

Gravitational collapse to extremal Reissner–Nordström and the third law of black hole thermodynamics

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MIT

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joint work with Ryan Unger (Stanford)

BLACK HOLE THERMODYNAMICS

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The Four Laws of Black Hole Mechanics

J. M. Bardeen★

Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking

Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics is a proposed close mathematical **analogy** between black hole dynamics and classical thermodynamics.

BLACK HOLE THERMODYNAMICS

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon

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BLACK HOLE THERMODYNAMICS

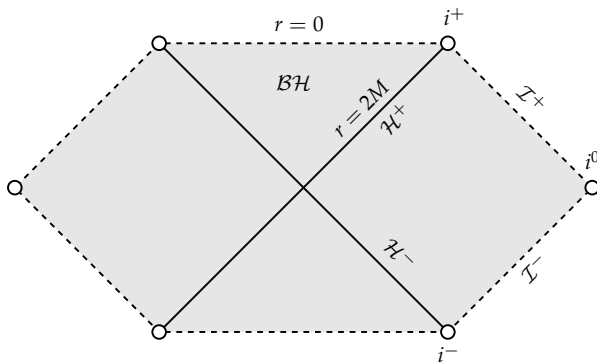
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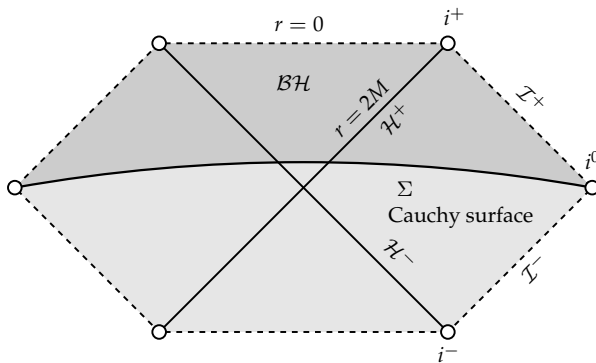
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- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

REFRESHER ON SCHWARZSCHILD

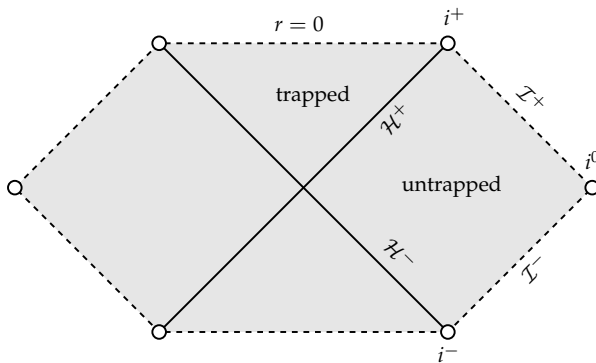


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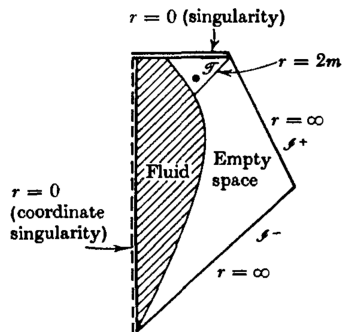
Maximally extended Schwarzschild is the unique maximal Cauchy development of the data induced on a spacelike hypersurface $\Sigma \cong \mathbb{R} \times S^2$ as depicted here.

REFRESHER ON SCHWARZSCHILD



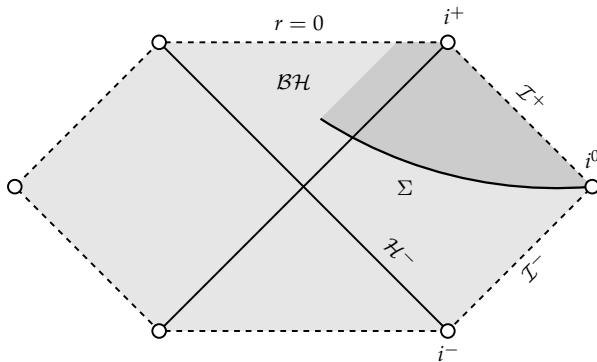
The black hole interior is foliated by **trapped spheres**
(both future null expansions negative).

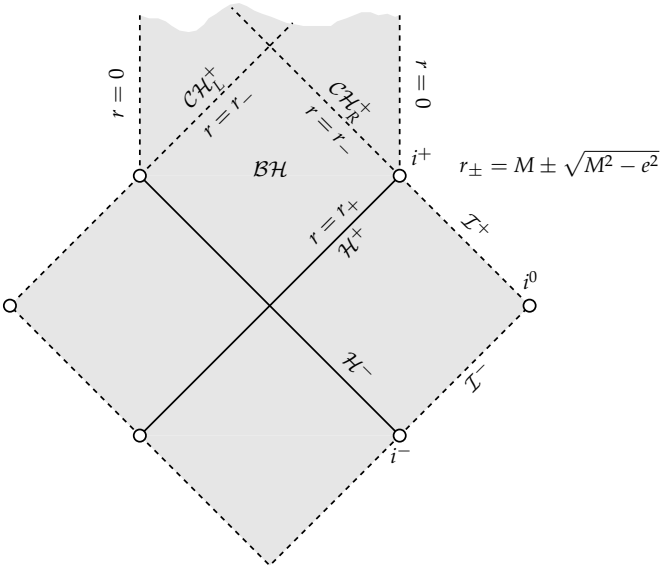
REFRESHER ON GRAVITATIONAL COLLAPSE

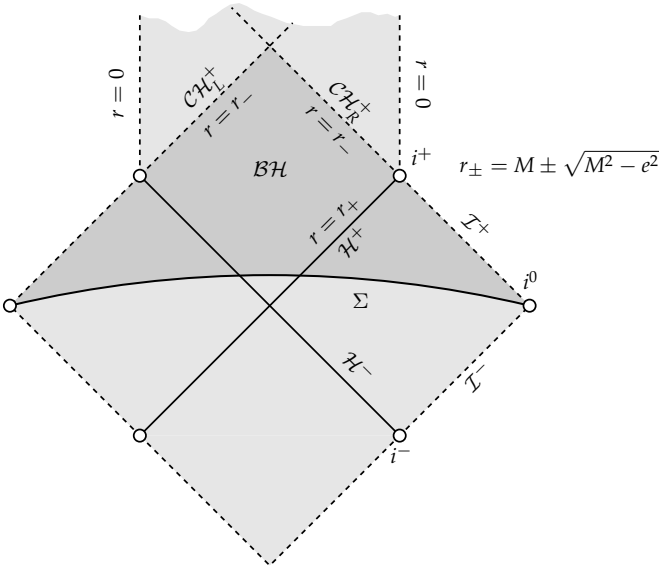


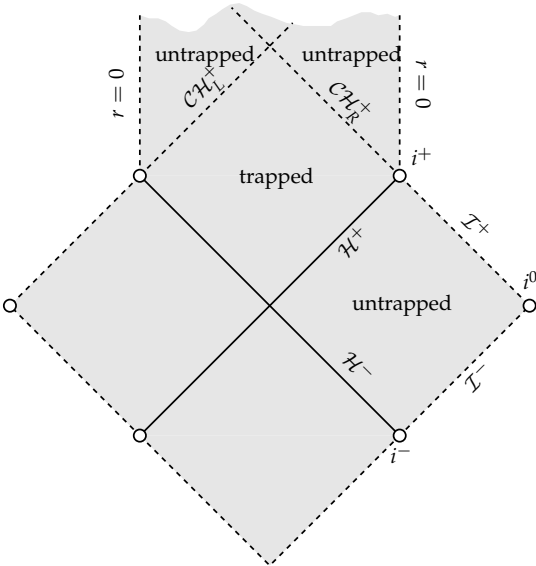
Penrose diagram of gravitational collapse. One-ended Cauchy data!

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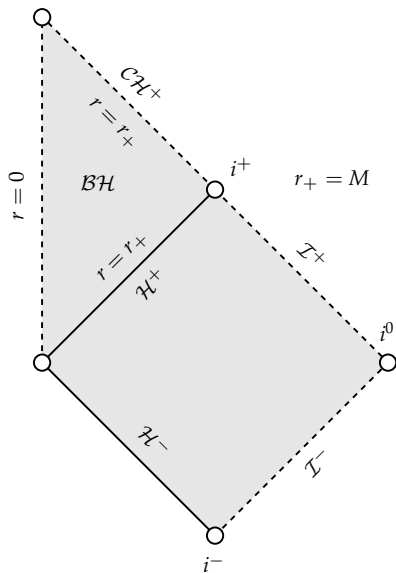


