$

with a similar expression for $\mathrm{Hor}_{(x,p)}(v)$:

$$\mathrm{Hor}_{(x,p)}(v) = v^\mu e_\mu - v^\mu p^\nu \Gamma^\lambda_{\mu\nu} \partial_{p^\lambda}.$$

We note that, after restricting X to $\mathcal{P} \subset T\mathcal{M}$, the Vlasov equation rewrites as

$$X|_{\mathcal{P}} f = p^\mu e^\mu(f) - \Gamma^{\hat{\lambda}}_{\mu\nu} p^\mu p^\nu \partial_{\bar{p}^{\hat{\lambda}}} f = 0.$$

Here $\hat{\lambda}$ runs over $1, 2, 3$ and $\bar{p}^1, \bar{p}^2, \bar{p}^3$ denote the restrictions of p^1, p^2, p^3 to $\mathcal{P}$, while $\partial_{\bar{p}^1}, \partial_{\bar{p}^2}, \partial_{\bar{p}^3}$ denote the partial derivatives with respect to the restricted coordinate system $(\bar{p}^1, \bar{p}^2, \bar{p}^3)$. It is easy to see that

$$\partial_{\bar{p}^A} = \partial_{p^A} + \frac{\not{g}_{AB} p^B}{2p^3} \partial_{p^4}, \quad \partial_{\bar{p}^3} = \partial_{p^3} - \frac{p^4}{p^3} \partial_{p^4}.$$

Estimating the first derivatives of f

For a vector field $V \in \Gamma(T\mathcal{P})$, the following equation holds (from now on we restrict X to $\mathcal{P}$) :

$$X(Vf) = [X, V]f,$$

where V denotes the Lie bracket.

- The idea is to break $T_{(x,p)}\mathcal{P}$ into a basis of vectors $V_{(i)}$ for which the size $|V_{(i)}f|$ along the geodesic flow is uniformly bounded by the initial data.
- The vector fields used are

$$V_{(A)} := \mathrm{Hor}(e_A) - \frac{p^3}{|u|}\partial_{\bar{p}^A}, \quad V_{(3)} := \mathrm{Hor}(e_4), \quad V_{(4)} := p^3\partial_{\bar{p}^3} - |u|\mathrm{Hor}(e_3),$$

$$V_{(4+A)} := \frac{p^3}{|u|^2}\partial_{\bar{p}^A}, \quad V_{(0)} = |u|\mathrm{Hor}(e_3).$$

Estimating the first derivatives of f

We introduce a frame F on the mass shell $\mathcal{P}$ given by

$$F_A = e_A,\ F_3 = |u|\partial_u,\ F_4 = \partial_v, F_{4+A} = \frac{p^3}{|u|}\partial_{\bar{p}^A}, F_7 = p^3\partial_{\bar{p}^3}.$$

- We make the bootstrap assumption $\sum_{i=1}^7 |F_j(f)(x,p)| \leq V$, where V is a large constant $\leq a$. What we show is that $\sum_{i=1}^7 |F_j(x,p)| \lesssim$ *initial data*.
- We do this by estimating, instead, the quantities $|V_i f|$ and then obtain the improvement by expressing the $F-$frame in terms of the V-frame.

Estimating derivatives of f using Jacobi fields

In preceding works in double null foliation, the $V_{(i_1)} \dots V_{(i_k)} f$ (or actually, vector fields very similar to our own) were estimated using Jacobi fields.

- In Taylor ('16), given a vector $V \in T_{(x,p)}\mathcal{P}$ and since
$$f(x,p) = f(\exp_s(x,p)),$$
where $\exp_s(x,p) = (\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s))$, there holds
$$Vf(x,p) = J(s)f(\exp_s(x,p)).$$
- If s is taken so that $\pi(\exp_s(x,p))$ lies on the initial surface, the above expression relates a derivative of f at (x,p) to a derivative of f at the initial surface.
- Taylor then proceeds to carefully estimate the components of $J(s) = d\exp_s|_{(x,p)} V$ in a suitable frame for $\mathcal{P}$, using the Jacobi equation $\hat{\nabla}_X \hat{\nabla}_X J = \hat{R}(X,J)X$.

