The Einstein-Euler system with a physical vacuum boundary in spherical symmetry.

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Department of Mathematics, Vanderbilt University.
Joint work Jared Speck.

General Relativity Seminar,

Center of Mathematical Sciences and Applications, Harvard University,

Cambridge, MA, November 2024

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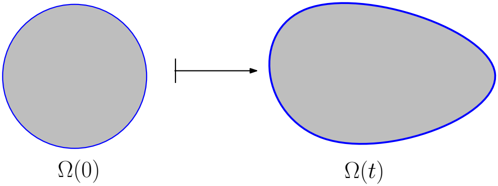

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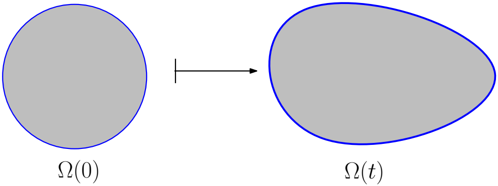

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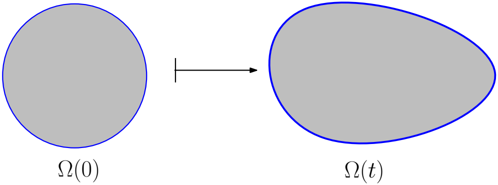

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Fluids of this type are called **free-boundary** fluids.

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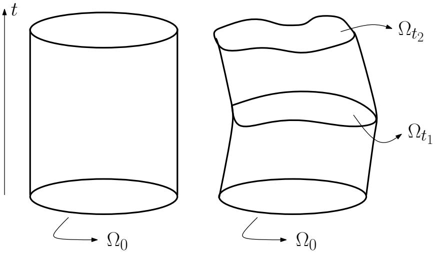

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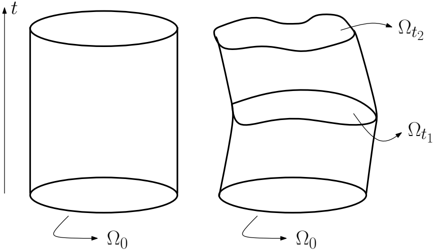

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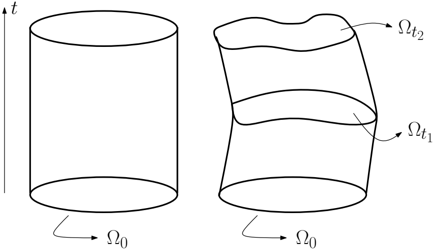

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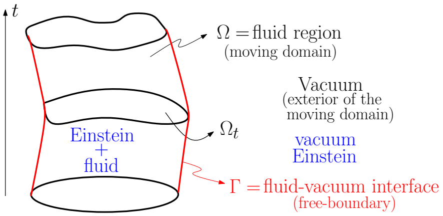
Understanding the dynamics of Γ is essential in free-boundary problems.

In general relativity, stars are modeled as free-boundary fluids coupled to gravity (Einstein's equations).



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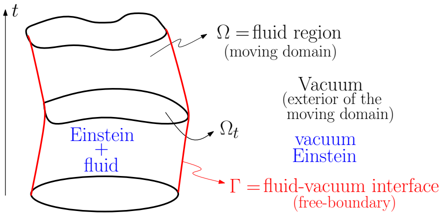

Equations, **free-boundary Einstein-fluid system:**

$$\mathrm{Ric}(g) - \frac{1}{2}R(g)g = \mathcal{T} \quad \text{in } \Omega,$$

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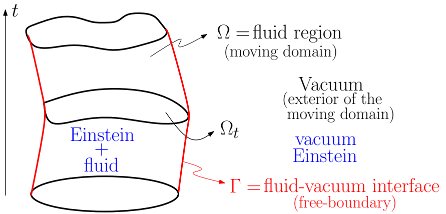

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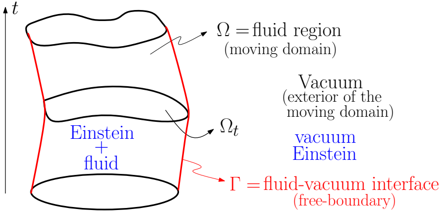

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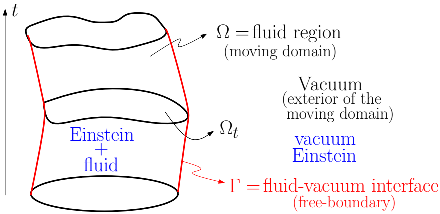

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- Oppenheimer-Snyder ('39): gravitational collapse (and black-hole formation) of a pressureless spherically symmetric star.

