

Abstract

A Few Applications of Seismic Waves: Anisotropy Tomography and All That

Puskar Mondal

2021

Seismic anisotropy, the variation of seismic wave speed with direction, is an extremely important physical phenomena. When a certain type of seismic wave (shear wave) propagates in an anisotropic medium, the component polarized parallel to the fast direction (along which the speed is higher) begins to lead and the component polarized to the slow direction lags behind (analogous to the optical birefringence). This observation of seismic anisotropy may be used to infer several physical properties of the medium through which these waves are propagating. Fortunately, Earth's upper mantle shows significant seismic anisotropy due to preferred crystallographic orientation of the constituent minerals. Therefore, it can provide crucial information regarding the convective flow and stress patterns in the upper mantle. To be more precise, seismic anisotropy can shed light on detail inner working of several geodynamic processes which are inherently anisotropic in nature and therefore insensitive to isotropic seismology.

Owing to its simplicity, the classical ray theory based formulation is widely used to infer anisotropic structures of the upper mantle. However, due to the lack of vertical resolution of infinite frequency ray theory based methods and its numerous other shortcomings even in the simplified studies assuming isotropy, it is undesirable to use a ray theory based method in a fully anisotropic framework. The major portion of this thesis is devoted to developing anisotropy tomography method in a perturbative framework where the 'finite-frequency' or the full 'wave' feature is taken into account. Such technique is proven to be a substantial improvement in terms of localization of the anisotropy of upper mantle. After benchmarking, it is applied to infer the anisotropic structures beneath the High Lava Plains of Oregon and as such was able to provide an avenue for reconciling apparently contradictory constraints on anisotropic structures from different measurements.

In the last part of the thesis, we briefly discuss a technique (slightly tangential to the main theme of anisotropy however seems to enjoy a connection at a more fundamental level) we develop to obtain an effective description of the physical properties of a general heterogeneous medium (including pure randomness). This is motivated by the fact that when propagating through small heterogeneities, seismic waves naturally average the elastic properties of the medium and therefore only an effective physics is realized.

A Few Applications of Seismic Waves: Anisotropy Tomography and All That

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Puskar Mondal

Dissertation Director: Prof. Maureen D. Long and Prof. Jun Korenaga

December, 2021

Copyright © 2021 by Puskar Mondal
All rights reserved.

Contents

Acknowledgements	v
Introduction	vi
1. A model space search approach to finite-frequency SKS splitting intensity tomography in a reduced parameter space	1
1.1. Introduction	1
1.2. Theoretical formulation of sensitivity kernels	3
1.3. Parameterization	7
1.4. Examples and properties of sensitivity kernels	11
1.5. Inversion Strategy: Model space search approach	16
1.6. Synthetic tests	18
1.6.1. Single layer case	19
1.6.2. Fully heterogeneous case with 3D structure	22
1.6.3. Totally heterogeneous case	28
1.6.4. A note on model agreement and number of parameters	30
1.7. Discussion	33
1.8. Summary and outlook	35
2. Strong seismic anisotropy in the deep upper mantle beneath the Cascadia backarc: Constraints from probabilistic finite-frequency SKS splitting intensity tomography	37
2.1. Introduction	37
2.2. Probabilistic finite frequency SKS splitting intensity tomography: Methodology . . .	39
2.3. Data and Measurements	41
2.4. Results	44
2.4.1. Anisotropy model of the NW-SE array	46

2.4.2.	Anisotropy model of the N-S array	48
2.4.3.	Consistency checks and a pseudo-3D model	48
2.5.	Resolution tests	51
2.6.	Discussion	55
2.6.1.	Probabilistic finite-frequency SKS splitting intensity tomography: Proof of concept	55
2.6.2.	Tectonic implications	56
2.6.3.	Pervasive anisotropy in the deep upper mantle beneath the Cascadia backarc	57
2.6.4.	Implications for reconciling surface wave and SKS constraints on anisotropy .	58
2.7.	Summary	60
3.	Finite frequency coupled SKS-Rayleigh wave anisotropy tomography	61
3.1.	Introduction	61
3.2.	Theoretical Formulation	63
3.2.1.	Kernel Examples	67
3.2.2.	Inversion Strategy	70
3.3.	Resolution Tests	71
3.4.	Application to real data	76
3.5.	Results and discussions	78
4.	An effective theory for elastic medium with complex heterogeneity	84
4.1.	Introduction	84
4.2.	The statement of the problem	86
4.3.	Method	87
4.4.	Functional representation of 4.4: derivation of a generating functional	89
4.5.	Mode Elimination: Course graining	102
4.5.1.	Effective Action and graph expansion	102
4.6.	RG flow of effective constants	110
4.7.	Discussion	114
5.	Appendix	115
5.0.1.	Graph expansion for the quadratic part	127
5.0.2.	Quartic part	132
	Bibliography	137

Acknowledgements

I Am Because We Are

Introduction

Elastic waves generated by Earthquakes are one of the most powerful tools to scan the Earth's interior. They provide us access to the topology and geometry of the structures and the spatial distribution of the mechanical properties of the interior of the Earth. As a whole, they lead to a better understanding of the geodynamic processes of the planet. Since the beginning, Seismology, the study of these elastic waves, has provided a tremendous amount of important information regarding the structure of the Earth. A few of such contributions include identification of the layered structure and composition of the Earth [1], delineating several anomalies at different depths [2,3,4]. With the progressing computational technology, it generated a high-resolution image of important tectonic phenomena such as plate subduction, etc and not to mention its impact on the industrial aspects. Most of the first-order structure of Earth's interior is obtained by so-called isotropic tomography which assumes that the medium is only heterogeneous. However, to delve into the detailed working of some important geodynamic processes such as discerning the overall patterns of mantle convection, detail of subduction dynamics, continental deformation, flow at the base of the mantle, etc. one needs to break the assumption of the isotropy. This is mainly because these physical processes are intrinsically anisotropic and therefore not detected by the isotropic tomography.

The study of seismic anisotropy essentially deals with the directional variation of the elastic wave speed. Such dependence is caused by the aforementioned physical processes. Earth's upper mantle shows significant seismic anisotropy due to the preferred crystallographic orientation of the constituent minerals [5]. Therefore, it can provide crucial information regarding the flow and stress patterns in the upper mantle. Experimental studies revealed the simple shear to be the most dominant mode of deformation in the upper mantle. Under the action of simple shear, the $[1\ 0\ 0]$ axis of *A* type olivine aligns parallel to the slip direction, while the other two directions ($[0\ 1\ 0]$ and $[0\ 0\ 1]$) tend to remain perpendicular to the slip direction [6]. Therefore, delineation of the directional variation of the elastic property (which controls the speed of the associated elastic waves) would seem to help one to unfold the detailed mechanism of the associated physical processes. Despite this fairly straightforward plan of attack, one is met with few obstructions while dealing with anisotropy.

Firstly, the theoretical study of wave propagation through a general anisotropic medium is slightly complicated compared to the purely isotropic case primarily due to the increase in the number of independent elasticity constants. Secondly, with the increase in the number of elasticity constants, the number of unknowns that are to be estimated in the tomography problem increases. This in turn necessitates the required number of observed data. Therefore, an observational limitation arises. Lastly, the introduction of anisotropy introduces strong non-linearity in the associated ‘inversion’ process making it extremely difficult to uniquely determine the model parameters. These last two problems are the most challenging ones since these are practical issues and in most cases difficult to overcome. Therefore, one needs to improve the theoretical study since that is the only thing under our control.

A common method for upper mantle anisotropy is to measure shear-wave splitting parameters (delay time and fast direction) [15, 29, 34]. However, the interpretation is not straightforward, because splitting measurements represent an integration of structure along the ray path. The high delay time may yield several interpretations in terms of strength of anisotropy: a thin highly anisotropic layer or a thick layer of weak anisotropy. In addition to being non-unique in terms of strength, the details causing anisotropy (LPO or SPO in presence of melt) may not be understood clearly. To avoid such ambiguity of interpretations, therefore, a tomographic approach that allows for localization of anisotropy is desirable. Secondly, such interpretation is dependent on the ray theory description of wave propagation, which in the context of shear wave propagation through the upper mantle is a rather crude assumption. This is primarily because such a geometric optical description of the wave propagation is valid in the regime where the frequency is high. To be precise, such approximation holds when the frequency (f) and the travel-time (T) satisfies a relation of the type $fT \gg 1$ (simply as a consequence of the phase space uncertainty). For SKS waves traveling through the upper mantle, this entity is of the order of 10 and therefore one needs to address the ‘finite frequency effect’. Besides, since the ray paths are one-dimensional, one needs in principle an infinite number of rays to completely sample a higher dimensional heterogeneity (dimension ≥ 2). Another severe problem with the geometrical optics assumption is the generation of infinite strain at caustics (the volumetric strain $\Theta := \text{div } u$, u being the displacement field, diverges) which is unphysical. This is precisely a consequence of the non-renormalizability of the elasticity theory at the level of very low wavelengths (λ) in the presence of complicated (possibly random) heterogeneity. The propagator diverges at low wavelength as $O(\frac{1}{\lambda^3})$. Notice that even though the geometric optical calculations are suitable for several applications in the context of EM wave propagation, one needs to remember that EM theory is *not* non-renormalizable like elasticity in the presence of complex heterogeneity.

In order to address these issues, we want to develop a self-consistent theory that addresses the underlying physics fairly accurately while yielding a better result from a practical point of view (i.e., able to resolve structures using a reasonable amount of observations). To avoid the ambiguity associated with the classical ray theory, we implement the so-called finite frequency tomography originally developed by [7] in the context of isotropic tomography. This approach was later generalized to the anisotropic framework by [8]. Building on previous work on finite frequency shear wave splitting intensity tomography [8,9,10], here we develop a framework for tomographic inversion of SKS splitting intensity measurements from dense seismic networks, with a new parametrization of anisotropy and a model space search approach. The model space search approach rather than a linearized inversion is utilized to address the non-linearity associated with the anisotropy. The problem of ‘under-sampling of heterogeneity’ associated with the ray theory-based tomography problem is circumvented using the finite frequency sensitivity kernels which sample the first few Fresnel zones. The use of overlapping kernels enhances the robustness of the model estimation substantially. In the first chapter of this thesis, we address the theoretical and computational detail of our study of finite frequency SKS splitting intensity tomography along with several synthetic examples to benchmark it. In the second chapter, we move on to applying this theory to real data associated with the High Lava Plain of Oregon, USA. A particular reason for choosing this data set is the availability of a fairly dense seismic network along with fairly good azimuthal coverage. Using the inversion results coming from two dense arrays of stations, we set a possible connection between the observed anisotropy and geodynamic processes associated with this particular geophysical setting.

Even though this finite frequency SKS splitting intensity tomography advances the anisotropy tomography up to a certain degree, one would ideally require further constraints. This is primarily because the shallow to upper-middle part of the model remains unresolved. After all, the sensitivity kernels do not have sufficient overlap in this portion. Therefore, we need additional constraints which are likely to improve the model. As it happens, we do have the surface waves which are sensitive to the anisotropy and have control over the depth resolution within the upper mantle. In a suitable sense, surface waves possess the ability to scan several depths in a telescopic manner. In order to take advantage of this property of the surface waves, we couple the SKS splitting intensity tomography with the surface wave phase velocity tomography in the sense of a ‘joint’ inversion. The third chapter deals with the theoretical framework of this study as well as an application to the High Lava Plain of Oregon (HLP).

Lastly, in the fourth and last chapter of the thesis, we develop a theory that is slightly tangential to the main theme of seismic anisotropy. However, it seems to enjoy a connection with the tomog-

raphy (possibly anisotropic) at a more fundamental and deeper level. We shortly describe such a connection. This involves an effective field theory (EFT) approach to address the physics of wave propagation in heterogeneous (possibly random) media, their homogenization, and the occurrence of possible critical phenomena. The occurrence of critical phenomena is quite common in a range of physical systems. Roughly the critical phenomena occur when there are two competing forces present which nullify each other. Therefore, if one starts with a heterogeneous medium that has a degree of randomness as well as a correlation, then the wavefield passing through the medium responds accordingly; it may be localized or may be correlated over a large distance displaying an ‘order’ depending on the relative strengths of the random and correlated heterogeneities. Wave propagation (or simply elastostatic deformations) in these heterogeneous media is by itself an extremely rich topic and we only address a quite small fraction. Often it is important to obtain an effective homogeneous description of these heterogeneous media that respects the underlying physics. This is physically relevant and important because of the following fact. Consider the heterogeneities present in the mantle. They do have a large range of length scales (few *mm* to few *km*) and when seismic waves propagate through such a heterogeneous medium, a multitude of processes occur at a wide range of length scales. However, the observations we make on the surface are effective ones since the seismic waves have limited spectra. Therefore it is necessary to develop an effective theory of heterogeneous medium which is to be incorporated appropriately while performing seismic tomography. Secondly, the critical phenomena are relevant in the current context because it is scale invariant. The complex processes associated with each length scale combine to yield an effective scale-invariant phenomena where the system acquires a certain level of ‘order’. In summary, we develop a homogenization technique of heterogeneous medium (including purely random medium) and describe possible critical phenomena occurring in complex elastic systems containing purely random and correlated heterogeneity. We discuss a potential application of this critical phenomena to the Earthquake theory.

Chapter 1

A model space search approach to finite-frequency SKS splitting intensity tomography in a reduced parameter space

1.1 Introduction

Relating shear wave splitting observations to anisotropic structure in the Earth's interior is a classical problem in seismology [11,12,13,14]. Shear wave splitting arises when shear waves travel through an anisotropic medium, and refers to the phase lead of the quasi-shear wave component polarized in the fast direction with respect to the component polarized in the slow direction. Observations of shear wave splitting (or other wave phenomena that reflect anisotropy) are important in understanding various geodynamic phenomena, such as overall patterns of mantle convection [15,16], subduction dynamics [17,18,19,20], oceanic spreading centers and continental rift systems [21,22,23], continental deformation [24,25], and flow at the base of the mantle [26,27,28].

A commonly used method to infer upper mantle anisotropy - and therefore upper mantle flow patterns - is to measure shear wave splitting parameters (fast direction, ϕ , and delay time, dt), typically for phases such as SKS that have been refracted through the outer core [29]. However, the interpretation of SKS splitting measurements is not straightforward, because they represent an integration of anisotropic structure along the ray path. Large delay times, for example, may yield several interpretations in terms of strength of anisotropy (a thin highly anisotropic layer vs. a thick layer with weak anisotropy). Similarly, in the presence of multiple anisotropic layers, the apparent fast direction reflects a complicated (and non-commutative) [12,30] combination of layer parameters, and isolating the depth of different anisotropic regions is a major challenge.

To lessen the ambiguity of shear wave splitting interpretation, a tomographic approach that allows for the localization of anisotropy is desirable. However, tomographic inversion for anisotropic structure is a daunting task, since 21 parameters are needed to describe the most general form of anisotropy, and such a large parameter space does not allow for a tightly constrained tomographic

inversion given limited data. Furthermore, the inverse problem for anisotropy is highly non-linear, as the sensitivities of splitting observations to anisotropy depend on the starting model used to compute them [31]. Finally, shear wave splitting tomography is further complicated by the measurements themselves, which are challenging for noisy data [32], resulting in much smaller data sets than traditional traveltime tomography. The measurement problem can be somewhat ameliorated by making measurements of the splitting intensity [33] rather than the traditionally estimated splitting parameters (ϕ, dt) , as this quantity can be more robustly estimated from noisy data [32, 34] and has the advantage of being commutative along the ray path [8, 30, 33].

Substantial effort has been devoted to developing theoretical frameworks and practical strategies for inverting shear wave splitting data for anisotropic structure, much of it in the context of finite-frequency sensitivity kernels that relate model perturbations to splitting observations [8, 9, 10, 35, 36]. However, there have only been a few published examples of the application of splitting tomography to actual data sets to study mantle anisotropy. [31] applied a ray theoretical framework to invert for anisotropic structure in a subduction zone setting, taking advantage of seismicity within the subducting slab to provide ray coverage for local S phases; this approach was also applied by [37]. [38] used data from dense seismic networks in southern California to apply finite-frequency SKS splitting tomography, producing a model of lateral and depth variations of anisotropy parameters. Lin et al. (2014a) applied a full-wave approach to the computation of sensitivities and a multiscale parameterization to the inversion of a similar dataset from southern California, based on insights from full-wave modeling of SKS phases (Lin et al. 2014b).

While these tomographic approaches have enabled substantial progress in understanding the 3D distribution of seismic anisotropy in well-sampled regions, they all share a significant limitation in that the inversion strategy is based on a linearization around a background model, which may result in bias in the model estimation. Sensitivity kernels for anisotropic structure depend strongly on the background model used to calculate them, and are substantially different for heterogeneous anisotropic starting models than for the homogeneous ones that are often used to calculate them [9]. Because of this limitation, the implementation of a model space search approach to shear wave splitting is desirable; such an approach, while computationally intensive, avoids the explicit linearization of a highly nonlinear inverse problem, and yields constraints on the ranges of model parameters allowed by the observations [39].

Building on previous work on finite frequency shear wave splitting intensity tomography [8, 9, 10], here we develop a framework for tomographic inversion of SKS splitting intensity measurements from dense seismic networks, with a new parameterization of anisotropy and a model space search

approach. We reparameterize the full elasticity tensor based on insights from olivine elasticity and fabric development, reducing the number of parameters to three (a measure of anisotropy strength, plus the dip and azimuth of the fast symmetry axis). We compute Born-approximation finite frequency sensitivity kernels relating model perturbations to splitting intensity, allowing us to sample the first few Fresnel zones of sensitivity instead of assuming sensitivity along an infinite frequency ray. This finite frequency approach allows us to obtain overlapping sensitivity kernels for observations at adjacent stations, even for nearly vertically propagating SKS waves. We implement a Markov chain Monte Carlo (McMC) technique to sample model space and obtain a most likely model for anisotropic structure, and sample the resulting probability distributions for various anisotropic parameters.

The structure of the paper is as follows. We begin with a development of the theoretical formulation necessary for the finite-frequency anisotropy tomography problem in a three-dimensional, Cartesian geometry, building on previous work by [8] and [9]. We discuss a simplified parameterization for anisotropy based on elasticity and fabric development in olivine. We then test our forward formulation against various analytical solutions for the behavior of splitting intensity observations for simple anisotropic models. Next, we develop and implement an McMC approach to probing parameter space, and perform inversion tests with synthetic data sets to evaluate computational cost and infer the resolving power of our algorithm for synthetic models with multiple anisotropic layers. Finally, we discuss the prospects for implementation of our tomography strategy to actual SKS splitting data sets.

1.2 Theoretical formulation of sensitivity kernels

Equation of motion for the infinitesimal displacement along with the constitutive relation for linear elasticity lead to the wave equation

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot (\mathbf{C} : \nabla \mathbf{u}), \quad (1.1)$$

where $\mathbf{u}$, $\mathbf{C}$, and ρ are the displacement, elasticity tensor, and density, respectively. Given the initial displacement and velocity (i.e., the Cauchy data), equation (1.1) has a unique time evolution (in $C^0(I \subset \mathbf{R}; W^{1,2} \times L^2(V \subset \mathbf{R}^3))$), which may be computed using Green's function [40]. Computation of the Green's function ($\mathbf{G}$) involves the solution of the following inhomogeneous partial differential equations [41]

$$\rho \partial_t^2 \mathbf{G} = \nabla \cdot [\mathbf{C} : \nabla \mathbf{G}] + \mathbf{I} \delta(\mathbf{r} - \mathbf{r}_s) \delta(t - t_s), \quad (1.2)$$

where $\mathbf{I}$, $\mathbf{r}_s$, and t_s are the identity operator, seismic source location, and source time, respectively.

To focus on anisotropic tomography, we express the wave related entities as first order perturbations from the background isotropic reference state:

$$\mathbf{u} = \mathbf{u}_0 + \epsilon \delta \mathbf{u}, \quad (1.3)$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 + \epsilon \sigma, \quad (1.4)$$

$$\mathbf{C} = \mathbf{C}_{Iso} + \epsilon \delta \mathbf{C}, \quad (1.5)$$

where $[\mathbf{u}_0, \boldsymbol{\Sigma}_0, \mathbf{C}_{Iso}]$, $[\delta \mathbf{u}, \sigma, \delta \mathbf{C}]$, and ϵ represent the isotropic reference state, the perturbed state, and the perturbation parameter, respectively. Subtracting the isotropic reference state and neglecting the advection term from the total derivative and higher order terms ($O(\epsilon^2)$) leads to

$$\rho \frac{\partial^2 \delta \mathbf{u}}{\partial t^2} = \nabla \cdot (\mathbf{C}_{Iso} : \nabla \delta \mathbf{u}) + \nabla \cdot (\delta \mathbf{C} : \nabla \mathbf{u}_0) \quad (1.6)$$

which resembles the equation of the motion of the unperturbed state, with $\nabla \cdot (\delta \mathbf{C} : \nabla \mathbf{u}_0)$ being the source term (forcing term). Therefore, we may now invoke the same Green's function in order to compute the perturbed displacement field. However, before doing so, we transform the entities to the frequency domain for convenience in computation

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(t') e^{-i\omega t'} dt' \right) e^{i\omega t} d\omega, \quad (1.7)$$

where ω is the angular frequency. Here, we assume that $g \in L^1(\mathbf{R})$ i.e., $\int_{-\infty}^{\infty} |g(t)| dt < \infty$ with $\mathbf{R}$ being the real line in order to guarantee the existence of Fourier transform of Dirac's delta function (absolutely integrable, but not square integrable). Under such transformation, elliptic equations defining the perturbed state may be written as

$$-\rho \omega^2 \delta \mathbf{u} = \nabla \cdot (\mathbf{C}_{Iso} : \nabla \delta \mathbf{u}) + \nabla \cdot (\delta \mathbf{C} : \nabla \mathbf{u}_0), \quad (1.8)$$

and existence of its solutions is well established [42]. The eigenfrequency spectrum is unique and may be computed as follows. Taking dot product of both sides of equation (1.8) by $\delta \mathbf{u} \in W^{1,2}(V) - \{0\}$ and integrating over the spatial domain and assuming that the perturbation vanishes on the boundary

of the manifold (if it has a boundary), we obtain the following expression for ω :

$$\omega^2 = \frac{\int_V (\nabla \delta \mathbf{u} : \mathbf{C}_{Iso} : \nabla \delta \mathbf{u} + \nabla \delta \mathbf{u} : \delta \mathbf{C} : \nabla \mathbf{u}_0) dV}{\int_V \rho |\delta \mathbf{u}|^2 dV}, \quad (1.9)$$

which guarantees the existence of the solution of equation (1.8) given that the numerator is positive.

We may write the solution in the frequency domain in terms of the Green's function as follows

$$\delta \mathbf{u}(\omega) = \int_{\Omega} \mathbf{G} \cdot (\nabla \cdot (\delta \mathbf{C} : \nabla \mathbf{u}_0)) dV, \quad (1.10)$$

which may further be simplified

$$\delta \mathbf{u} = \int_{\Omega} \nabla \cdot (\mathbf{G} \cdot (\delta \mathbf{C} : \nabla \mathbf{u}_0)) dV - \int_{\Omega} (\delta \mathbf{C} : \nabla \mathbf{u}_0) : \nabla \mathbf{G} dV, \quad (1.11)$$

$$\begin{aligned} &= \int_{\partial \Omega} (\mathbf{G} \cdot (\delta \mathbf{C} : \nabla \mathbf{u}_0)) \cdot \mathbf{n} dS - \int_{\Omega} (\delta \mathbf{C} : \nabla \mathbf{u}_0) : \nabla \mathbf{G} dV, \\ &= - \int_{\Omega} (\delta \mathbf{C} : \nabla \mathbf{u}_0) : \nabla \mathbf{G} dV, \end{aligned} \quad (1.12)$$

where we have assumed that the perturbation vanishes on the boundary. Now we move on to use the expressions of the useful entities to obtain the observables of seismic anisotropy.

In a general, three dimensional elastic continuum, three body waves exist: one primary wave (P) and two shear waves (S). In an isotropic medium, $\mathbf{SO}(3)$ symmetry of the elasticity tensor forces the eigenspectrum of the wave operator (equation 1.8) to be degenerate. In other words, both the shear waves (SV and SH) travel with the same speed. However, in the presence of anisotropic scatterer, $\mathbf{SO}(3)$ symmetry is broken and therefore, the degeneracy is lost; two shear waves have different speeds. Thus, a phase delay between two such shear wave phases provides an observational constraint on the anisotropic properties of the medium.