An apparent loss of regularity

As Taylor also points out, there is an apparent issue with directly commuting the Vlasov equation with various vector fields and estimating the error terms arising along trajectories of the geodesic flow.

- We recall that the Vlasov equation reads

$$X(f) = p^\mu e_\mu f - \Gamma^\lambda_{\mu\nu} p^\mu p^\nu \partial_{p^\lambda} f = 0.$$

- If V is a vector field, commuting will give $X(Vf) = E$, where E is an error involving terms of the form $V(\Gamma^\mu_{\nu\lambda})$.
- At first glance this seems promising, as derivatives of Ricci coefficients should lie at the same level of differentiability as Ψ.
- However, this is not the case for all $\Gamma^\mu_{\nu\lambda}$. For example, if V involves an angular derivative then $V(\Gamma^B_{4A})$ will contain two angular derivatives of the vector field b.

An apparent loss of regularity II

- Itself, b is estimated through an equation of the form

$$\nabla\!\!\!/_3 b = \Gamma + \Gamma \cdot b,$$

  whence commuting twice with angular derivatives and using elliptic estimates will only give estimates for two angular derivatives of b in terms of first order derivatives of Ψ and $\mathcal{T}$.
- A similar issue appears with angular derivatives of the spherical Christoffel symbols $\Gamma\!\!\!/$, which also appear when commuting the Vlasov equation.
- In this work we propose a way around these difficulties.

A different way of estimating the matter part

We will illustrate the idea by estimating $[X, V_{(A)}]$, where

$$V_{(A)} = \mathrm{Hor}(e_A) - \frac{p^3}{|u|}\partial_{\bar{p}^A}.$$

- For one derivative of f, there holds $XVf = [X, V]f$. We estimate the commutators in a holonomic basis

  $$C := (\partial_{\theta^A}, \partial_u, \partial_v, \partial_{\bar{p}^A}, \partial_{\bar{p}^3}).$$

  This is done because, in a holonomic basis, the commutator enjoys the simple formula $[X, V]^i = X(V^i) - V(X^i)$.
- There holds $XV_{(A)}f = \sum_{i=1}^{7}[X, V_{(A)}]^i C_i f$.

For the first two components in the C basis, there holds

$$
\begin{aligned}
&[X, V_{(A)}]^B \\
=&X(\delta_A{}^B) - V_{(A)}(p^4 b^B + p^B) \\
=&-p^4 e_A(b^B) + \Gamma^B_{A\nu} p^\nu + \Gamma^C_{A\nu} p^\nu \frac{\not{g}_{CD} p^D}{2p^3} b^B - \Gamma^3_{A\nu} p^\nu \frac{p^4}{p^3} b^B + \frac{p^3}{|u|}(\delta_A{}^B + \frac{\not{g}_{AC} p^C}{2p^3} b^B) \\
=&\Gamma^B_{A\nu} p^\nu + \frac{p^3}{|u|}\delta_A{}^B + o(|u|^{-2}),
\end{aligned}
$$

However, it is at this point that a cancellation occurs, since

$$\Gamma^B_{A\nu} p^\nu = \frac{1}{2}\mathrm{tr}\underline{\chi}\delta_A{}^B p^3 + o(|u|^{-2})$$

and hence

$$\Gamma^B_{A\nu} p^\nu + \frac{p^3}{|u|}\delta_A{}^B + o(|u|^{-2}) = \frac{1}{2}p^3\delta_A{}^B\left(\mathrm{tr}\underline{\chi} + \frac{2}{|u|}\right) + o(|u|^{-2}) = o(|u|^{-2}).$$

Similar cancellations occur for the third and fourth components.

