Mass for the LARGE and for the small

CMSA General Relativity seminar

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SL Math

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Outline

Curvatures I

Let $p \in (M^n, g)$. Scalar curvature R reflects volume. For small ε,

$$\frac{Vol_g(B_\varepsilon(p))}{Vol_{g_{Euc}}(B_\varepsilon)} = 1 - \frac{R(p)}{6(n+2)}\varepsilon^2 + O(\varepsilon^4).$$

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i.e. if $R > 0$

$$Vol_g(B_\varepsilon(p)) < Vol_{g_{Euc}}(B_\varepsilon).$$

Curvatures II

Let Σ^{n-1} be a closed hypersurface in (M^n, g), let $\{\Sigma_t\}$ be a smooth family of surfaces such that $\Sigma_0 = \Sigma$. Mean curvature H reflects the change of area with unit speed ν,

$$\frac{d}{dt}|_{t=0} Area(\Sigma_t) = \int_\Sigma H.$$

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Hence, both the ambient space and the direction matter.

Initial data sets

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1. M: an n-dimensional manifold,
2. g: a Riemannian metric,
3. k: a symmetric (0,2)-tensor.

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Cauchy problem is to construct a Lorentzian manifold $(\overline{M}^{n+1}, \overline{g})$ such that

1. $\bar{g}$ satisfies the Einstein Equation

$$G_{\mu\nu}(:= \overline{R}_{\mu\nu} - \frac{1}{2}\overline{R}_{\overline{g}}\overline{g}_{\mu\nu}) = T_{\mu\nu},$$

2. there exists an isometric embedding $(M, g) \subset (\bar{M}, \bar{g})$,
3. k is the second fundamental form of M with respect to $\bar{M}$.

Notations

Let (M^n, g) be a Riemannian manifold and $\Sigma^{n-1} \subset M^n$ be a 2-sided submanifold.

Definition

1. *2nd fundamental form:* $h(X, Y) = g(\nabla_X N, Y)$ *for* $X, Y \in T\Sigma$, *where* N *is the normal of* Σ *with respect to* (M, g);
2. *Mean curvature:* $H = tr_g h$;
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3. Σ *is called a minimal surface if* $H \equiv 0$ *on* Σ*;*
4. *If* k *is taken into consideration,* Σ *is called*
    1. *a marginally outer trapped surface (MOTS) if* $H + tr_\Sigma k \equiv 0$ *on* Σ*,*
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5. $\pi = k - (tr_g k)g$ *is called the conjugate momentum tensor.*

Energy Density

Definition

Given an initial data set (M, g, k), the mass density μ and the current density J are defined by

$$\mu = T_{00}|_M,$$
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Dominant energy condition (DEC) is satisfied if $\mu \geq |J|_g$.

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$$\begin{aligned} \mu = G_{00} =& \frac{1}{2}(R_g - |k|_g^2 + (\mathrm{tr}_g k)^2), \\ J = G_{0\cdot}|_M =& \mathrm{div}_g k - d(\mathrm{tr}_g k) \\ =& \mathrm{div}_g \pi. \end{aligned}$$

Dominant energy condition (DEC) is satisfied if $\mu \geq |J|_g$.

Asymptotical Flatness

An initial data set (M, g, k) is asymptotically flat if

1. there is a compact set $K \subset M$ and a coordinate chart (x^i) such that
$$M \setminus K \cong \mathbb{R}^n \setminus B$$
for some closed ball $B \subset \mathbb{R}^n$, under which as $|x| \to \infty$,
$$g \to g_{\mathbb{E}}, k \to 0.$$
2. $\mu, J \in L^1(M)$.

ADM energy-momentum vector

For asymptotically flat (M, g, k), the *ADM energy* E and the *ADM momentum* P can be defined as

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_0^j \, d\mathcal{H}_0^{n-1}$$

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{j=1}^{n} \pi_{ij}\nu_0^j \, d\mathcal{H}_0^{n-1}, \ \ i = 1, 2, \ldots, n.$$

The integrals are computed in the asymptotically flat coordinates,

1. $\nu_0^j = \frac{x^j}{|x|}$,
2. $\mathcal{H}_0^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and
3. ω_{n-1} is the volume of the standard unit sphere in $\mathbb{R}^n$.

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Bartnik and Chruściel independently proved the definition is independent of the choice of A.F. coordinates.