REFRESHER ON SUBEXTREMAL REISSNER–NORDSTRÖM: $0 < |e| < M$ 

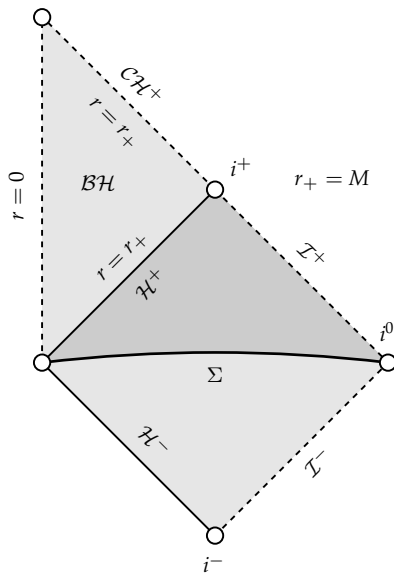
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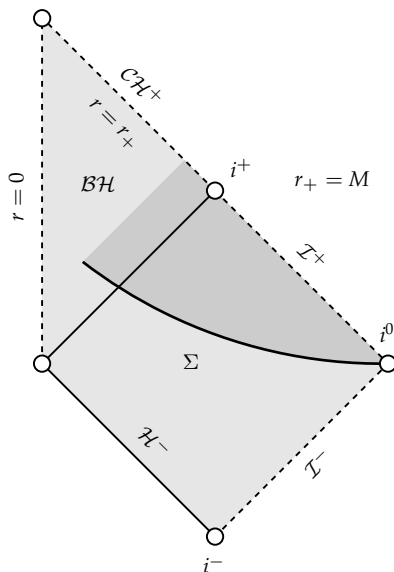
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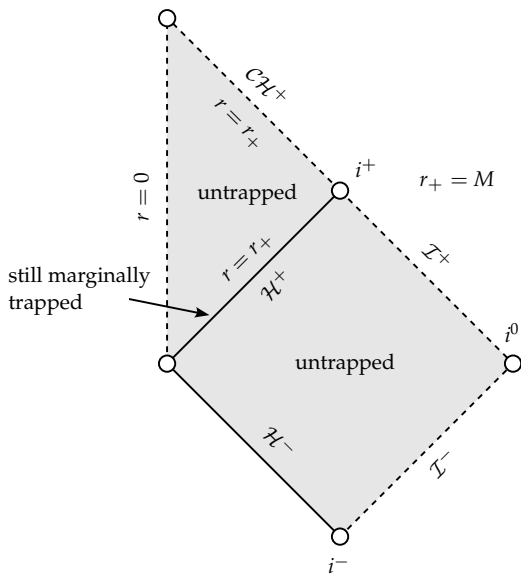


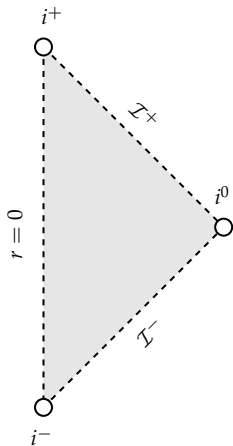
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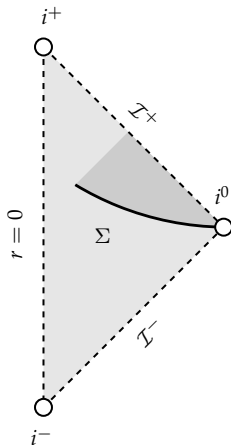
Non-negative Hawking mass requires $r \geq \frac{e^2}{2M}$.

REFRESHER ON EXTREMAL REISSNER–NORDSTRÖM: $0 < |e| = M$





REFRESHER ON SUPEREXTREMAL REISSNER–NORDSTRÖM: $0 < M < |e|$



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SURFACE GRAVITY κ OF REISSNER–NORDSTRÖM

- ▶ RN with mass M and charge e , $|e| \leq M$, has

$$\kappa = 2\pi T = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- ▶ **Subextremal:** $\kappa > 0$
- ▶ **Extremal:** $\kappa = 0$

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 - ▶ If singularities allowed, counterexample using **massive dust shell**. [FARRUGIA-HAJICEK '79]
4. Weak energy condition must be enforced.
 - ▶ Otherwise: counterexample using **charged null dust**. [SULLIVAN-ISRAEL '80]

RETIRING THE THIRD LAW

Conjecture (The third law, BCH '73, Israel '86).

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*There exists a precisely defined process in which a **subextremal** black hole becomes **extremal** in **finite time**, evolving from **regular** initial data in the Einstein–Maxwell charged scalar field system. In particular, the “third law of black hole thermodynamics” is **false**.*

Third Law of Black-Hole Dynamics: A Formulation and Proof

W. Israel^(a)

Research Institute for Fundamental Physics, Yukawa Hall, Kyoto University, Kyoto 606, Japan

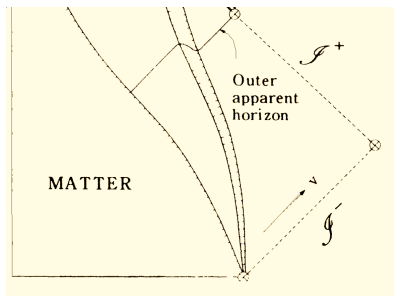
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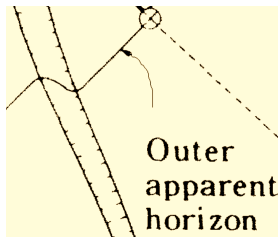
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Israel argues by contradiction. Assume:

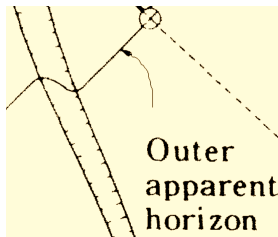
- First incoming matter flux creates (dynamical) subextremal apparent horizon.
- Second matter flux pushes the horizon to become extremal.

ISRAEL'S ARGUMENT II



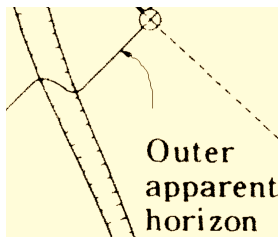
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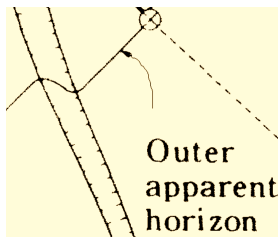
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Implicit assertion: regular solution $\Rightarrow$ connected outer apparent horizon.

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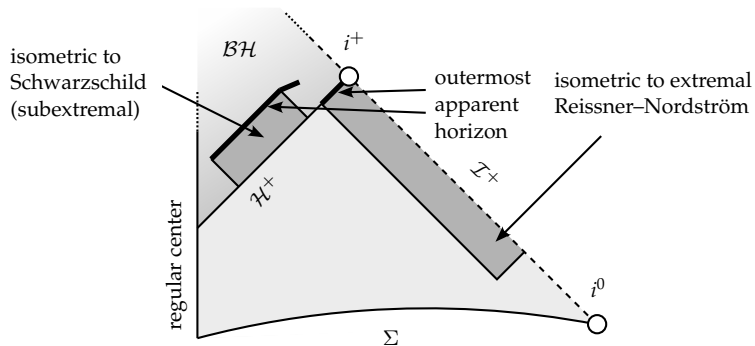
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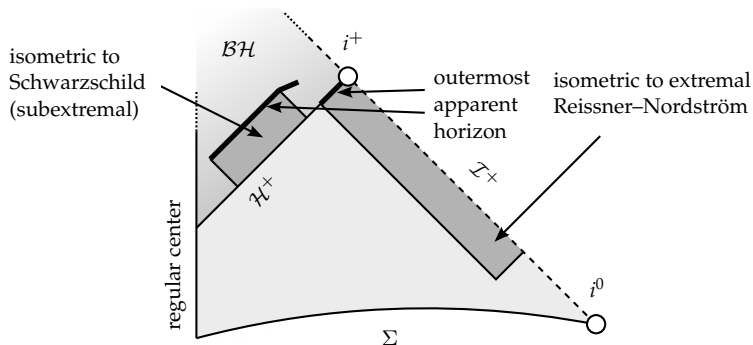
However, outer apparent horizon **can jump in smooth spacetimes.**

COUNTEREXAMPLE TO THE THIRD LAW



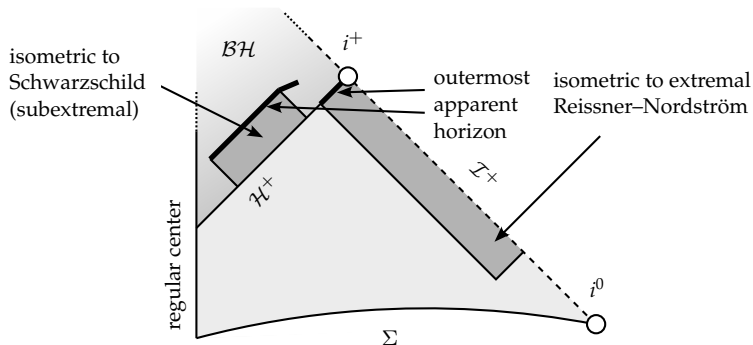
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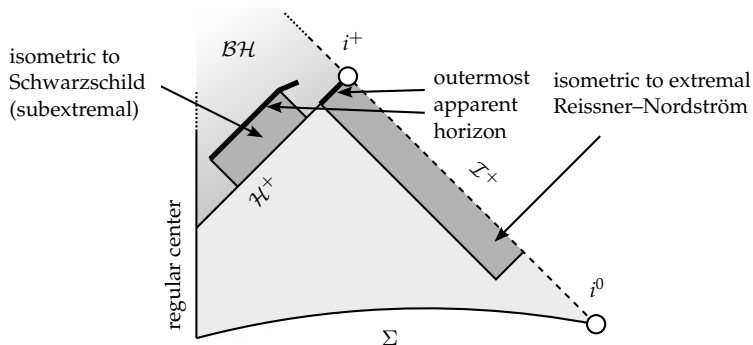
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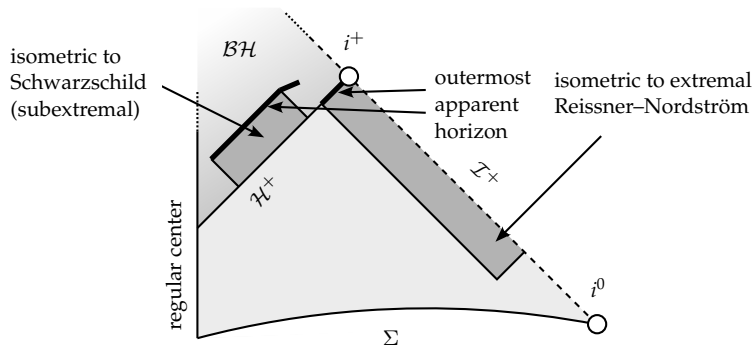
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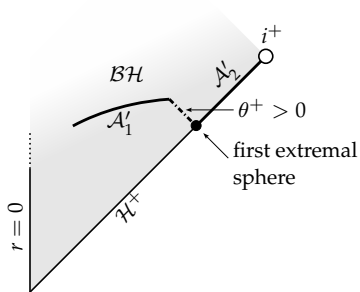
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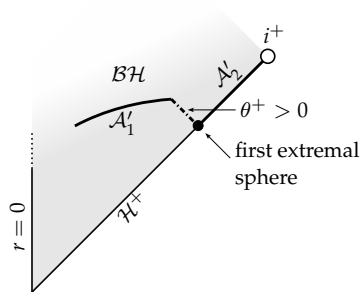
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- Dominant energy condition ($\Rightarrow$ weak energy condition)

ISRAEL'S PAPER REINTERPRETED



Outermost apparent horizon becomes disconnected the instant the black hole becomes extremal!