In this study we consider the SKS phase. The SKS phase is generated from a P to S conversion at the core-mantle boundary and therefore, it is supposed to be radially polarized (that is, polarization in the source-receiver plane) in an isotropic medium. The presence of anisotropy in the mantle, however, can transfer energy from the radial part to the transverse part of the wave displacement field via scattering leading to a non-zero amplitude of the transverse component. Thus, we may use this scattered transverse amplitude as a measure to describe shear wave splitting. Following [33], shear wave splitting intensity(S) is defined as

$$S(\mathbf{r}, \mathbf{r}_s) = 2 \frac{\text{Re} \int \omega \delta u^T(\mathbf{r}, \mathbf{r}_s; \omega) u_0^{*R}(\mathbf{r}, \mathbf{r}_s; \omega) d\omega}{\int \omega^2 |u_0^R(\mathbf{r}, \mathbf{r}_s; \omega)|^2 d\omega}, \quad (1.13)$$

where δu^T and u_0^R are the transverse and radial components of the displacement, respectively. Using this particular measure of anisotropic characteristic of the SKS wave form, we construct an integral map between the anisotropic perturbation to the elasticity tensor and the physical observable such as the one we have just defined. Now we move on to compute the integral kernels relating the derived observable (splitting intensity) and the unknown anisotropic scatterers.

Let us denote the unit vectors in the radial and transverse directions by $\hat{\mathbf{r}}$ and $\hat{\mathbf{t}}$, respectively. Here $\hat{\mathbf{r}}$ is the particle displacement direction (not to be confused with the radial direction in spherical polar co-ordinate system) associated with the SKS waves polarized in the radial plane (source-station plane). Then the transverse component of the scattered wave field is $\delta \mathbf{u} \cdot \hat{\mathbf{t}}$, which may be written in the following form using equation (3.7)

$$\delta u^T = \delta \mathbf{u} \cdot \hat{\mathbf{t}} = - \left(\int_{\Omega} (\delta \mathbf{C} : \nabla \mathbf{u}_0) : \nabla \mathbf{G} dV \right) \cdot \hat{\mathbf{t}}, \quad (1.14)$$

$$= - \int_{\Omega} ((\delta \mathbf{C} : \nabla \mathbf{u}_0) : \nabla \mathbf{G}) \cdot \hat{\mathbf{t}} dV, \quad (1.15)$$

$$= - \int_{\Omega} \hat{\mathbf{t}} \cdot (\nabla \mathbf{u}_0 \otimes \nabla \mathbf{G}) :: \delta \mathbf{C} dV. \quad (1.16)$$

We may neglect the perturbation to the radial component, as such perturbation has only second order effect on the splitting intensity. This could be seen immediately from equation (2.1) if one were to consider the perturbation to the radial component i.e., $u^R = u_0^R + \delta u^R$: the contribution of such perturbation takes the form $\delta u^T \delta u^{*R}$, which is clearly second order and thus may be neglected. Under such assumption, the radial component is written as

$$u_0^R = \mathbf{u}_0 \cdot \hat{\mathbf{r}}. \quad (1.17)$$

Now that we have obtained each component required to compute the splitting intensity, we may substitute equations (1.16) and (1.17) into equation (2.1)

$$S(\mathbf{r}, \mathbf{r}_s) = -2 \frac{Re \int \omega \left(\int_{\Omega} \hat{\mathbf{t}} \cdot (\nabla \mathbf{u}_0 \otimes \nabla \mathbf{G}) :: \delta \mathbf{C} dV \right) (\mathbf{u}_0 \cdot \hat{\mathbf{r}})^* d\omega}{\int \omega^2 |(\mathbf{u}_0 \cdot \hat{\mathbf{r}})|^2 d\omega}, \quad (1.18)$$

$$= -2 \frac{\int_{\Omega} Re \left(\int \omega (\mathbf{u}_0 \cdot \hat{\mathbf{r}})^* \hat{\mathbf{t}} \cdot (\nabla \mathbf{u}_0 \otimes \nabla \mathbf{G}) d\omega \right) :: \delta \mathbf{C} dV}{\int \omega^2 |(\mathbf{u}_0 \cdot \hat{\mathbf{r}})|^2 d\omega}, \quad (1.19)$$

$$= \int_{\Omega} \mathbf{K} :: \delta \mathbf{C} dV, \quad (1.20)$$

where the sensitivity kernel $\mathbf{K}$, which relates perturbations in $\delta \mathbf{c}$ to splitting observation S , is given

as follows

$$\mathbf{K} = -2 \frac{\text{Re} \left(\int \omega (\mathbf{u}_0 \cdot \hat{\mathbf{r}})^* \hat{\mathbf{t}} \cdot (\nabla \mathbf{u}_0 \otimes \nabla \mathbf{G}) d\omega \right)}{\int \omega^2 |(\mathbf{u}_0 \cdot \hat{\mathbf{r}})|^2 d\omega}. \quad (1.21)$$

Here, ‘ $\otimes$ ’ represents tensor product operation on vector bundles (tensors are treated as sections of vector bundles). In deriving the previous equation, we have considered the fact that in purely elastic medium, the elasticity tensor is independent of frequency.

In order to calculate the sensitivity kernels, we need the Green’s function. For a homogeneous, isotropic wave speed model, the Green’s functions can be solved analytically [41]. However, for a velocity structure that varies with depth, a semi-analytical technique such as a propagator matrix method works well [43, 44, 45]. In our case, we experimented with two different methods for approximating the Green’s function in the kernel computation. First, we used analytical solutions for the Green’s function for a single-layer model, using average isotropic P and S wave speeds for the upper mantle ($c_p = 7.9 \text{ km s}^{-1}$ and $c_s = 4.4 \text{ km s}^{-1}$ [46]). Second, we implemented the semi-analytical propagator matrix technique for a depth-dependent vertical velocity profile (ak135) [47]. In the propagator matrix technique, the fields are expanded in the Fourier series in the horizontal direction (no horizontal variation of seismic wave speed), and the frequency domain wave equation is converted to a set of ordinary differential equations in z . Further details can be found in [41]. In our kernel computations presented in the paper, we used the propagator matrix technique applied to the depth dependent velocity model ak135; because it is a semi-analytical technique, is very efficient and allows for exploration of the model space with reasonable number of iterations. We point out, however, that because the variations in isotropic wave speed within the upper mantle are not dramatic, the differences between our propagator matrix computations for the Green’s functions in a depth-dependent velocity model and the analytical Green’s function for a homogeneous, single-layer model are slight. We compared kernels using the two different methods of computing the Green’s function, and found that they were nearly identical.

1.3 Parameterization

Once we have derived the expression for the sensitivity kernels (equation 24), in theory we may invert equation (23) to obtain the full anisotropic perturbation of the elastic medium given the splitting intensity (S). However, such an inversion would involve solving for each independent elastic constant in a fourth-rank tensor (21 in the most general case), which represents an impractically large number of unknown parameters given the limitations of observational data sets. If the symmetry of anisotropy is known or assumed, then the number of independent parameters is reduced (for

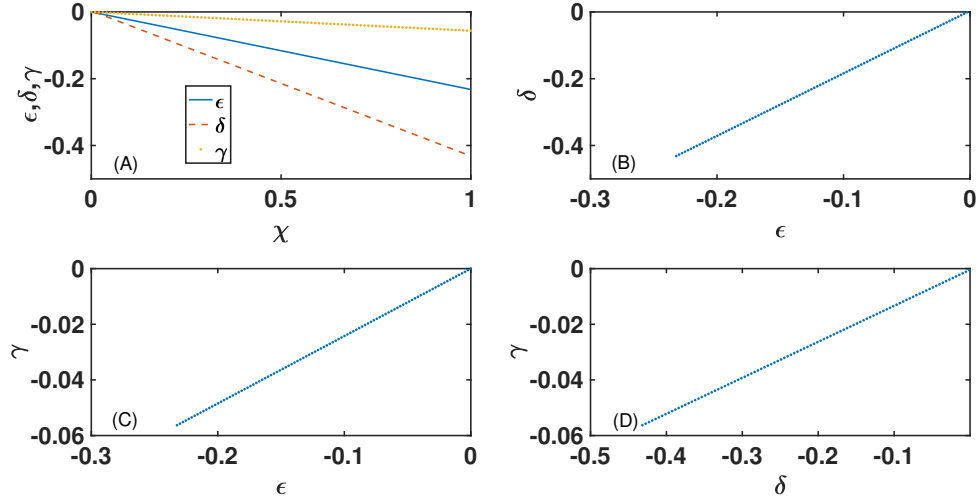


Figure 1.1: (A) Variation of the three dimensionless Thomsen parameters (ϵ , δ , γ) with the strength parameter χ , (B) Correlation between ϵ and δ , (C) Correlation between ϵ and γ , (D) Correlation between δ and γ . Our parameterization scheme uses χ instead of the Thomsen parameters to define the strength of upper mantle anisotropy, taking advantage of the linear relationships between χ and ϵ , δ , and γ .

example, for an olivine crystal with orthorhombic symmetry, the number of independent elastic constants is 9; in practice, this is still too large a number for an anisotropic tomography problem).

In previous work, [8] suggested a vectorial tomography which employs a linear inversion technique in order to estimate three Thomsen's anisotropy parameters (ϵ , δ , and γ). These three dimensionless parameters are defined as follows for hexagonal anisotropy (for the case in which anisotropy is weak i.e., $\|\delta \mathbf{c}\|_{L^p} \ll \|\mathbf{C}_{iso}\|_{L^p}$, for some $p \geq 1$) [48]

$$\epsilon = \frac{c_{11} - c_{33}}{2\rho\alpha^2}, \quad (1.22)$$

$$\delta = \frac{c_{13} - c_{33} + 2c_{44}}{\rho\alpha^2}, \quad (1.23)$$

$$\gamma = \frac{c_{66} - c_{44}}{2\rho\beta^2}, \quad (1.24)$$

where ρ , α , and β are the density, isotropic P, and isotropic S wave speed, respectively. Together with the orientation of the symmetry axis in space (dip angle from the horizontal, θ , and azimuth from north, ϕ), in this formulation there are a total of five parameters to invert, and additional simplifications are needed to reduce this number. Because typical shear wave splitting data sets

cannot recover all five parameters simultaneously without producing a bias towards some parameters, such additional simplifications are necessary for our problem.

Our approach is to reparameterize the strength of anisotropy using insights from mineral physics, specifically experimental and observational constraints on the development of crystallographic preferred orientation (CPO) in olivine aggregates. Seismic anisotropy of the upper mantle is thought to be mainly controlled by this mechanism [6]; furthermore, olivine aggregates that have been deformed to high strains develop CPO such that the elasticity of textured aggregates has a geometry (though not a strength) that is generally similar to the elasticity of single-crystal olivine [6]. We therefore assume that olivine aggregates in the upper mantle will exhibit behavior characteristic of a linear mixture of two endmembers: an isotropic equivalent, in which the individual grains are randomly oriented, and a perfectly aligned aggregate, in which all grains exhibit the same (preferred) orientation and the elasticity is equivalent to a single crystal of olivine. In this scheme, the strength of anisotropy can be described with a single parameter that varies from 0 (random alignment) to 1 (complete alignment). We further assume that the elasticity of the aligned olivine endmember can be well described by a hexagonal approximation to the actual (orthorhombic) symmetry, an assumption that is justified by the elastic characteristics of single-crystal olivine, along with experimental results on deformation of polycrystalline olivine aggregates [6].

Our goal is to obtain a hexagonal equivalent of the orthorhombic olivine tensor. There are several ways of doing so; one way is to use the projection method described in [49]. An alternative (but conceptually equivalent) method to obtain the hexagonal part of the orthorhombic tensor is to rotate about the [100] (fast) olivine axis and average the elements of the elasticity tensor describing the intermediate and slow velocities, as described below. We have tested both schemes and they yield virtually identical results; results shown in this paper were obtained using the latter approach. We carry out hexagonal averaging of the elasticity tensor from the lower (orthorhombic) symmetry class by taking the angular average of each of the three sets (c_{11}, c_{22}) , (c_{44}, c_{55}) , and (c_{13}, c_{23}) . This averaging scheme utilizes the fact that elements of each set do not differ substantially from each other (that is, the orthorhombic symmetry is weak on the plane perpendicular to the c axis). The spherical harmonic averaging is done using

$$C_{11hexagonal} = \frac{\sum_{l=0}^{\infty} \int_0^{2\pi} (C_{11orthorhombic}(\Phi) + C_{22orthorhombic}(\Phi)) P_l(\cos \Phi) d\Phi}{2 \sum_{l=0}^{\infty} \int_0^{2\pi} P_l(\cos \Phi) d\Phi}, \quad (1.25)$$

where P_l 's are the ordinary Legendre polynomials and Φ is angle of rotation about c axis. Similarly, other sets $((C_{44}, C_{55})$ and (C_{13}, C_{23})) may also be averaged to obtain the hexagonal equivalence. Fol-

lowing homogeneous stress approximation (Reuss average; [50]), the elastic constants of the mixture C_{ij} may be expressed as

$$C_{ij} = \chi C_{ij\text{single}} + (1 - \chi) C_{ij\text{Isotropic}}, \quad (1.26)$$

where χ is the fraction of pure hexagonal crystal present in the mixture. This formulation uses the technique of [49] in order to extract the isotropic part of the elasticity tensor from the full olivine anisotropic tensor. Using this particular approximation, we may compute the three Thomsen's parameters based on equations (1.22)-(1.24) and note their evolution as a function of χ (Fig. 1.1). When the mixture consists of completely aligned crystals ($\chi = 1$), the Thomsen's parameters are that of (hexagonally averaged) olivine, and for the purely isotropic crystal ($\chi = 0$), they are identically zero.

We checked whether the Thomsen parameters that we computed for the hexagonally averaged crystal are close to those presented in [51]. This comparison is imperfect, however, because [51] used an aggregate of olivine and enstatite, rather than the pure olivine aggregate that we use in our work. The values for the Thomsen parameters in our work are therefore not identical, but they are similar. Thomsen's parameters (ϵ, δ, γ) vary linearly with χ , as shown in Figure 1.1(a). Figure 1.1(b), c, and d show the correlation between the Thomsen parameters. We also considered the correlations among different Thomsen parameters, both for our result and the one presented in [51], and again the relationships were generally similar. Because in our approach we used a linear mixture between the isotropic and (hexagonal) anisotropic crystals, a linear correlation between the anisotropy parameters is expected, and this is indeed what is observed.

Using equations (1.20), (1.21), (1.22-1.24), and (1.26) we may obtain the following simplified equation relating the splitting intensity and the 'strength' of anisotropy

$$S(\mathbf{r}) = \int_{\Omega} K_{\chi}(\mathbf{r}, \theta, \phi; \mathbf{r}_s) \chi(\mathbf{r}_s) d^3 \mathbf{r}_s, \quad (1.27)$$

where K_{χ} is the corresponding sensitivity kernel, which is itself a non-linear function of the orientation parameters (θ, ϕ) . One important thing is to note is that the equation (2.2) is only valid under the assumptions made previously; for the case of relatively weak anisotropy due to CPO of olivine in the upper mantle, this equation yields a good approximation for the relationship between anisotropic model perturbations and the splitting intensity S . Under these conditions, we can represent the anisotropic geometry and strength with three parameters (θ, ϕ, χ) , instead of the 21 independent parameters needed for the most general case of anisotropy.

1.4 Examples and properties of sensitivity kernels

The strong dependence of the sensitivity kernel on the background model is an important issue in shear wave splitting tomography [9], and this dependence makes the linearization of the inverse problem difficult, as discussed in section 5. In this section, we explore the behavior of sensitivity kernels and provide some simple examples of their strong dependence on the background model. The kernel behavior is illustrated via figures that show the variation of the sensitivity in the model space as functions of the background model properties (azimuth and dip of the symmetry axes of the anisotropic medium). In addition to representing linear maps between the model space and the splitting intensity observable, sensitivity kernels provide insights into the first-order behavior of the wavefield in the anisotropic medium.

We choose the power spectrum of the incoming SKS wave to be similar to that of [9], i.e.,

$$|\mathbf{u}_0^R(\omega)|^2 = \frac{\omega^2 \tau^2}{4\pi} e^{-\frac{\omega^2 \tau^2}{8\pi^2}}, \quad (1.28)$$

where τ is the characteristic period of the wave, which is taken to be 10 s for SKS phase. The integral over the frequency in equation (1.21) is performed over the frequency range of 0.02 – 0.125 Hz which is the same as used by [9] and typical for SKS splitting studies [29]. We approximate the spectrum as the second derivative of a Gaussian; this is reasonable for SKS-type phases where the records have been deconvolved from the radial component waveform [33, 34]. With this choice of unperturbed wave spectrum, we compute the sensitivity kernels with different background anisotropic model. The kernels are computed on a fine grid ($(5 \text{ km})^3$ box) for two different cases: (a) variation of the dip of the symmetry axis, and (b) variation of the azimuth of the symmetry axis. The incoming polarization direction is kept constant in both of these cases. The width of the maximum sensitivity zone is calculated as $\approx \sqrt{\lambda L}$, where λ and L are the characteristic wavelength and the propagation distance respectively. In our case, with the characteristic period of 10 s and approximate propagation distance of 400 km, we find the width of the maximum sensitivity zone to be ≈ 300 km. However, the nature of the sensitivity varies with each cases discussed below.

We compute a series of sensitivity kernels for different values of the azimuth of the symmetry axis (ϕ) while the dip ($\theta = 0^\circ$) and the incoming polarization direction ($\alpha = 0^\circ$) are kept fixed. The anisotropic perturbation is homogeneous in nature. The spatial distribution of the sensitivity is not substantially modified with the variation of azimuth (ϕ) of horizontal symmetry axis. Figure 1.2(B) and (C) show near symmetric sensitivity kernels corresponding to $\phi = 30^\circ$, and $\phi = 60^\circ$,

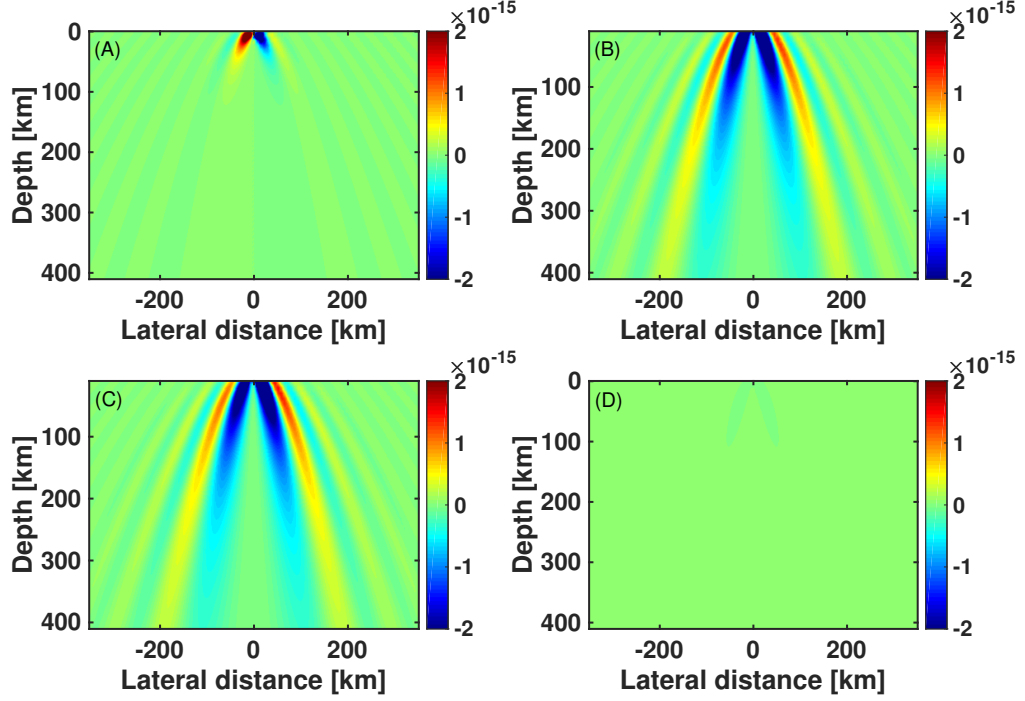


Figure 1.2: Sensitivity kernel examples: 2-D slice along the plane of propagation. Plots show behavior of the sensitivity kernels at different values of the azimuth of the symmetry axis (ϕ) while the dip of the symmetry axis ($\Theta = 0^\circ$), and the incoming polarization direction ($\alpha = 0^\circ$) remain fixed. (A) symmetry axis is parallel to the polarization direction ($\phi = 0^\circ$): the kernel is perfectly antisymmetric leading to a zero splitting intensity for a homogeneous anisotropic model, (B) $\phi = 30^\circ$, (C) $\phi = 60^\circ$: symmetric kernel, and (D) symmetry axis is perpendicular to the polarization direction ($\phi = 90^\circ$): trivially zero sensitivity kernel yielding zero splitting intensity for arbitrary spatial distribution of anisotropic strength. Unit of the sensitivity is $m^{-3}s$.

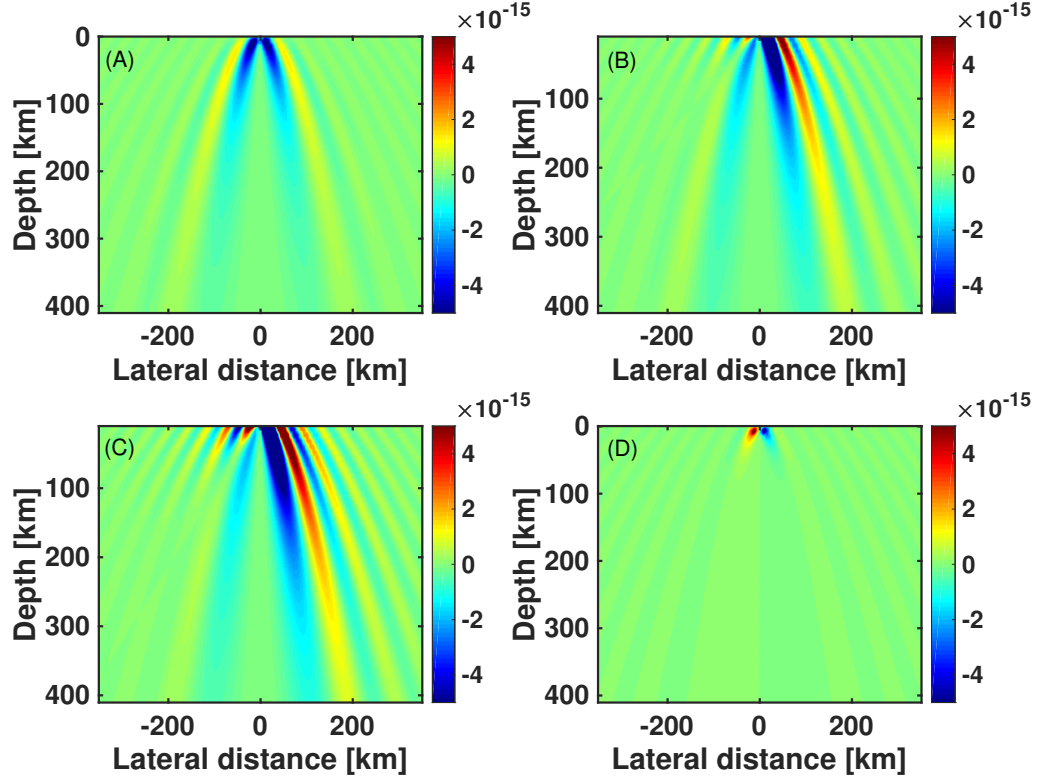


Figure 1.3: Sensitivity kernel examples: 2-D slice along the plane of propagation. Plots show behavior of the sensitivity kernels at different values of the dip of the symmetry axis (Θ), while the azimuth of the symmetry axis ($\phi = 45^\circ$) and the polarization of the incoming wave ($\alpha = 0^\circ$) are kept fixed. (A) 2D slice through a 3D sensitivity kernel for a model with a horizontal symmetry axis ($\Theta = 0^\circ$): almost fully symmetric kernel and thus sensitivity is evenly distributed, (B) tilted symmetry axis ($\Theta = 30^\circ$), (C) $\Theta = 60^\circ$, and (D) vertical symmetry axis ($\Theta = 90^\circ$): perfectly antisymmetric sensitivity kernel yielding zero splitting intensity for homogeneous anisotropic model. Unit of the sensitivity is $m^{-3}s$.

respectively. However, the end member cases of $\phi = 0^\circ$ (polarization is parallel to the symmetry axis) and $\phi = 90^\circ$ (polarization is perpendicular to the symmetry axis) are quite interesting. While the latter corresponds to a trivial sensitivity kernel (Figure 1.2(D)) leading to a zero splitting as expected, the former still has a non-trivial but perfectly antisymmetric nature (Figure 1.2(A)), leading to a zero splitting intensity while integrated.