Positive Mass Theorem

Theorem

Let (M^3, g, k) be an asymptotically flat initial data set. If the dominant energy condition ($\mu \geq |J|_g$) is satisfied, then

$$E \geq |P|,$$

where (E, P) is the ADM energy $-$ momentum vector of (M, g, k).

If $E = |P|$, then $E = |P| = 0$ and (M, g, k) is a slice of $\mathbb{R}^{3,1}$.

Positive Mass Theorem

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Let (M^3, g) *be an asymptotically flat manifold with* $R_g \geq 0$*, then*

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If $E = 0$*, then* (M, g) *is diffeomorphic to* $\mathbb{R}^3$.

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Schoen-Yau, Witten, Bartnik, Jezierski-Kijowski, Huisken-Ilmanen, Eichmair-Huang-Lee-Schoen, Y. Li, Bray-Kazaras-Khuri-Stern, Hirsch-Kazaras-Khuri, Agostinani-Mazzieri-Oronzio, Miao...

Schoen-Yau Proof I: Area minimisers

1. By density theorem, we can further assume g is with $R_M > 0$ outside a compact set and asymptotically Schwarzschild, i.e.

$$g_{ij} = (1 + \frac{E}{2|x|})^4 \delta_{ij} + O^2(|x|^{-2})$$

with $R_M > 0$ on $\{|x| \geq C\}$ for some $C >> 1$.

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2. Assume on the contrary that $E < 0$, with respect to ∂_3,

$$\begin{aligned} H_{\{x^3=\Lambda\}} =& 4(1 + O(|x|^{-1}))[-\frac{E}{2}|x|^{-3}x^3|_{\{x^3=\Lambda\}} + O(|x|^{-3})] \\ =& -2E|x|^{-3}\Lambda + O(|x|^{-3}) > 0 \end{aligned}$$

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if $\Lambda >> 1$.
Similarly, $H_{\{x^3=-\Lambda\}} < 0$. These planes thus form barriers.

Schoen-Yau Proof I: Area minimisers

1. By geometric measure theory, one can obtain an area minimising surface, which is asymptotic to a horizontal plane.
2. Stability and Gauss equation implies for all $\psi \in C_c^1(\Sigma)$,

$$\int_\Sigma |\nabla^\Sigma \psi|^2 - \frac{1}{2}\left(R_M - R_\Sigma + |h|^2\right)\psi^2 d\mathrm{vol}_\Sigma \geq 0. \qquad (1)$$

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3. By logarithmic cut-off trick ("take $\psi \equiv 1$"), we have

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4. On the other hand, the total geodesic curvature of $\Sigma \cap \{|x| = r\} \to 2\pi$ as $r \to \infty$. Hence, by Gauss-Bonnet Theorem $\int_\Sigma \frac{1}{2} R_\Sigma d\mathrm{vol}_\Sigma = 0$. Contradiction.

Schoen-Yau Proof II: the Jang equation

Recall $(M, g \to g_{Euc}, k \to 0)$. By density theorem, $\mu > |J_g|$. E almost the same.

1. Considering the model case, a spacelike slice in Minkowski space with f being the height. W.L.O.G, $f \to 0$.

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4. Ideally, we want to solve f such that the graph would have second fundamental form the same as k.

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3. One can explicitly closes up the cylinder $\Sigma_{\check{B}} \times \mathbb{R}_+$ to obtain an AF manifold $(\tilde{M}, \tilde{g})$ with $R_{\tilde{g}} > 0$ while $\tilde{E}$ is close to $\hat{E}$.

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4. $\tilde{E} \geq 0$ by the Riemannian PMT.

Witten Proof: Dirac operator

Theorem

Let (M, g) be an asymptotically flat and spin manifold. If $R_g \geq 0$, then there exists asymptotically constant harmonic spinor $(D\psi = 0)$ such that

$$c_n E \geq \int_M |\nabla\psi|^2 + \frac{1}{4}R|\psi|^2.$$

Localisation of mass

For (Σ^2, g), we have Gauss-Bonnet Theorem

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Assuming $\Sigma^2 \cong \mathbb{D}^2$, rearranging,

$$\int_\Sigma K = 2\pi - \int_{\partial\Sigma} \kappa = \int_{\partial\Sigma} \kappa_0 - \int_{\partial\Sigma} \kappa,$$

where κ_0 is the geodesic curvature of a length preserving embedding of $\partial\Sigma$ into $\mathbb{R}^2$.