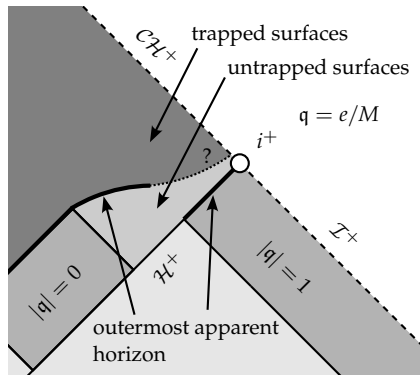
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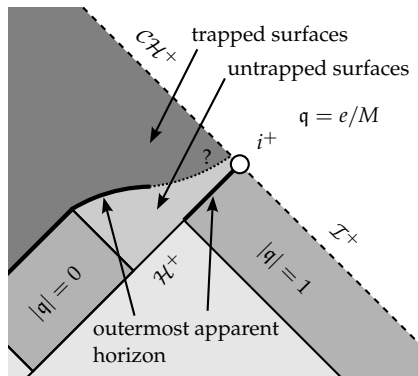
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This is a feature, not a glitch!

INTERIOR STRUCTURE OF THIRD LAW VIOLATING SOLUTIONS

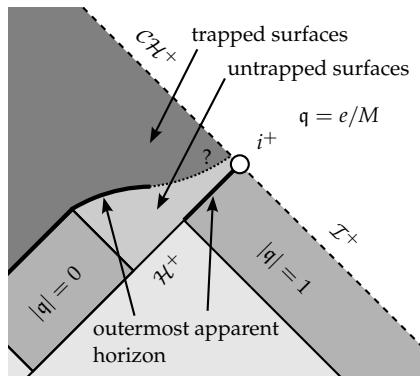


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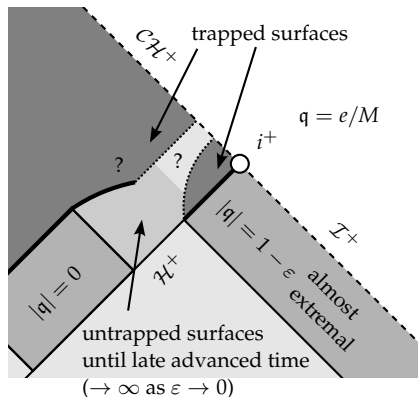
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- The outermost apparent horizon becomes disconnected, yet the spacetime is regular.
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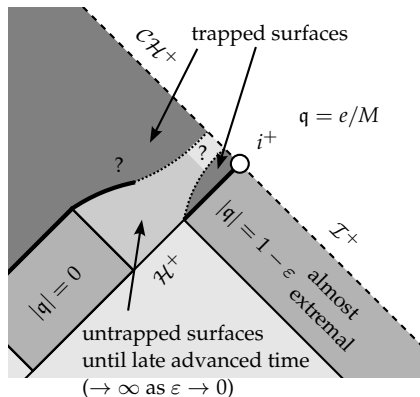
SO WHERE DO THE TRAPPED SURFACES GO?

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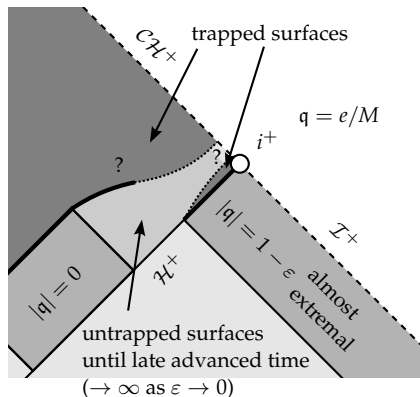
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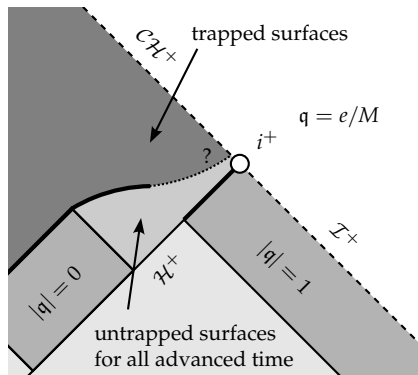
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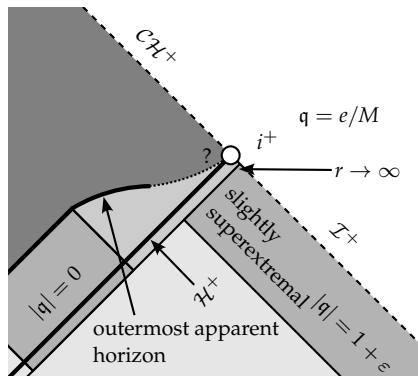


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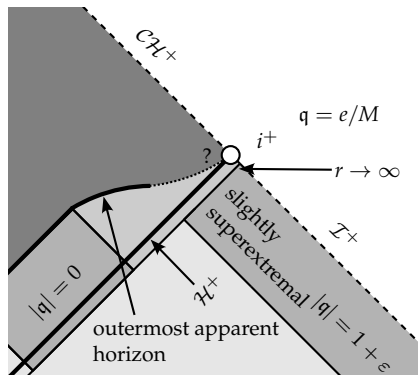
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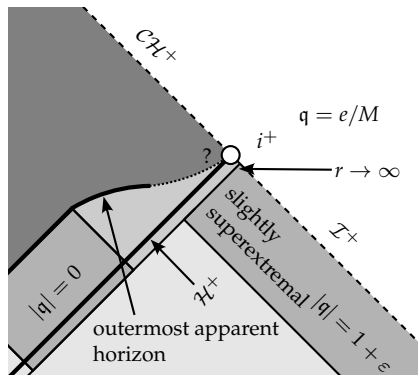


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Critical behavior: The event horizon jumps inwards the moment the exterior becomes superextremal. There is **no naked singularity**.

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The event horizon jumping associated to extremal horizons and the stability of this local critical behavior was conjectured by [DAFERMOS–HOLZEGEL–RODNIANSKI–TAYLOR '21].

ASIDE: OVERCHARGING

Bardeen–Carter–Hawking:

Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity.

This has led to the paradigm of **overcharging** and **overspinning**.

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- ▶ Such an attempt is a doomed endeavor.
- ▶ Overcharging has been definitively **disproved** in sph. symmetry [KOMMEMI '13].
- ▶ Part of the spacetime is isometric to superextremal Reissner–Nordström $\nRightarrow$ there exists a naked singularity!

EINSTEIN-MAXWELL-CHARGED SCALAR FIELD SYSTEM

- ▶ Lorentzian manifold $(\mathcal{M}^{3+1}, g)$
- ▶ 2-form $F = dA$ (electromagnetism)
- ▶ Charged (complex) scalar field ϕ

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- 2-form $F = dA$ (electromagnetism)
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$$R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = 2 \left(T_{\mu\nu}^{\text{EM}} + T_{\mu\nu}^{\text{CSF}} \right)$$

$$\nabla^\mu F_{\mu\nu} = 2\mathfrak{e} \operatorname{Im}(\phi \overline{D_\nu \phi})$$

$$g^{\mu\nu} D_\mu D_\nu \phi = 0$$

$$T_{\mu\nu}^{\text{EM}} = g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu}$$

$$T_{\mu\nu}^{\text{CSF}} = \operatorname{Re}(D_\mu \phi \overline{D_\nu \phi}) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} D_\alpha \phi \overline{D_\beta \phi}$$

EINSTEIN-SCALAR FIELD IN SPHERICAL SYMMETRY

► $\mathcal{M}^{3+1} = \mathcal{Q}^{1+1} \times S^2$

$$g = -\Omega^2 du dv + r^2 g_{S^2}$$

► $\Omega(u, v) > 0$ lapse, $r(u, v) > 0$ area-radius

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► $\Omega(u, v) > 0$ lapse, $r(u, v) > 0$ area-radius

► Wave equations

$$\partial_u \partial_v \phi = -\frac{\partial_u \phi \partial_v r}{r} - \frac{\partial_u r \partial_v \phi}{r}$$

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EINSTEIN-SCALAR FIELD IN SPHERICAL SYMMETRY

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► Raychaudhuri's equations (constraints)