Figure (1.3) shows the variation of the sensitivity kernel with different values of the dip of the symmetry axis while the incoming polarization direction ($\alpha = 0^\circ$) and the azimuth of the symmetry ($\phi = 45^\circ$) are fixed. The presence of the inclined symmetry axis causes an uneven spatial distribution of the sensitivity, as expected (Fig. 1.3(B) and (C)). The striking property of the sensitivity kernel is

the non-triviality even when the propagation direction is parallel to the symmetry axis (Fig. 1.3(D)). Such a physical scenario should lead to a zero splitting intensity and the perfect antisymmetry of the sensitivity kernel (Fig. 1.3(D)) confirms this well known result [13, 29, 33]. This non-trivial antisymmetry of the sensitivity kernel is a remarkable property and it may help us to understand the underlying physics in more details.

The kernel examples shown in Figures (1.2)-(1.3), which show the sensitivity for a range of angles between the fast direction and the incoming polarization direction, illustrate some interesting and subtle features for the case in which the initial polarization (equivalent to the backazimuth for SKS waves) is aligned either parallel or perpendicular to the fast axis of the medium. For the case in which the wave propagation direction (i.e., the wave vector $\mathbf{k}$) is parallel to the symmetry axis, we would observe a zero splitting intensity [13, 29, 33]. Zero splitting intensity may also be obtained in different scenarios, such as symmetry axis being parallel or perpendicular to the polarization direction. Therefore, each of these three cases predicts the same observation (zero splitting intensity). However, the sensitivity kernels in Figure (1.2) illustrate that interactions of waves with the anisotropic perturbations corresponding to these three different cases are different. In case of a trivial sensitivity kernel (Figure 1.2(D)), the constructive and destructive interferences cancel each other in either side of the vertical ray-path. However, in the case of a perfectly antisymmetric kernel (Figure. 1.3(D) and Figure 1.2(A)), the opposite wave phases accumulate in both sides of the raypath with opposite polarity and thus, the sensitivity is non-vanishing. Therefore, in such special case, the finite frequency approach would be able to sense the anisotropic perturbation even if the observable is zero (corresponds to a non-trivial solution to the system of linear equations described by equation (2.2)). Such different interactions lead to ambiguous interpretations while ordinary ray theory is employed, because of the integration of the structure along the ray path. However, using a tomographic approach, we may be able to delineate the three distinct cases.

We may explain the importance of the non-trivial sensitivity kernel yielding zero sensitivity in tomography by invoking a simple concept of linear maps. We provide here a simplified description of the concept, and interested readers may consult [53] for more involved concepts of linear maps and quotient spaces. We may write the equation (2.2) as a linear map between the model space X (containing all possible anisotropic perturbations) and space of observable Y (all possible splitting intensity in this case), i.e., $\Psi : X \rightarrow Y$, where Ψ is the linear operator $\int_V dV K_X$. In case of a trivial sensitivity kernel $\Psi(x) = 0, \forall x \in X$ i.e., both sides of the equation (2.2) are identically zero and thus, one cannot estimate the model parameters at all. However, zero splitting intensity may

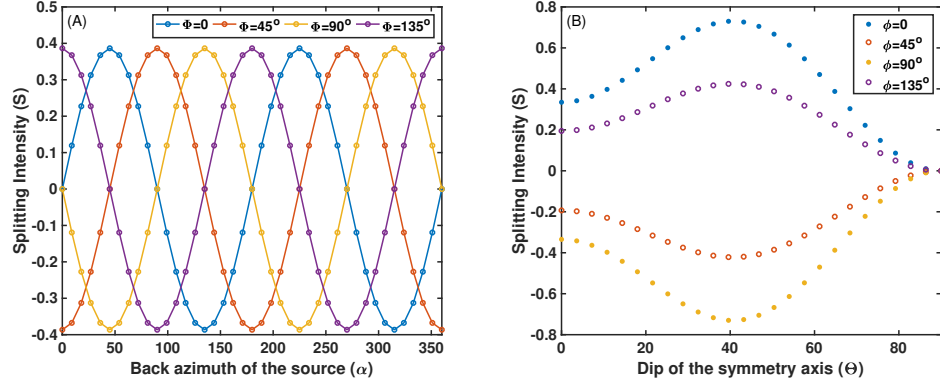


Figure 1.4: Behavior of sensitivity kernels against analytical solutions. We compute splitting intensity by integrating the sensitivity kernels (2.2) for a series of event backazimuths for a model with single layer of horizontal anisotropy. Kernel predictions (dots) are plotted against the predicted splitting intensity from analytical solutions (line) with a horizontal symmetry axis (1.29) as a function of (A) incoming polarization back-azimuth (α) of the wave at different azimuth of the symmetry axis (ϕ). The variation of splitting intensity as a function of dip of the symmetry at (B) different azimuths of the symmetry axis. In (B), maximum splitting is obtained at $\theta \approx 45^\circ$ consistent with the result of [52]. The starting model used for these computations has Thomsen parameters of $\epsilon = 0.03$, $\delta = 0.04$, and $\gamma = 0.01$ (corresponding to $\chi = 0.1$, Figure 1.1) and with a layer thickness of 400 km.

also be obtained by integrating the perfectly antisymmetric kernels with a heterogeneous strength of anisotropy (in other words, if the strength of the anisotropy is distributed in such a way that the integration of equation (2.2) yields zero). Mathematically, in this case, we get a non-trivial anisotropic perturbation which consists the kernel of linear operator Ψ (kernel of a map is given as follows: $\Psi : X \rightarrow Y$, the kernel of Ψ is the subset of X for which Y vanishes i.e., $\text{kernel}(\Psi) = \{x \in X | \Psi(x) = 0\}$). In the splitting measurements, the null data (zero splitting intensity) are neglected. However, we do not know a-priori the orientation of the symmetry axis (we wish to know this), and ray theory does not distinguish between these different situations in which the splitting intensities turn out to be zero. Thus, in ray theory, we eventually neglect a part of the model space or just resolve the quotient space $X/\text{ker}(\Psi)$. In our method, whenever sensitivity kernel is non-trivial, kernel of sensitivity kernel is estimated uniquely. Thus, the method overcomes the disadvantage of neglecting a portion of model space.

In order to benchmark our framework for computing sensitivity kernels, we compare our predictions for splitting intensity observations with known (ray theoretical) solutions for simple model cases, such as that of a single anisotropic layer with a horizontal symmetry axis. When the symmetry axis is horizontal, simple analytical solutions exist [33] and we compare our solution with

such limiting cases. Specifically, for the case of a single horizontal layer of anisotropy, the splitting intensity is predicted to vary as:

$$S = \delta t \sin(2(\alpha - \phi)), \quad (1.29)$$

where δt , α , and ϕ are the delay time, incoming polarization back-azimuth, and azimuth of the symmetry axis, respectively. We compute sensitivity kernels for a series of initial polarization directions for a simple, single layer model and predict splitting intensities by integrating over the model volume. Figure 1.4(A) and (B) show these modeled splitting intensities, along with predictions from equation (1.29), demonstrating that our kernels capture the expected behavior for the simplest models well. Specifically, we predict sinusoidal behavior of the splitting intensity with backazimuth, as predicted by [33]. We also compute predicted splitting intensities for homogeneous models with a non-horizontal horizontal symmetry axis to demonstrate the effect of a tilted symmetry axis (Fig. 1.4(C), (D)). These figures show that the splitting intensity maximum corresponds to a dip of 45° , consistent with the results of [52].

1.5 Inversion Strategy: Model space search approach

We have discussed in the previous section the non-linear relation between the splitting intensity and the anisotropy parameters (θ, ϕ) through the sensitivity kernel examples. Thus, any attempt in linearizing such parameters would severely bias the model estimation. The application of a non-linear search which does not rely on the starting model becomes desirable. Therefore, our inversion strategy invokes a model space search technique based on a Markov chain Monte Carlo sampling algorithm. It is an efficient model space search technique based on Bayes' theorem and Markov process. Bayes' theorem describes how to update the probabilities of hypotheses given the evidence, and follows simply from the axioms of conditional probability. Let us assume that we have a data vector d which is given or observed, and several models can reproduce this data within different error limits. Therefore, we may construct a conditional probability: given the data d , what is the probability that a particular model m would satisfy it? The probability of the assumed model m satisfying the given data d is the conditional probability $P(m/d)$. Clearly, if this conditional probability is high enough, we may assert that the model fit is reasonably good. Now, from Bayes' theorem, we may write the following identity

$$P(m/d) = \frac{P(d/m)P(m)}{\sum_m P(d/m)P(m)}, \quad (1.30)$$

where $P(d/m)$ is the probability of achieving the given data with the chosen model parameter m , and $P(m)$ is probability of the model parameter m to exist. The denominator describes the total conditional probability of observing this particular data. Clearly, if we choose the model parameters from the physically plausible set, then $P(m)$ is identically 1, and 0 if it is chosen from outside of the set. Bayes' theorem is applied in conjunction with the Markov process, which states that in a sequence of N events, the outcome of i th event depends only on the outcome of $(i - 1)$ th event. Following these two basic notions, several sampling algorithms have been developed, including the Metropolis-Hastings algorithm, the Gibbs sampling algorithm, and several adaptive algorithms. We implement the Gibbs sampling technique [54], which allows us to sample the models with higher probability efficiently. Such technique does not involve the concept of a starting model, as the model space is sampled randomly following a particular probability distribution. The details of the Gibbs algorithm are not presented here, as it has been extensively applied in computational physics [55,56], computational chemistry [57], and geophysics [58,59,60]. The misfit is chosen to be the ordinary Euclidean distance between the observed data and synthetic data produced by the chosen model sets weighted with the squared data error vector:

$$E^2 = \sum_{i=1}^N \frac{(d_i - f_i(\mathbf{m}))^2}{2\sigma_i^2} \quad (1.31)$$

where σ_i is the data error vector. The Bayesian likelihood function $P(d/m)$ is just the Gaussian distribution (adopted in accordance with the central limit theorem):

$$P(d/m) = \exp(-E^2). \quad (1.32)$$

In our approach, we avoid the Metropolis-Hastings algorithm, as the convergence of such algorithm is extremely sensitive to the step size of the random jump. Gibbs sampling, on the other hand, has an advantage of sampling all misfit wells (or, correspondingly, probability hills) [59]. However, in problems such as tomographic inversions, it is extremely difficult to resolve small scale features as the number of parameters increases. In such cases, inversion becomes computationally expensive (for example, three parameters defined at each spatial point leads to a total of 300 parameters with a grid size of $50 \text{ km} \times 50 \text{ km} \times 50$ in a $500 \text{ km} \times 500 \text{ km} \times 50 \text{ km}$ domain) and thus the Markov chain requires larger number of iteration to converge. The power of our approach, and the number of parameters that can reasonably be resolved, thus depend on the available computational resources. We will discuss the advantages and limitations of model space search technique in the following

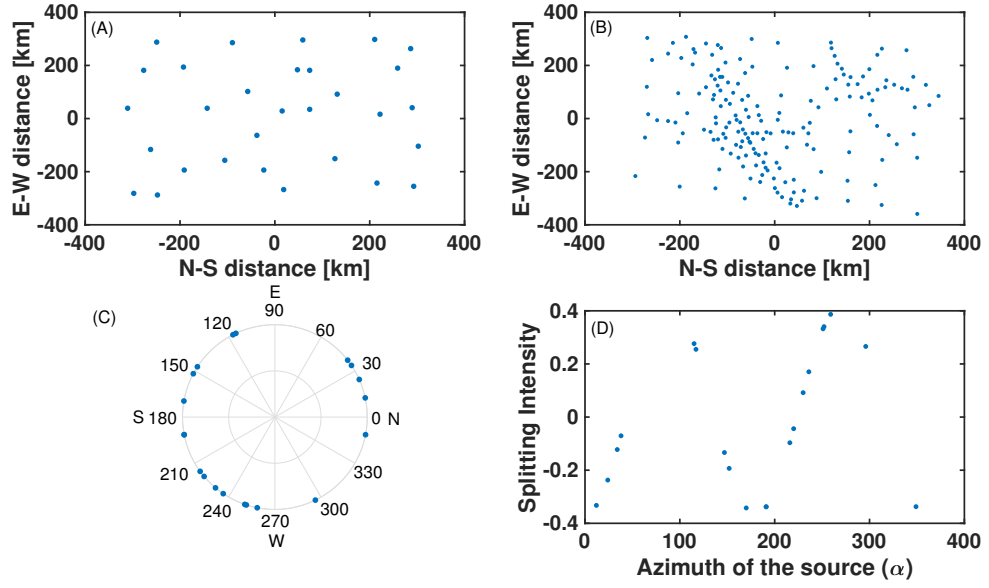


Figure 1.5: (A) The station distribution is shown on the horizontal plane (blue dots); the average station spacing is roughly 100 km. (B) The station distribution of the dense high lava plain seismic network. (C) Azimuthal distribution of the randomly chosen 20 seismic events. (D) Synthetic splitting intensity as a function of azimuth of the events for a particular station.

sections using the results of resolution tests.

1.6 Synthetic tests

In this section, we provide several examples of synthetic tests to show how well our method can handle different anisotropic scenarios. We start with the simplest model (a single homogeneous layer of anisotropy) and increase the complexity thereby increasing size of the parameter space. This sequential increase in complexity in our resolution tests helps us to illustrate the possible length scales and size of the parameter space which can be resolved with confidence. In this study we restrict ourselves to a grid based parametrization (discrete). There are other possible parameterizations as well. One such possibility would be representing the anisotropic strength by linear combination of basis functions in proper function space (e.g., L^2 orthonormal basis) and solving for the coefficients. Such alternative parameterization will be explored in future work. A summary of the specifications for each resolution test presented in this paper can be found in Table 1.

Table 1.1: A summary of the specifications for each resolution tests presented in this paper

Test number	Type of variability	Station configuration	Parameters varied	Grid size	Type/size of input anomalies	Figure number
1	one layer	random	χ, θ, ϕ	$\approx 800^2 \times 400 \text{ km}^3$	single layer	1.6
2	2D (lateral)	random	χ	$\approx 200^2 \times 400 \text{ km}^3$	$\approx 400 \text{ km}$ checkerboard	1.9(A)
3	2D (lateral)	HLP	χ	$\approx 100^2 \times 400 \text{ km}^3$	$\approx 400 \text{ km}$ checkerboard	1.10(A)
4	2D (lateral)	HLP	χ, ϕ	$\approx 200^2 \times 400 \text{ km}^3$	$\approx 400 \text{ km}$ checkerboard	1.12(A)
5	2D (lateral)	HLP	χ	$\approx 50^2 \times 400 \text{ km}^3$	$\approx 100 \text{ km}$ spike	1.14(A)
6	2D (vertical)	HLP	χ	$\approx 50^2 \times 400 \text{ km}^3$	$\approx 50 \text{ km}$ checkerboard	1.14(B)
7	Fully 3D	random	χ	$\approx 200^2 \times 200 \text{ km}^3$	$\approx 400^2 \times 200 \text{ km}^3$ checkerboard	1.9(B),(C),(D)
8	Fully 3D	HLP	χ	$\approx 200^2 \times 100 \text{ km}^3$	$\approx 400^2 \times 200 \text{ km}^3$ checkerboard	1.10(B),(C),(D)
9	Fully 3D	HLP	χ, ϕ	$\approx 200^2 \times 100 \text{ km}^3$	$\approx 400^2 \times 200 \text{ km}^3$ checkerboard	1.12(B)

1.6.1 Single layer case

A single layer model is our simplest scenario, but it is important in order to understand the underlying complexity of the data-model parameter relation. We need to estimate three parameters: the strength of anisotropy (χ), the dip (θ), and the azimuth (ϕ) of the symmetry axis. We choose a physical domain with size $\approx 800 \text{ km} \times 800 \text{ km}$ (in the horizontal dimensions) $\times 400 \text{ km}$ (in the vertical dimension) (Fig. 1.5(A)), designed to be typical of a regional anisotropy inversion. We generate a set of 25 randomly distributed station locations (Figure 1.5), and compute a synthetic data set of splitting intensities using an azimuthally random distribution of 20 sources and a starting model with parameters ($\chi = 0.1, \theta = 45^\circ (0.7854 \text{ rad}), \phi = 30^\circ (0.5236 \text{ rad})$). We use this set of synthetic splitting intensities (see example in Figure 1.5(D)) as the input for our inverse model (i.e., to invert equation (2.2) for (χ, θ, ϕ)). Using this input model, we ran a model space search for a total of 8000 iterations (length of the Markov chain); this number of iterations was chosen after a series of tests to investigate how many iterations were needed for the algorithm to converge. For this single-layer test in which all three parameters were allowed to vary, the sensitivity kernels were re-computed at each step for each candidate starting model; the inversion thus comes at a non-trivial computational cost, despite the small number of parameters. This particular initial test ran for 6 hours on a laptop computer with 8 cores.

The results of our initial single-layer recovery test are shown in Figure 1.6, which shows the

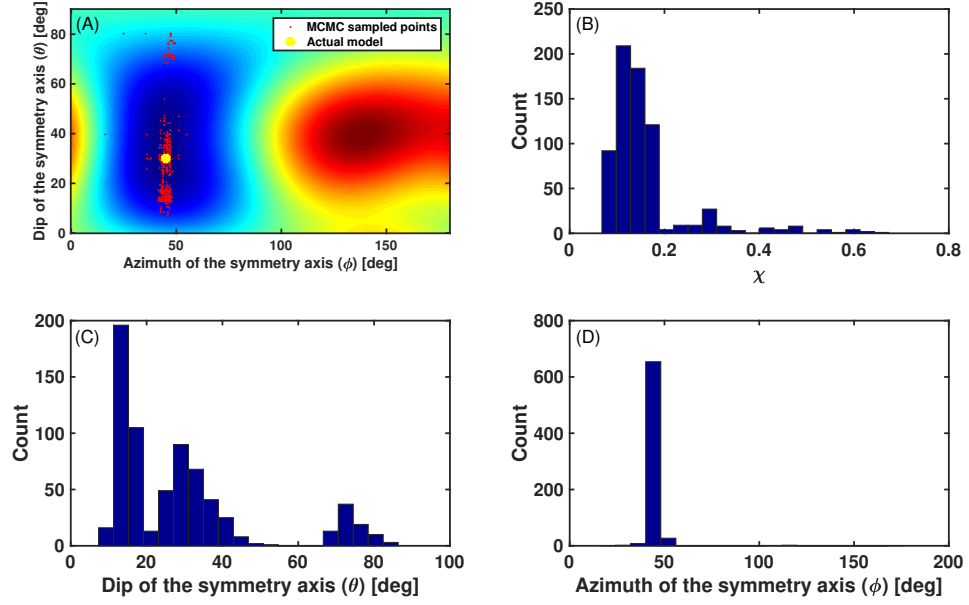


Figure 1.6: Single layer inversion result. The input model consists of a single layer of anisotropy with $\chi = 0.1$, $\theta = 0.52$ rad (30°), and $\phi = 0.78$ rad (45°). (A) 2-D section of the normalized squared misfit (colours) and the points (red dots) sampled by the MCMC inversion algorithm. The yellow dot corresponds to the location of the true input model. (B) The posterior distribution of the strength parameter χ with a mean ($\bar{\chi}$) of 0.1075 and standard deviation (σ) of 0.0277. (C) The posterior distribution of the dip of the symmetry axis (θ) with a mean of 0.4419 (25.4°) and standard deviation of 0.1658 (9.5°). (D) The posterior distribution of the azimuth of the symmetry axis (ϕ) with a mean of 0.7933 (45.4527°) and standard deviation (σ) of 0.1152 (6.6°).

misfit function for a fixed χ (projection of the 3-D error hyper-surface onto $\chi = \text{constant}$ plane is used in order to represent graphically) and the points sampled by the MCMC algorithm. The shape of the "valley" of misfit indicates that multiple choices of θ (dip of the symmetry axis) explain the data equally well. The posterior distribution also shows that the maximally sampled points may not correspond to the actual dip (θ) (Fig. 1.6(C)). However, the anisotropy strength (χ) and the azimuth of the symmetry axis (ϕ) are extremely well resolved (Fig. 1.6 (B) & (C)) for the single-layer case.

Despite the fact that splitting intensity depends on the strength of anisotropy (χ) in a linear fashion (equation (2.2)), the non-zero covariance among parameters makes a non-linear search crucial, even for a hypothetical inversion cases in which χ is the only parameter of interest. These parameter covariances are demonstrated in Figure 1.7(B), which shows a power law relation $\theta \approx 0.1\chi^{-1.8}$ between χ and θ when χ is small (within a few standard deviations from the original value of 0.1). This demonstrates that the presence of a dipping symmetry axis impedes the convergence of the model space search algorithm, as well as presenting a source of non-uniqueness in the inversion. We emphasize that our inability to constrain the dip of the symmetry axis in our synthetic test is not unique to our study and is not a consequence of our model space search approach; rather, it is due to our use of nearly vertically propagating SKS waves. On the other hand, the resolved azimuth (ϕ) is insensitive to (that is, does not trade off with) the strength (χ) and the dip of the symmetry axis (θ) (fig. 1.7(C) & (D)). This implies that SKS splitting observations can distinguish variability in ϕ from variability in other parameters, and furthermore suggests that strong constraints on ϕ can be achieved very fast through the MCMC technique. In many geophysical studies of anisotropy, a horizontal symmetry axis is either suggested by the data or (more commonly) assumed. Therefore, we suggest a strategy that assumes a horizontal symmetry axis the most appropriate configuration for many problems; if this is assumed, our approach can effectively provide a robust estimation of the strength of the anisotropy and the fast axis direction for simple models.

The test shown in Figure 1.6, like most of the test cases we present in our paper, was carried out with noise-free synthetic data. In order to investigate the effect of random noise on our inversion scheme, we carried out a second version of the single-layer test with the same parameters describing the input model, but with Gaussian-distributed noise (standard deviation of 0.1) applied to the synthetic splitting intensity observations. For noisy data, it is necessary to run the MCMC algorithm for a larger number of iterations to achieve convergence; for the example shown in Figure 8, we ran the algorithm for 12000 iterations (12000 vs. 8000 for Figure 6). Even with a larger number of iterations, this test demonstrates that the MCMC algorithm demonstrates output with a greater

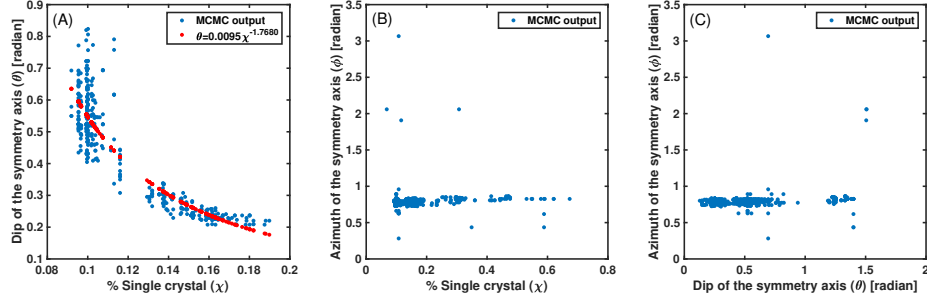


Figure 1.7: Estimation of tradeoffs among parameters. (A) MCMC sampled dip (θ) is plotted against the MCMC sampled strength (χ), clearly showing a correlation and thus a parameter tradeoff. (B) The nearly horizontal trend between the MCMC sampled azimuth (ϕ) and the MCMC sampled strength (χ) depicts that the former is insensitive to the later, and (C) similar trend is observed between MCMC sampled azimuth (ϕ) and the MCMC sampled dip (θ).

uncertainty than that of noise-free data, as expected. For this test, we found that the most likely model for the strength parameter χ is somewhat different (0.129), from the input strength (0.1), with larger uncertainty than for the corresponding noise-free case. This test demonstrates that even for modest amounts of noise, the convergence of the MCMC algorithm will be slower, which can represent a practical limitation for real data.