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If $K \geq 0$, then $\int_{\partial\Sigma} \kappa_0 - \kappa \geq 0$.

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What happens in higher dimensions?

Quasilocal mass

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Quasilocal mass

General relativity is different from Newtonian mechanics, "local" mass cannot be defined.
By Hamiltonian formulation with the choice of $V = 1, W = \vec{0}$, Brown-York defined a quasilocal mass as follows.

Definition (Brown-York mass)

Let (Ω^n, g) be a compact Riemannian manifold with smooth boundary $\partial\Omega$. If $\partial\Omega$ can be isometrically embedded into $\mathbb{R}^n$, then

$$m_{BY}(\Omega) := \frac{1}{(n-1)|\mathbb{S}^{n-1}|} \int_{\partial\Omega} H_0 - H \, dA,$$

where H and H_0 respectively denote the mean curvature of g and g_{Euc} with respect to the outward pointing normal.

Hamiltonian formulation

Let (Ω^n, g, k) be a compact initial data set with boundary Σ.

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Hamiltonian formulation

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A spacetime $(\bar{\Omega}^{n+1}, \bar{g})$ with boundary $\bar{\Sigma}$ can be constructed by infinitesimally deforming the initial data set (Ω, g, k, Σ) in a transversal, timelike direction $\partial_t = V\vec{n} + W$, where

1. V is the lapse function,
2. $\vec{n}$ is the timelike unit normal of Ω in $\bar{\Omega}$, and
3. $W \in T\Omega$ is the shift vector.

Hamiltonian formulation

Further assume that Ω meets $\bar{\Sigma}$ orthogonally. The purely gravitational contribution $\mathcal{H}_{grav}$ to the total Hamiltonian at the slice Ω is given by

$$c(n)\mathcal{H}_{grav}(V, W) = \int_\Omega (\mu V + \langle J, W \rangle) - \int_\Sigma (HV - \pi(\nu, W)),$$

where H is the mean curvature of Σ with respect to ν, the outward normal of Ω, and π is the conjugate momentum tensor.

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where H is the mean curvature of Σ with respect to ν, the outward normal of Ω, and π is the conjugate momentum tensor.
In particular, if $k \equiv 0$,

$$c(n)\mathcal{H}_{grav}(1, \vec{0}) = \int_\Omega \frac{1}{2} R_g - \int_\Sigma H.$$

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$$c(n)\mathcal{H}_{grav}^{ref}(V, W) = \int_{\Omega}(\mu V + \langle J, W\rangle) - \int_{\Sigma}(H^{ref} V - \pi^{ref}(\nu, W)).$$

Quasilocal mass

By Hamiltonian formulation, Wang-Yau defined a quasilocal mass with the choice of $V = \sqrt{1 + |\nabla^\Sigma \tau|^2}$, $W = -\nabla^\Sigma \tau$.

Definition (Wang-Yau mass, cf. Kijowski-Liu-Yau mass)

Let $\Sigma \subset L$ be a spacelike 2 surfaces which bounds a compact spacelike surface. If Σ has a shadow Σ_τ which can be isometrically embedded into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$, then

$$\begin{aligned}
&8\pi m_{WY}(\Sigma) \\
:= &\inf_\tau \Big[\int_{\Sigma_\tau} H_\tau - \int_\Sigma \inf_{e_3} -\sqrt{1 + |\nabla^\Sigma \tau|^2}\langle \vec{H}, e_3\rangle - \langle \nabla_{-\nabla^\Sigma \tau} e_4, e_3\rangle \, dA\Big] \\
= &\inf_\tau \Big[\int_{\Sigma_\tau} H_\tau - \int_\Sigma \inf_{e_3} (HV - \pi(W, e_3)) \, dA\Big],
\end{aligned}$$

where H_τ and H respectively denote the mean curvature of $\Sigma_\tau \subset (\mathbb{R}^3, g_{Euc})$ and Σ with respect to the outward pointing normal and e_3.

Quasilocal mass

By Hamiltonian formulation, there could be a more direct comparison with a choice of a null-vector, i.e. "$(V, W) = (1, 1)$".