The fifth and sixth term behave as follows:

$$
\begin{aligned}
&[X, Hor(e_A) - \frac{p^3}{|u|}\partial_{\bar{p}^A}]^{4+B} \\
= &- X(\Gamma^B_{A\nu}p^\nu) + Hor(e_A)(\Gamma^B_{\mu\nu}p^\mu p^\nu) - X\left(\frac{p^3}{|u|}\delta_A{}^B\right) \\
&- \frac{p^3}{|u|}\partial_{\bar{p}^A}(\Gamma^B_{\mu\nu}p^\mu p^\nu) \\
= &\underbrace{\left(e_A(\Gamma^B_{\mu\nu}) - e_\mu(\Gamma^B_{A\nu})\right)p^\mu p^\nu}_{\mathrm{I}} - \underbrace{\Gamma^C_{A\lambda}p^\lambda\Gamma^B_{\mu\nu}\partial_{\bar{p}^C}(p^\mu p^\nu)}_{\mathrm{II}} \\
&- \underbrace{\frac{(p^3)^2}{|u|^2}\frac{1}{\Omega^2}\delta_A{}^B}_{\mathrm{I}} - \underbrace{\frac{p^3}{|u|}\partial_{\bar{p}^A}(p^\mu p^\nu)\Gamma^B_{\mu\nu}}_{\mathrm{II}} + o(|u|^{-3}).
\end{aligned}
$$

Each pair of terms in brackets labelled (I) and (II) leads to a cancellation that improves the order of decay.

- For the terms in bracket (I), it is here where not only the issue of integrability, but also the issue of regularity in the terms is overcome. Indeed, notice that

$$e_A(\Gamma^B_{\mu\nu}) - e_\mu(\Gamma^B_{A\nu}) = R^B{}_{\nu A\mu} - \Gamma^\epsilon_{\mu\nu}\Gamma^B_{A\epsilon} + \Gamma^\epsilon_{A\nu}\Gamma^B_{\mu\epsilon} + (\Gamma^\epsilon_{A\mu} - \Gamma^\epsilon_{\mu A})\Gamma^B_{\epsilon\nu}.$$

- In other words, throughout the entire argument, **derivatives of Christoffel symbols will always appear in differences that define Riemann curvature components (up to terms of the form $\Gamma\Gamma$, which are harmless, at least in terms of integrability)**.
- This observation means one can avoid, if one wishes, the use of Jacobi fields altogether. This is true in any data regime, not just in the large data case.

A few more examples

$$
\begin{aligned}
[X, V_{(3)}]^{4+B} =& X(-\Gamma^B_{4\nu} p^\nu) + Hor(e_4)(\Gamma^B_{\mu\nu} p^\mu p^\nu) \\
=& (e_4(\Gamma^B_{\mu\nu}) - e_\mu(\Gamma^B_{4\nu}))p^\mu p^\nu + \Gamma^C_{\mu\nu} p^\mu p^\nu \Gamma^B_{4\lambda} \partial_{\bar{p}^C}(p^\lambda) \\
&+ \Gamma^3_{\mu\nu} p^\mu p^\nu \Gamma^B_{4\lambda} \partial_{\bar{p}^3}(p^\lambda) - \Gamma^C_{4\lambda} \Gamma^B_{\mu\nu} p^\lambda \partial_{\bar{p}^C}(p^\mu p^\nu) - \Gamma^3_{4\lambda} \Gamma^B_{\mu\nu} p^\lambda \partial_{\bar{p}^3}(p^\mu p^\nu) \\
=& \left(R^B{}_{\nu 4\mu} - \Gamma^\epsilon_{\mu\nu} \Gamma^B_{4\epsilon} + \Gamma^\epsilon_{4\nu} \Gamma^B_{\mu\epsilon} + (\Gamma^\epsilon_{4\mu} - \Gamma^\epsilon_{\mu 4}) \Gamma^B_{\epsilon\nu} \right) p^\mu p^\nu \\
&+ \Gamma^C_{\mu\nu} p^\mu p^\nu \Gamma^B_{4\lambda} \partial_{\bar{p}^C}(p^\lambda) + \Gamma^3_{\mu\nu} p^\mu p^\nu \Gamma^B_{4\lambda} \partial_{\bar{p}^3}(p^\lambda) \\
&- \Gamma^C_{4\lambda} \Gamma^B_{\mu\nu} p^\lambda \partial_{\bar{p}^C}(p^\mu p^\nu) - \Gamma^3_{4\lambda} \Gamma^B_{\mu\nu} p^\lambda \partial_{\bar{p}^3}(p^\mu p^\nu) = o(|u|^{-3}),
\end{aligned}
$$