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2$$

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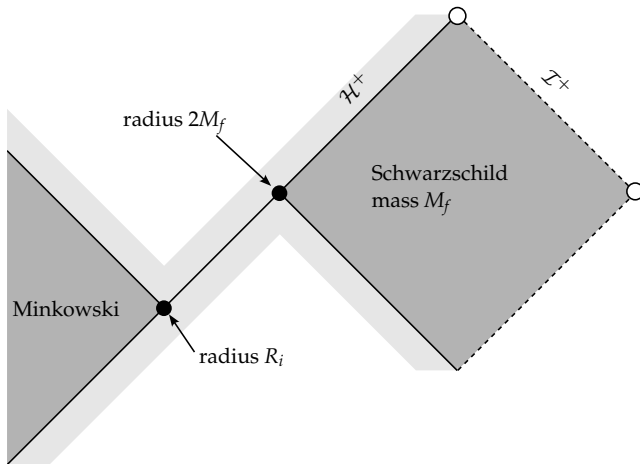
Hawking mass $m \doteq \frac{r}{2} (1 + 4\Omega^{-2} \partial_v r \partial_u r)$:

$$\partial_v m = 2r^2 \Omega^{-2} (-\partial_u r) (\partial_v \phi)^2$$

MINKOWSKI TO SCHWARZSCHILD GLUING

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See [ARETAKIS–CZIMEK–RODNIANSKI, CHRUŚCIEL–CONG] for Einstein vacuum equations



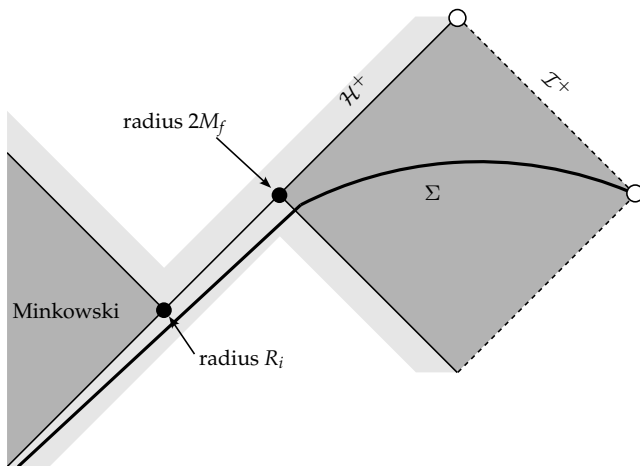
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Set up characteristic data such that radii and Hawking masses have **a priori specified values** and $\phi, \partial_v^j \phi, \partial_u^j \phi$.

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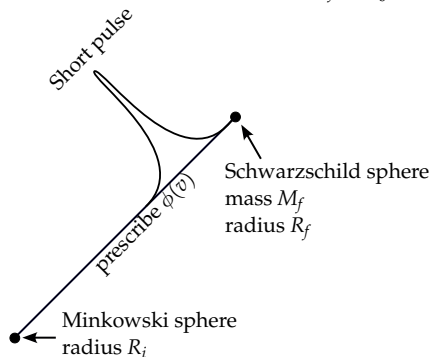
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Theorem (K.–Unger '22).

For any $k \in \mathbb{N}$, $M_f > 0$ and $0 < R_i < R_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild sphere with radius R_f and mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.

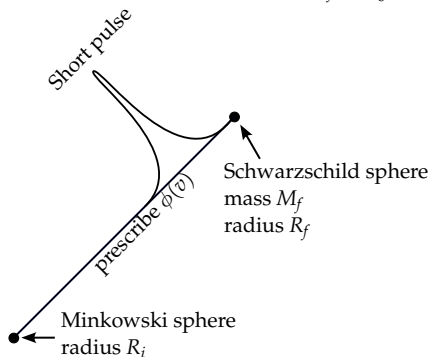
A FIRST APPROACH AND THE ISSUE OF TRANSVERSE DERIVATIVES

- ▶ On $v \in [0, 1]$ use gauge $\Omega^2 = 1$ we impose $-\partial_u r(1) \gg 1 \Rightarrow |\partial_v r|, \Delta r \ll 1$ (short pulse [CHRISTODOULOU])
- ▶ Intermediate value thm: $\exists$ amplitude of ϕ such that $M_f = \int_0^1 2r^2(-\partial_u r)(\partial_v \phi)^2 dv$



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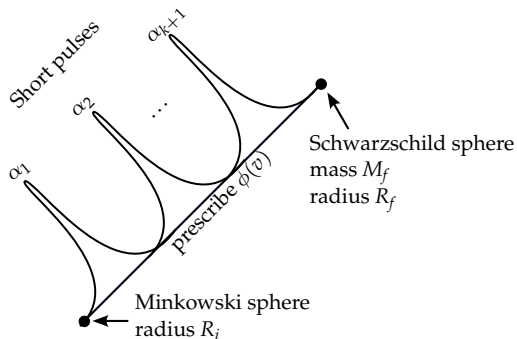
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This is not enough because:

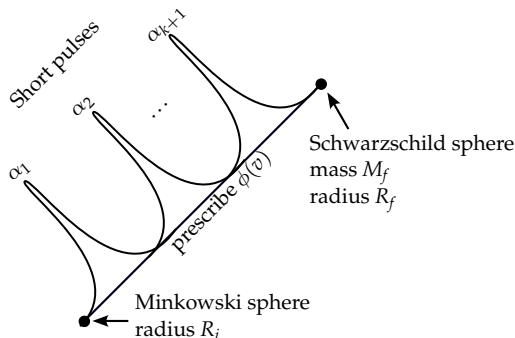
- ▶ Transverse derivative $\partial_u \phi$ is transported and sourced by ϕ along outgoing cone:
 $\partial_v(\partial_u \phi) = -\partial_u \phi \partial_v \log r - \partial_v \phi \partial_u \log r$.
- ▶ Generic choice of profile can only satisfy **either** $\partial_u \phi(0) = 0$ **or** $\partial_u \phi(1) = 0$.
- ▶ However, **gluing requires both** and also higher transverse derivatives.

IDEA OF THE PROOF: SCHWARZSCHILD



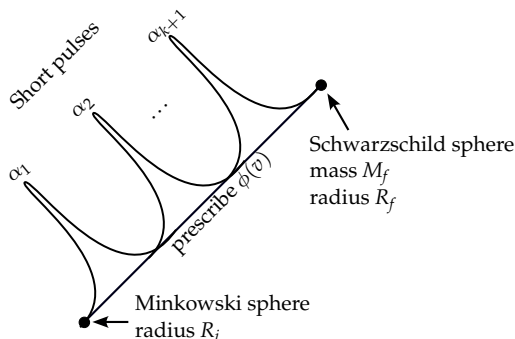
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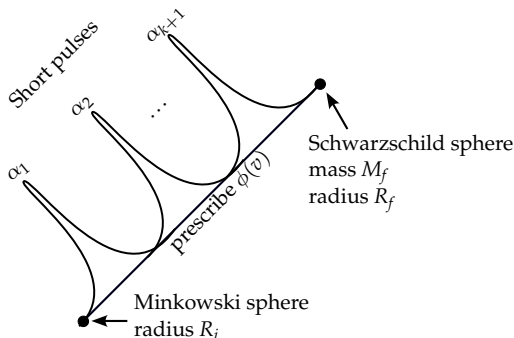
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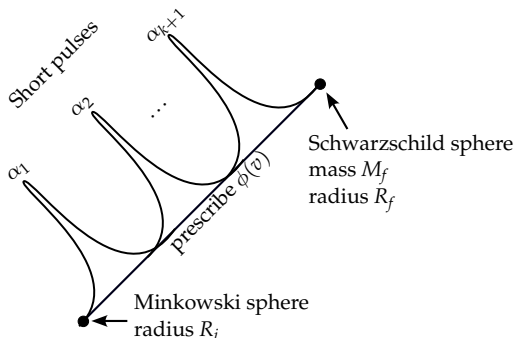
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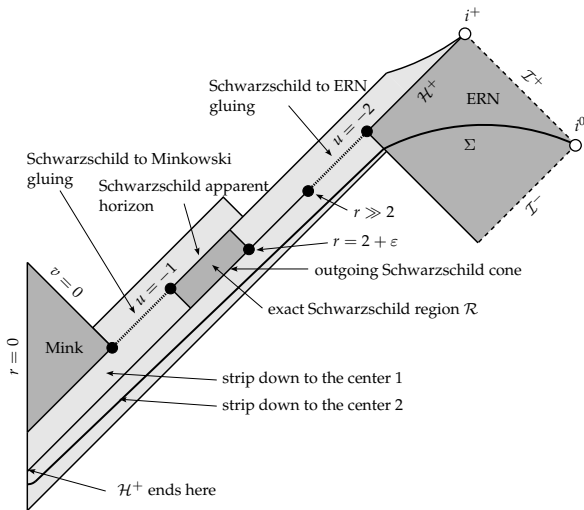
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- ▶ **Borsuk–Ulam theorem:** there exists α_* such that

$$\left(\partial_u \phi_{\alpha_*}(1), \dots, \partial_u^k \phi_{\alpha_*}(1) \right) = 0.$$

DISPROOF OF THE THIRD LAW



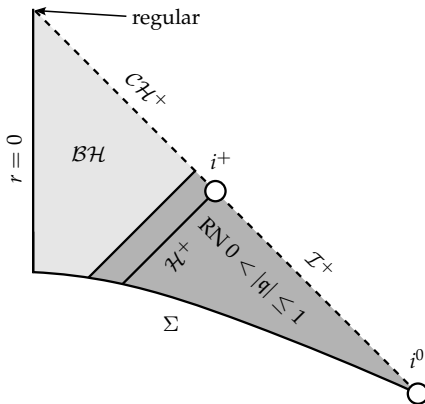
Poincaré inequality obstruction: $\partial_v m \sim (-\partial_u r)r^2(\partial_v \phi)^2$ but $\partial_v Q \sim r^2 \phi \partial_v \phi$
 $\Rightarrow$ A short pulse **cannot** produce an extremal black hole.

