Formation of Trapped Surfaces in Geodesic Foliation

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General Relativity

General relativity is a theory proposed by Einstein that describes gravity as the curvature of spacetime.

- Spacetime is defined as a 1+3-dimensional Lorentzian manifold $(\mathcal{M}, g)$.
- An example is Minkowski spacetime $(\mathbb{R}^{1+3}, m)$:

$$m = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

- In general, a spacetime only locally resembles Minkowski spacetime, but this is sufficient to define timelike, null, and spacelike vector fields:
  - $g(X, X) < 0$ timelike;
  - $g(X, X) = 0$ null (lightlike);
  - $g(X, X) > 0$ spacelike.

Einstein equation

Einstein field equation for a Lorenzian manifold

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$$

T is the energy-momentum tensor of matter fields:

- Scalar fields: $T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \, \partial_\sigma \phi$
- Electromagnetic fields: $T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$
- Yang-Mills, Vlasov, Euler, ...

$T = 0$: Einstein vacuum equation (EVE) $\iff$ $R_{\mu\nu} = 0.$

Minkowski is a solution to the vacuum equation.

More solutions

Schwarzschild: Spherically symmetric solution

$$g_{Schw} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

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Kerr family $K(m, a)$

$$g_{m,a} = -\left(1 - \frac{2mr}{\rho^2}\right)dt^2 - \frac{4mar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2ma^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2mr + a^2.$$

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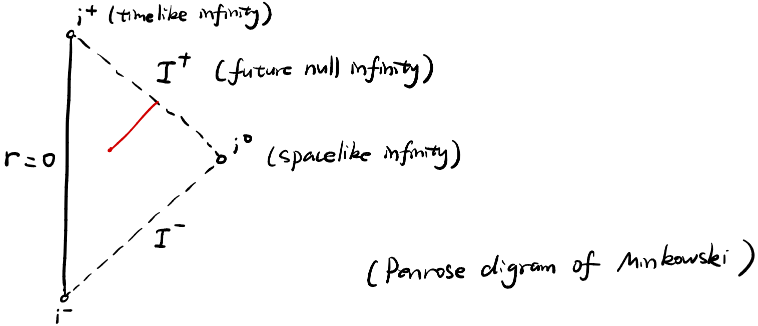
Reissner-Nordström, Kerr-Newmann: Solutions to the Einstein-Maxwell equations

Penrose Diagram

A conformally compactified picture of the spacetime; Reflects the direction of lights and captures the causal relations.

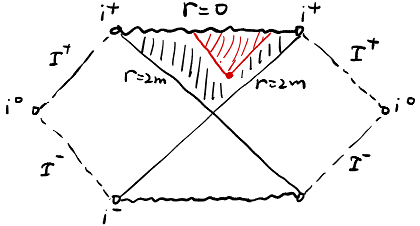


Note: This would be an oversimplification for non-spherically symmetric situations. Still helpful as it provides a quick understanding of some information of the spacetime.

Black Hole

For the Schwarzschild solution, it turns out that $r = 2m$ is a removable singularity; However, $r = 0$ is a genuine singularity.

Penrose diagram of Schwarzschild spacetime (Maximal extension)



Future Event horizon $\mathcal{H}^+ = \{r = 2m\}$

$r < 2m$: the black hole region

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All these solutions are invariant under the flow of ∂_t; "stationary", non-dynamical.

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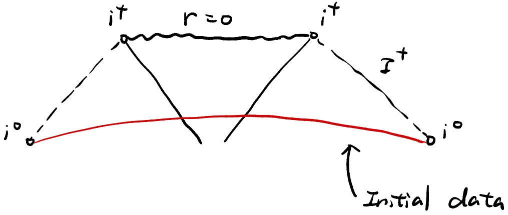
- Otherwise (any nontrivial data): Schoen-Yau positive mass theorem (1979); mr^{-1} tail as $r \to \infty$ $(m > 0)$.
- Christodoulou-Klainerman (1993) Minkowski stability: Initial small perturbation of Minkowski in a certain sense $\implies$ unique globally hyperbolic, geodesically complete solution close to Minkowski

Dynamical formation of black holes?

Question: Will black holes form dynamically from "regular" data?

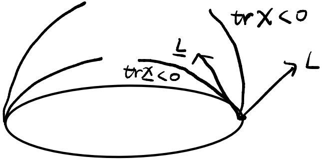
Note: Should define Schwarzschildian initial data as "irregular"! However, there are no singularities.