1.6.2 Fully heterogeneous case with 3D structure

Next we perform a series of synthetic inversion tests with increasing model complexity, including lateral heterogeneities in the anisotropic strength parameter χ . For the first set of these tests, we focus on input models that allow for lateral variations in χ , but not in ϕ ; in all of our subsequent tests, the dip parameter is fixed at zero, such that the symmetry axis is horizontal. Initially, we choose an input model with four distinct blocks within the model domain ($2 \times 2 \times 1$) with alternating strength χ (i.e., two subsequent blocks have different strength). Each block of this input model is of size $\approx 350 \times 350 \times 400 \text{ km}^3$. Using this initial model, we compute a series of splitting intensities with the station-source distribution same as that used in our one-layer test (Section 6.1). For this series of tests, we parameterize the model space with a grid size of $\approx 200 \times 200 \times 400 \text{ km}^3$ i.e., 16 blocks model ($4 \times 4 \times 1$).

Figure 1.9(A) shows the retrieved anisotropic strength and the posterior distribution corresponding to the best and worst resolved grid for this checkerboard test (with 4 input blocks). Our result suggests that the distribution of stations greatly affects the resolution. In this particular test, the

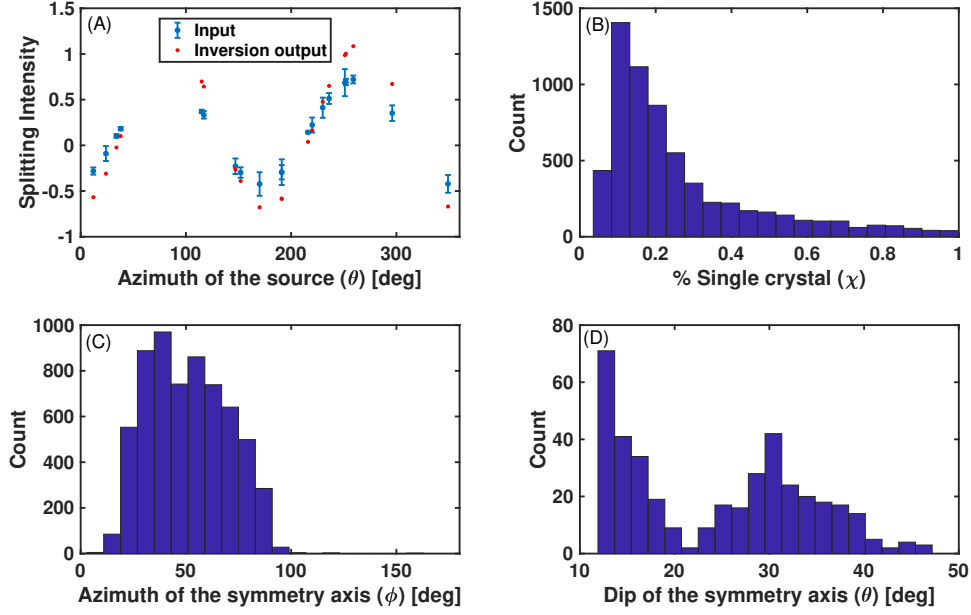


Figure 1.8: Single layer inversion result with Gaussian-distributed (standard deviation of 0.1) noisy input data. The input model consists of a single layer of anisotropy with $\chi = 0.1$, $\theta = 30^\circ$, and $\phi = 45^\circ$. (A) Synthetic splitting intensity as a function of azimuth of the events for a particular station (blue dots with error bar) and the splitting intensity computed using the most likely model obtained from MCMC algorithm (red dots). (B) The posterior distribution of the strength parameter χ with a mean ($\bar{\chi}$) of 0.1291 and standard deviation (σ) of 0.1086. (C) The posterior distribution of the azimuth of the symmetry axis (ϕ) with a mean of 46° and standard deviation (σ) of 9° . (D) The posterior bimodal distribution for the dip of the symmetry axis (θ).

average station spacing of ≈ 100 km yields a fairly good resolution for the chosen grid size, but there is variability: grid blocks that happen to have good station sampling are well resolved (see "best resolved" histogram in Figure 1.9(A)), while grid blocks with few stations that happen to be located along the edges of the block are poorly resolved. This is demonstrated by a similar test that has the same input model but a different (and much denser) station distribution. In addition to our tests with a random spatial distribution of 25 stations, we also carried out resolution tests using the actual station locations of the dense High Lava Plains seismic network [61, 62], which operated Oregon, Nevada, and Idaho (USA) between 2006-2009. Specifically, we used the identical station configuration as [61], which incorporated HLP stations as well as surrounding USArray Transportable Array stations and the nearby Wallowa Mountains array. The station distribution for this synthetic experiment (Figure 1.5(B)) is irregular, with station spacing as fine as 10-15 km along the dense HLP lines (oriented WNW-ENE and N-S) but as coarse as 70 km in the portions of the model domain that are sampled mainly by TA stations. We carried out a test with the same input model as that shown in Figure 1.9(A), but with a finer grid parameterization ($8 \times 8 \times 1$). The results of this test are shown in Figure 1.10(A), which presents the input-output pattern along with the station distribution and the posterior distribution corresponding to the best and worst resolved grid. As with the random station distribution, this test makes it clear that regions of the model with poor station sampling yield highly uncertain parameter estimates, while model blocks with good station sampling are well resolved (compare best-resolved and worst-resolved posterior distributions in Figure 1.10(A)).

The tests shown in Figures 1.9(A) and 1.10(A) only involved variability in the strength parameter in the input model; however, it is desirable to carry out inversions for χ and ϕ simultaneously. In Figure 1.12(A), we show the results of such a synthetic test, in which we vary both χ and ϕ , keeping $\theta = 0$, with a $4 \times 4 \times 1$ block model; the total number of parameters is thus $16 \times 2 = 32$. The station distribution is again taken to be the that of the HLP experiment and surrounding stations, following [61]. The input-output model is presented in Figure 1.12(A). As in other laterally heterogeneous cases (Figures 1.9(A) and 1.9(A)), grid blocks with a dense station distribution (number of station ≥ 4) are resolved fairly well, while grid blocks devoid of stations do not produce any constrained result. This is shown in detail in Figure 1.13, which presents the posterior distribution of the symmetry axis azimuth (ϕ) for four different grid blocks in the inversion, some of which are well constrained and others of which are poorly constrained.

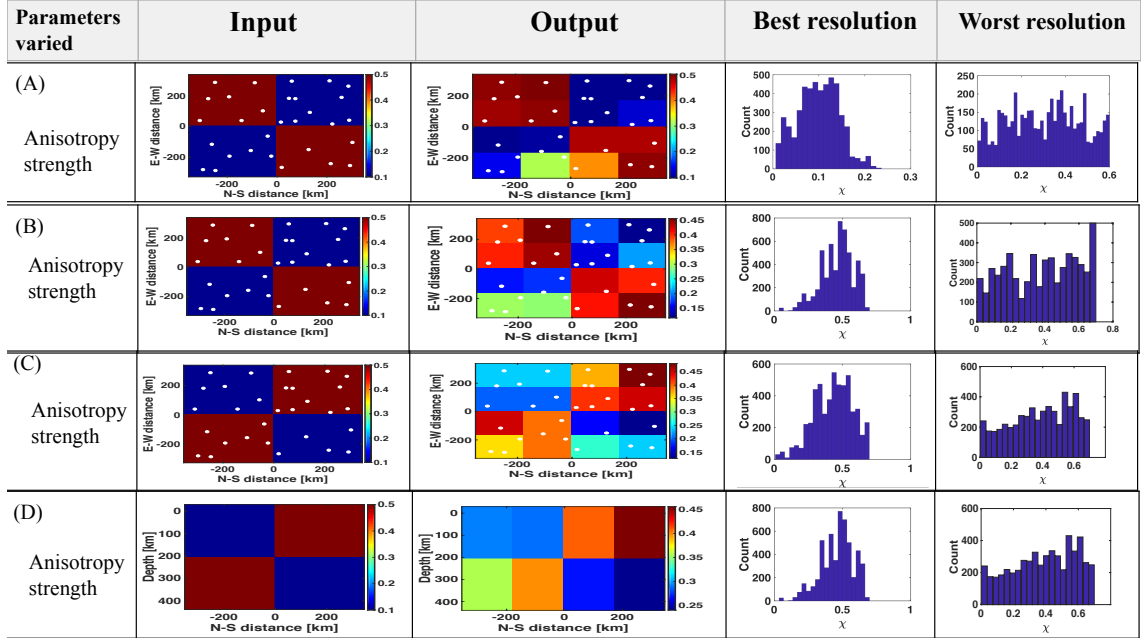


Figure 1.9: Results of checkerboard resolution tests with several different input model configurations for a random station configuration. The station locations are shown by white dots. In each of these tests, the strength parameter (χ) is varied while keeping $\theta = 0$, and $\phi = 45^\circ$. (A) Horizontal slice of input-output model and the posterior distributions of the $4 \times 4 \times 1$ checkerboard test in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$, (B) horizontal slice of Input-output model at $z = 100 \text{ km}$ and the posterior distributions of the $4 \times 4 \times 2$ checkerboard test in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$, (C) horizontal slice of Input-output model at $z = 300 \text{ km}$ and the posterior distributions of the $4 \times 4 \times 2$ checkerboard test in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$, and (D) vertical slice of the input output model at $y = 100 \text{ km}$ and the posterior distributions of the $4 \times 4 \times 2$ checkerboard test in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$. The best and worst resolutions correspond to the grids for which the inversion algorithm yields outcomes closest to the input model.

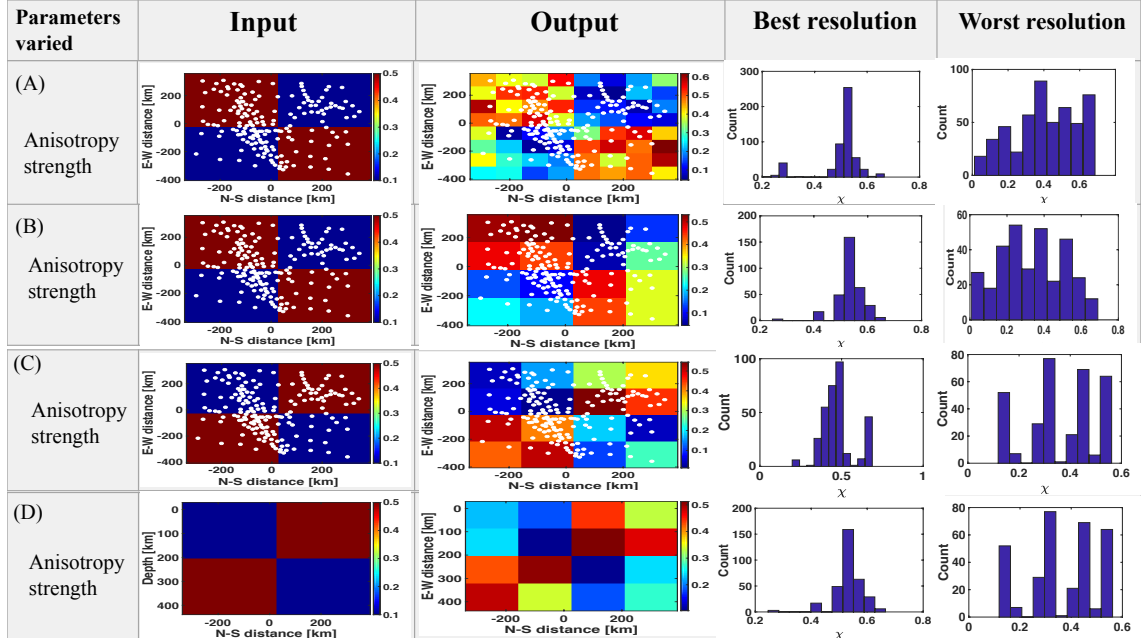


Figure 1.10: Results of the checkerboard tests with several different input model configurations for the HLP station geometry. The station locations are shown by white dots. In all of these tests, the strength parameter (χ) is varied while keeping $\theta = 0$, and $\phi = 45^\circ$. (A) Horizontal slice of Input-output model and the posterior distributions of the $8 \times 8 \times 1$ checkerboard test in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$, (B) horizontal slice of Input-output model at $z = 50 \text{ km}$ and the posterior distributions of the $4 \times 4 \times 4$ checkerboard test in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$, (C) horizontal slice of Input-output model at $z = 250 \text{ km}$ and the posterior distributions of the $4 \times 4 \times 4$ checkerboard test in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$, and (D) vertical slice of the input output model at $y = 100 \text{ km}$ and the posterior distributions of the $4 \times 4 \times 4$ checkerboard test in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$. The best and worst resolutions correspond to the grids for which the inversion algorithm yields outcomes closest to the input model

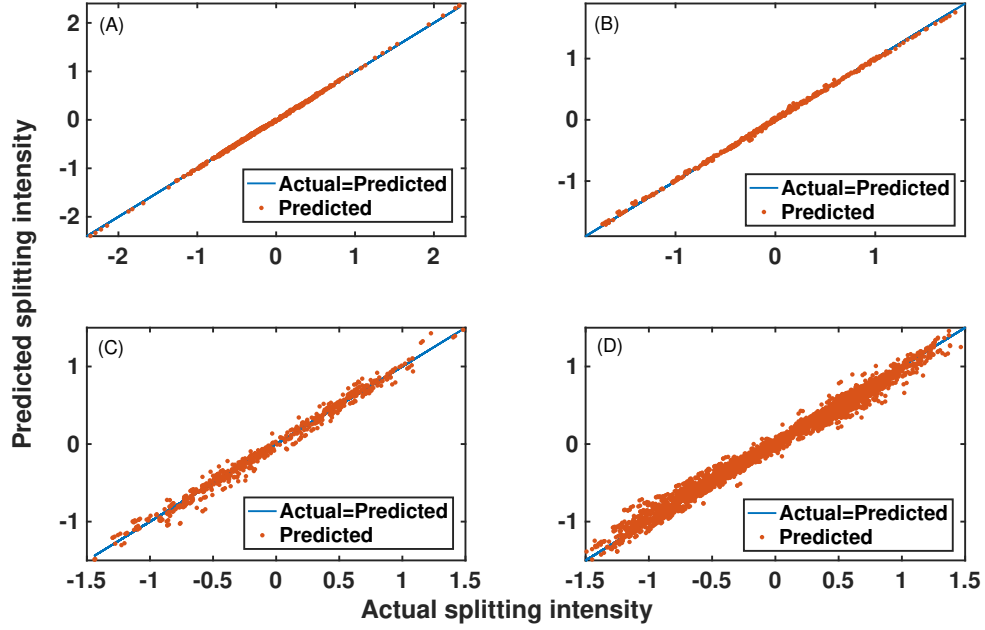


Figure 1.11: Actual (synthetic) and predicted observations for selected sensitivity tests. Correspondence between the predicted (using the most likely model from the output of MCMC algorithm) and the actual (using the chosen input model) splitting intensities for (A) single layer model; number of parameters being 3, (B) model with $4 \times 4 \times 1$ blocks in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$; number of parameters being $4 \times 4 = 16$, (C) model with $4 \times 4 \times 2$ blocks in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$; number of parameters being $4 \times 4 \times 2 = 32$, and (D) model with $4 \times 4 \times 4$ blocks in a model domain of $\approx 700 \times 700 \times 400 \text{ km}^3$; number of parameters being $4 \times 4 \times 4 = 64$. The solid line represents the exact one-one correspondence between the actual and predicted splitting intensities i.e., exact model fit. An increase in the number of model parameters (A through D) increases the deviation of the predicted data from the one-one correspondence.

Our last depth-independent recovery test consists of a spike test, in which two distinct highly anisotropic anomalies of dimension $150 \text{ km} \times 200 \text{ km}$ inserted in the model space, and we explore how well the algorithm can recover them. For this test, θ and ϕ are kept constant, with $\theta = 0$ and $\phi = 45^\circ$, and only χ is varied in the input model. The HLP station configuration is used. Figure 1.14(A) presents the input-output model with a $8 \times 8 \times 1$ model grid parameterization (note that we plot the most likely model result given by the McMC algorithm after smoothing). Because both ”spike” anomalies were inserted in the model in regions with dense station spacing, the algorithm correctly identifies the input model parameters. This test shows that a dense seismic network can easily resolve single-layered structures with lateral dimension of 150 km , and the finer grid resolution suggests that structures on the length scale of $\approx 50 \text{ km}$ can be resolved.

1.6.3 Totally heterogeneous case

In this section, we present synthetic tests with input models that include vertical heterogeneity in addition to the lateral heterogeneity. These fully heterogeneous cases illustrate a key advantage of moving to a tomographic approach in SKS splitting studies. In traditional shear wave splitting studies, which measure and interpret the fast splitting direction and splitting delay time (ϕ, dt) , there is little or no depth resolution, as splitting is a path-integrated measurement and SKS phases propagate nearly vertically. In our approach, however, the finite width of the sensitivity kernels enables us to obtain depth constraints, even with nearly vertical SKS phases, as long as the station configuration is dense enough to yield overlapping sensitivity kernels. For these tests with depth-dependent starting models, we implement a series of two-layer input models, involving the same number of horizontal blocks as in previous tests. First we implement a test with a total number of 32 blocks (i.e., $4 \times 4 \times 2$) with the randomly generated station distribution; we vary the strength parameter (χ) while keeping θ (0) and ϕ (45°) fixed. Figure 1.9(B) and (C) presents the results of this test in map view (showing the horizontal resolution) at two different depths ($z = 100$ km and $z = 300$ km), while Figure 1.9(D) presents the input-output model of a vertical section, illustrating the vertical resolution. We find that the horizontal resolution deteriorates with increasing depth (compare Figures 1.9(B) and 1.9(C)); this is a consequence of the decaying strength of sensitivity kernel with depth ($\approx \frac{1}{r^a}$ for some $a > 0$), as shown in Figures 1.2 and 1.3.

We repeated this multiple-layer model test a four-layer input model; in this test the total number of model parameters is 64 ($4 \times 4 \times 4$). For this increased amount of model complexity, we use the dense HLP station distribution. The results of this test are shown in Figure 1.10(B) and (C), which show the horizontal resolution at two different depths, and in Figure 1.10(D), which presents the resolution of the vertical section. This test demonstrates that for a dense station configuration, finite-frequency SKS splitting tomography is capable of resolving depth-dependent anisotropic structure throughout the upper mantle, and the vertical resolution length scale is roughly 100 km (the thickness of the layers in the four-layer input model). This increased vertical resolution is a crucial advantage to SKS splitting tomography, as opposed to traditional SKS splitting analysis.

In our final depth-dependent case, we consider two cases involving a 2-D inversion for depth-dependent anisotropic structure along the WNW-ENE dense line of stations of the HLP experiment (station spacing along this line is roughly 10 km on average). Because of the reduction in dimensions (of physical domain) from three to two, we choose a finer input model for this test (16×16). The

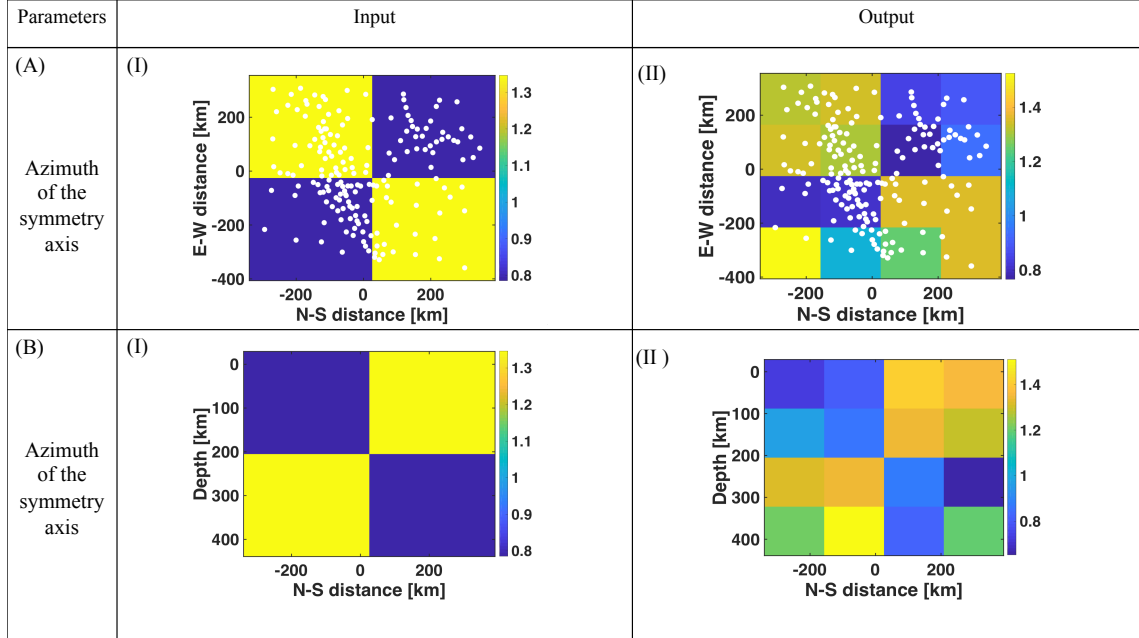


Figure 1.12: Results of checkerboard recovery tests for input models that vary the fast axis orientation. **A(I)** Spatial distribution of the azimuth of the symmetry axis used as input while only lateral variation is considered in an one layer model of $700 \times 700 \times 400 \text{ km}^3$. High: 80° or 1.35 rad , Low: 45° or 0.78 rad , **A(II)** recovered spatial distribution of the azimuth with a $4 \times 4 \times 1$ checkerboard test. Unit of the angle is given in radian here. The red circles denote the station locations. **(B)(I)** Spatial distribution of the azimuth of the symmetry axis used as input while only vertical variation is considered in a 2-D section below densely distributed stations. High: 80° or 1.35 rad , Low: 45° or 0.78 rad , **B(II)** recovered spatial distribution of the azimuth. Unit of the angle is given in radian here.

number of parameters is therefore $16 \times 16 = 256$, with input anomalies with length scale of 50 km both laterally and with depth. Our pattern of input anomalies involves a checkerboard pattern in χ , with the other parameters fixed ($\theta = 0$ and $\phi = 45^\circ$). Figure 1.14(B) shows the input and output models, demonstrating the potential for a dense line of stations such as that included in the HLP experiment to resolve depth-dependent anisotropic structure in the upper mantle. We also performed a similar test with simultaneous variation of χ and ϕ at $\theta = 0$ (i.e., horizontal axis of symmetry) in the input model. However, due to the increased number of unknown parameters, we used a coarser grid in our model space parameterization (total number of blocks 4×4 ; total number of parameters 16×2). Figure 1.12(B) shows the input-output pattern of ϕ for this 2-D, depth dependent inversion. Taken together, our two synthetic 2D inversions for depth-dependent structure show that structure is generally well-resolved to a depth of approximately 100-150 km. This may be explained by the properties of the sensitivity kernels; specifically, the strength of the kernel decays with the depth and the maximum power of the kernels is confined within the first 100-150 km depth range. As a consequence, we expect that the resolution is best within the region where kernel holds its maximum power. However, the observations contain at least some information about anisotropic structure down to the base of the mantle.