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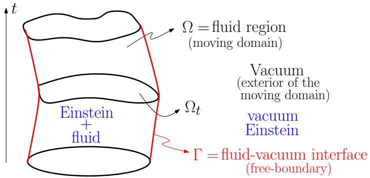

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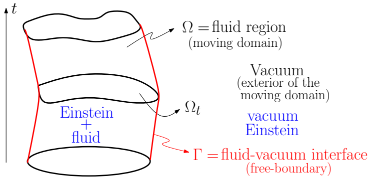

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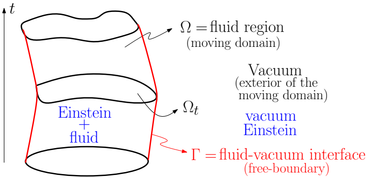

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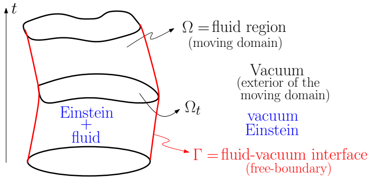

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(1b) is a constraint propagated by the flow.

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(Kinematic boundary condition: u is tangent to Γ.)

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- A faster decay would cause the boundary to simply move independently and linearly with the outer particle's speed.
- A slower decay rate would lead to a very singular problem, likely causing an infinite initial acceleration of the boundary.

Statement of the problem

Given (initial data):

- $\mathring{\varrho}\colon \mathbb{R}^3 \to \mathbb{R}_+$, $\mathring{\Omega} := \{\mathring{\varrho} > 0\}$; $\mathring{c}_s^2 \approx \mathrm{dist}(\cdot, \partial\mathring{\Omega})$ ($c_s^2 = p'$, $p = p(\varrho)$).
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- $\varrho\colon [0, T] \times \mathbb{R}^3 \to \mathbb{R}_+$,
- $u\colon \Omega \to \mathbb{R}^4$, $\Omega := \{\varrho > 0\} \subset [0, T] \times \mathbb{R}^3$,
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such that the Einstein-Euler system with a physical vacuum boundary,

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Punchline of the talk: we solve this problem in spherical symmetry.

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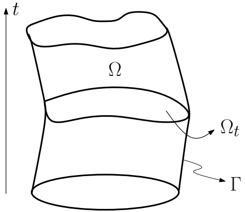

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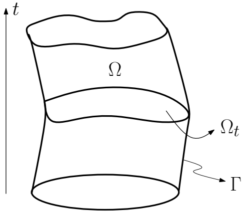

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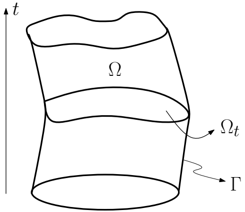

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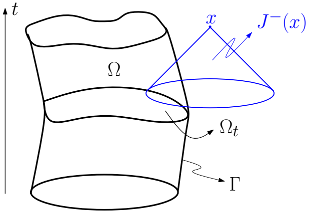

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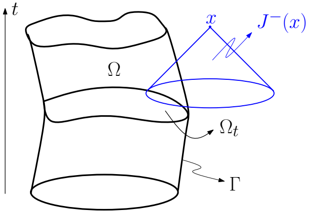

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No boundary condition for $g|_\Gamma$; need to understand it dynamically.

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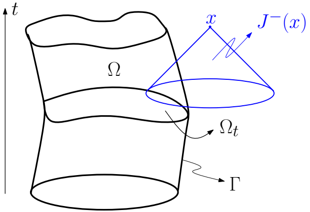

Sharp transition from matter to vacuum. $\mathrm{Ric}(g) - \frac{1}{2}R(g)g = \chi_\Omega \mathcal{T}$.

Quasilinear problem with a discontinuous (or Hölder) source term.

Domain of dependence: coupling between fluid and vacuum region.

No boundary condition for $g|_\Gamma$; need to understand it dynamically.

Further difficulties for the fluid dynamics ($\mathrm{div}_g \mathcal{T} = 0$ in Ω).