Trapped Surfaces

Definition: An embedded 2-sphere in the spacetime, with both of its null expansions being negative everywhere.

- Null expansion $\mathrm{tr}\,\chi$, $\mathrm{tr}\,\underline{\chi}$: Trace of null second fundamental forms.



Penrose incompleteness theorem (1965): For spacetimes solving EVE, if a non-compact spacelike hypersurface contains a trapped surface, then the spacetime is future geodesically incomplete.

Correct problem: Can initial data free of trapped surfaces form trapped surfaces dynamically?

Solving EVE in a non-perturbative regime is extremely hard! Therefore, people started with symmetry assumptions.

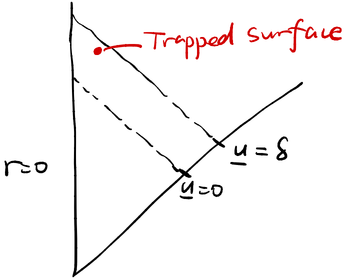
Birkhoff theorem (1923): Any spherically symmetric solution of EVE must be static asymptotically flat.

The proxy model: Spherically symmetric Einstein-scalar field system.

Christodoulou (1991): If

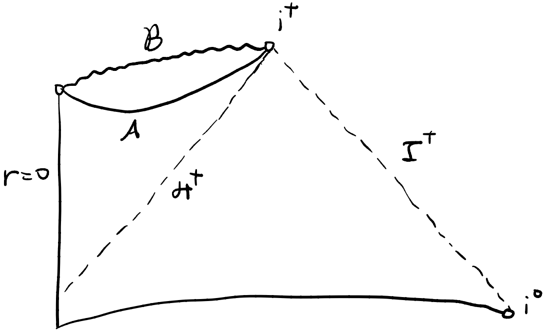
$$\int_0^\delta (\partial_{\underline{u}}\phi)^2 d\underline{u} \geq C_0 \delta \log(1/\delta),$$

then a trapped surface must form (regardless of the incoming data).



Generic Penrose Diagram

Christodoulou gives a generic Penrose diagram of this process:



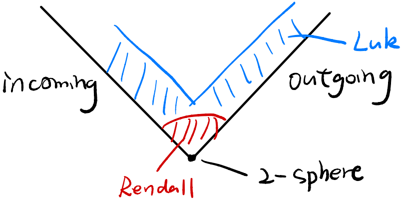
Apparent Horizon $\mathcal{A}$: Foliated by MOTS (marginally outer trapped surfaces), i.e., spheres with $\operatorname{tr}\chi = 0$

Characteristic data for EVE

In the above case: The free data is $\partial_{\underline{u}}\phi$ (for the outgoing null cone).

In vacuum case, the analogue is the shear tensor $\widehat{\chi} := \chi - \frac{1}{2}\mathrm{tr}\,\chi \not{g}$: Can be thought of as the "free data" for a characteristic initial value problem

- $\widehat{\chi}$ determines the entire geometry of the initial outgoing null cone.
- Local Wellposedness: Rendall ('90), Luk ('12).



Theorem (Christodoulou '08, finite region version)

For the characteristic initial value problem, if $\widehat{\chi} \sim \delta^{-\frac{1}{2}}$ *(roughly) on* H_{-1} *and the data is Minkowskian on* $\underline{H}_0$*, then the solution can be extended to a region that contains a trapped surface.*

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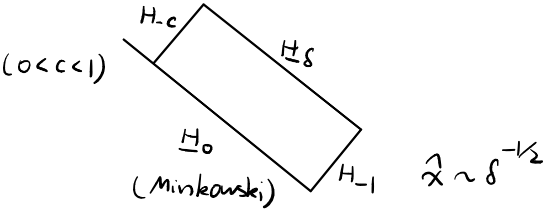
- The extension (hyperbolic part) is the main part. With the hyperbolic estimates, the formation of trapped surfaces follows from a simple estimate.



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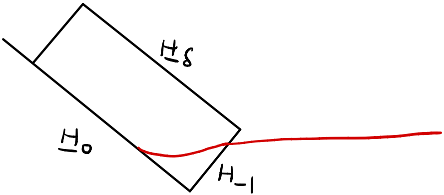

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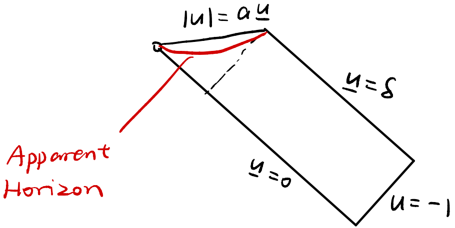
Extensions and simplifications: Klainerman-Rodnianski, Klainerman-Rodnianski-Luk, An-Luk, An, ...