1.6.4 A note on model agreement and number of parameters

The results of our synthetic tests, shown in Figures 1.9-1.14, show the input and output models as well as selected views of the posterior distribution for some parameters, helping us to understand the quality of the resolution for different cases. A direct comparison of input-output models and posterior distributions among different synthetic tests is difficult, however, because the inverse problem is configured differently across different tests. A different view of the inversion scheme's success comes from a comparison between the (synthetic) splitting intensity observations computed using the input model, and the set of predicted splitting intensities generated from the output of the inversion (that is, from the most likely model). Figure 1.11 shows the actual synthetic splitting intensity (using input model) and the predicted splitting intensity (using output model of inversion) for a suite of resolution tests of increasing complexity. This demonstrates a reasonable correspondence between the predicted and actual values of the model outputs for the aforementioned calculations, showing that our inversion scheme is capable of identifying models that are consistent with the synthetic "observations", even for fairly complex cases. However, the deviation from the one to one correspondence depicts the extent of failure to fit the model exactly. Figure 1.11 shows that the

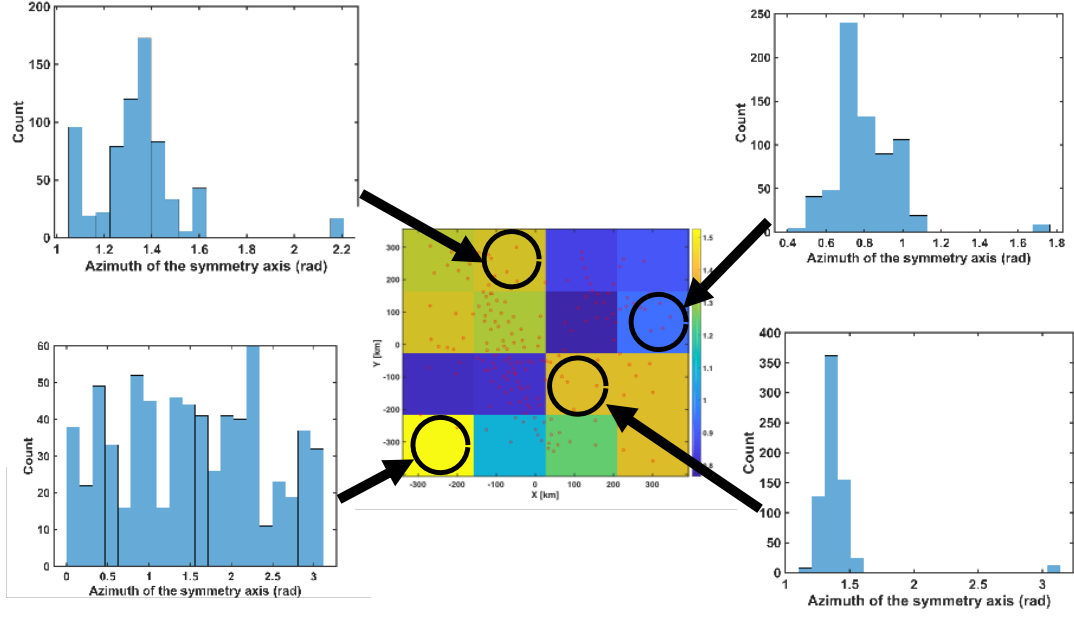


Figure 1.13: Posterior distributions of the recovered azimuth (ϕ) of the symmetry axis. This corresponds to $4 \times 4 \times 1$ checkerboard test in a model domain of $700 \times 700 \times 400 \text{ km}^3$ and χ and ϕ are varied simultaneously; total number of parameters being $4 \times 4 \times 2 = 32$ with $\theta = 0$. Red dots denote the station locations. This test demonstrates that the input is well resolved where station coverage is good, and poorly resolved where it is sparse.

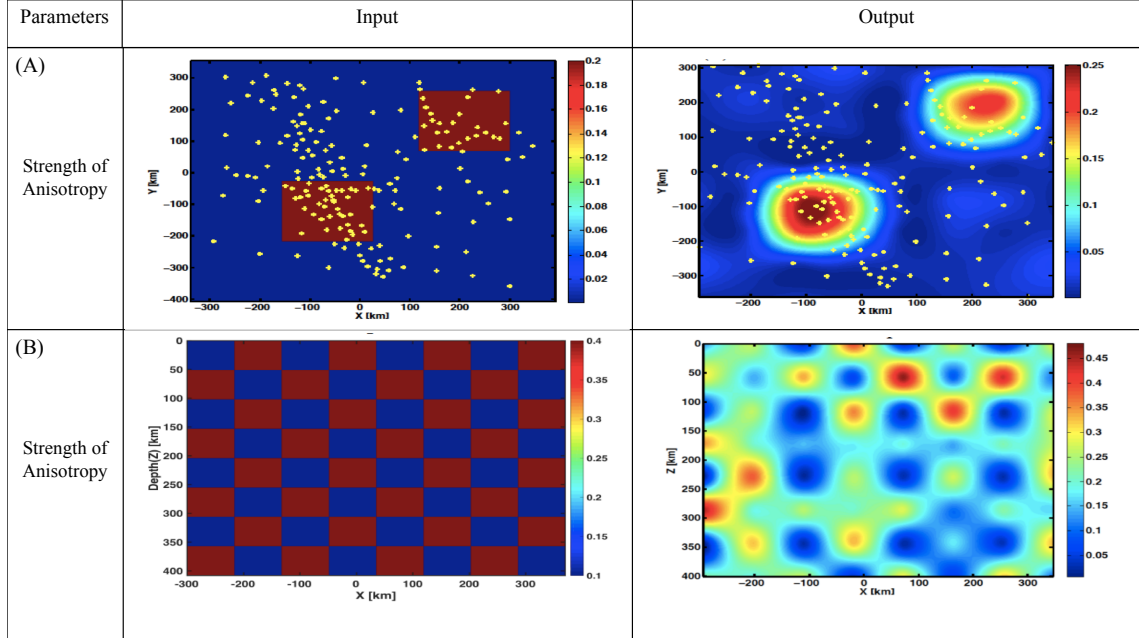


Figure 1.14: Results of the sensitivity tests using the HLP station configuration, with two different input model configurations. In both of these tests, the strength parameter (χ) is varied while keeping $\theta = 0$, and $\phi = 45^\circ$. (A) Horizontal slice of input-output model of the two block spike test in a physical domain of $\approx 800 \times 800 \times 400 \text{ km}^3$. The inversion is performed on 8×8 horizontal grids and output is smoothed. The station locations are denoted by yellow dots. (B) 2-D input-output model of the 8×8 checkerboard test in a physical domain of $\approx 700 \times 400 \text{ km}^2$ along the dense linear array of the HLP experiment. This test demonstrates the promise of 2D splitting intensity tomography for dense linear arrays (station spacing of 10-15 km in this example). X and Y denote N-S and E-W distance, respectively. Z denotes depth.

deviation from one-to-one correspondence increases with increasing number of unknown parameters in the inversion. This highlights a limitation of the model space search approach in the case of large number of parameters or a highly complex model.

1.7 Discussion

The theoretical and practical framework for the forward and inverse problem associated with anisotropy tomography with SKS observations developed here highlights both the promise and the challenge of SKS splitting tomography. In our implementation, the size of the parameter space (in general 21 parameters for a homogeneous, general anisotropic model) is reduced to three based on the results from mineral physics regarding the elasticity and fabric development in olivine, the dominant mineral in the upper mantle. This parameterization scheme addresses one of the major challenges in shear wave splitting tomography, namely the unrealistically large number of parameters needed to describe anisotropy. Our implementation of finite-frequency theory enables us to address another of the major challenges, namely how to achieve the overlapping sensitivity for nearly vertically propagating SKS waves (see also [8]). In our implementation, we achieve satisfactory depth resolution shallower than roughly 100-150 km depth, which lessens the ambiguity of traditional shear wave splitting interpretation.

Our implementation also addresses a third great challenge of anisotropy tomography, namely that the non-linear relation between the observed splitting intensity and the anisotropy parameters (χ, θ, ϕ) of the underlying medium results in a bias in the model estimation while implementing an inversion strategy based on a linearization around a background model. Sensitivity kernels for anisotropic structure depend strongly on the background model used to calculate them, and are substantially different for heterogeneous anisotropic starting models than for the homogeneous ones that are often used to calculate them [9]. Our technique based on a model space search accounts for this non-linear nature of the sensitivity kernels, and is able to resolve two parameters (namely the strength of anisotropy, χ , and the azimuth of the symmetry axis, ϕ) with generally high confidence. In contrast, the uncertainty in estimating the dip parameter (θ) is much higher, due to the use of only (nearly) vertically propagating SKS waves. This challenge may be ameliorated by using teleseismic direct S waves [34] in conjunction with SKS/SKKS phases; in this case, however, contamination from anisotropy near the earthquake source is a concern [63].

The synthetic tests presented in this paper demonstrate that the quality of the resolution is greatly

affected by the station configuration. This limitation is a general aspect of the seismic tomography inverse problem; however, given the complexity associated with anisotropic tomography, including the large dimensionality of the parameter space and the non-linearity issue, the station geometry is particularly crucial. Specifically, the presence of overlapping sensitivity kernels in the tomography problem is essential to provide confidence on the estimated model parameters. Our sensitivity tests demonstrate that with a dense station configuration, such as that obtained by the High Lava Plains (HLP) experiment (average station spacing 10-15 km in its densest portions), anisotropic structure in the upper mantle may be resolved at a length scale of roughly 50 km with teleseismic SKS waves in a 2D inversion (Figure 1.14(B)).

Our analysis also underscores the importance of a deep understanding of links between seismic anisotropy and mantle geodynamic processes. The orientation of the symmetry axis is closely related to the flow velocity field, but the relationship depends on the type of olivine fabric (i.e., A, B, C, D, and E) as well as the strain geometry. For example, under the application of simple shear, the [100] axis of A type olivine is parallel to the shear direction [5,6]; however, the details of fabric development are different for other fabric types. Therefore, the interpretation of the seismic anisotropy to infer mantle geodynamic processes critically hinges not only on the results from experimental mineral physics but also on the ability to resolve the details of seismic anisotropy at depth with a tomographic approach.

The results of the resolution tests for depth-dependent models presented in this study highlight the potential for constraining the depth distribution of anisotropy in the upper mantle, a particularly exciting avenue for future application of SKS splitting intensity tomography to actual data. We illustrate this potential by discussing results from the High Lava Plains SKS splitting data set, which was presented in [61] and [62]. These authors documented striking variability in SKS splitting delay times along the HLP dense line (see Figure 6 of [62]), with an increase in measured delay times from 1.5 sec to 2.5 sec across a lateral distance of 200 km. This lateral variability in delay times was interpreted to represent small-scale lateral variability in the thickness of the anisotropic layer, the strength of olivine LPO in the upper mantle, or the presence of partial melt (in a shape preferred orientation, or SPO) in the shallowest upper mantle. Each of these possible scenarios would have important implications for the structure and dynamics of the High Lava Plains region, and for the origin of anomalous, young volcanism in the Cascades backarc [61]. However, because SKS splitting observations themselves do not constrain the depth distribution of the anisotropy, [61] were unable to distinguish among the three different possible models for the lateral variations in delay times. The resolution test shown in Figure 12B, which was constructed using the station distribution of

the HLP experiment, points to a resolution of this outstanding problem, as it demonstrates that finite-frequency SKS splitting tomography applied to the HLP data set can resolve depth-dependent anisotropic structure in the upper mantle beneath this region.

1.8 Summary and outlook

Here we have developed and presented a theoretical and practical framework for the application of finite-frequency SKS splitting tomography with a sharply reduced number of parameters. Our approach relies on a Markov chain Monte Carlo with Gibbs sampling algorithm to explore parameter space. Our strategies for parameterizing the problem, calculating sensitivity kernels, and searching parameter space allow us to ameliorate some of the well-known challenges inherent in the inversion of shear wave splitting observations for anisotropic structure. These challenges include the large number of parameters needed to describe anisotropy, the difficulty of obtaining overlapping sensitivity zones (crossing rays, for the ray theoretical case) for nearly vertically propagating SKS phases, and the highly nonlinear nature of the inversion, in which the sensitivity kernels strongly depend on the starting model. We have benchmarked our sensitivity kernel computations against well-known analytical solutions for simple models, demonstrating that our parameterization and kernel computations are valid. We carried out a suite of synthetic tests for input models of varying complexity to establish the effectiveness of our approach, to explore tradeoffs among different parameters, and investigate the length scale of structure that can reasonably be resolved by our approach. We find that SKS phases are not well suited to constraining the dip of the symmetry axis, consistent with previous work, but that the strength and azimuth of the anisotropy can be reliably constrained for a series of increasingly complex input models. A test of resolution for a two-dimensional inversion along a dense linear array of stations (using the station configuration from the High Lava Plains experiment in the Pacific Northwest USA) shows that structure in the upper mantle (depths up to 150 km) can be reliably retrieved at length scales of ~ 50 km.

The synthetic experiments and proof of concept presented in this paper pave the way for future applications of our method to actual SKS splitting datasets, which have only been used in tomographic inversions in a few previous studies that have focused on Southern California [32, 64]. Based on our results, the implementation of two-dimensional inversions along dense linear arrays is particularly likely to be successful, given their relatively modest computational requirements. Given the substantial increase in availability of data from dense broadband seismic deployments since the mid-2000s, when the early papers developing the theory underpinning finite-frequency SKS splitting

tomography were published [8, 9, 10, 35], the time is right for substantial progress on the application of SKS splitting tomography to real data. We are currently implementing our approach to data from the HLP SKS splitting data set [61, 62], and other recently deployed experiments that include dense linear arrays represent obvious candidates for future work (e.g., the PULSE and CAUGHT experiments in South America; [65]; [25]). Broadband seismic experiments with dense station spacing within North America that were enabled by the recently completed EarthScope USArray data gathering effort also represent prime targets for future application of our technique. Examples include the SPREE [66], OIINK [67], and NELE [68] experiments in the mid-continent, as well as the SESAME [69], MAGIC [70], and QM-III [71] experiments in eastern North America.

Chapter 2

Strong seismic anisotropy in the deep upper mantle beneath the Cascadia backarc: Constraints from probabilistic finite-frequency SKS splitting intensity tomography

2.1 Introduction

Located in the tectonically active western United States [72], the High Lava Plains (HLP) of Oregon stands out as a distinctive feature of the Cascadia subduction zone backarc (Figure 1). Notable characteristics of the HLP include bimodal volcanism over the past ~ 12 Ma, encompassing both age-progressive rhyolitic volcanism and basaltic volcanism that is widespread in space and time [73,74,75]. Notably, the spatiotemporal progression of the rhyolites is oblique to plate motion, and volcanism in the HLP is contemporaneous with, but in a different geometry than, the Yellowstone/Snake River Plain trend to the west [73,75]. Several hypotheses have been proposed to explain the origin and evolution of volcanic trends in the Cascadia backarc [76,77]. These include models that emphasize the effect of a deep mantle plume that impinged on the western North American lithosphere around 17 Ma [77,78,79], those that invoke the rollback, steepening, tearing, and/or fragmentation of the Juan de Fuca slab at depth [76,80,81], those that appeal to lithospheric delamination as a main driver for volcanism [82,83], and those that foreground the effect of preexisting lithospheric structures [84]. In order to discriminate among these different hypotheses, a thorough understanding of the present-day mantle flow field, and the driving forces for that flow, is necessary [85].

Over the past decade, a wealth of geophysical observations has become available for the Cascadia backarc region, including those collected as part of the EarthScope USArray effort (including the Transportable Array, or TA) and those that resulted from the interdisciplinary High Lava Plains Project. Taken together, these data sets have yielded new insights into the structure and dynamics of the crust and upper mantle beneath the HLP, and include analyses based on SKS splitting [61,62,86,87], body wave tomography [88,89], surface wave tomography [62,90,91], ambient noise tomography [90], receiver function analysis [92], and scattered wave imaging [93]. One of the most

striking geophysical characteristics of the HLP is the strong seismic anisotropy in the upper mantle beneath the region. Specifically, SKS splitting with delay times of up to ~ 2.5 s has been documented beneath the region [61,62,86]; these delay times are among the highest documented in western North America [87] or in any continental region worldwide. Evidence for relatively strong anisotropy in the upper mantle also comes from surface wave analysis [62,94].

While the particularly strong upper mantle anisotropy beneath the HLP is notable, its interpretation remains ambiguous, largely due to the path-integrated nature of SKS splitting measurements and the challenges inherent in reconciling SKS splitting and surface wave-based measures of anisotropy [62]. Nearly vertically propagating SKS waves, when interpreted in a ray theoretical framework, have little or no depth resolution, leading to ambiguity in their physical interpretation [18]. (In some cases indirect inferences can be made based on a spatial coherency argument; e.g., [95]). Large splitting delay times, for example, may yield several different interpretations in terms of the strength and mechanism for anisotropy (i.e., a thin highly anisotropic layer vs. a thicker layer with weaker anisotropy). Similarly, in the presence of multiple layers of anisotropy with varying geometries, individual apparent fast direction measurements reflect a complicated (and non-commutative) combination of layer parameters [96], and isolating the depth extent of different anisotropic layers is a major challenge. In the context of the scientific questions underpinning the study of the origin and evolution of the HLP, the depth distribution of upper mantle anisotropy is a crucial question. [61] proposed several different scenarios that could explain the spatial variability in delay times observed beneath the HLP, variously invoking strong anisotropy in the uppermost mantle due to aligned partial melt, variations in the thickness of the anisotropic layer, or locally enhanced strength of lattice preferred orientation (LPO) in olivine. Discriminating among these scenarios is vital for our understanding of present-day mantle flow beneath the HLP region; furthermore, given that isotropic tomography models are biased in the presence of strong anisotropy [97], detailed knowledge of the depth distribution of anisotropy is necessary to obtain accurate images of upper mantle structure.

Substantial effort has been devoted to developing theoretical frameworks and practical strategies for inverting SKS splitting observations for anisotropic structure, much of it in the context of finite-frequency sensitivity kernels that relate model perturbations to splitting observations [8,9,10,35,38,64]. While these tomographic approaches have enabled progress in understanding the 3D distribution of seismic anisotropy in a few well-sampled regions, they all share a significant limitation in that the inversion strategy is based on linearization around a starting model, which may bias the model estimation. Furthermore, shear wave splitting tomography has only been applied to actual data sets

in a very few regions, including subduction zones that have abundant seismicity that can be sampled with local S waves [31] and regions with exceptionally dense station coverage that can be sampled with SKS waves in a finite-frequency framework [38, 64].

Here we present an application of a model space search approach to finite frequency SKS splitting tomography, developed by [98], to the densely sampled HLP region of the Cascadia backarc. The station spacing of the broadband HLP array, which included over 100 stations, is approximately 10-15 km in its densest portions, allowing us to produce a robust model for upper mantle anisotropy beneath the HLP that is well resolved both laterally and with depth. We reprocessed data from the HLP array and USArray TA stations to obtain high-quality measurements of splitting intensity [33], which were used as input into our probabilistic inversion. The resulting models allow us to distinguish among the different scenarios proposed by [61].

2.2 Probabilistic finite frequency SKS splitting intensity tomography: Methodology

We implement the model space search technique for SKS splitting intensity tomography in a finite frequency framework developed by [98]. Here we briefly describe our methodology; a more detailed description of the underlying theory is contained in the online Supplementary Information, and a full explanation of the method is contained in [98]. Due to the nearly vertical propagation of SKS phases, the application of finite frequency theory is necessary in SKS splitting tomography [8]. In this framework, the superposition of multiple finite-frequency sensitivity kernels of adjacent stations provides a means of localizing structure. We rely on observations of the splitting intensity [33], a measure of the amount of energy on the transverse component waveform; in the absence of shear wave splitting, the SKS phase will be radially polarized. Following [33], shear wave splitting intensity (S) is defined as

$$S(\mathbf{r}) = 2 \frac{\text{Re} \int \omega \delta u^T(\mathbf{r}, \omega) u_0^{*R}(\mathbf{r}, \omega) d\omega}{\int \omega^2 |u_0^R(\mathbf{r}; \omega)|^2 d\omega}, \quad (2.1)$$

where δu^T , u_0^R , ω , and $\mathbf{r}$ are the transverse and radial component of the displacement, angular frequency, and station locations, respectively.

Our parameterization of anisotropy uses insights from experimental and observational constraints on lattice preferred orientation (LPO) in olivine aggregates, thought to be primarily responsible for seismic anisotropy in the upper mantle [6]. We assume that olivine aggregates can be represented as a linear mixture of two endmembers: an isotropic equivalent, in which the individual grains are randomly oriented, and a perfectly aligned aggregate, in which the elasticity is equivalent to a single crystal of olivine. In this scheme, the strength of anisotropy can be represented with a

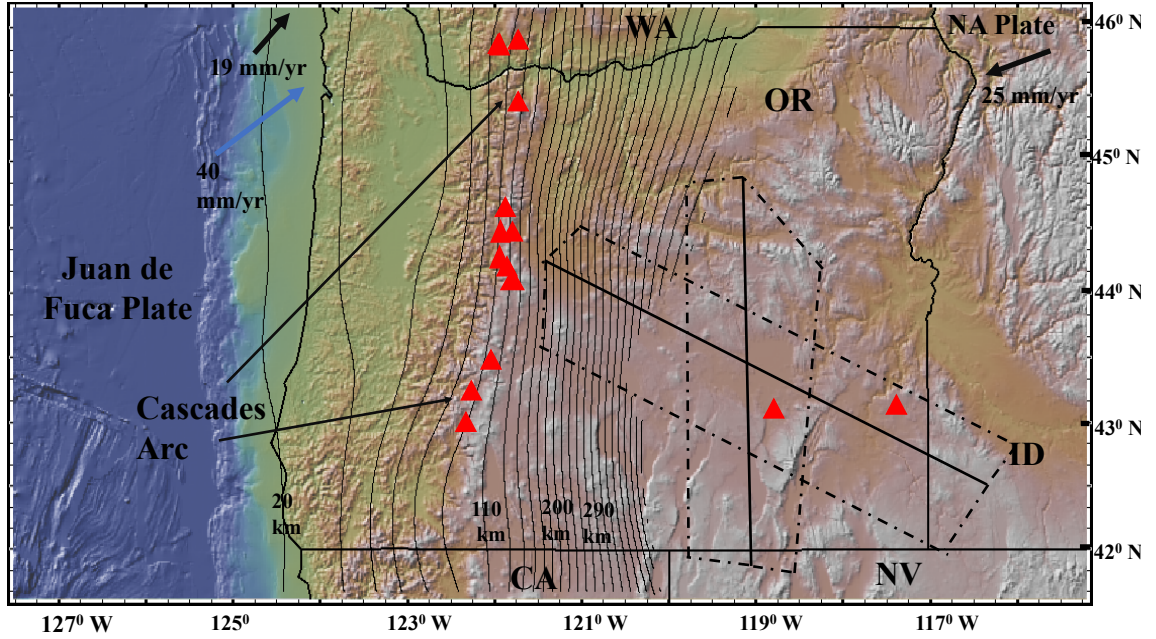


Figure 2.1: General tectonic setting and geographic features of the Cascadia subduction zone. Background colors show topography (onshore) and bathymetry (offshore); Holocene volcanoes are marked with red triangles (Smithsonian Global Volcanism Program; volcano.si.edu). Thin black lines show the outlines of US states (Washington, Oregon, California, Idaho, and Nevada are marked with two-letter abbreviations). The North American (NA) and Juan de Fuca Plates are marked. Thick black arrows show absolute motion of the major plates in a hotspot reference frame, and thick blue arrow shows the relative motion (convergence direction) between the Juan de Fuca and North American plates. The vertical contours represent the depths of the Juan de Fuca plate varying between 20 and 410 km with an interval of 15 km (Slab 2.0 model, [99]). Dashed boxes represent the regions under study, with black lines representing the location of the dense lines of the HLP seismic experiment. The box oriented NW-SE roughly corresponds to the location of the High Lava Plains province. Plate motion directions and rates (shown) were calculated using the UNAVCO plate motion calculator (<https://www.unavco.org/dxdt/model>).

parameter (χ) that varies from 0 (random alignment) to 1 (complete alignment). We further assume that the elasticity of the perfectly aligned olivine endmember can be well described by a hexagonal approximation.

Using this parameterization, we established the following relation between the splitting intensity and the anisotropy parameters [98]:

$$S(\mathbf{r}) = \int_{\Omega} K_{\chi}(\mathbf{r}, \theta, \phi; \mathbf{r}') \chi(\mathbf{r}') d^3 \mathbf{r}', \quad (2.2)$$

where K_{χ} is the corresponding sensitivity kernel, which is itself a non-linear function of the orientation parameters (θ, ϕ) . Here, θ and ϕ represent the dip and the azimuth of the symmetry axis of the anisotropy. We do not require sensitivity kernels for the orientation parameters θ and ϕ as we do not perform a linearized inversion. Instead, the sensitivity kernel for the strength parameter depends on θ and ϕ and therefore, with each iteration the sensitivity kernel is updated as one moves through parameter space. Examples of sensitivity kernels are shown in Figure 2. There is a strong non-linear relationship between the splitting intensity and the anisotropy parameters (θ, ϕ) . Rather than constructing a linearized inversion, then, our inversion strategy invokes a Markov chain Monte Carlo search technique based on the adaptive Gibbs sampling algorithm. Such a technique satisfies the requirement that the outcome does not depend on the starting model, as the model space is sampled randomly following the particular probability distribution. In order to represent the most likely model from our probability distributions, we rely on an ensemble average [100] which is suitable for multimodal distributions.

2.3 Data and Measurements

The High Lava Plains project and the EarthScope USArray Transportable Array (TA) provide a wealth of broadband seismic data in and around the HLP region. Between the HLP and TA deployments, there are 137 densely distributed broadband stations in the region, operated between 2006 and 2009. The HLP station configuration consists of two dense lines surrounded by a “cloud” of stations (Figure 3). The first line spans from southwest Idaho to northwest Oregon, ending just to the west of the Cascades Range. This line follows the time-progressive track in rhyolitic volcanism [75]. The second dense line is aligned almost N-S from Basin and Range extension (south) through HLP volcanism to the accreted terranes of Oregon’s Blue Mountains (Fig. 2.3a). The average spacing between each station along the dense lines is approximately 10 – 15 km, appropriate for our analysis.