Beyond the disproof of the third law,
the gluing method allows us to construct further interesting behavior.

BLACK HOLES WITHOUT TRAPPED SURFACES

Theorem (K.-Unger '22).

There exist black holes *without* trapped surfaces.



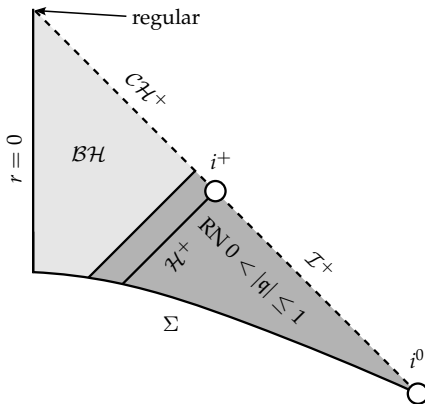
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Penrose's theorem **does not** guarantee the stability of their black hole-ness.

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Such black holes could be natural candidates for **critical solutions**!

Living Rev. Relativity, **10**, (2007), 5
<http://www.livingreviews.org/lrr-2007-5>
(Update of lrr-1999-4)

Critical Phenomena in Gravitational Collapse

Carsten Gundlach

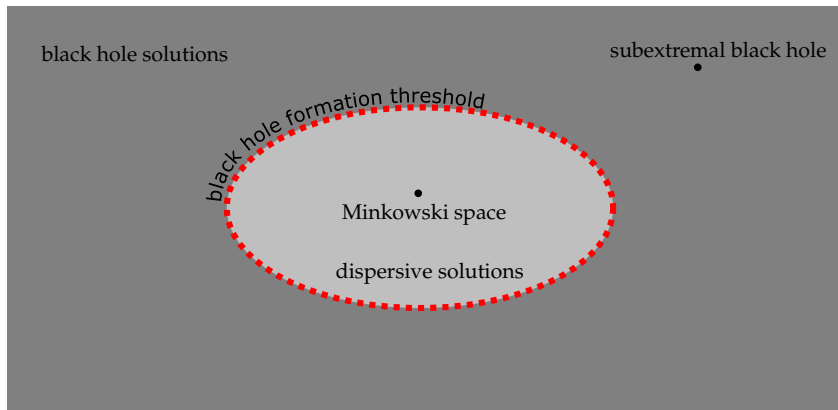
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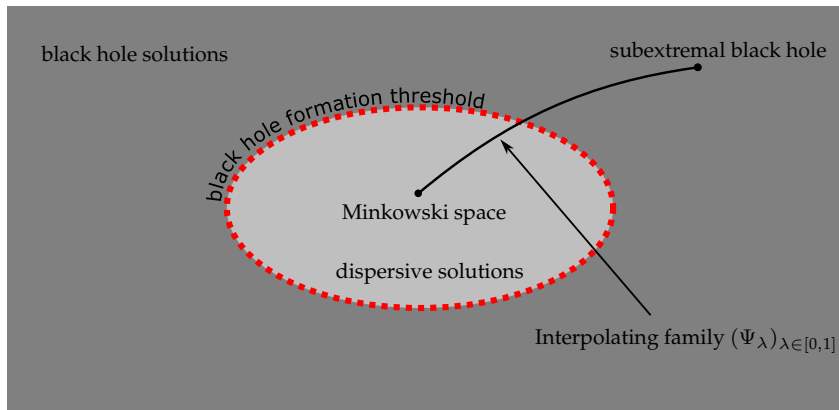
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98 bis boulevard Arago, 75014 Paris, France
Laboratoire Univers et Théories CNRS & Université Paris Diderot,
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Published on 11 December 2007
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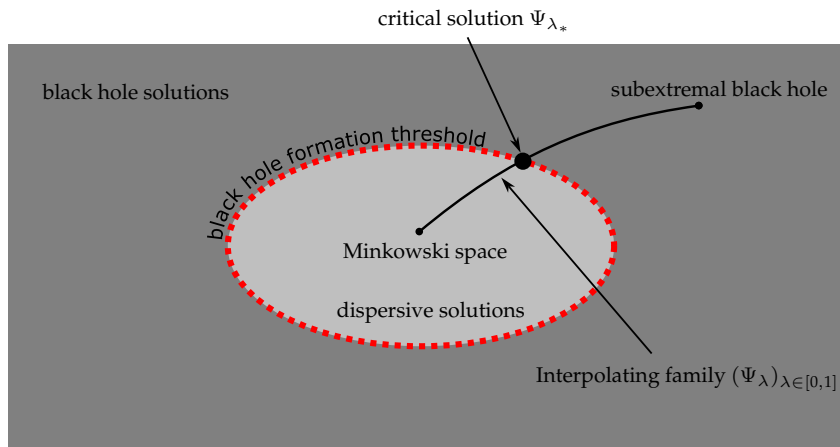
CARTOON PICTURE OF MODULI SPACE



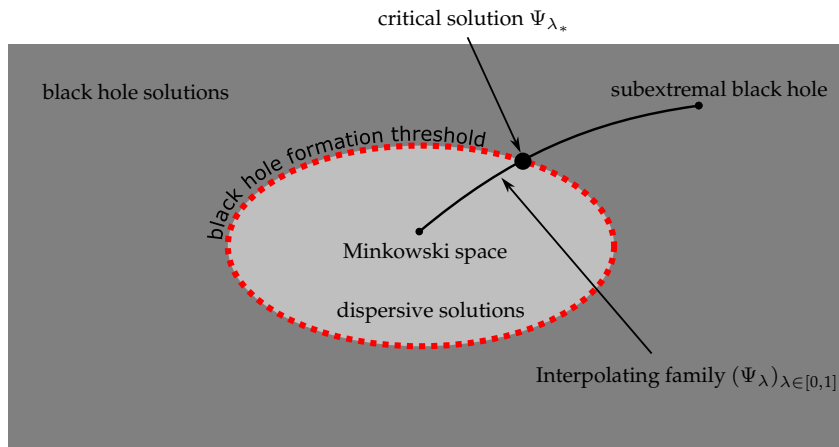
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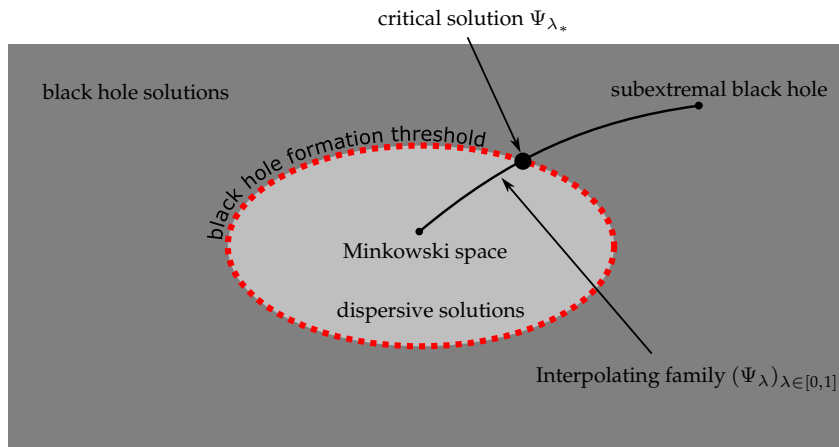


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Numerics for sph. symm. Einstein-scalar field: Ψ_{λ_*} leads to a **naked singularity**
[CHOPTUIK '93, ...]

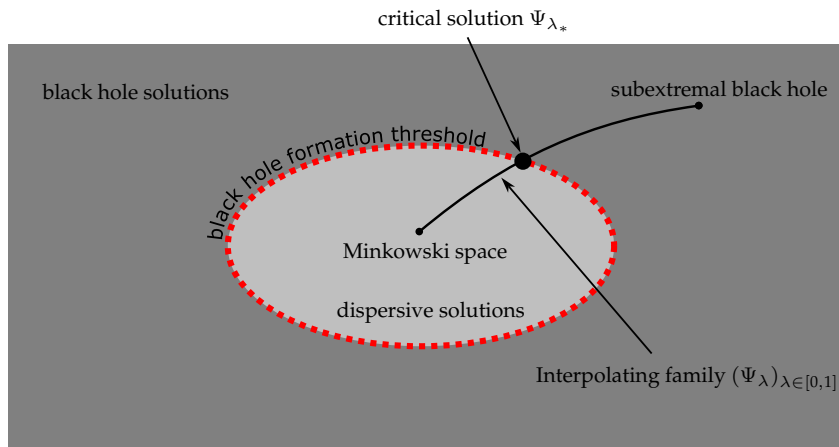
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It is an open problem to make any of these numerics rigorous!