Einstein equation coupled with matter fields: Maxwell (Yu, An-Athanasiou), Yang-Mills (Athanasiou-Mondal-Yau), Scalar field (Zhao-Hilditch-Kroon), ...

Scale-critical result by An-Luk

Au-Luk shows that the largeness of $\widehat{\chi}$ could be described by another parameter a and one could allow $a \ll \delta^{-1}$.

- The data is only large in the critical $H^{\frac{3}{2}}$ norm of the Einstein vacuum equation. Christodoulou's data is large in H^1.
- This allows a full description of the apparent horizon (An '19, An-Han '20, An '24)



Null frame formalism

A null frame $\{e_3, e_4, e_a\}_{a=1,2}$: e_3 and e_4 are null with $g(e_3, e_4) = -2$; $\{e_a\}$ is an orthonormal frame of $\{e_3, e_4\}^\perp$.

Ricci coefficients

$$\begin{aligned}
\chi_{ab} &= g(D_a e_4, e_b), \quad \underline{\chi}_{ab} = g(D_a e_3, e_b), \quad \omega = \frac{1}{4} g(D_4 e_4, e_3), \\
\underline{\omega} &= \frac{1}{4}(D_3 e_3, e_4), \quad \eta_a = \frac{1}{2} g(D_3 e_4, e_a), \quad \underline{\eta}_a = \frac{1}{2} g(D_4 e_3, e_a), \\
\zeta_a &= \frac{1}{2} g(D_a e_4, e_3), \quad \xi_a = \frac{1}{2} g(D_4 e_4, e_a), \quad \underline{\xi}_a = \frac{1}{2} g(D_3 e_3, e_a),
\end{aligned}$$

Curvature components

$$\begin{aligned}
\alpha_{ab} &= R_{a4b4}, \quad \beta_a = \frac{1}{2} R_{a434}, \quad \rho = \frac{1}{4} R_{3434} \\
{}^*\rho &= \frac{1}{4}\, {}^*R_{3434}, \quad \underline{\beta}_a = \frac{1}{2} R_{a334}, \quad \underline{\alpha}_{ab} = R_{a3b3}.
\end{aligned}$$

Remark: R is the same as its Weyl part if $R_{\mu\nu} = 0$

Null structure equations

Structure equations: Relations between $d\Gamma$ and R

- Transport equations, e.g.,

$$\nabla_3 \mathrm{tr}\,\chi = -\underline{\widehat{\chi}} \cdot \widehat{\chi} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi + 2\underline{\omega}\mathrm{tr}\,\chi + 2\xi \cdot \underline{\xi} + 2\mathrm{div}\,\eta + 2|\eta|^2 + 2\rho$$

$$\nabla_4 \eta - \nabla_3 \xi = -\widehat{\chi} \cdot (\eta - \underline{\eta}) - \frac{1}{2}\mathrm{tr}\,\chi(\eta - \underline{\eta}) - 4\underline{\omega}\xi - \beta$$

  Note: In general, they lose derivatives

- Constraint equations, e.g,

$$\mathrm{div}\,\widehat{\chi} + \zeta \cdot \widehat{\chi} = \frac{1}{2}\nabla \mathrm{tr}\,\chi + \frac{1}{2}\mathrm{tr}\,\chi\zeta - \beta$$

Null Bianchi equations

Comes from the Bianchi equation $D^\mu R_{\mu\nu\lambda\delta} = 0$

Different Bianchi pairs (α, β), $(\beta, (\rho, \ {}^*\rho))$, $((\rho, \ {}^*\rho), \underline{\beta})$, $(\underline{\beta}, \underline{\alpha})$

E.g.,

$$\begin{aligned} \nabla_3 \alpha - \nabla \widehat{\otimes} \beta &= -3\rho\widehat{\chi} + \cdots \\ \nabla_4 \beta - \operatorname{div} \alpha &= 3\xi\rho + \cdots \end{aligned}$$

This is a hyperbolic system; We have the "energy estimate" (integrate over spacetime regions; obtain the control of flux on hypersurfaces)

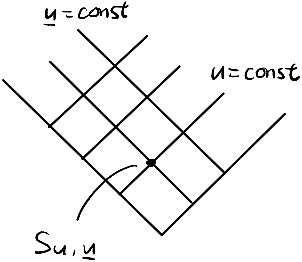
- Energy estimate can be induced by the Bel-Robinson tensor;
- Signature conservation: The total signatures are the same in each term

$$s(\alpha) = 2, \quad s(\nabla_3\alpha) = 1, \quad s(\rho) = 0, \quad s(\widehat{\chi}) = 1, \quad s(\rho\widehat{\chi}) = 1.$$

Double null foliation

Used in Klainerman-Nicolo, Christodoulou, Luk-Rodnianski, Rodnianski-Shlapentokh-Rothman, Dafermos-Holzegel-Rodnianski-Taylor, ...