Definition (T)

Let (Ω^n, g, k) be a compact initial data set with smooth boundary $\partial\Omega$. If $\partial\Omega$ can be isometrically embedded into $\mathbb{R}^n$, then

$$\mathcal{W}(\Sigma) := \frac{1}{(n-1)\omega_{n-1}} \int_{\partial\Omega} H_0 - (H - |\pi(\cdot, \nu)|)\, dA,$$

where H and H_0 respectively denote the mean curvature with respect to the outward pointing normal of g and g_{Euc}.

Quasilocal mass

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Theorem (Shi-Tam 02)

Let (Ω^n, g) be a compact Riemannian (spin) manifold with smooth boundary $\Sigma := \partial\Omega$. If

1. $H > 0$ *on* Σ,
2. Σ *can be isometrically embedded into* $\mathbb{R}^n$ *as a strictly convex hypersurface, and*
3. $R_g \geq 0$ *in* Ω,

then

$$m_{BY}(\Omega) := \frac{1}{(n-1)|\mathbb{S}^{n-1}|}\int_{\partial\Omega} H_0 - H \, dA \geq 0.$$

Moreover, if $n = 3$, $\lim_{r\to\infty} m_{BY}(S_r) = E$.

Idea of proof

1. Σ can be isometrically embedded into $\mathbb{R}^n$ as a strictly convex hypersurface. (For n=3, Pogorelov, Nirenberg.)
2. Hence, the flat metric on the exterior can be expressed as $d\rho^2 + \rho^2 g_\Sigma$, $\rho \in [1, \infty)$.

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3. Look for $u(\rho)^2 d\rho^2 + \rho^2 g_\Sigma =: g_{BST}$ such that $R_{g_{BST}} = 0$, $u(1) = \frac{H_0}{H}$ and $\lim_{\rho \to \infty} u(\rho) = 1$.

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4. $Q(\rho) := \int_{\Sigma_\rho} H_\rho^0 (1 - \frac{1}{u})$ is monotonically decreasing in ρ. Note that $Q(1) = m_{BY}(\Omega)$ and $\lim_{\rho\to\infty} Q(\rho) = E_{g_{BST}}$.

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4. $Q(\rho) := \int_{\Sigma_\rho} H^0_\rho (1 - \frac{1}{u})$ is monotonically decreasing in ρ. Note that $Q(1) = m_{BY}(\Omega)$ and $\lim_{\rho\to\infty} Q(\rho) = E_{g_{BST}}$.
5. $(\Omega \cup M_{ext}, g \cup g_{BST})$ has a corner along $\partial\Omega$. (cf. Miao)

Quasilocal mass

One also wants to show Kijowski-Liu-Yau mass and Wang-Yau mass under certain condition.

1. $\mu \geq |J|_g$.
2. $H > |tr_\Sigma k|$.

The Jang equation is used to reduce to usual time symmetric case. Then use spinor (recall: $R_g + 2div_g X \geq 2|X|^2$) on Bartnik-Shi-Tam construction.

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1. Unfortunately, we cannot always get energy as we like. (Corvino, Corvino-Huang, e.g. Miao, cf. Schoen-Yau, EHLS)
2. May not capture the positivity of a Schwarzschild spacetime.

PMT with corners with corners

Theorem (T)

If (Ω^3, g_-, k_-) and $(M^3 \setminus \Omega, g_+, k_+)$ satisfy the following:

1. $(g_-, k_-), (g_+, k_+)$ *are* $C^2 \times C^1$ *up to* $\Sigma := \partial\Omega$,
2. (g_+, k_+) *is asymptotically flat,*
3. g_-, g_+ *Lipschitz across* Σ, *and*
4. $H_2(M, \mathbb{Z}) = 0$,

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then there exists a spacetime harmonic coordinate u *such that*

$$
\begin{aligned}
8\pi(E - |P|) \geq & \frac{1}{2}\int_{M\setminus\Sigma} \frac{|\overline{\nabla\nabla} u|^2}{|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u\rangle \, dV \\
& + \int_{\Sigma} (H_- - H_+)|\nabla u| - (\pi_- - \pi_+)(\nabla u, \nu) dA.
\end{aligned}
$$

If $\mu \geq |J|_g$, $H_- - H_+ \geq |\pi_- - \pi_+(\cdot, \nu)|$ *and* $E = |P| = 0$, *then* M *is topologically* $\mathbb{R}^3$ *and can be isometrically embedded into* $\mathbb{R}^{3,1}$.