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We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

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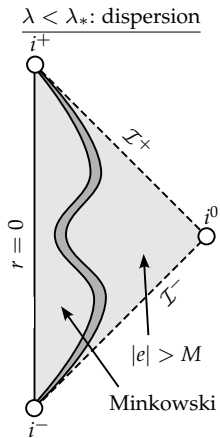
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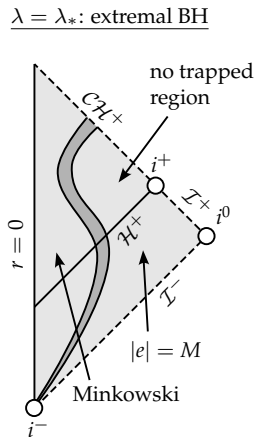
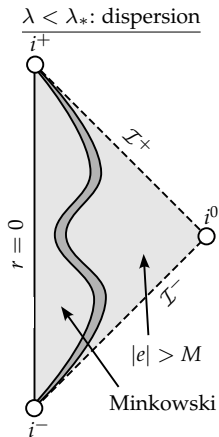
There exist **extremal** black holes on the black hole formation threshold!

We call this phenomenon **Extremal Critical Collapse**.

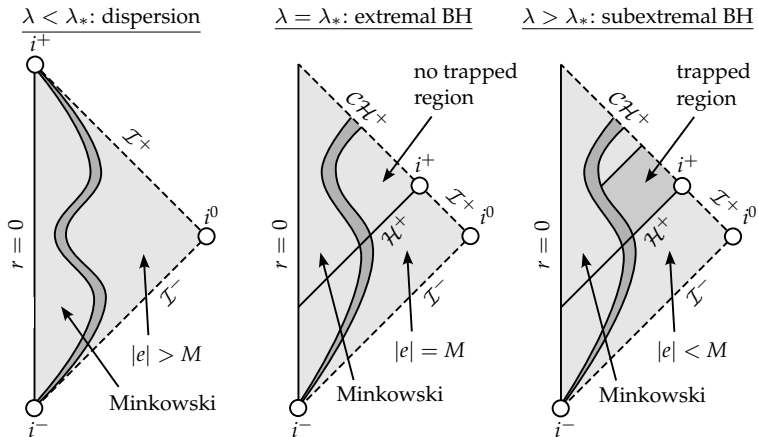
PENROSE DIAGRAM: EXTREMAL CRITICAL COLLAPSE



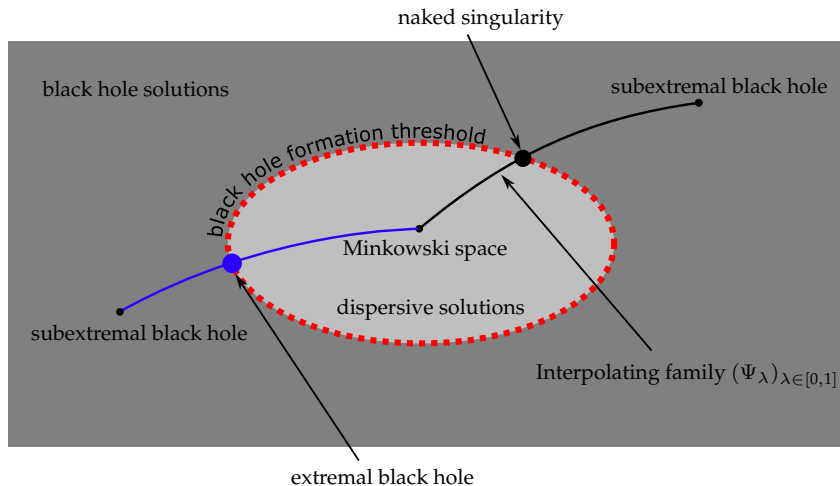
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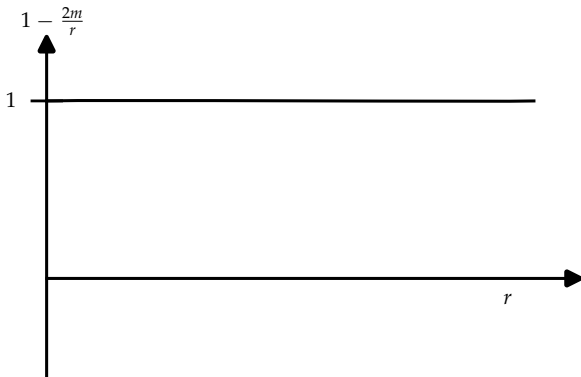
CARTOON PICTURE OF MODULI SPACE



EXTREMAL CRITICAL COLLAPSE: $1 - \frac{2m}{r}$ ALONG LATE INGOING CONE

In spherical symmetry: **trapped** sphere if and only if $1 - \frac{2m}{r} < 0$.

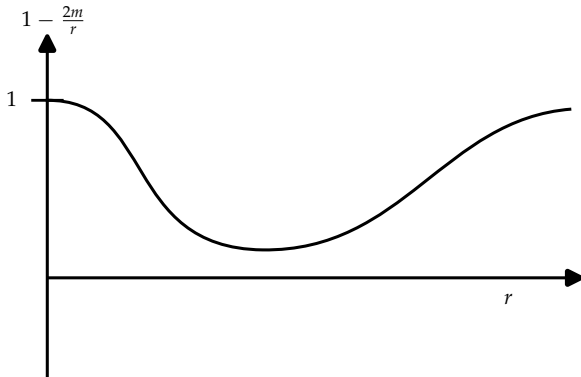
$\lambda = 0$: Minkowski



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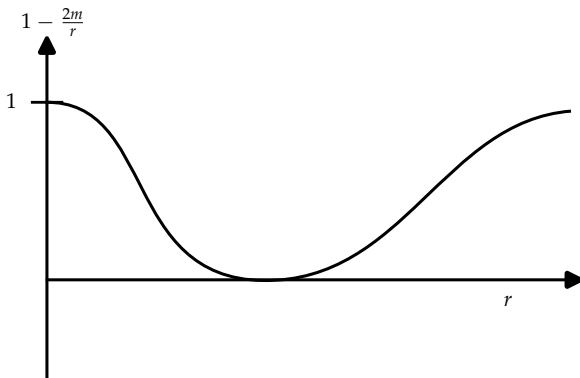
$\lambda < \lambda_*$: dispersion



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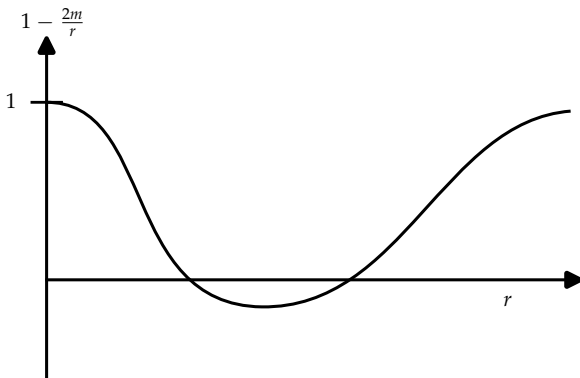
$\lambda = \lambda_*$: extremal black hole



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$\lambda > \lambda_*$: subextremal black hole



ASPECTS ABOUT THE PROOF

Consider a singular toy model: Einstein–Maxwell–charged null dust

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2 \left(T_{\mu\nu}^{\text{EM}} + T_{\mu\nu} \right),$$

$$\nabla^\alpha F_{\mu\alpha} = \mathfrak{e}\rho k_\mu,$$

$$k^\nu \nabla_\nu k^\mu = \mathfrak{e}F^\mu{}_\nu k^\nu,$$

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The system is **not well-posed** but an explicit, **singular** solution can be written down in terms of the ingoing charged Vaidya solution (Ori '91) and “free” functions $\varpi_{\text{in}}, Q_{\text{in}}$

with $\dot{\varpi}_{\text{in}} \geq 0$ and $\dot{Q}_{\text{in}} \geq 0$ for $D(V, r) \doteq 1 - \frac{2\varpi_{\text{in}}(V)}{r} + \frac{Q_{\text{in}}^2(V)}{r^2}$:

$$\begin{aligned} g_{\text{in}}[\varpi_{\text{in}}, Q_{\text{in}}] &\doteq -D(V, r) dV^2 + 2 dV dr + r^2 \gamma, \\ F &\doteq -\frac{Q_{\text{in}}}{r^2} dV \wedge dr, \\ k &\doteq \frac{\mathfrak{e}}{\dot{Q}_{\text{in}}} \left(\dot{\varpi}_{\text{in}} - \frac{Q_{\text{in}} \dot{Q}_{\text{in}}}{r} \right) (-\partial_r), \quad \rho \doteq \frac{(\dot{Q}_{\text{in}})^2}{\mathfrak{e}^2 r^2} \left(\dot{\varpi}_{\text{in}} - \frac{Q_{\text{in}} \dot{Q}_{\text{in}}}{r} \right)^{-1} \end{aligned}$$

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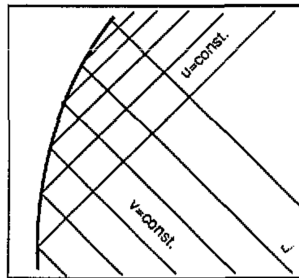
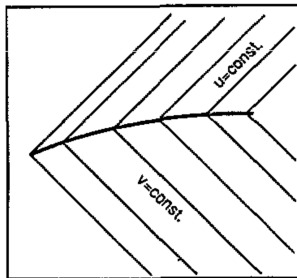
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$$\text{Bounce radius: } r_b \doteq \frac{Q_{\text{in}} \dot{Q}_{\text{in}}}{\dot{\varpi}_{\text{in}}}$$

Note: $T_{\mu\nu} = \rho k_\mu k_\nu$ violates null energy condition if $r < r_b$.

Ori's interpretation: Once an ingoing fluid trajectory hits the bounce hypersurface $\Sigma_b = \{r = r_b\}$, it has to change direction from **ingoing** to **outgoing**.