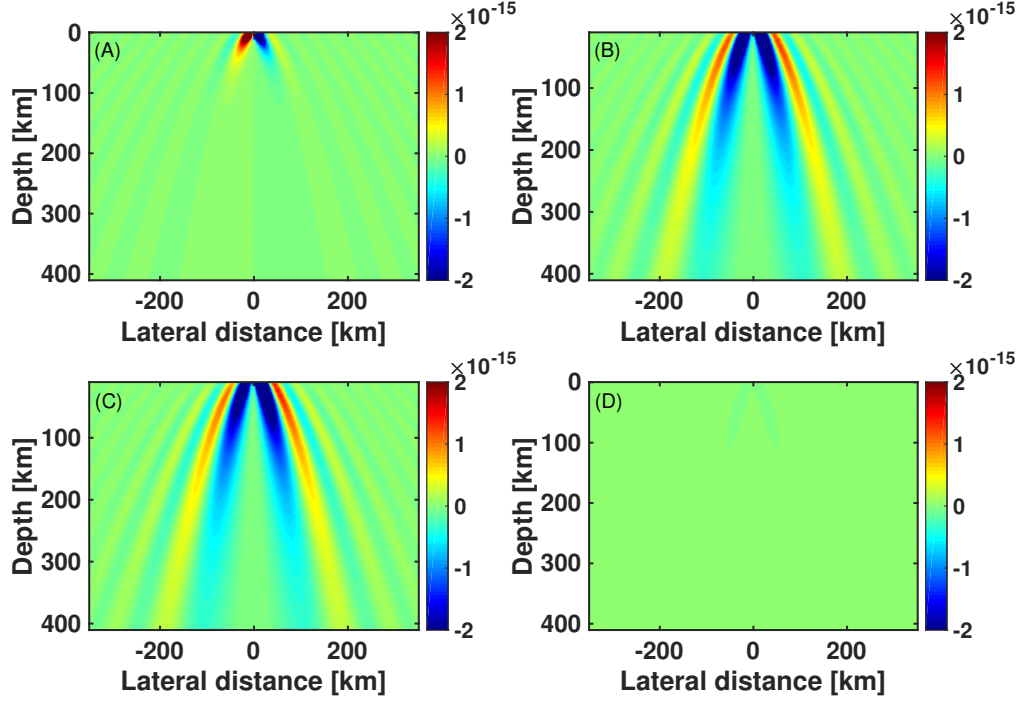


Figure 2.2: Sensitivity kernel examples showing a 2-D slice along the plane of wave propagation. Plots show the behavior of the sensitivity kernels for the anisotropy strength at different values of the azimuth of the symmetry axis (ϕ) while the dip of the symmetry axis ($\theta = 0^\circ$), and the incoming polarization direction ($\alpha = 0^\circ$) remain fixed. (A) Symmetry axis is parallel to the polarization direction ($\phi = 0^\circ$): the kernel is perfectly antisymmetric, leading to a zero splitting intensity for a homogeneous anisotropic model. (B) $\phi = 30^\circ$, (C) $\phi = 60^\circ$. For these examples the kernels are symmetric. (D) Symmetry axis is perpendicular to the polarization direction ($\phi = 90^\circ$): the sensitivity kernel is trivially zero, yielding zero splitting intensity for arbitrary spatial distribution of anisotropic strength. Unit of the sensitivity is $m^{-2}s$. The frequency range associated with the computation of these kernels is 0.04-0.125 Hz.

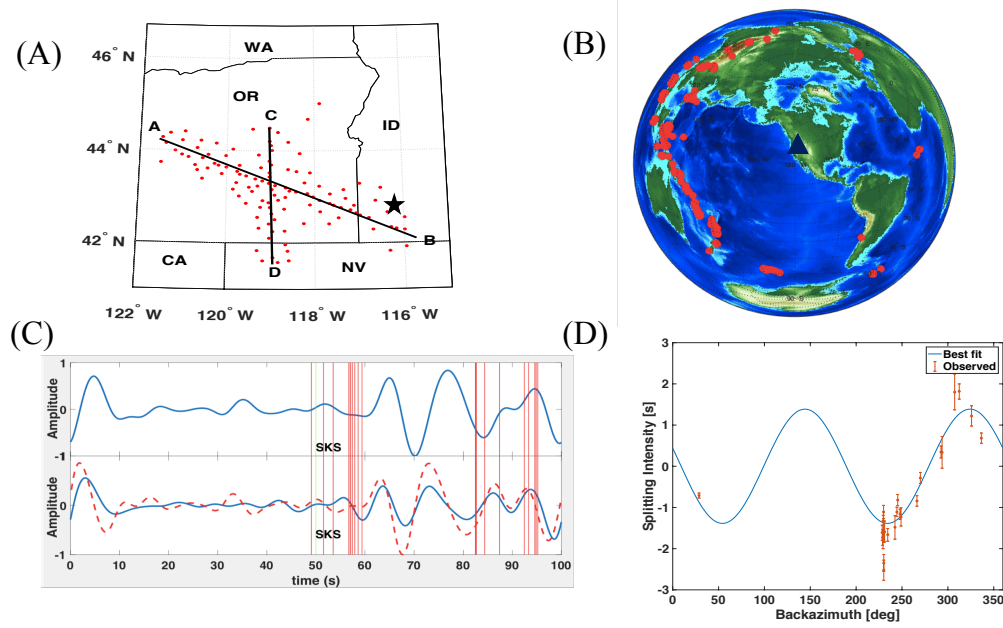


Figure 2.3: (A) The station distribution used in this study, shown by the red dots. The two dense arrays are marked as AB and CD. The star indicates the location of station ID005, shown in panel D. (B) Map of events used. (C) An example of the SKS split phase (in the red time window) observed at the station ID005. The blue trace in the top panel represents the radial component of the SKS waveform while the blue trace in the bottom panel shows the transverse component. The red dotted line is the time derivative of the radial component. The red lines indicate a selection of the multiple time windows automatically chosen by the splitting algorithm, as described by Reiss and Rumpker (2017). (D) The splitting intensity data plotted as a function of the backazimuth of the events observed at station ID005. The red bars denote the observed measurements while the blue line is the best fit of a $\sin(2\theta)$ curve to this observed data. Black star on (A) indicates the position of the station ID005.

Shear wave splitting measurements are a routine procedure in the study of seismic anisotropy. SKS splitting parameters (fast directions and delay times) for stations of the HLP experiment and nearby TA stations have been measured previously by [61] and [62]. In our analysis we require measurements of splitting intensity, a scalar entity obtained by computing zero-lag cross-correlation between the transverse component and the time differentiated radial component of the seismogram. Therefore, we reprocessed the HLP and TA datasets using the Split-Racer software, recently developed by [101]. Split-Racer estimates splitting intensity and associated errors and allows rapid processing of a large number of waveforms.

We processed and measured approximately 2000 waveforms with high quality SKS phases. We selected events of magnitude $M_w \geq 6$ located at epicentral distances between 90° and 130° . An event map is shown in Figure 2.3. Each station has recorded approximately 20 high quality events on average during the specified time interval. We applied a bandpass filter to retain energy at periods between 8 and 25 s, and visually examined the horizontal components of the SKS waveform to retain waveforms with a high signal to noise ratio ($\text{SNR} > 2.5$) and with a transverse component that takes the shape of the time derivative of the vertical component. After the quality checks, we obtained 1292 robust SKS splitting intensity measurements which were found to be exceptionally high quality. Examples of an SKS splitting intensity measurement and the splitting intensity curve for station ID005 are presented in Figure 2.3. Additional examples of splitting intensity curves for HLP stations are shown in Supplementary Figure S1.

2.4 Results

Here we present the results from our two 2-D inversions along the dense lines of the HLP experiment (Figure 2.3). These two inversions were carried out separately; given the computational limitations of a model space search approach, a 2-D inversion approach is tractable [98]. A check of the consistency between the two models at the intersection point is discussed in section 4.3, which also presents a pseudo-3D model for the region based on the 2-D inversions. We restrict our attention to the inversion of the strength parameter (χ) and the azimuth of the fast axis (ϕ); we assume that the symmetry axis is horizontal. Providing a reasonable constraint on the dip (θ) of the fast axis is unlikely due to the fact that the SKS wave propagation direction is nearly vertical, as discussed in [98]. In addition, we restrict the anisotropy strength to be less than 10%, a reasonable upper limit for most of the upper mantle.

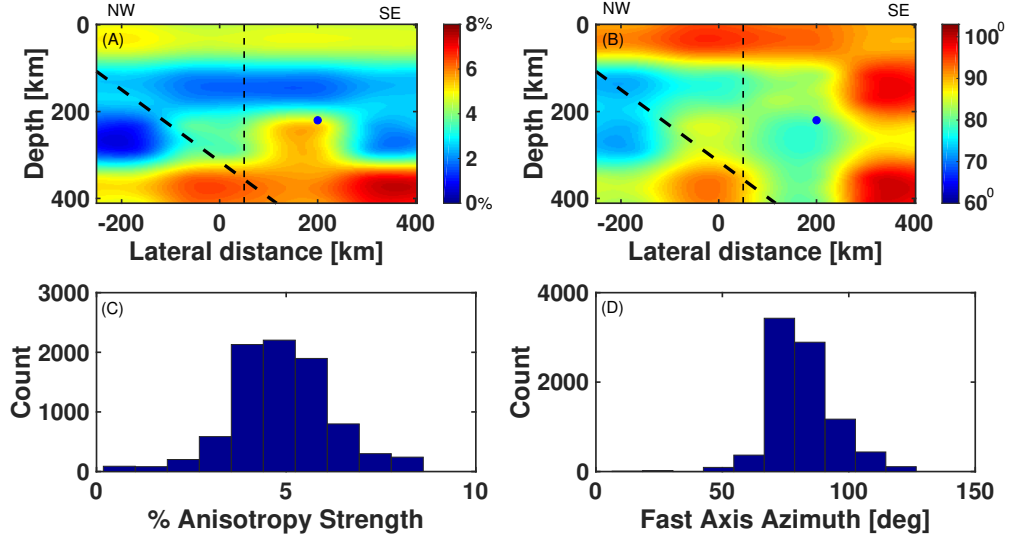


Figure 2.4: Most likely model for the NW-SE line (labeled AB on Figure 2). The spatial domain vertical section is discretized by $50 \text{ km} \times 50 \text{ km}$ grids. The image is obtained after smoothing the original $50 \times 50 \text{ km}^2$ discretized model for visual presentation. (A) Spatial distribution of the anisotropy strength, showing strong anisotropy in the deep upper mantle. Considerable spatial variation is observed, with anisotropy values ranging from 1% to 7%. (B) The spatial distribution of the fast axis azimuth, which shows only limited variation. The vertical dashed line shows the intersection of the arrays AB and CD (see Figure 2). The inclined dashed line shows the intersection of the subducting Juan de Fuca slab with the vertical section passing through NW-SE line, from the Slab 2 model of [99]. (C) Histogram showing the posterior distribution of anisotropy strength, corresponding to a well-resolved grid node marked by the dot on (A). (D) Histogram showing the posterior distribution of fast axis orientation corresponding to well-resolved grid node marked by the dot on (B).

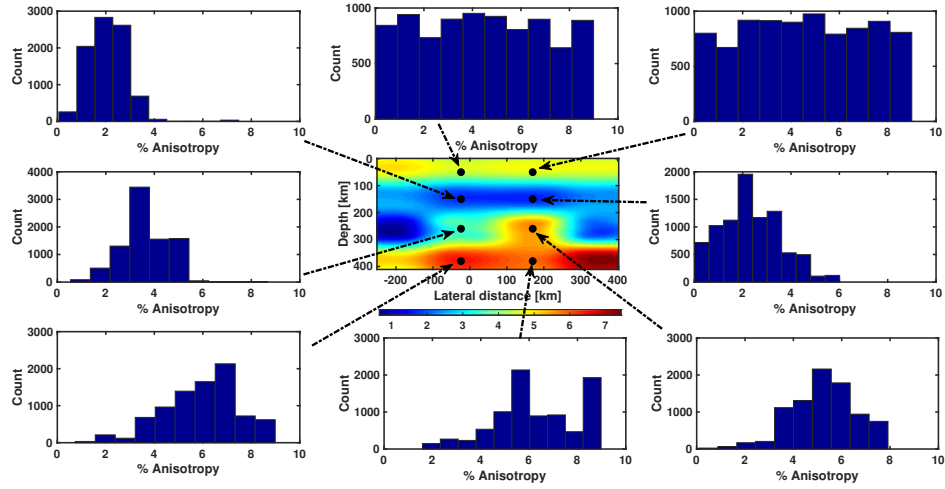


Figure 2.5: Posterior distributions of the most likely recovered model for anisotropy strength along the NW-SE line, shown in Figure 4a. These demonstrate that the shallow part of the model (< 50 km) is poorly constrained, whereas the deeper parts, with considerable overlapping of the sensitivity kernels, yield a fairly good resolution.

2.4.1 Anisotropy model of the NW-SE array

Striking lateral variations in SKS splitting delay times were measured along the NW-SE trending dense line of the HLP experiment, along with more modest variations in measured fast directions. The highest delay times (up to ~ 2.5 s) are roughly in the central portion of the transect ([62]; see their Figure 6). This variability suggests variation in the strength of anisotropy beneath the HLP, laterally and/or with depth, but previous investigations have left the depth range of this variability ambiguous.

Our preferred inversion was carried out with a cell size of $50 \text{ km} \times 50 \text{ km}$, based on previous work [98] and on resolution considerations that are discussed in Section 5. A total of 822 splitting measurements from 63 stations were used as the input for our inversion and convergence was attained after approximately 140,000 iterations (a total of 256,000 iterations were performed). While we solve for anisotropy parameters throughout the model domain, down to a depth of 400 km, our focus is on the depth range between 50-400 km, as resolution tests show that the shallow portion of our model is not well resolved. Because our tomographic approach is probabilistic, we obtain posterior probability distributions for each parameter for each grid point, rather than a single estimate. We present our models by plotting the most likely parameter values (that is, the values that were most

heavily sampled by the Markov chain, as expressed by the value of the ensemble average), along with histograms showing the probability distributions for selected portions of the model. A full set of histograms for regions throughout the model space are shown in Supplementary Figures S2-S3; similar figures for the N-S model (discussed in section 4.2) are shown in Supplementary Figures S4-S5.

Our model for the NW-SE dense line, shown in Figure 2.4, reveals intriguing variations in both the strength of anisotropy (plotted as % anisotropy in Figure 2.4a) and, to a lesser extent, the fast direction orientation (plotted as azimuth from N in Figure 2.4b). Anisotropy strength ranges from relatively weak values ($\sim 1\text{-}2\%$) up to a maximum of $\sim 8\%$, while fast directions exhibit comparatively weak variability (values ranging from $\sim 60^\circ$ to 105° east of north). The model features include anisotropy that is relatively uniform (with a strength of $\sim 4\text{-}5\%$ and a fast direction that is close to E-W) in the uppermost mantle (depths between 50-100 km). Between 100-200 km depth, anisotropy is surprisingly weak, with values of $\sim 2\%$, and fast orientations that range from $\sim 70^\circ$ in the NW to $\sim 95^\circ$ in the SE. Below 200 km, there is considerably stronger anisotropy, with values of $\sim 6\text{-}8\%$ in the 300-400 km depth range. For the most part, the region of strong anisotropy in the deep upper mantle lies above the slab surface (Figure 4a). In the 200-300 km depth range, there is notable variation in both fast directions and delay times, with a region in the central portion of the array that exhibits stronger anisotropy, and a counterclockwise rotation in fast axis orientation, compared to the adjacent regions.

While Figure 4 shows the most likely model for anisotropy beneath the NW-SE line, it is crucial to understand the character of the probability distributions for anisotropy parameters in order to appreciate how well constrained different portions of the model are. We have examined these distributions for our entire model space (Supplementary Figures S2-S3), and illustrate selected histograms for the anisotropy strength parameter in Figure 2.5. As expected, our model is poorly resolved in the shallowest part (above 50 km); in this depth range we do not achieve overlapping sensitivity kernels with the station density available, and we therefore do not interpret this portion of the model. Elsewhere, the histograms reveal that the distributions are much tighter, and parameters are much better constrained. In particular, both in the depth range associated with weak anisotropy (100-200 km) and in the deeper portions with strikingly strong anisotropy (300-400 km), the histograms of model parameters show that they are generally tightly constrained by the observations.

2.4.2 Anisotropy model of the N-S array

A total of 470 splitting measurements from 47 stations were used as the input for our inversion beneath the N-S line (corresponding to the line CD in Figure 3a) and convergence is attained after approximately 140,000 iterations (a total of 256,000 iterations were performed). As with the NW-SE line, we used a grid spacing of 50 km x 50 km. Again as with the NW-SE line, this line cuts across the region of particularly large splitting delay times identified by [61] and [62], so we expect to see lateral and/or depth variations in anisotropy strength in our model. This expectation is borne out by the model in Figure 6, which exhibits strong anisotropy in the deep upper mantle (200-400km depth range) in the central portion of the model, with weaker anisotropy in this depth range at the edges. Furthermore, in the asthenospheric upper mantle (depth range 100-200 km), there are dramatic lateral variations in anisotropy strength. We document a region of strong anisotropy located just to the south of the crossing point between the NW-SE and N-S dense lines. Interestingly, this region of locally strong anisotropy ($\sim 7\%$) is juxtaposed with a region of much weaker anisotropy ($\sim 2\%$) in the same depth range just to the north. The strong anisotropy in the 100-200 km depth range coincides geographically with a region of particularly high delay times documented by [61], suggesting that constructive interference between anisotropy layers at different depths may contribute to the particularly large δt observed in the central portion of the HLP.

Our model for the spatial variability in fast axis orientations shows more consistent layering than the corresponding model for the NW-SE line (Figure 4b), but as with the NW-SE model, the actual variability in orientations is relatively minor. We see fast directions that are nearly E-W in the shallower portions of the model (depths shallower than 200 km). In the deeper portions of the model, the most likely fast axis orientations are rotated slightly counterclockwise from the shallower portions ($\sim 70^\circ$ - 80° east of north); however, the difference is slight, and is likely not dramatic enough to cause strong variability in apparent splitting parameters with backazimuth at most azimuthal ranges [96]. This is consistent with the observation that apparent splitting parameters generally do not vary with backazimuth at HLP stations (e.g., [62]; see their Figure 2).

2.4.3 Consistency checks and a pseudo-3D model

Because we have carried out two separate 2D inversions, a natural check on our results is to compare the variation in anisotropy parameters with depth at the point at which the two models cross. Of course, the two models are not completely independent, in the sense that a portion of the

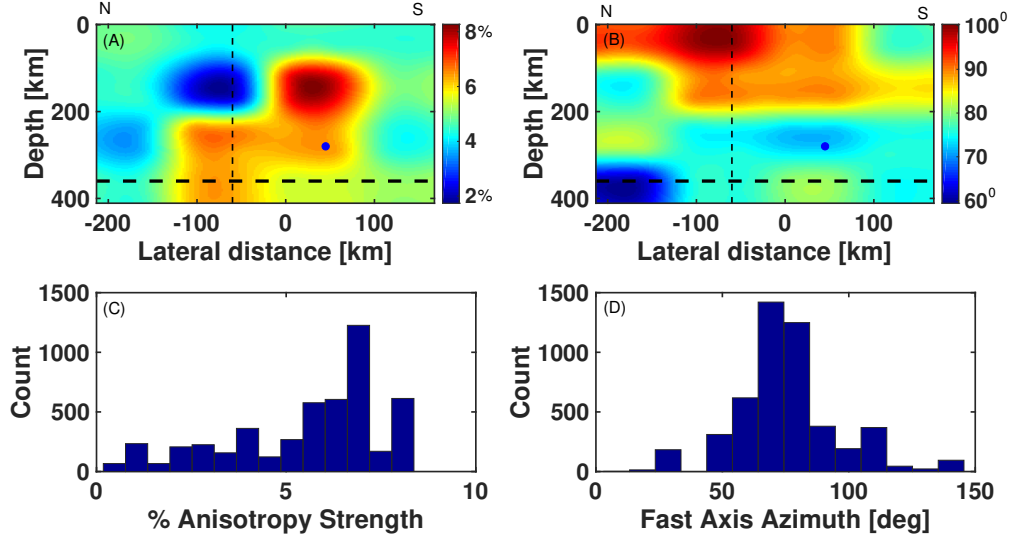


Figure 2.6: Most likely model for the N-S line (labeled CD on Figure 2). The spatial domain vertical section is discretized by $50 \text{ km} \times 50 \text{ km}$ grids. The image is obtained after smoothing the original $50 \times 50 \text{ km}^2$ discretized model for visual presentation. (A) Spatial distribution of the anisotropy strength showing strong anisotropy in the deep upper mantle, as well as a localized strong shallow (100-200 km) anisotropy feature, between 0 and 100 km. Considerable spatial variation is observed, with anisotropy values ranging from 1% to 7%. (B) The spatial distribution of the fast axis azimuth, which shows only limited variation. The horizontal dashed line shows the intersection of the subducting Juan de Fuca slab with the vertical section passing through N-S line, from the Slab 2 model of [99]. The vertical dashed line shows the intersection of the arrays AB and CD (see Figure 3)(C) Histogram showing the posterior distribution of anisotropy strength, corresponding to a well-resolved grid node marked by the dot on (A). (D) Histogram showing the posterior distribution of fast axis orientation corresponding to well-resolved grid node marked by the dot on (B).

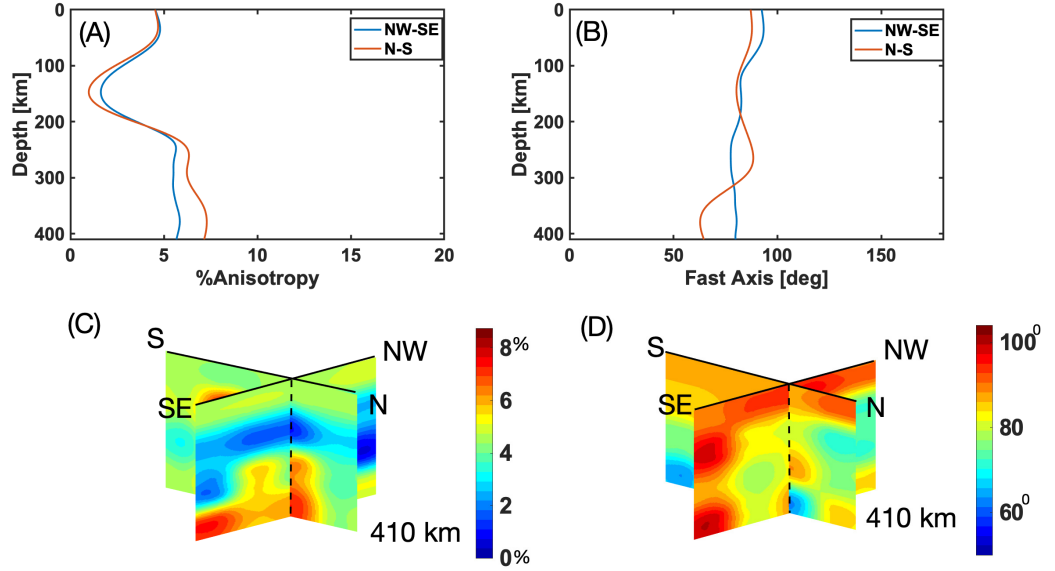


Figure 2.7: Comparison of the (A) strength of anisotropy and the (B) fast axis orientation between the results obtained from two different 2D inversions (corresponding to the NW-SE (blue curves) and N-S (red curves) arrays) at the intersection point. Panels (C) and (D) show a pseudo 3-D representation of the (C) strength and (D) orientation parameter, with the orientations of each line shown.

splitting intensity observations were included in both inversions; specifically, data from 13 stations, representing a total of 119 measurements, were used in both inversions. Still, a consistency check between the two independent inversions is useful; while we do not expect the most likely models at the crossing point to be identical, we do expect them to be generally consistent. As Figure 2.7 shows, this expectation is borne out, particularly for the anisotropy strength parameter, which shows an excellent match for both models. The models for fast axis orientation are not quite as consistent, but they do show generally similar behavior, with orientations near 80° - 90° in the shallow upper mantle and a slight counterclockwise rotation in the deep upper mantle. At depths below 200 km, the two models diverge somewhat, with differences in the most likely fast axis orientation of up to 20° .