Solution I

Theorem (T)

Let (Ω^n, g, k) be a compact initial data set with smooth boundary $\Sigma := \partial\Omega$. If

1. $H - |\pi(\cdot, \nu)| > 0$ *on* Σ *(boundary strict DEC),*
2. Σ *can be isometrically embedded into* $\mathbb{R}^n$ *as a strictly convex hypersurface, and*
3. $\mu - |J|_g \geq 0$ *in* Ω*,*
4. $H_2(\Omega^3, \mathbb{Z}) = 0$ *or* Ω^n *is spin.*
5. Ω *contains apparent horizons.*

then

$$\mathcal{W}(\Sigma) := \frac{1}{(n-1)|\mathbb{S}^{n-1}|} \int_{\partial\Omega} H_0 - (H - |\pi(\cdot, \nu)|)\, dA \geq 0.$$

Solution II: Get asymptotics on the product

Andersson-Yau-Zhao:

1. Existence of spinors by weighted space (Parker-Taubes, Bartnik)
2. If the MOTS is strictly stable, then by Metzger, Yu, $\Sigma_{\check{B}} \times \mathbb{R}_+$ has an explicit asymptotics.
3. Design weighted space carefully.

Solution III: XL Positive mass theorem

Theorem

If (M, g) is compete with an asymptotically flat end $\mathcal{E}$ and $R_g \geq 0$, then $E(\mathcal{E}) \geq 0$.

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1. Comes from Schoen-Yau Liouville theorem for conformally flat manifolds.
2. Different approaches and strategies:
    1. Lesourd-Unger-Yau, Lee-Lesourd-Unger (soap bubbles)
    2. Cecchini-Ziedler (spinor)
    3. Chen-Liu-Shi-Zhu, Zhu (reduction to Geroch conjecture with completeness)

Solution III: XL Positive mass theorem

Based on Wang-Yau and Cecchini-Ziedler, we have

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If(M, g) is compete with corners and an asymptotically flat end $\mathcal{E}$, and satisfies $R_g + 2div_g X \geq 2|X|_g^2$, then $E(\mathcal{E}) \geq 0$.

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Note: With Cecchini-Lesourd-Ziedler, one can discuss W on complete spin initial data sets.

Idea of proof

Key: construct a suitable function h for the barrier on exhaustion M_j.

$$c_n E(g) + ||\mathcal{B}_h \psi||^2_{L^2(M_j)} \geq \frac{1}{2}||\nabla\psi||^2_{L^2(M_j)} + \int_{M_j} \theta_h^X |\psi|^2 + \int_{\partial M_j} \eta_h^X |\psi|^2.,$$

where

1. $\mathcal{B}_h = D + he_0\cdot$,
2. $\theta_h^X = \frac{1}{4}(R_g + 2div_g X - 2|X|_g^2) + h^2 - |\nabla h|^2$,
3. $\eta_h^X = \frac{1}{2}\langle X, \nu\rangle + h$.

Corollary

Weighted positive mass theorem ($X = \nabla f$) (Baudalf-Ozuch) and positive mass theorem with electric fields ($X = E$) (Gibbons-Hawking-Horowitz-Perry) with arbitrary complete ends.

THANK YOU!!

Boundary and interior geometry

For (Σ^2, g), we have Gauss Bonnet Theorem

$$\int_\Sigma K + \int_{\partial\Sigma} \kappa = 2\pi\chi(\Sigma).$$

If (Σ, g) has $K \geq 0$, then $\int_{\partial\Sigma} \kappa$ is bounded.

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Q: What about higher dimensions?

Gromov fill-in conjecture

Conjecture (Gromov)

Let (M, g) be a compact Riemannian manifold with scalar curvature $R \geq \sigma$. Then there exists Λ depending only on σ and the intrinsic geometry of $(\partial M, g|_{T(\partial M)})$ such that

$$\int_{\partial M} H \leq \Lambda,$$

where H is the mean curvature of the boundary ∂M in (M, g) with respect to the outward unit normal vector.