SPACELIKE BOUNCE HYPERSURFACE



$\Sigma_b := \{r = r_b\}$ is **spacelike** $\Rightarrow$ Explicit surgery with an outgoing Vaidya solution is possible such that second fundamental form is continuous. (Ori '91)

However, solution is still singular across Σ_b :

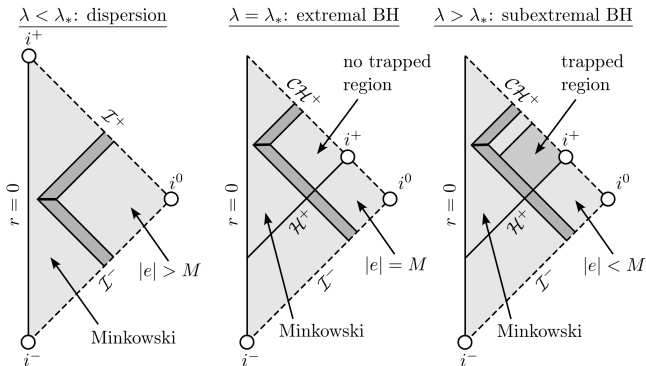
$$\rho \notin L^\infty, \quad N := \rho k \notin C^0$$

$\Sigma_b := \{r = r_b\}$ being **spacelike** is a **teleological** assumption!

EXTREMAL CRITICAL COLLAPSE IN NULL DUST MODEL

Theorem (K.–Unger '24).

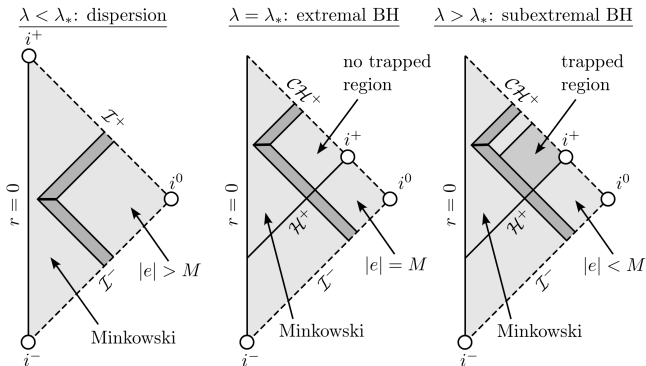
The charged null dust model exhibits extremal critical collapse.



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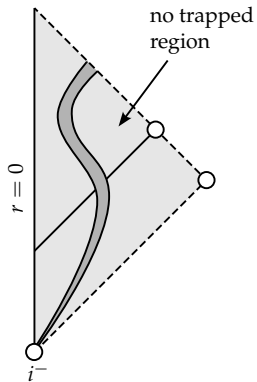
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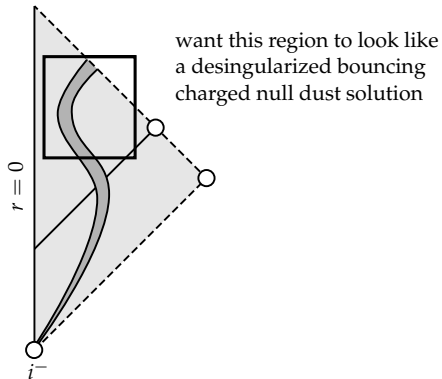


Proof idea: Instead of prescribing free function ϖ , Q as in Ori's model, we **directly** prescribe the geometry of Σ_b : find solutions to a system of ODEs and differential inequalities.

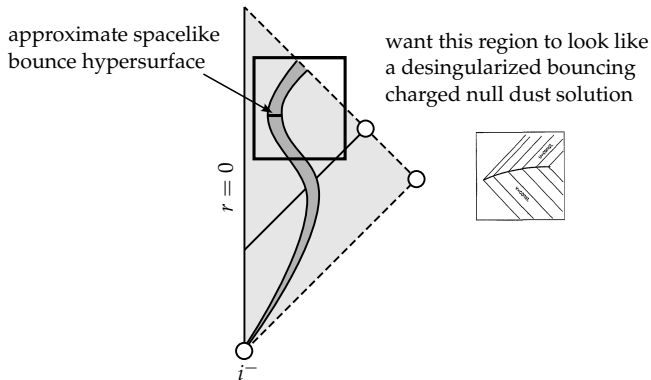
SMOOTH EXTREMAL CRITICAL COLLAPSE: VLASOV CASE



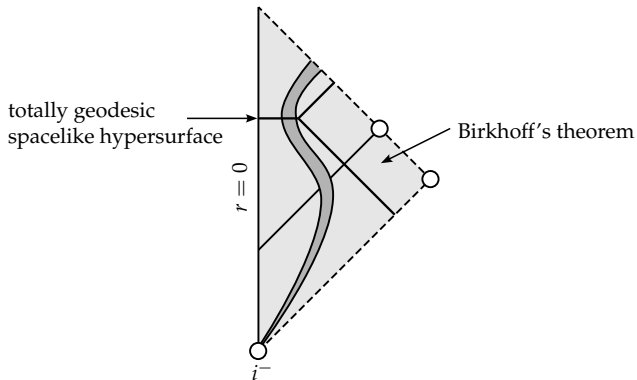
CONSTRUCTION OF BOUNCING CHARGED VLASOV BEAMS



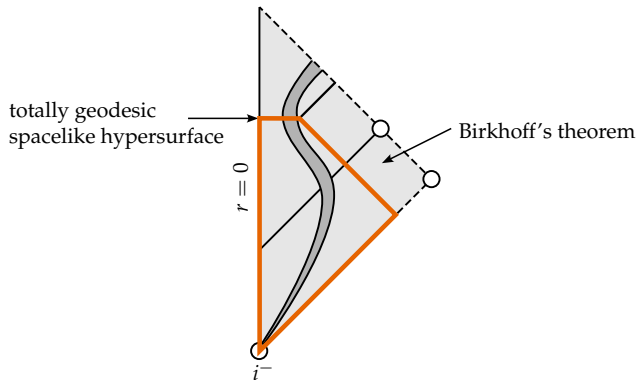
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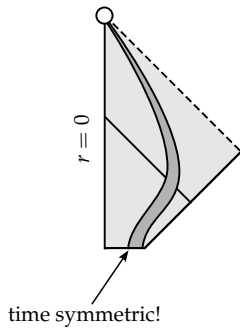
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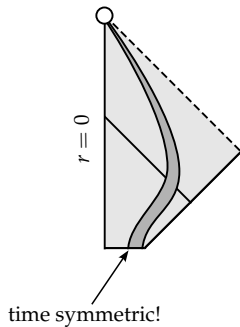


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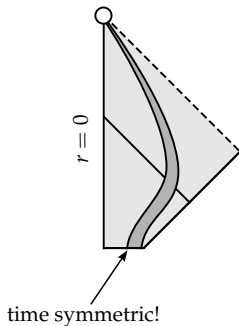
particles have conserved
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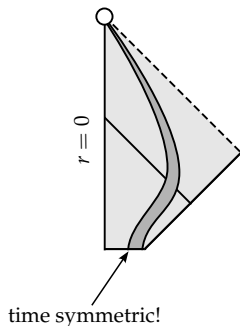
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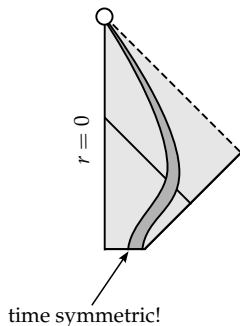


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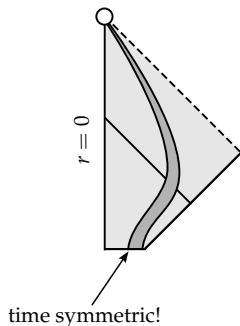
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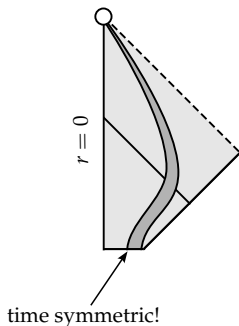
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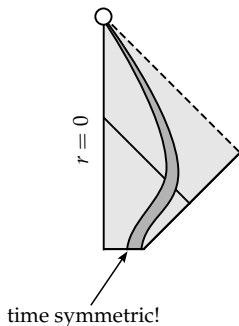
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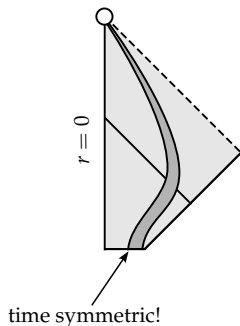
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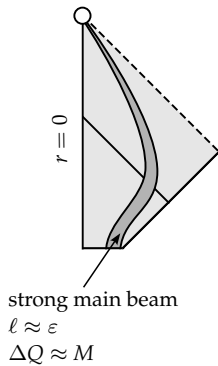
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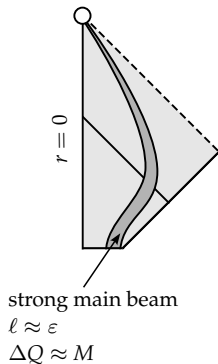
Difficulty: **Instability of em-geodesic flow** at the inner edge of the beam, where charge repulsion is arbitrarily small.