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New variables: Convenient to introduce $h := \frac{1}{\kappa} c_s^2 = \frac{\kappa+1}{\kappa} \varrho^\kappa$ and $v := \langle h \rangle^{1+\frac{1}{\kappa}} u$, where $\langle h \rangle := 1 + \frac{\kappa}{1+\kappa} h$, as a variable and work with (h, v, g) instead of (ϱ, u, g).

Admissible data

An initial-data set $\mathcal{I} = (\Sigma, \mathring{h}, \mathring{v}, \mathring{g}, \mathring{K})$ for the spherically symmetric Einstein-Euler system is called **admissible** if:

- $(\Sigma, \mathring{g})$ is spherically symmetric.

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- For the radius and mass of the matter region at $t = 0$,

$$\mathring{R} = \text{radius of } \mathring{\Omega} := \{\mathring{h} > 0\}, \quad \mathring{M} = 4\pi \int_0^{\mathring{R}} r^2 \mathring{\varrho}(r)\, dr,$$

(recall $h := \frac{1}{\kappa} c_s^2 = \frac{\kappa+1}{\kappa} \varrho^\kappa$), we have that $1 - \frac{2\mathring{M}}{\mathring{R}} > 0$.

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For admissible data:

- Schwarzchild gauge is well defined.
- $\mathring{M} \leq \frac{4\mathring{R}}{9}$, a well-known limit derived form the TOV equations.

Function spaces for our *degenerate* problem

Weighed Sobolev space $H_h^{N,\sigma}(\Omega_t)$: distributions on Ω_t with finite norm

$$\|\mathrm{f}\|^2_{H_h^{N,\sigma}(\Omega_t)} := \sum_{|\alpha|\leq N} \|h^\sigma \partial^\alpha \mathrm{f}\|^2_{L^2(\Omega_t)}.$$

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For triplets (h, v, g), set (part of g only in the moving domain Ω):

$$\mathcal{H}_h^{2N}(\Omega_t) = H_h^{2N, N+\frac{1-\kappa}{2\kappa}}(\Omega_t) \times H_h^{2N, N+\frac{1-\kappa}{2\kappa}+\frac{1}{2}}(\Omega_t) \times H_h^{2N, N+\frac{1-\kappa}{2\kappa}+\frac{1}{2}}(\Omega_t)$$

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- κ comes form the equation of state $p(\varrho) = \varrho^{\kappa+1} \sim h^{1+\frac{1}{\kappa}}$.
- $+\frac{1}{2}$ in the second factor comes from structure of the equations.

Theorem: local well-posedenss, I

Let $\mathcal{I} = (\Sigma, \mathring{h}, \mathring{v}, \mathring{g}, \mathring{K})$ be an admissible initial-data set for the Einstein-Euler system with a physical vacuum boundary in spherical symmetry, with $\mathring{\Omega} := \{\mathring{h} > 0\}$ bounded and $\mathring{h} \approx \mathrm{dist}(\cdot, \partial\mathring{\Omega})$. Suppose that $(\Sigma \backslash \mathring{\Omega}, \mathring{g})$ is isometric to the exterior region of a slice of a Schwarzschild spacetime with mass $\mathring{M}$ and radius $\mathring{R}$, where $\mathring{M}$ is the mass of $\mathcal{I}$ and $\mathring{R}$ is the radius of $\mathring{\Omega}$. Assume, for $N > 2.5 + \frac{1}{\kappa}$,

$$(\mathring{h}, \mathring{v}, \mathring{g}) \in \mathcal{H}^{2N}_{\mathring{h}}(\mathring{\Omega}),\ \mathring{g} \in C^{1,\frac{1}{\kappa}}(\Sigma), \quad \mathring{K} \in \mathcal{H}^{2N-1}_{\mathring{h}}(\mathring{\Omega}),\ \mathring{K} \in C^{0,\frac{1}{\kappa}}(\Sigma)$$

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Then, there exist a spherically symmetric globally hyperbolic spacetime $(\mathcal{M}, g)$, an embedding $\iota \colon \Sigma \to \mathcal{M}$, and maps $h \colon \mathcal{M} \to \mathbb{R}_+$, $v \colon \Omega \to T\mathcal{M}$, where $\Omega := \{h > 0\}$, such that the (spherically symmetric) Einstein-Euler system

$$\mathrm{Ric}(g) - \frac{1}{2}R(g)g = \chi_\Omega \mathcal{T},$$
$$\mathrm{div}_g(\mathcal{T}) = 0,$$

is satisfied in $\mathcal{M}$ and $\iota^*(h, v, g, K) = (\mathring{h}, \mathring{v}, \mathring{g}, \mathring{K})$, where K is the second fundamental form of $i(\Sigma) \subset \mathcal{M}$. Moreover, $h \approx \mathrm{dist}(\cdot, \partial\Omega)$.