Recall: Hamiltonian formulation

The purely gravitational contribution $\mathcal{H}_{grav}$ to the total Hamiltonian at the slice Ω is given by

$$c(n)\mathcal{H}_{grav}(V, W) = \int_\Omega (\mu V + \langle J, W\rangle) - \int_\Sigma (HV - \pi(\nu, W)).$$

Bartnik data set and Fill-in

Definition

$(\Sigma^{n-1}, \gamma, \alpha, H, \beta)$ is called a spacetime Bartnik data set (D_{SB}),

1. *(Σ, γ, α) is an oriented closed null-cobordant initial data set.*
2. *H is a smooth function on Σ.*
3. *β is a 1-form on $T\Sigma$.*

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Definition

A compact initial data set (Ω^n, g, k) is called a fill-in of D_{SB} if there is an isometry $\phi : (\Sigma^{n-1}, \gamma) \to (\partial\Omega, g|_{\partial\Omega})$ such that

1. *$\phi^* H_g = H$, where H_g is the mean curvature of $\partial\Omega$ to g with respect to the outward unit normal ν,*
2. *$\phi^* tr_{\partial\Omega} k = tr_\Sigma \alpha$, and*
3. *$\phi^*(k(\nu, \cdot)) = \beta$.*

Non-existence of NNSC Fill-in

Theorem (Shi-Wang-Wei-Zhu)

For $3 \leq n \leq 7$ and $k \geq 5$, let γ be a smooth metric isotopic to γ_{std} in $\mathcal{M}^k_{psc}(S^{n-1})$. Then there exists a constant $h_0 = h_0(\gamma)$ such that (S^{n-1}, γ, H) admits no fill-in of nonnegative scalar curvature whenever $H > 0$ and $\int_{S^{n-1}} H \, d\mu_\gamma > h_0$.

Theorem (Shi-Wang-Wei)

Let $3 \leq n \leq 7$ and Σ^{n-1} be a closed manifold which can be smoothly embedded into $\mathbb{R}^n$. For any metric γ on Σ, there is a constant $C_0(\Sigma, \gamma)$ depending only on (Σ, γ) such that for any smooth function H on Σ satisfying $H > C_0(\Sigma, \gamma)$, (Σ, γ, H) admits no fill-in of nonnegative scalar curvature.

Non-existence of NNSC Fill-in

Theorem (Shi-Wang-Wei)

Let Ω^n be a compact n-dimensional manifold with boundary Σ. Then any metric γ on Σ can be extended to a Riemannian metric g on Ω with positive scalar curvature.

Theorem (Miao)

Let Σ^{n-1} be the boundary of some compact n-dimensional manifold Ω. Given any Riemannian metric γ on Σ, there exists a constant $C_0 = C_0(\gamma, \Omega)$ such that, if $\min_\Sigma H \geq C_0$, there do not exist NNSC fill-ins of (Σ, γ, H).

Extended Gromov's conjecture

Conjecture

Let (M, g, k) be a compact initial data set satisfying $\mu - |J|_g \geq \sigma$. Then there exists Λ depending only on σ and $g|_{T(\partial M)}$ and $k|_{T(\partial M)}$ such that

$$\int_{\partial M} H - |\pi(\cdot, \nu)| \leq \Lambda,$$

where H is the mean curvature of the boundary ∂M in (M, g) with respect to the outward unit normal vector.

Non-existence of DEC Fill-in

Theorem (T)

Let $D_{SB} := (\Sigma^2, \gamma, \alpha, H, \beta)$ be a spacetime Bartnik data set where Σ^{n-1} can be smoothly embedded into $\mathbb{R}^3$ and γ is smooth. There exists a constant $C_0 = C_0(\Sigma, \gamma) > 0$ such that if

$$H - f \geq C_0,$$

where $f := \sqrt{(\mathrm{tr}_\Sigma \alpha)^2 + |\beta|^2_\gamma} = \phi^*(|\pi(\cdot, \nu)|)$, then D_{SB} cannot admit a fill-in satisfying the dominant energy condition.

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Built upon Cecchini-Lesourd-Ziedler, fill-ins can be complete non-compact or with an energy shield. (cf. Lee-Lesourd-Unger).

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Raulot: 1st eigenvalue estimate of the Dirac operator to establish non-existence of DEC spin compact fill-in.

Energy and Geometry

1. C^0 characterisation of dominant energy condition
2. Boundary and interior geometry
3. Localised mass
4. *ADM* mass

THANK YOU!!

Sketch of proof: cf. Shi-Wang-Wei

1. Let $F : \Sigma^{n-1} \hookrightarrow \mathbb{R}^n$ denote an embedding. There exists a $\lambda_0 > 0$ such that $\gamma_1 := \lambda_0^2 F^*(g_{Euc}) > \gamma$, where g_{Euc} is the Euclidean metric on $\mathbb{R}^n$.