CONSTRUCTION OF BOUNCING CHARGED VLASOV BEAMS



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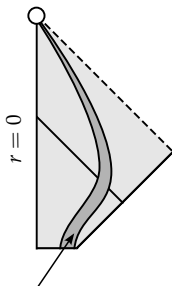
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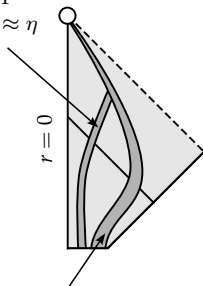
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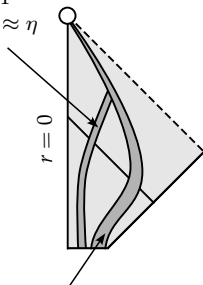
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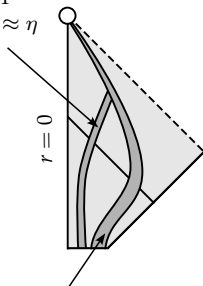
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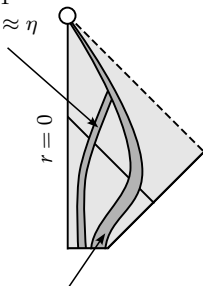
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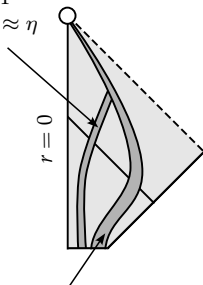
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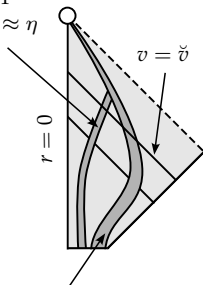
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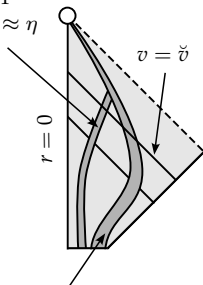
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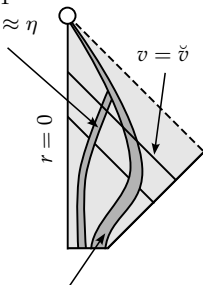
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For Vlasov we make fundamental use of the repulsive effects of **charge** and **angular momentum**.

STABILITY OF EXTREMAL CRITICAL COLLAPSE

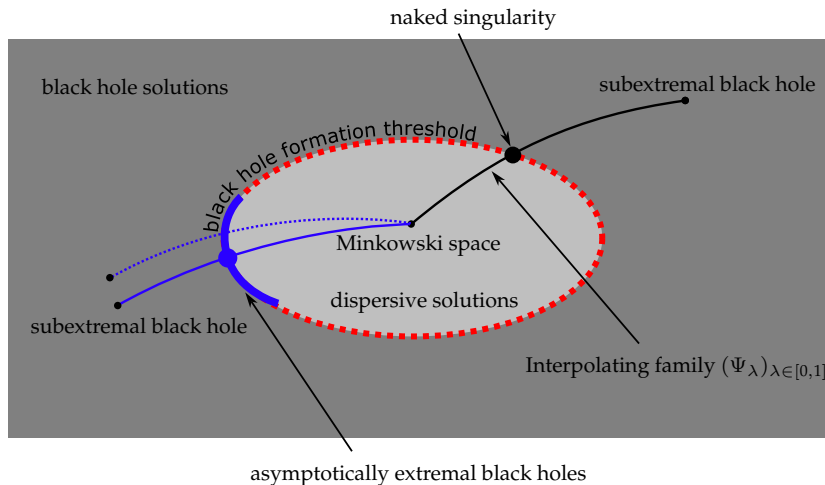
Conjecture.

*Extremal critical collapse is a **stable** phenomenon.*

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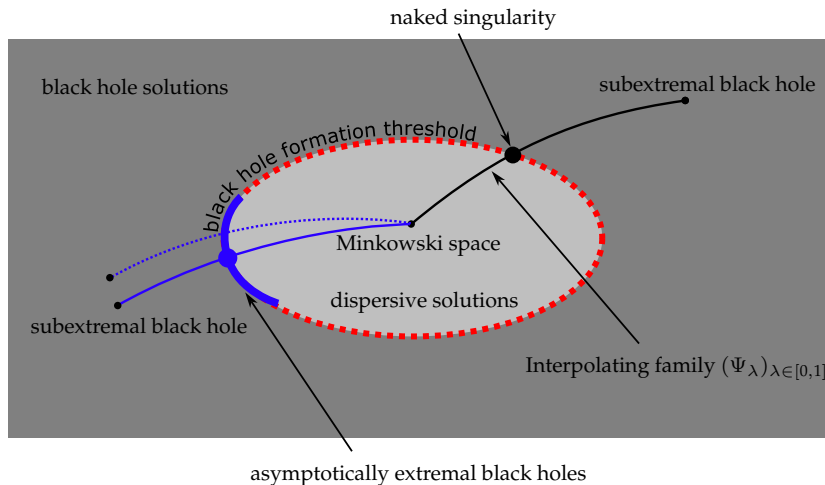
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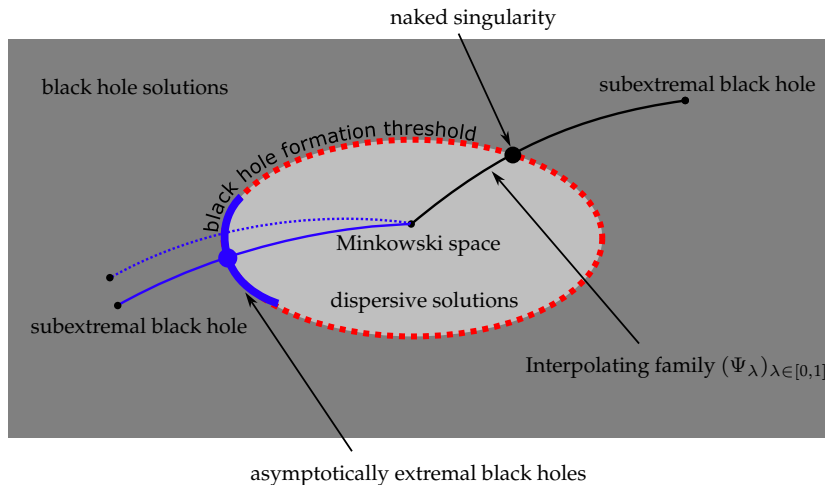


- This is also a non-trivial statement about the **interiors** of black holes.

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- This is also a non-trivial statement about the **interiors** of black holes.
- Further difficulty: **Aretakis instability** associated to extremal horizons

THE VACUUM CASE: THE THIRD LAW

Far less is known in **vacuum** and even the third law has not yet been disproved.

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Conjecture.

There exist Cauchy data for the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

*which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time. In particular, **already in vacuum**, the “third law of black hole thermodynamics” is false.*

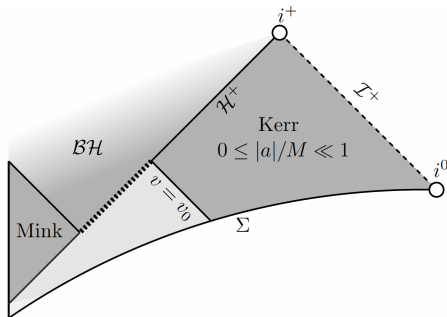
THE VERY SLOWLY ROTATING CASE

Theorem (K.–Unger, '23).

For any $0 \leq |a| \ll M$, there exist Cauchy data for the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

which undergo gravitational collapse and form an **exactly** Kerr event horizon at a finite advanced time with specific angular momentum a and mass M .



THE VACUUM CASE: EXTREMAL CRITICAL COLLAPSE

In principle, however, *extremal critical collapse*, its *stability*, and the *revised picture of moduli space*

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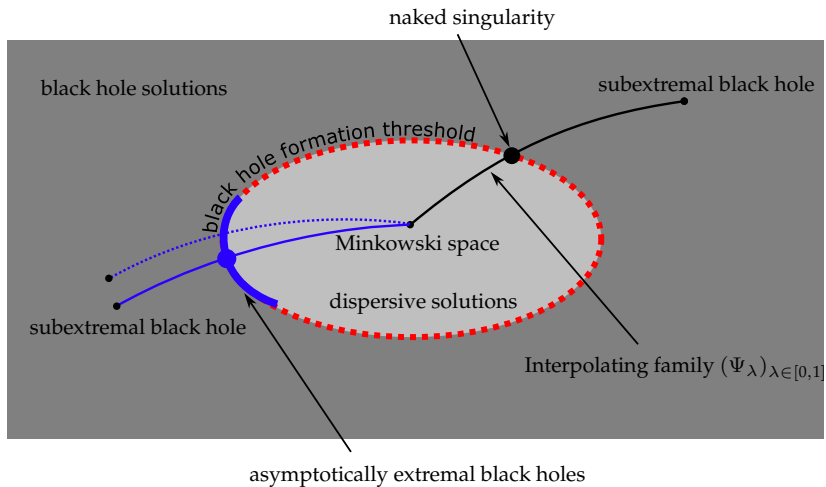
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THE VACUUM CASE: EXTREMAL CRITICAL COLLAPSE

In principle, however, *extremal critical collapse*, its *stability*, and the *revised picture of moduli space* can be conjectured to also hold true in **vacuum** with extremal Reissner–Nordström replaced by **extremal Kerr**.

However, this is a very difficult open problem and also relates to understanding

- ▶ the codimension stability and stability of extremal and near-extremal black holes
[DAFERMOS–HOLZEGEL–RODNIANSKI–TAYLOR]
- ▶ the nonlinear ramifications of horizon instabilities associated to extremal Kerr
[ARETAKIS, GAJIC].
- ▶ See the forthcoming essay by M. Dafermos on: “The stability problem for extremal black holes.”



Thank you!