Theorem: Local well-posedness, II

Furthermore, there exists a time function $t\colon \mathcal{M} \to \mathbb{R}_+$ and a foliation of $\mathcal{M}$ by constant-time slices Σ_t of t, $\mathcal{M} = \cup_{0\leq t\leq T}\{t\} \times \Sigma_t$, such that $\Omega = \cup_{0\leq t\leq T}\{t\} \times \Omega_t$, where $\Omega_t := \Omega \cap \Sigma_t$. Then, in Schwarzschild gauge,

$$g = -e^{2\phi}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

we have

$$(h, v, g)(t, \cdot) \in \mathcal{H}_h^{2N}(\Omega_t),\ g(t, \cdot) \in C^{1,\frac{1}{\kappa}}(\Sigma_t),$$

and the solution (h, v, g) is unique within this class. Finally, (h, v, g) depends continuously on the initial data $(\mathring{h}, \mathring{v}, \mathring{g})$.

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Remark: Different solutions are defined in different spaces. We need to introduce an appropriate function space and suitable topology in order to compare different solutions and talk about continuous dependence on the data.

Previous results

Fixed background, liquid: Trakhinin; Oliynyk; Ginsberg; Miao, Shahshahani, and Wu.

Fixed background, gas, physical vacuum: Jang, LeFloch, and Masmoudi; Hadžić, Shkoller, and Speck; D-, Ifrim, and Tataru.

Fixed background, gas, not physical vacuum: LeFloch and Ukai; Brauer and Karp.

Einstein-Euler, liquid: Miao and Shahshahani; Hao, Huo, and Miao.

Einstein-Euler, static, gas, physical vacuum: Lindblom, Rendall and Schmidt; Makino; Heilig.

Einstein-Euler, gas, not physical vacuum: Rendall.

Einstein-Euler, gas, physical vacuum: Ours is the first result outside static solutions.

Highlights of the proof

Use Schwarzschild gauge $g = -e^{2\phi} dt^2 + e^{2\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

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- ϕ determined by λ, fluid, and Schwarzschild's boundary condition,

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- Use the constraints to write Einstein as a first-order transport equation (adapted to the fluid) for the metric,

$$D_t \lambda = h, v, \lambda,$$

$D_t := \partial_t + \frac{v}{v^0}\partial_r$, velocity: $(v^0, v, 0, 0)$. Transport estimates for λ.

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  $D_t := \partial_t + \frac{v}{v^0}\partial_r$, velocity: $(v^0, v, 0, 0)$. Transport estimates for λ.
- Fluid equations sourced by Christoffel symbols $\partial(\phi, \lambda)$,

$$D_t h + h\partial_r v = \partial_r \phi, \partial_t \lambda, \partial_r \lambda, \quad D_t v + \partial_r h = \partial_r \phi, \partial_t \lambda, \partial_r \lambda.$$

  Lost of derivatives; use constraints to algebraic eliminate $\partial_t \lambda, \partial_r(\phi, \lambda)$ (no $\partial_t \phi$). Original equations recovered a posteriori.

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Compare solutions in different domains: boundary layer estimates in $\Omega^{(1)} \cap \Omega^{(2)}$.

— Thank you for your attention —

Appendix: To know more

D-. Recent developments in mathematical aspects of relativistic fluids. Living Reviews in Relativity, Vol. 27, No. 6, 218 pages (2024). arXiv: 2308.09844 [math.AP].