Another way to visualize our tomography results is through the construction of a pseudo-3D model, which merges the two 2D models in the appropriate geographic reference frame (Figure 7). This view of our most likely models again demonstrates the major features of our images, particularly

the pervasive strong anisotropy in the deep upper mantle and the relatively weak anisotropy at shallower depths (100-200 km), with the exception of a region in the southern portion of the N-S line. This view also illustrates the generally good match between the models at their intersection point.

Finally, we visualize our most likely models in map view using a series of stick diagrams (Figure 2.8), which employ a common convention for representing anisotropy, with bars whose orientations correspond to the fast axis orientation and whose lengths correspond to the strength of anisotropy. We show the variation in anisotropy parameters for depth slices at 100 km, 200 km, 300 km, and 400 km. These maps again demonstrate the significant variation in strength of anisotropy with depth, with the deep upper mantle (300 km and 400 km depth slices) generally exhibiting considerably stronger anisotropy than the shallower upper mantle. The region of strong anisotropy in the central and southern portion of the study area at relatively shallow depths (100 km and 200 km depth slices) is particularly prominent. The fast axis orientations, in contrast, do not vary dramatically across our study region, either laterally or with depth. This is consistent with previous suggestions that the mantle flow field beneath the Cascadia backarc is generally simple, and may be controlled by the rollback of the Cascadia slab [76, 85, 102].

2.5 Resolution tests

As with any tomographic inversion, the question of resolution is paramount in the interpretation of the results. Given the challenges inherent in resolving anisotropy in tomographic inversions, particularly for nearly vertically propagating SKS phases [8, 9], it is important to evaluate how well our anisotropy models are resolved, and to restrict our interpretations to robust features that are well-resolved and required by the observations. A series of resolution tests for synthetic data sets was presented in [98]; many of these tests were aimed at highly simplified scenarios, but a subset of them used the actual HLP experiment station configuration as an example of a plausible real-world data set. These previous tests demonstrated that the HLP station configuration should allow for the resolution of anisotropic structure on length scales of ~ 50 km in most portions of the model, although there is little or no resolution of structure shallower than ~ 50 km depth. Here we build on the tests presented in [98], using the actual distribution of stations and events represented in our splitting intensity data set (rather than a synthetic, random distribution of earthquakes over all azimuths). We use our input model to create a synthetic data set of splitting intensity “observations” by computing sensitivity kernels for each event-station pair in our real data set, as

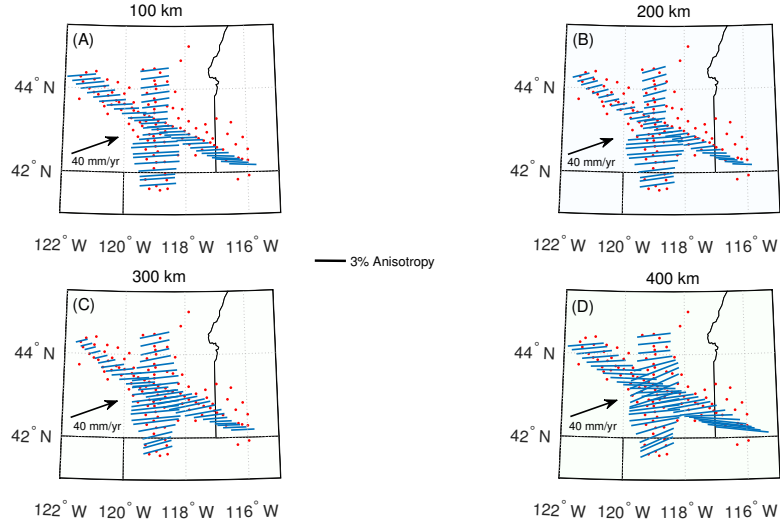


Figure 2.8: Diagram of the anisotropy strength and fast axis orientation (from our most likely models) in map view along the two dense lines at different depths. The length of the bar is scaled to anisotropy strength, while the orientation of each bar shows the local orientation of fast axis. (A) At depth 100 km: the models exhibit generally relatively weak anisotropy (less than 2%) beneath most of the study region, although there is a region of locally stronger anisotropy just to the south of the crossing point between the two lines.. (B) At depth 200 km: models exhibit relatively weak anisotropy $\sim 2\%$, again with generally uniform fast axis orientation. An exception is the region of notably strong anisotropy (up to 6%) just to the south of the crossing point of the two lines. (C) At depth 300 km: models exhibit stronger anisotropy with maximum anisotropy reaching up to 5%. The fast axis orientation along the NW-SE array varies from 70° in the NW to $\sim 90^\circ$ in the SE, while it remains almost uniform along the N-S line. (D) At depth 400 km: Both models exhibit strong anisotropy throughout, up to 7%. The fast axis orientation along the NW-SE line changes from $\sim 80^\circ$ in the NW to $\sim 95^\circ$ in the SE. It also varies somewhat along the N-S line, from 65° in the south to $\sim 90^\circ$ in the north. Thick black arrow shows the relative motion (convergence direction) between the Juan de Fuca and North American plates.

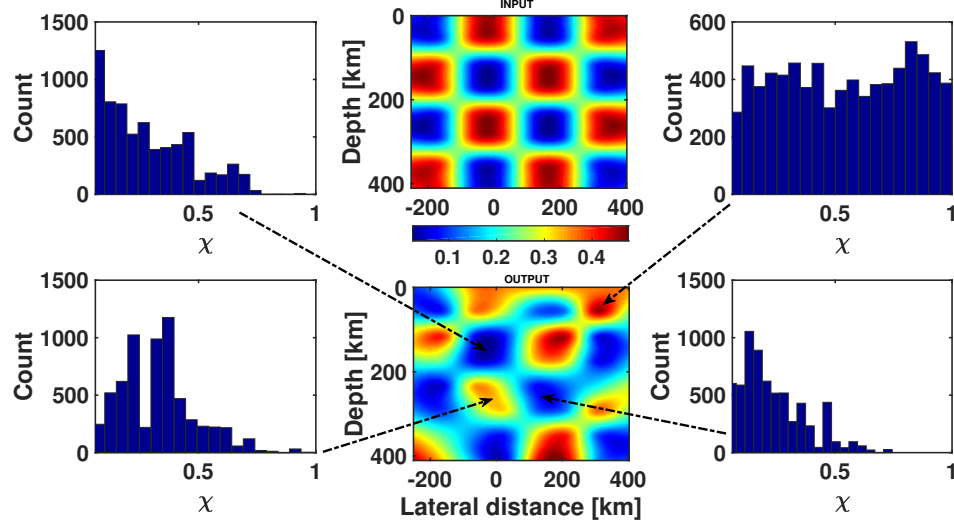


Figure 2.9: Results of checkerboard resolution test. Input model is shown in the top central panel, while the bottom central panel shows the most likely recovered model, and the histograms show posterior distributions of the recovered strength of anisotropy at selected points in the model. The input pattern corresponds to a $100 \times 100 \text{ km}^2$ checkerboard of alternating strong and weak anisotropy in a model domain of $600 \times 400 \text{ km}^2$ with 50 km grid spacing. We only solve for strength χ ; the other parameters are held fixed with $\theta = 0$ and $\phi = 80^\circ$. This test demonstrates that the input is well resolved in portions of the model domain where the sensitivity kernels overlap. As expected, the shallow part of the model lacks such overlap and remains unresolved. Note that here we show values of the strength parameter χ , which takes values between 0 and 1, instead of % anisotropy.

described in section 2, and integrating those kernels over the model volume, according to equation (2.2).

We first perform a checkerboard test, shown in Figure 2.9, with input anomalies that are 100 km square. Additional histograms showing probability distributions throughout the model space for this test are shown in Figure S6. Because our preferred models themselves tend to exhibit variations on a spatial length scale of about 100 km (Figures 2.4 and 2.6), this is a reasonable choice for an input pattern. Because most of the variability in our recovered models is in anisotropy strength, with little variability in the fast axis orientation, in this test we hold ϕ fixed at a reasonable average value of 80° . We calculate synthetic splitting intensity values for the input pattern and then use these values in the inversion scheme. The results of this test reveal several interesting properties of our inversion scheme. First, as expected, we observe significant smearing in the shallowest part of the model (depth $< 50 \text{ km}$), and the posterior distributions for these grid nodes are spread widely throughout parameter space. This is due to the lack of substantial overlap among the sensitivity

kernels at shallow depths, and is limited by the station spacing in the experiment. At depths greater than 50 km, however, the resolution test successfully recovers each of the grids of the input pattern with sufficient accuracy. As in any tomographic inversion, there is some smearing of structure, and the posterior distributions do show some spread in their values. Nevertheless, this test demonstrates that our inversion scheme is capable of resolving anisotropic structure at depth, even using only (nearly) vertically propagating SKS waves.

In the second test, shown in Figure 2.10, we carry out a recovery test, in which we use the most likely model from our actual inversion (in this case, beneath the NW-SE line) as an input pattern. We calculate synthetic splitting intensity values for this most likely model, and invert those synthetic data using the same grid parameterization (50 x 50 km) as in the actual inversion. The results of this test show that the output model strongly resembles the input model in its first-order characteristics, although there are some differences in the detail. The synthetic inversion successfully reproduces the major features, including the relatively weak anisotropy in the 100-200 km depth range, and the particularly strong anisotropy in the deep upper mantle (particularly in the SE portion of the model). Moreover, histograms that show the posterior probability distributions for selected grid points (Figure 10) show that these major features are generally well resolved.

Because SKS kernels have their maximum sensitivity in the depth range extending down to roughly 300 km (Figure 2), the question of how well our observations constrain structure in the deepest portions of the model is particularly crucial. As shown by the resolution tests in Figures 9 and 10 and Supplementary Figure S6, the inversion does retrieve structure in the 300-400 km depth range, although the amplitude is imperfectly estimated. We note that there are some tradeoffs in our inversion between anisotropy in the shallowest part of the model (shallower than 50 km, where we effectively have no resolution) and anisotropy in the deepest part of the model. One question, then, is whether the strong anisotropy that we image in the deepest parts of the model might actually reside in the shallowest part of the model, where we have poor resolution. We argue that this scenario is highly unlikely, because surface wave phase velocity maps at the periods that have maximum sensitivity in this shallow depth range do not show strong azimuthal anisotropy (Wagner and Long, 2013). This argument, together with the resolution tests presented here, give us confidence that the first-order features of our most likely models for anisotropy strength beneath the HLP - namely, the strong anisotropy in the deep upper mantle, the relatively weak anisotropy between 100-200 km depth beneath the NW-SE line, and the locally strong anisotropy anomaly between 100-200 km depth beneath the southern portion of the N-S line - are in fact required by the data and well-resolved by the inversion, and can be interpreted in terms of physical processes.

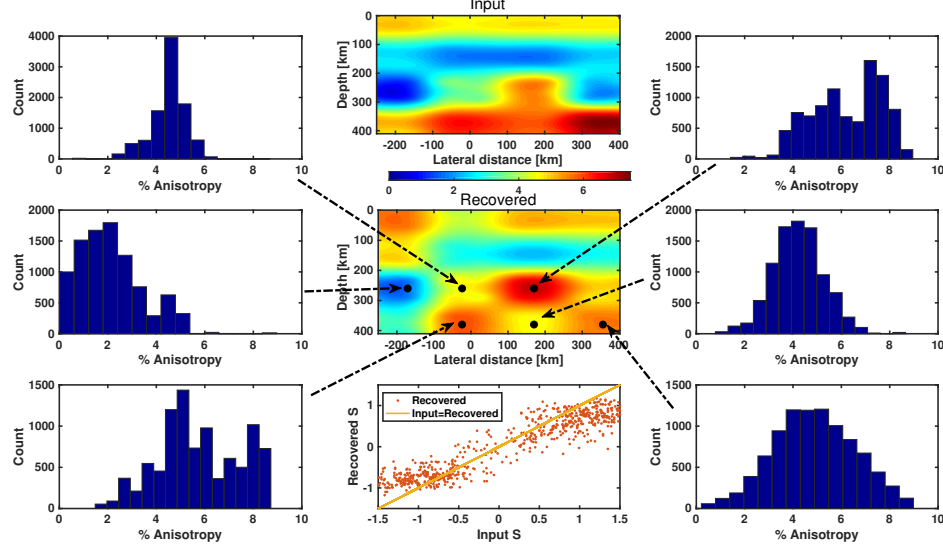


Figure 2.10: Results of the recovery resolution test. Top middle panel shows the input model (the most likely model achieved from the inversion along the NW-SE line) for anisotropy strength (in %, represented by the color bar). The central panel shows the most likely recovered model. Bottom middle panel shows the correspondence between input (the synthetic splitting intensity values calculated for the input model) and output (splitting intensity values predicted for the most likely output model) splitting intensities. In general, the recovered model reproduced the input splitting intensity values well. Other panels show posterior distributions of the recovered anisotropy strength at selected points of the model (indicated by arrows), plotted as histograms.

2.6 Discussion

2.6.1 Probabilistic finite-frequency SKS splitting intensity tomography: Proof of concept

We have presented a first practical application of probabilistic finite-frequency splitting intensity tomography to the Cascadia backarc region. This densely distributed seismic network yields a generally robust model for upper mantle anisotropy at depths between 50-400 km. The resolution tests (Figures 2.9 and 2.10) show that SKS splitting intensity data collected during a temporary broadband deployment are capable of constraining depth-dependent anisotropy in the upper mantle. This provides an exciting new avenue for the interpretation of SKS splitting datasets, and builds on the few previous studies that have applied finite-frequency SKS splitting tomography approaches to actual data [38, 64]

One caveat to the successful application of probabilistic finite-frequency SKS splitting intensity tomography presented here is that the HLP splitting data set is one that displays significant variability in observed splitting delay times, but much less in observed fast splitting directions ([61], [62]).

This makes this set of observations an easier target for our tomographic approach, since the tomography problem for anisotropy strength (for the case in which the anisotropy geometry varies little) is more nearly linear than the corresponding problem in which both the strength and geometry of anisotropy varies widely. Work to apply our analysis approach to datasets for which we expect both the geometry and strength of anisotropy to vary substantially is ongoing.

2.6.2 Tectonic implications

A key finding from this work is that SKS splitting intensity observations beneath the HLP prefer models that include strong anisotropy (up to $\sim 6\text{--}8\%$) in the deep upper mantle (200-400 km depth range) beneath much of the Cascadia backarc. This finding contradicts the simplest expectations that might be formed from visual inspection of the traditionally measured SKS splitting parameters, as discussed in [61]. Variations in shear wave splitting behavior over short length scales are often interpreted to require variations in anisotropy at shallow depths, based on an argument that invokes overlapping Fresnel zones [95]. However, our work shows that the generally smooth variations in measured δt across the HLP stations (e.g., Figure 6 in [62]) can result from the integrated effects of anisotropic layers with different strengths, including (in this case) a layer of strong anisotropy in the deep upper mantle. Specifically, the particularly large delay times beneath the central part of the HLP study area result from the combined effect of strong anisotropy in the deep upper mantle and a localized region of strong anisotropy in the 100-200 km depth range evident in the N-S model (Figure 6).

This finding has implications for discriminating among the different scenarios proposed as possible explanations for the large SKS splitting delay times beneath the HLP. [61] discussed three possible explanations for the δt values (and their spatial variability): a particularly thick anisotropic layer(s) beneath the region, particularly strong olivine LPO (perhaps due to specific deformation conditions in terms of temperature or water content), or a contribution from shape preferred orientation of partial melt in the uppermost mantle. Based on the models presented in this paper, the last of these three possibilities can be effectively ruled out: there is no evidence in our models for strong anisotropy in the shallow depth ranges at which partial melt may be present, and instead our inversions prefer models that include strong anisotropy in the deep upper mantle. While the large delay times beneath the HLP do likely reflect the constructive interference of multiple layers of anisotropy over several depth ranges, our finding that strong ($\sim 6 - 8\%$ anisotropy) is required in much of the deep upper mantle, and in a localized region of the asthenospheric upper mantle along the N-S line, is consistent with the earlier suggestion by [61] that particularly strong olivine LPO

may be present. What is surprising, however, is the extent to which this strong LPO is forming in the deep upper mantle.

What do our models imply for our understanding of the tectonic evolution of the HLP and the Cascadia backarc more generally? First, our finding of generally E-W fast axis orientations, with little variability in the geometry of anisotropy, is consistent with previous suggestions that the flow field beneath the Cascadia backarc is generally simple, and likely controlled by rollback subduction of the Juan de Fuca slab (e.g., Druken et al., 2011; Long et al., 2012; Long, 2016), perhaps in combination with other factors [103]. A comparison between our anisotropy models and isotropic velocity models for the same region derived from tomography [88] does not demonstrate any striking correlations between anisotropy strength and shear wave velocity (Supplementary Figure S7), as might be expected if the strength of olivine LPO was controlled by a factor such as temperature. The region of strong anisotropy in the asthenospheric upper mantle evident along the N-S line (Figure 6) corresponds roughly to the location of the Steens Mountain basalts (which are contemporaneous with the ~16 Myr old Columbia River basalts; see [61], and references therein) and the Holocene volcanic activity of Diamond Craters (Figure 1). This possible spatial relationship is suggestive, but its nature remains obscure, given that our model has ruled out the possibility that anisotropy strength is controlled by variations in aligned partial melt in the shallowest upper mantle.

2.6.3 Pervasive anisotropy in the deep upper mantle beneath the Cascadia backarc

Our finding of strong anisotropy, and by inference strong olivine LPO, in the deep upper mantle (200-400 km depth) beneath the Cascadia backarc region has potentially important implications. Numerous studies on the rheology and deformation of the upper mantle [6, 104] have suggested that olivine LPO is the primary mechanism for generating seismic anisotropy. However, there are indirect arguments, mainly based on previous observations of generally weaker anisotropy at depths greater than 200 km, that deformation in this depth range may be dominated by diffusion creep, rather than the dislocation glide that is necessary to form LPO (e.g. Karato & Wu, 1993). Our finding of strong anisotropy in the deep upper mantle implies that there is, in fact, strong olivine LPO in this depth range, at least locally. This in turn suggests that the deep upper mantle is deforming in the dislocation glide regime, or, alternatively, that olivine LPO is being generated via a different mechanism at these depths. [105] recently proposed that diffusion creep may produce LPO in the upper mantle, although this idea remains controversial.

Recent results on deformation of olivine at relatively high pressure conditions may shed additional light on possible mechanisms for strong anisotropy in the deep upper mantle. Couvy et al. (2004)

and Ohuchi and Irifune (2013) presented experimental evidence for the formation of olivine LPO at the pressures associated with the deep upper mantle; in particular, Ohuchi and Irifune (2013) argued for the development of A-type olivine fabric under water-rich conditions. More recently, Masuti et al. (2019) carried out deformation experiments on olivine single crystals and found evidence for the development of C-type olivine fabric under water-rich conditions, although their experiments were carried out at lower pressures than those of Ohuchi and Irifune (2013). Given the geodynamic setting of our model in a subduction zone backarc, it is plausible that deformation is taking place under water-rich conditions, and we can envision a few possible scenarios that are consistent with our model. For the most part, the strong anisotropy in the deep upper mantle in our model is located above the slab surface (Figure 4). If A-type olivine LPO dominates, as suggested by Ohuchi and Irifune (2013), then strong azimuthal anisotropy would suggest dominantly horizontal shearing (with a generally E-W shear direction). On the other hand, if C-type olivine LPO dominates, as suggested by Masuti et al. (2019), then a component of upwelling flow may align olivine C axes nearly vertically, producing strong azimuthal anisotropy. Geodynamic modeling results for the HLP region (Druken et al., 2011; Long et al., 2012) do suggest a component of upwelling associated with the rollback and steepening of the Cascadia slab, although significant horizontal flow is present as well.

2.6.4 Implications for reconciling surface wave and SKS constraints on anisotropy

An interesting aspect of our model is that it can be compared with models based on independent constraints, namely surface wave dispersion measurements. We are particularly interested in comparisons with the models of Yuan and Romanowicz (2010) and [106] for the western U.S., because these models are based on joint inversion and/or interpretation of surface wave data and station-averaged SKS splitting measurements, and the surface wave model of Wagner and Long (2013), because that model was based on measurements made at HLP stations.

The model of Yuan & Romanowicz (2010) covers the entire western U.S. and has relatively coarse (~ 250 km) lateral resolution; we focus our comparison on the region beneath the HLP itself. This model features multiple layers of anisotropy beneath this region, with fast directions that vary from NE-SW fast directions in the shallow upper mantle (~ 70 - 100 km depth) to more nearly NW-SE fast directions in the mid-upper mantle (~ 200 - 250 km) to nearly E-W fast directions in the deepest upper mantle (~ 350 - 400 km). This model features relatively strong anisotropy in the deepest upper mantle, in that the anisotropy strength here is comparable to the anisotropy strength at shallower depths, but it is considerably weaker than in our preferred model (~ 2 - 3 % vs. up to ~ 6 - 8 %).

This difference may reflect the fact that the model of Yuan & Romanowicz (2010) is based on a regularized inversion that may underestimate amplitudes, or it may be a consequence of their coarser lateral resolution. Interestingly, a comparison of models obtained by Yuan & Romanowicz (2010) (see their Figure 7) that include only surface waveform constraints vs. those that include both surface waveforms and SKS splitting shows that when the SKS measurements are included, anisotropy in the deeper portions of the model (150 to 500 km depth) beneath the HLP is stronger and has a more nearly E-W fast direction. Furthermore, when a model is produced based only on surface waveforms, it strongly underpredicts the SKS splitting delay times. This is generally consistent with the inference from our study that the SKS observations themselves require a major contribution from anisotropy in the deep upper mantle. The agreement between our model and that of Yuan & Romanowicz (2010) is only partial, however, as our model does not include the considerable variability in fast axis orientations that is exhibited in the Yuan & Romanowicz (2010) model beneath the HLP.

The model of Lin et al. (2011), based on a combination of ambient noise and event-based surface waves, does not extend into the deep upper mantle, but at asthenospheric depths the model shows generally E-W fast directions, consistent with our model. At crustal depths (at which our model does not have resolution), the Lin et al. (2011) model shows relatively weak ($\sim 2\%$) and variable anisotropy. Lin et al. (2011) predicted SKS splitting fast directions and delay times based on their model; their predictions for the HLP region match the fast direction observations, but substantially underpredict the delay times (predicted times of up to ~ 1.2 s vs. observed times of up to ~ 2.5 s). Our finding of a substantial contribution to SKS splitting from anisotropy in the deep upper mantle is therefore consistent with the predictions of the Lin et al. (2011) model.

Our model provides an avenue for reconciling what had been thought of apparently contradictory constraints on upper mantle anisotropy provided by surface wave and SKS splitting observations made at stations of the HLP experiment. Wagner & Long (2013) presented azimuthally anisotropic phase velocity maps based on surface wave dispersion measurements and a major focus of this paper was the apparently contradictory views of upper mantle anisotropy provided by the different types of data. The surface wave phase velocities, which have little or no sensitivity at depths greater than ~ 250 km, showed generally uniform fast orientations beneath the HLP with modest anisotropy strengths (values of up to $\sim 2.5\%$). Wagner & Long (2013) found that the anisotropy strengths suggested by the phase velocity maps were much smaller than needed to explain the SKS splitting delay times if all the anisotropy was distributed in the upper ~ 200 km of the mantle; furthermore, they argued that a strong contribution from crustal anisotropy (shallower than ~ 50 km) would have

strongly affected the phase velocities at 33 and 40 s period, which was not observed. Wagner & Long (2013) left the resolution of this discrepancy to future work, and the model presented here now resolves the apparent paradox. Our preferred model for anisotropy beneath the HLP invokes strong anisotropy in the deep upper mantle, at a depth range that is beyond the sensitivity of the observations (at periods up to 143 s) used in the Wagner & Long (2013) surface wave model. The presence of deep upper mantle anisotropy can explain the apparent discrepancy between surface wave and SKS splitting constraints documented by Wagner & Long (2013).

2.7 Summary

We have presented a pseudo-3D model for the distribution of seismic anisotropy beneath the High Lava Plains of Oregon and its surrounding region in the backarc of the Cascadia subduction zone. This model was obtained in the framework of a probabilistic, finite-frequency SKS splitting intensity tomography approach applied to a data set of newly processed SKS splitting intensity measurements for stations of the HLP broadband array. The dense station spacing of this array allowed us to resolve anisotropic structure, at length scales of ~ 50 km, in the depth range between 50-400 km. We found evidence for strong seismic anisotropy in the deep upper mantle beneath the Cascadia backarc, with generally weaker anisotropy in the 100-200 km depth range (with some local exceptions). There is little variability in the fast axis orientations, consistent with previous suggestions of a relatively simple mantle flow field. This finding is consistent with the idea of mantle flow beneath the Cascadia backarc that is driven by rollback subduction, as suggested by previous workers, and with the presence of lateral variations in olivine LPO strength within that simple flow field. Resolution tests demonstrate that our method is capable of resolving anisotropic structure throughout the model domain, and posterior probability distributions provide a sense for uncertainties in the model parameter estimations. Our finding of strong anisotropy in the deep upper mantle beneath the HLP helps to reconcile previously conflicting views of upper mantle anisotropy from surface wave constraints and SKS splitting data, and may suggest that deep upper mantle anisotropy is more important to the interpretation of shear wave splitting data sets than is often thought.