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2. Let $\tilde{h}$ denote the mean curvature of γ_1 with respect to the outward normal in $\mathbb{R}^n$. Denote the unbounded region of $\mathbb{R}^n$ outside $\lambda_0 F(\Sigma)$ by M_+.

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3. By Shi-Wang-Wei Lemma 2.1, we know that there exists a cobordism $(\Sigma \times [0,1], \hat{g})$ and $h_0, h_1 \in C^\infty(\Sigma)$ such that
    1. $\hat{g}|_{\Sigma\times\{0\}} = \gamma$ and $\hat{g}|_{\Sigma\times\{1\}} = \gamma_1$,
    2. With respect to $\hat{g}$ and the outward normal, the mean curvature of $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ are respectively h_0 and h_1,
    3. $h_1 > \tilde{h}$ and
    4. $R_{\hat{g}} > 0$.

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1. Pick $C_0 = \max(-h_0)$. Assume on the contrary that there exists (Ω, g, k) be a fill-in of $(\Sigma, \gamma, \alpha, H, \beta)$, where $H - f \geq C_0$. Then glue $\partial\Omega$ to $\Sigma \times [0,1]$ along $\Sigma \times \{0\}$ and further glue $\Sigma \times [0,1]$ along $\Sigma \times \{1\}$ to M_+.

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2. Altogether, we have an A.F. initial data set with a flat end and hence with $E_{ADM} = 0$. Across the corners $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$, we respectively have $H - (-h_0) - f \geq 0$ and $h_1 - \tilde{h} > 0$. Since (Ω, g, k) satisfies DEC, $E_{ADM} > 0$. Contradiction arises.

Connection with Spacetime PMT

With Lohkamp's construction of $R_g > 0$ island. (cf. Schoen-Yau 17)

Theorem (Li)

Dihedral rigidity implies the Riemannian positive mass theorem.

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Proposition (T)

Dihedral rigidity implies the spacetime positive mass theorem.

Alternative proof of SPMT in dimension 3

1. Assume on the contrary, $E < |P|$. Furthermore, by Christodoulou and O'Murchadha's boost argument, w.lo.g. $E < 0 \leq |P|$. Following Lohkamp's argument, one can construct a $(\tilde{\mu} - |\tilde{J}|_{\tilde{g}}) > 0$-island $(\tilde{M}, \tilde{g}, \tilde{k})$, i.e. there exists a non-empty open set $U \subset M$ with compact closure such that
    1. $\tilde{\mu} - |\tilde{J}|_{\tilde{g}} > 0$ on U, and
    2. $(M \setminus U, \tilde{g}, \tilde{k}) \equiv (\mathbb{R}^n \setminus B_r(0), g_{Euc}, 0)$.

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2. Now, the construction of the generalised exterior region (Hirsch-Kazaras-Khuri, Andersson-Dahl-Galloway-Pollack (Eichmair-Huang-Lee-Schoen 16, Schoen-Yau 81)) can be carried out on $(\tilde{M}, \tilde{g}, \tilde{k})$. In particular, there exists $(\hat{M}, \hat{g}, \hat{k})$ with boundary $\partial\hat{M}$ composed of MOTS and MITS such that $H_2(\hat{M}, \partial\hat{M}, \mathbb{Z}) = 0$.

Alternative proof of SPMT in dimension 3

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2. Then, let P^3 be a cube. As aforementioned, consider a large scaling of P such that ∂P can be isometrically embedded into $(\hat{M} \setminus \hat{U}, \hat{g}, \hat{k})$ and encloses $\hat{U}$. Let $\hat{\Omega}$ denote the region in $\hat{M}$ bounded by ∂P. Hence, we have $H_2(\hat{\Omega}, \partial \hat{M}, \mathbb{Z}) = 0$.

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3. By controlling the normal derivative, we can control the sign of the extra boundary integral. $\hat{\mu} = |\hat{J}|_{\hat{g}} = 0$. Contradiction.

Charged initial data sets

For a charged initial data set $(M, g, \mathcal{E})$, where $\mathcal{E}$ is a divergence free electric field. By considering charged harmonic functions proposed by Bray-Hirsch-Kazaras-Khuri-Zhang,

$$\Delta u - \langle \mathcal{E}, \nabla u \rangle = 0,$$

the ideas can be applied.

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