Appendix: Equations in spherical symmetry

Gauge:

$$g = -e^{2\phi}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Einstein's equations ($tt, rr, tr, \theta\theta = \phi\phi$, respectively):

$$\begin{aligned}
2re^{-2\lambda}\partial_r\lambda - e^{-2\lambda} + 1 &= r^2(\varrho + (p+\varrho)e^{2\lambda}u^2),\\
2re^{-2\lambda}\partial_r\phi + e^{-2\lambda} - 1 &= r^2(p + (p+\varrho)e^{2\lambda}u^2),\\
2\partial_t\lambda &= -r(p+\varrho)e^{\phi+2\lambda}\sqrt{1+e^{2\lambda}u^2}\,u,\\
e^{-2\phi}(\partial_t^2\lambda + (\partial_t\lambda)^2 - \partial_t\lambda\partial_t\phi) - e^{2\lambda}(\partial_r^2\phi + (\partial_r\phi)^2 + \frac{1}{r}\partial_r\phi&\\
-\partial_r\lambda\partial_r\phi - \frac{1}{r}\partial_r\lambda) &= -p.
\end{aligned}$$

Appendix: Equations in spherical symmetry

Euler's equations, with $(u^0, u^1, 0, 0) = (e^{-\phi}\sqrt{1+e^{2\lambda}u^2}, u, 0, 0)$:

$$\partial_t \varrho + \frac{e^{\phi}}{\sqrt{1+e^{2\lambda}u^2}} u \partial_r \varrho + (\varrho + p)\Big[\partial_t \lambda + \frac{e^{\phi} u}{\sqrt{1+e^{2\lambda}u^2}}(\partial_r \lambda + \partial_r \phi + \frac{2}{r})$$
$$+ \frac{e^{\phi}}{\sqrt{1+e^{2\lambda}u^2}} \partial_r u + \frac{e^{2\lambda}}{1+e^{2\lambda}u^2}(\partial_t \lambda u + \partial_t u)\Big] = 0,$$

$$(\varrho + p)\Big[e^{2\lambda}(\partial_t u + 2\partial_t \lambda u) + e^{\phi}\sqrt{1+e^{2\lambda}u^2}\partial_r \phi + \frac{e^{\phi+2\lambda}u}{\sqrt{1+e^{2\lambda}u^2}}(\partial_r u + \partial_r \lambda u)\Big]$$
$$+ e^{\phi}\sqrt{1+e^{2\lambda}u^2}\partial_r p + e^{2\lambda} u \partial_t p = 0.$$

Appendix: New variables and equations

Define (recall $p(\varrho) = \varrho^{\kappa+1}$, $\kappa > 0$).

$$h := \frac{c_s^2}{\kappa} = \frac{\kappa+1}{\kappa}\varrho^{\kappa}, \quad v := \langle h \rangle^{1+\frac{1}{\kappa}} u,$$

where $\langle h \rangle := 1 + \frac{\kappa}{1+\kappa}h$. Reduced system:

$$\begin{aligned} D_t h + hG\partial_r v + ha_1 v\partial_r h &= \mathcal{F}^h, \\ D_t v + a_2 \partial_r h &= \mathcal{F}^v, \\ D_t \lambda &= \mathcal{F}^\lambda, \end{aligned}$$

where $D_t := \partial_t + \frac{v}{v^0}\partial_r$.

Appendix: Phase space

Define the phase space

$$\mathbf{H}^{2N} := \{(h, v, g) \mid (h, v, g)(t, \cdot) \in \mathcal{H}^{2N}(\Omega_t),\ \Omega_t = \{h(t, \cdot) > 0\}\}$$

A sequence (h_n, v_n, g_n) converges to (h, v, g) in $\mathbf{H}^{2N}$ if:

(i) Uniform nondegeneracy: $|\nabla h_n(t, \cdot)| \geq C$ for some constant $C > 0$ independent of n.

(ii) Domain convergence: $\|h_n(t, \cdot) - h(t, \cdot)\|_{Lip(\mathbf{R}^3)} \to 0$, h_n and h view as Lipschitz functions in $\mathbb{R}^3$ by extending them to be zero outside their respective moving domains.

(iii) Norm convergence: for each $\varepsilon > 0$, there exist smooth $(\tilde{h}, \tilde{v}, \tilde{g})$ defined in a neighborhood of Ω such that

$$\|(h, v, g) - (\tilde{h}, \tilde{v}, \tilde{g})\|_{\mathcal{H}^{2N}(\Omega_t)} \leq \varepsilon$$
$$\limsup_{n \to \infty} \|(h_n, v_n, g_n) - (\tilde{h}, \tilde{v}, \tilde{g})\|_{\mathcal{H}^{2N}(\Omega_t^{(n)})} \leq \varepsilon,$$

where Ω and $\Omega^{(n)}$ are the moving domains for (h, v, g) and (h_n, v_n, g_n).

(iv) The above conditions hold uniformly on t.