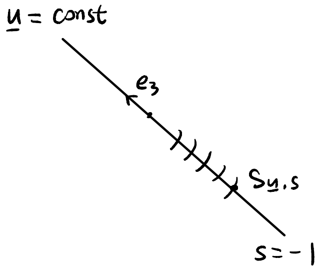
- The spacetime is foliated by constant leaves of two optical functions u, $\underline{u}$
- $L' = -2\mathrm{grad} u$, $\underline{L}' = -2\mathrm{grad}\underline{u}$
- $2\Omega^{-2} = -g(L', \underline{L}')$, $e_3 = \Omega\,\underline{L}'$, $e_4 = \Omega L'$.
- Special relations under this gauge:
  $\xi = \underline{\xi} = 0$; $\omega = -\frac{1}{2}\nabla_4(\log\Omega)$,
  $\underline{\omega} = -\frac{1}{2}\nabla_3(\log\Omega)$.
  $\eta_a = \zeta_a + \nabla_a(\log\Omega)$,
  $\underline{\eta}_a = -\zeta_a + \nabla_a(\log\Omega)$.
- Symmetric in e_3, e_4



(Incoming) Geodesic Foliation

A foliation induced by a geodesic affine parameter

- Principal in $e_3 = -2\mathrm{grad}\underline{u}$;
- $e_3(s) = 1$ determines spheres; The constant leaves of $\underline{u}$ and s then define spheres $S_{\underline{u},s}$. Take $\{e_1, e_2\}$ as an orthonormal basis of the sphere.



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- Again, some of them lose derivatives:

$$\nabla_3 \xi = \nabla_4 \eta + \cdots, \quad \nabla_3 \widehat{\chi} = \nabla \widehat{\otimes} \eta + \cdots$$

Change frame?

It turns out that, all loss of derivatives comes from the quantity η.

- Idea: Find a new (but related) frame such that $\eta' = 0$. All equations will then not lose derivatives anymore!

This, of course, is not free; we will now discuss the cost of this possible simplification.

Non-integrable formalism

Systematically used by Klainerman-Szeftel, Giorgi-Klainerman-Szeftel

- $\{e_1, e_2\}$ is not integrable as a distribution (not tangent to any spheres).
- The presence of anti-symmetric trace ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi}$.
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In our case, consider the frame transformation

$$e_4' = e_4 + f^a e_a + \frac{1}{4}|f|^2 e_3, \quad e_a' = e_a + \frac{1}{2} f_a e_3, \quad e_3' = e_3.$$

Then we have

$$\eta' = \eta + \frac{1}{2}\nabla_3' f + \cdots$$

This defines a transport equation of f if we want $\eta' = 0$.

Estimate of Ricci coefficients

The typical transport equation verified by all Ricci coefficients in the PT frame is of the form

$$\nabla_3 \psi - \frac{\lambda}{|s|}\psi = \cdots$$

Note that $s = -1$ initially and we wish to extend it until $|s| \sim a\delta$.

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ii. $\Gamma_g = \underline{\widehat{\chi}}, \mathrm{tr}\,\chi + 2/|s|$, ζ (signature 0, -1): $\lambda = 1$. They behave like $\delta a^{\frac{1}{2}}/|s|^2$. In fact $|s|^{-1}f$ also behaves like Γ_g.

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iii. The anomalous quantity: $\widehat{\chi}$ for which only have $\widehat{\chi}\sim a^{\frac{1}{2}}/|s|$. (The signature $+2$ quantity ξ behaves similarly.)

Estimate of curvatures

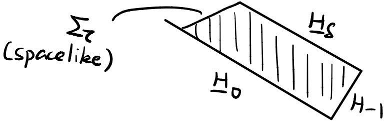
Similar to Dafermos-Rodnianski's r^p estimate, we perform the "s^p estimate".