$$
\begin{aligned}
[X, V_{(3)}]^7 =& X(-\Gamma^3_{4\lambda} p^\lambda) + Hor(e_4)(\Gamma^3_{\mu\nu} p^\mu p^\nu) \\
=& (e_4(\Gamma^3_{\mu\nu}) - e_\mu(\Gamma^3_{4\nu}))p^\mu p^\nu + \dots \\
=& \left(R^3{}_{\nu 4\mu} - \Gamma^\epsilon_{\mu\nu} \Gamma^3_{4\epsilon} + \Gamma^\epsilon_{4\nu} \Gamma^3_{\mu\epsilon} + (\Gamma^\epsilon_{4\mu} - \Gamma^\epsilon_{\mu 4}) \Gamma^3_{\epsilon\nu} \right) p^\mu p^\nu + \dots = o(|u|^{-2}).
\end{aligned}
$$

Higher derivatives

For higher derivatives, we obtain a schematic form for $XV_{(i_n)} \dots V_{(i_1)} f$. Denote by $\mathcal{C}_i^j := [X, V_{(i)}]^j$ the $j-$th component of the commutator in the basis $V_{(0)}, \dots, V_{(6)}$, meaning

$$[X, V_{(i)}] = \mathcal{C}_i^j V_{(j)}.$$

- By $V_{(n \dots 1)} f$ we will mean the expression $V_{(i_n)} \dots V_{(i_1)} f$.
- By $V_{(n \dots \hat{j} \dots 1)} f$ we will mean the expression $V_{(i_n)} \dots V_{(i_{j+1})} V_{(i_{j-1})} \dots V_{(i_1)} f$.
- By $V_{(n \dots \underline{k} \dots 1)} f$ we will mean the expression $V_{(i_n)} \dots V_{(i_{k+1})} V_{(j)} V_{(i_{k-1})} \dots V_{(1)} f$.

Then there holds

$$\begin{aligned} & XV_{(i_n)} V_{(i_{n-1})} \dots V_{(i_1)} f \\ = & \sum_{\lambda=1}^{n} \left(\sum_{1 \leq k_1 < \dots < k_\lambda \leq n} V_{(i_{k_\lambda})} \dots V_{(i_{k_2})} (\mathcal{C}_{i_{k_1}}^j) \, V_{(n \dots \hat{k}_\lambda \dots \hat{k}_{\lambda-1} \dots \hat{k}_2 \dots \underline{k_1} \dots 1)} f \right). \end{aligned}$$

Final comments

- The fact that, possibly, any number of $V-$ derivatives may fall on the $\mathcal{C}_i^j$ means that one needs estimates for all derivatives of Ricci coefficients and curvature components. We therefore obtain estimates for $\mathcal{D}^k(\psi, \Psi)$, where $\mathcal{D} \in \left\{|u|\nabla\!\!\!/_3, \nabla\!\!\!/_4, \nabla\!\!\!/\right\}$.
- At the highest level of derivatives, elliptic estimates are required for the Ricci coefficients (again, for all derivatives $\mathcal{D}^{\text{top}}$).
- Throughout, the hierarchies propagated are those given by the signature for decay rates.
- Once existence has been established, proving formation of trapped surfaces is straightforward.

Thank you

THANK YOU!