A model null Bianchi pair

$$\nabla_3\psi_1 - \frac{\lambda}{|s|}\psi_1 = \mathcal{D}^*\psi_2 + \cdots\ ,$$
$$\nabla_4\psi_2 = \mathcal{D}\psi_1 + \cdots\ ,$$

where $\mathcal{D}$, $\mathcal{D}^*$ represent horizontal Hodge operators that are formal adjoint of each other. The corresponding energy estimates for (ψ_1, ψ_2) is derived by integrating the divergence identity

$$\mathbf{Div}(|s|^{2(\lambda-1)}|\psi_1|^2 e_3) + \mathbf{Div}(|s|^{2(\lambda-1)}|\psi_2|^2 e_4) = \cdots$$



Linearized structure

Typical nonlinear term: $\Gamma_g \cdot \psi$ in the $\nabla_3\psi$ equation. We have

$$\Gamma_g \cdot \psi \sim \frac{\delta a^{\frac{1}{2}}}{|s|^2}\psi,$$

Integrating in e_3 essentially reduces one power of $|s|$ in the denominator; and when $|s| \gtrsim \delta a$, we have $\delta a^{\frac{1}{2}}/|s| \lesssim a^{-\frac{1}{2}} \ll 1$.

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- This is why bootstrap argument ("conceptual linearization") works.
- Here the linearized structure is triggered by the largeness of a.

Past null infinity perspective

In many aspects, this is similar to the analysis of gravitational waves towards future null infinity.

- One could think of this as incoming radiation from null infinity (Hence the roles of e_3 and e_4 are reversed). In fact one can obtain a parallel result by simple rescaling (An '19). In our proof, commutation with $s\nabla$ and $s\nabla_3$ are extensively used. (Analogies of $r\nabla$, $r\nabla_4$)

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- The curvature estimates are related to the peeling estimates of curvature components towards past null infinity:

$$\alpha \sim \frac{\delta^{-1}a^{\frac{1}{2}}}{|s|}, \quad \beta \sim \frac{a^{\frac{1}{2}}}{|s|^2}, \quad \rho, \ {}^*\rho \sim \frac{\delta a}{|s|^3}, \quad \underline{\beta} \sim \frac{\delta^2 a^{\frac{3}{2}}}{|s|^4}, \quad \underline{\alpha} \sim \frac{\delta^3 a^2}{|s|^5}.$$

  In fact, the hierarchy of different curvature components is much more fundamental; the Ricci coefficients rely more on the choice of gauge.

Write $\widehat{\chi}_{ab}|_{H_{-1}} = a^{\frac{1}{2}} I(\underline{u}, \theta)_{ab}$. We can show (for simplicity, in the integrable geodesic frame)

$$\operatorname{tr}\chi = 2|s|^{-1} - a \left(\int_0^{\underline{u}} |I(\underline{u}', \theta)|^2 \, d\underline{u}' \right) |s|^{-2} + O(a^{-\frac{1}{2}}),$$

- If $\inf_\theta \int_0^\delta |I(\underline{u}, \theta)|^2 d\underline{u} \geq \delta$, then a trapped surface will form by direct estimates

  $$\operatorname{tr}\chi < 2.1|s|^{-1} - a\delta|s|^{-2} < 0$$

  for $|s| \leq \frac{1}{4} a\delta$.

Formation of trapped surfaces

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for $|s| \leq \frac{1}{4} a\delta$.
- Anisotropic situations can also be recovered by directly applying the argument of Klainerman-Luk-Rodnianski (2013).

Remarks

Motivated by the proof of Kerr stability with small angular momentum.

- PG frame (the original geodesic frame): do not care about the loss of derivatives; good for getting also the decay in the transversal direction (which is non-existent in our case);
- PT frame: close the top order estimates for Ricci coefficients.

In our case, the PG frame is even less used as we do not need to be as precise with the decay in $\underline{u}$.

The PT frame in Klainerman-Szeftel satisfies (the interior part; similarly for the exterior part)

$$H^{(PT)} := \eta^{(PT)} + i \, {}^*\eta^{(PT)} = \frac{aq}{|q|^2}\mathfrak{J}$$

(the corresponding value in exact Kerr)

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- In the context of Kerr stability, the extreme curvature components α, $\underline{\alpha}$ are invariant under frame transformations at linear level;
- In our case, all curvature components are invariant at linear level (in our sense);
- Our proof only requires integration in e_4 on the initial null hypersurface H_{-1} and for energy estimates;
- As a result, only energy (curvature) bound is needed for the incoming initial data. One hope would be to generalize this to more complicated incoming data, e.g., naked singularity solutions.

Thank You!