Chapter 3

Finite frequency coupled SKS-Rayleigh wave anisotropy tomography

3.1 Introduction

Discerning the anisotropic structure of the Earth's interior via seismological observation is a classical problem of geophysics [11, 12, 13, 14]. Whereas the isotropic wave speed tomography provides with crucial information regarding the structure of Earth's interior, it is, in a broader sense, unable to yield information about physically interesting deformation processes which are intrinsically anisotropic in nature. Examples include understanding overall patterns of mantle convection [15, 16], subduction dynamics [17, 18, 19, 20], oceanic spreading centers and continental rift systems [21, 22, 23], continental deformation [24, 25], and flow at the base of the mantle [26, 27, 28].

Therefore, one requires to study the response of different seismic waves to anisotropic perturbations. A commonly used method to infer upper mantle anisotropy - and therefore upper mantle flow patterns - is to measure shear wave splitting parameters (fast direction, ϕ , and delay time, dt), typically for phases such as SKS that have been refracted through the outer core. However, the interpretation of SKS splitting measurements is not straightforward, because they represent an integration of anisotropic structure along the ray path. Large delay times, for example, may yield several interpretations in terms of strength of anisotropy (a thin highly anisotropic layer vs. a thick layer with weak anisotropy). Similarly, in the presence of multiple anisotropic layers, the apparent fast direction reflects a complicated (and non-commutative) combination of layer parameters, and isolating the depth of different anisotropic regions is a major challenge.

To lessen the ambiguity of shear wave splitting interpretation, a tomographic approach that allows for the localization of anisotropy is desirable. However, tomographic inversion for anisotropic structure is a daunting task, since 21 parameters are needed to describe the most general form of anisotropy, and such a large parameter space does not allow for a tightly constrained tomographic inversion given limited data. Furthermore, the inverse problem for anisotropy is highly non-linear,

as the sensitivities of splitting observations to anisotropy depend on the starting model used to compute them. Finally, shear wave splitting tomography is further complicated by the measurements themselves, which are challenging for noisy data, resulting in much smaller data sets than traditional traveltimes tomography. The measurement problem can be somewhat ameliorated by making measurements of the splitting intensity rather than the traditionally estimated splitting parameters (ϕ, dt) , as this quantity can be more robustly estimated from noisy data and has the advantage of being commutative along the ray path.

Substantial effort has been devoted to developing theoretical frameworks and practical strategies for inverting shear wave splitting data for anisotropic structure, much of it in the context of finite-frequency sensitivity kernels that relate model perturbations to splitting observations. However, there have only been a few published examples of the application of splitting tomography to actual data sets to study mantle anisotropy. Built on the work of [98], where a SKS splitting intensity tomography technique in a finite frequency setting was developed and subsequently tested in [107] by applying to the HLP data set, we incorporate a secondary constraint namely the Rayleigh waves. In the presence of anisotropy, the Rayleigh waves scatter and produces quasi-Rayleigh modes which in turn cause the perturbation to the phase velocity in the obvious sense. In addition, surface waves are very effective at constraining the shallow structures due to their telescopic property, that is, their frequency dependent depth sensitivity (even sensitivity of body waves do depend on the frequency, but, for surface waves it is stronger). Surface waves can therefore be an ideal candidate to provide a second constraint to our SKS splitting intensity tomography. We develop a joint inversion technique by coupling the SKS and Rayleigh waves (at a range of frequencies). Importantly, in order to assure a true coupling, we add an additional term which mixes the parameters suitably. This effectively constructs a search algorithm in the tensor product parameter space (between the surface waves degrees of freedom and SKS degrees of freedom) instead in a direct sum parameter space. Several resolution tests prove the robustness of the inversion scheme. Our inversion results for HLP data set suggests strong anisotropy at the deeper part while the shallow portion is dominated by weak anisotropy. This result is in agreement with the result of [107] for the deeper part where both results suggest strong anisotropy. However, [107] could not constrain the shallow part and that is the sole reason we wanted to study the problem from a 'joint' inversion perspective where Rayleigh waves provided extra constraint.

3.2 Theoretical Formulation

Anisotropic study of the surface waves goes back to the seminal paper [7], where a virial type argument in conjunction with tensor harmonic decomposition of relevant tensor fields were employed to obtain an azimuthal dependence of the phase velocity for both Rayleigh and Love waves in the regime of linear perturbation theory. Following this study, numerous studies have been performed over the years depicting a substantial progress in the field. In the current context though we are interested in utilizing the surface wave information as a constraint to the finite frequency splitting intensity tomography problem using SKS waves. Since, the SKS waves are not able to resolve the shallow structure ($\lesssim 50$ km), a secondary constrain is necessary and surface wave is such a possible candidate. We will focus on the Rayleigh (quasi-) wave throughout. The fundamental equations describing the elastic wave propagation are as follows

$$e_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i), \Sigma_{ij} = C_{ij}^{kl} e_{kl}, \rho(x) \frac{d^2 u_i}{dt^2} = \nabla_j \Sigma_i^j. \quad (3.1)$$

Here u, e, Σ, C, ρ denote the displacement field, the strain tensor field, the stress tensor field, the elasticity tensor field, and the medium density, respectively. For convenience, we prefer to work in the frequency domain since at the linear level different modes are essentially decoupled from each other. In the Fourier domain, the equation of motion yields

$$\nabla_j (C^{ijkl} \nabla_k u_l) = -\omega^2 u^i, \quad (3.2)$$

where ω is the frequency and since we are interested in purely elastic problem, C^{ijkl} s are time independent. Let us now briefly describe the exact theoretical calculations we want to perform. Inspecting equation (3.2), one naive guess would be to obtain a variational expression for the frequency ω . This is motivated by the fact that the structure of the equation suggests ω may be treated as a Lagrange multiplier. However, there are few subtleties one needs to address carefully in order to obtain a correct variational expression for the frequency. We will freeze ω and establish that each such ω yields a weak solution of the equation of interest (3.2). In other words, there exists a $\omega \in \mathbb{R}$ such that $\forall v \in H_c^1(\mathbb{R}^3)$ (this is nothing but the space of vector fields which along with its first derivatives are square integrable and the the vector fields are compactly supported i.e., vanishes

beyond some open set), the following holds

$$\int_{\mathbb{R}^3} C^{ijkl} \nabla_i u_j \nabla_k v_l d^3x = \omega^2 \int_{\mathbb{R}^3} u^i v_i d^3x, \quad (3.3)$$

where the norm $\|\cdot\|_{H^1(\mathbb{R}^3)}$ is defined as $\|v\|_{H^1} := (\int_{\mathbb{R}^3} C^{ijkl} \nabla_i v_j \nabla_k v_l d^3x)^{1/2}$, since C is positive definite and point-wise finite. Existence of a solution which along with its first derivative are square integrable is essentially equivalent to the existence of the following entity

$$\inf \left\{ \|u\|_{H^1(\mathbb{R}^3)}^2 \mid u \in H^1(\mathbb{R}^3), \|u\|_{L^2} = 1 \right\}. \quad (3.4)$$

How do we guarantee that this infimum is even attained in L^2 ? From standard Sobolev embedding theorem, we of course know that the space $H^1(\mathbb{R}^3)$ is continuously embedded in $L^2(\mathbb{R}^3)$. However, in order for this infimum to exist, we need a compact embedding. Compact embedding essentially guarantees that if we choose a minimizing sequence of functions, then after passing to a suitable sub-sequence, convergence is attained. But, H^1 embeds into L^2 compactly if and only if the vector field u^i is compactly supported. We therefore, impose the restriction that u^i of interest is compact supported. This is physically reasonable since the surface wave amplitude does exhibit decay with depth and it is fair to consider that such amplitude vanishes after some finite depth. Standard Lagrange multiplier technique yields a variational expression for ω

$$\omega^2 = \frac{\int_{\mathbb{T}^2 \times [0,d]} C^{ijkl} (\nabla_i u_j)^* \nabla_k u_l d^3x}{\int_{\mathbb{T}^2 \times [0,d]} |u|^2 d^3x} = \frac{\int_{\mathbb{T}^2 \times [0,d]} C^{ijkl} e_{ij}^* e_{kl} d^3x}{\int_{\mathbb{T}^2 \times [0,d]} |u|^2 d^3x} \quad (3.5)$$

Here, $\mathbb{T}^2$ indicates the periodicity in the horizontal direction and $*$ denotes the complex conjugate operation. Vertical extent is limited to d for compactness. Now that we have related the frequency ω to the the elasticity C^{ijkl} of the medium, we may perturb this equation (to first order) to yield the observable-model relation. Perturbation to the phase velocity $c^2(k) = \frac{\omega(k)^2}{k^2}$ may be computed as

$$\delta c(k) = \frac{c(k)}{2\omega^2} \delta \omega^2(k) \quad (3.6)$$

which, together with the perturbation of equation (3.5) yields

$$\delta c(k) = \frac{c(k)}{2\omega^2} \frac{\int_0^d \delta C^{ijkl} e_{ij}^* e_{kl} dx dy dz}{\int_0^d \rho(x, z) |u|^2 dx dy dz}. \quad (3.7)$$

Under the assumption of the horizontal periodicity, the displacement may be written explicitly as follows

$$u = (\xi(z) \cos \theta, \xi(z) \sin \theta, \eta(z)) e^{i(k \cos \theta x + k \sin \theta y)}, \quad (3.8)$$

θ being the polarization direction in a suitable coordinate system. Substituting the displacement (3.8) into equation (3.7) and integrating out the horizontal degrees of freedom, one immediately obtains

$$\delta c(k, \theta) = \frac{c(k) \int_0^d \delta C^{ijkl}(z) [B(z, \theta)]_{ij} [B(z, \theta)]_{kl} dz}{2\omega^2 \int_0^d (\xi^2 + \eta^2) dz}, \quad (3.9)$$

along the dispersion curves. Here B is a 2-tensor involving strain components evaluated by integrating out the horizontal degrees of freedom. Now let us write down the perturbations to the elasticity tensor for Hexagonal symmetry explicitly in order to implement the parameterization of our choice

$$\begin{aligned} \delta C_{ijkl} = & 2\rho\alpha^2 \epsilon (s_i s_j s_k s_l - \delta_{ij} s_k s_l - \delta_{kl} s_i s_j) + \rho\alpha^2 \delta (\delta_{ij} s_k s_l + \delta_{kl} s_i s_j - 2s_i s_j s_k s_l) \\ & + 2\rho\beta^2 \gamma (2\delta_{ij} s_k s_l + 2\delta_{kl} s_i s_j - \delta_{ik} s_j s_l - \delta_{il} s_j s_k - \delta_{jk} s_i s_l - \delta_{jl} s_i s_k), \end{aligned} \quad (3.10)$$

where $s = (\cos \phi, \sin \phi, 0)$ and ϕ being the azimuth of the symmetry axis in the same coordinate system chosen above. Notice that we are primarily interested in the azimuthal anisotropy and therefore turned off the ‘dip’ of symmetry axis. Here the parametrization we are interested in is as follows

$$C^{\mu\nu} = \chi C_{single}^{\mu\nu} + (1 - \chi) C_{isotropic}^{\mu\nu} \quad (3.11)$$

which would write ϵ , β , and δ in terms of χ in the expression of δC_{ijkl} . Using this parameterization, the observable-model relation may be written as follows

$$\frac{\delta c(k, \theta)}{c(k)} = \frac{1}{2\omega^2(k)} \int_0^d \mathcal{K}(\theta, z; \phi) \chi(z) dz, \quad (3.12)$$

where $\frac{\delta c(k, \theta)}{c(k)}$, χ , and $\mathcal{K}(\theta, z; \phi)$ are the perturbation to the phase velocity, the strength of anisotropy, and the sensitivity kernel, respectively. Here θ is simply the polarization of the incoming wave, while ϕ is the usual backazimuth of the symmetry axis. Similar calculations for the *SKS* phase may be performed to obtain the relevant sensitivity kernel. However, since we have already described it

in [98] in detail, we do not wish to repeat the same. Nevertheless, note an important point worth mentioning. The unperturbed displacement field is required for both the surface waves and the SKS waves in order to compute respective sensitivity kernel (for surface wave such calculation is required even at a more basic level to yield the dispersion relation). In order to calculate the sensitivity kernels, we need the Green's function. For a homogeneous, isotropic wave speed model, the Green's functions can be solved analytically [41]. However, for a velocity structure that varies with depth, a semi-analytical technique such as a propagator matrix method works well [43, 44, 45]. Since at the level of anisotropy, the effect of first order lateral heterogeneity is inconsequential, we simply consider the layered model **ak135** [47] for the seismic wave speed. In such a layered settings, the implementation of a semi-analytical technique namely the propagator matrix technique becomes most useful. In the propagator matrix technique, the fields are expanded in the Fourier series in the horizontal direction (no horizontal variation of seismic wave speed), and the frequency domain wave equation is converted to a set of ordinary differential equations in z . Further details can be found in [41]. In our kernel computations presented in the paper, we used the propagator matrix technique applied to the depth dependent velocity model ak135; because it is a semi-analytical technique, is very efficient and allows for exploration of the model space with reasonable number of iterations. Here we avoid describing the parameterization in detail since, it is exactly the same as the one used in [98]. One subtlety of course would be the validity of such parameterization in the case of surface waves since, we are primarily interested in constraining the anisotropy information of the shallow parts. Another possible objection would be the validity of the first order perturbation theory we are implementing to yield anisotropy structures. This is associated to the fact the larger deformation gradient associated to the shallow part ($\lesssim 100km$) might be responsible for the creation of large anisotropy falling beyond the regime of linear perturbation theory. However, these questions at one hand are theoretical in a sense that we need to develop a large data perturbation theory and on the other hand experimental in a sense that the precise deformation pattern and associated anisotropy need to be addressed carefully. We ignore such complexities all together and stick to the regime of linear perturbation theory and the simple parameterization developed in [98].

3.2.1 Kernel Examples

The sensitivity kernels play an important role in obtaining the data via forward calculations as evident from the basic equations

$$S_{surface}(k, \theta) = \int_0^d K(\phi, z; \theta, k) \chi(z) dz, \quad (3.13)$$

$$S_{SKS}(x) = \int_D K(\phi, x') \chi(x') dx'. \quad (3.14)$$

Surface waves do provide a good depth resolution via telescopic sampling at a number of frequencies. This is precisely reflected in the expression of observable-model relation via sensitivity kernel. Simultaneously, the SKS waves through their overlapping kernel provide a robust resolution for a reasonable depth range. Therefore, one would naturally hope that both of these waves combined would yield a better depth resolution almost through the whole upper mantle. Since, sensitivity kernels are the ones controlling the depth resolution, their spatial structure would essentially be informative in a sense that they would provide a rough indication whether the model in a particular spatial location is well resolved or not. Strong (and overlapping for SKS phase) sensitivity kernel roughly correlates with good resolution.

In this section, we explore the behavior of sensitivity kernels and provide some simple examples of their strong dependence on the background model. The kernel behavior is illustrated via figures that show the variation of the sensitivity in the model space as functions of the background model properties (azimuth and dip of the symmetry axes of the anisotropic medium).

We choose the power spectrum of the incoming SKS wave to be similar to that of [9], i.e.,

$$|\mathbf{u}_0^R(\omega)|^2 = \frac{\omega^2 \tau^2}{4\pi} e^{-\frac{\omega^2 \tau^2}{8\pi^2}}, \quad (3.15)$$

where τ is the characteristic period of the wave, which is taken to be 10 s for SKS phase. The integral over the frequency in equation (1.21) is performed over the frequency range of 0.02 – 0.125 Hz which is the same as used by [9] and typical for SKS splitting studies [29]. We approximate the spectrum as the second derivative of a Gaussian; this is reasonable for SKS-type phases where the records have been deconvolved from the radial component waveform [33, 34]. With this choice of unperturbed wave spectrum, we compute the sensitivity kernels with different background anisotropic model. The kernels are computed on a fine grid ((5 km)³ box) for two different cases: (a) variation of the dip of the symmetry axis, and (b) variation of the azimuth of the symmetry axis. The incoming

polarization direction is kept constant in both of these cases. The width of the maximum sensitivity zone is calculated as $\approx \sqrt{\lambda L}$, where λ and L are the characteristic wavelength and the propagation distance respectively. In our case, with the characteristic period of 10 s and approximate propagation distance of 400 km, we find the width of the maximum sensitivity zone to be ≈ 300 km. However, the nature of the sensitivity varies with each cases discussed below.

We compute a series of sensitivity kernels for different values of the azimuth of the symmetry axis (ϕ) while the dip ($\theta = 0^\circ$) and the incoming polarization direction ($\alpha = 0^\circ$) are kept fixed. The anisotropic perturbation is homogeneous in nature. The spatial distribution of the sensitivity is not substantially modified with the variation of azimuth (ϕ) of horizontal symmetry axis. Figure 1.2(B) and (C) show near symmetric sensitivity kernels corresponding to $\phi = 30^\circ$, and $\phi = 60^\circ$, respectively. However, the end member cases of $\phi = 0^\circ$ (polarization is parallel to the symmetry axis) and $\phi = 90^\circ$ (polarization is perpendicular to the symmetry axis) are quite interesting. While the latter corresponds to a trivial sensitivity kernel (Figure 1.2(D)) leading to a zero splitting as expected, the former still has a non-trivial but perfectly antisymmetric nature (Figure 1.2(A)), leading to a zero splitting intensity while integrated.

Figure (1.3) shows the variation of the sensitivity kernel with different values of the dip of the symmetry axis while the incoming polarization direction ($\alpha = 0^\circ$) and the azimuth of the symmetry ($\phi = 45^\circ$) are fixed. The presence of the inclined symmetry axis causes an uneven spatial distribution of the sensitivity, as expected (Fig. 1.3(B) and (C)). The striking property of the sensitivity kernel is the non-triviality even when the propagation direction is parallel to the symmetry axis (Fig. 1.3(D)). Such a physical scenario should lead to a zero splitting intensity and the perfect antisymmetry of the sensitivity kernel (Fig. 1.3(D)) confirms this well known result [13, 29, 33]. This non-trivial antisymmetry of the sensitivity kernel is a remarkable property and it may help us to understand the underlying physics in more details.

Surface wave sensitivity kernel is extremely sensitive to the frequency. Naturally, at low frequencies the depth of sensitivity is higher compared to high frequency. We present the sensitivity kernels corresponding to the fundamental mode as well as the first overtone. Fig. 3.1 and Fig. 3.2 show several examples of sensitivity kernels at different frequencies corresponding to fundamental mode and first overtone. The overlap of several maximum amplitudes is expected to produce a higher resolution. Note the the maximum amplitudes are primarily restricted to the shallow part of the model. This leads to the expectation that we might be able to resolve the shallow part better. Note that even though we show kernels for higher overtones, in the actual inversion strategy, we only use

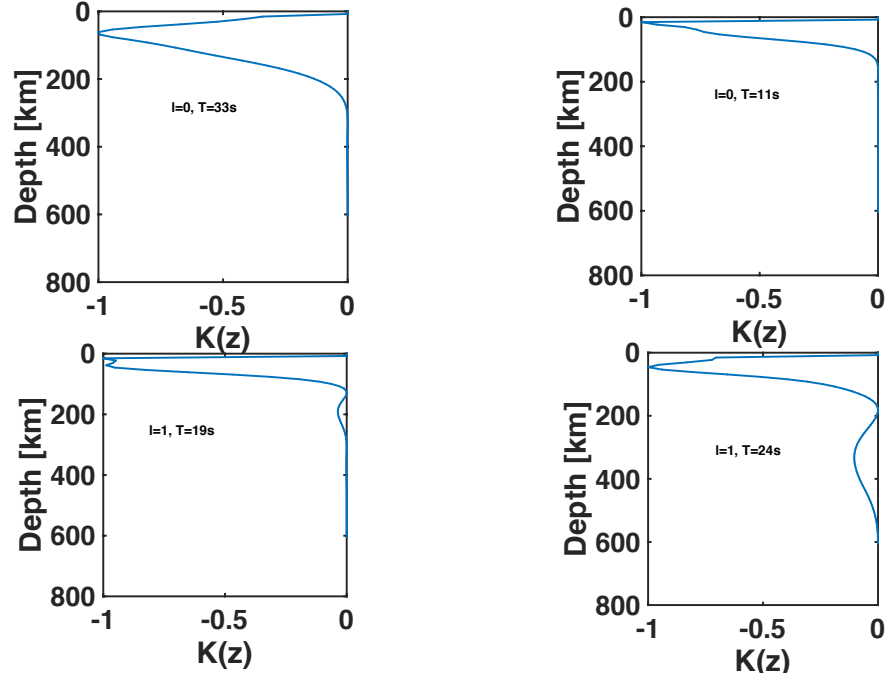


Figure 3.1: Surface wave sensitivity kernels for different four frequencies and for $l = 0$ and $l = 1$. The scale is normalized for visualization.

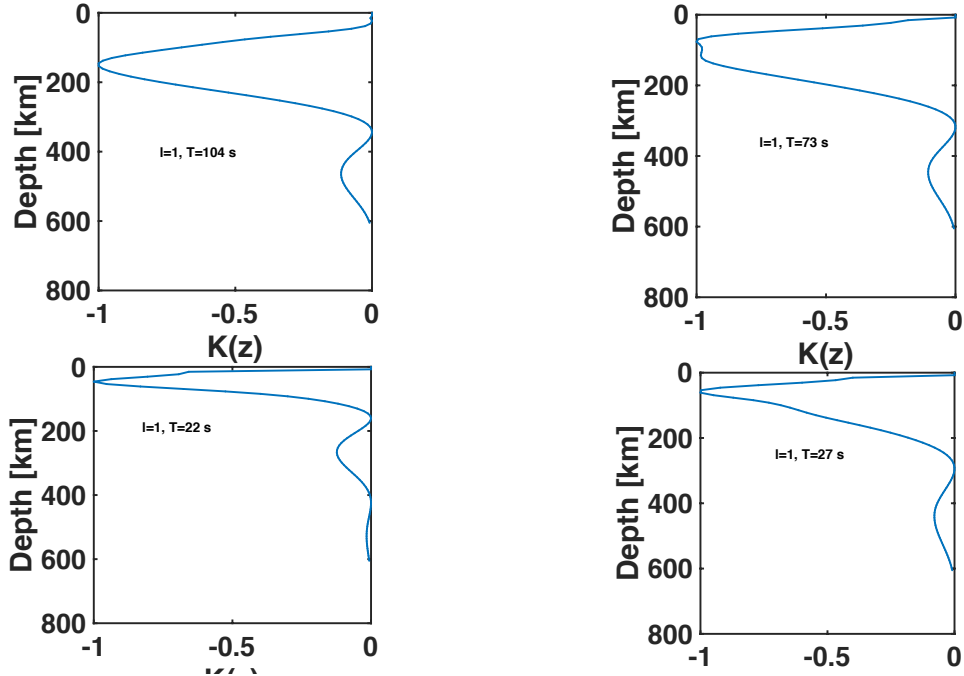


Figure 3.2: Surface wave sensitivity kernels corresponding to four different frequencies and for $l = 1, 2$. The scale is normalized for visualization.

the fundamental mode since the data we use corresponds to the fundamental mode only. Since the figures are self explanatory, we do not spend much time describing the kernel and these are much simpler in a sense due to their one dimensional nature.

3.2.2 Inversion Strategy

Our inversion strategy invokes a model space search technique based on a Markov chain Monte Carlo sampling algorithm. It is an efficient model space search technique based on Bayes' theorem and Markov process. Bayes' theorem describes how to update the probabilities of hypotheses given the evidence, and follows simply from the axioms of conditional probability. Let us assume that we have a data vector d which is given or observed, and several models can reproduce this data within different error limits. Therefore, we may construct a conditional probability: given the data d , what is the probability that a particular model m would satisfy it? The probability of the assumed model m satisfying the given data d is the conditional probability $P(m/d)$. Clearly, if this conditional probability is high enough, we may assert that the model fit is reasonably good. Now, from Bayes' theorem, we may write the following identity

$$P(m/d) = \frac{P(d/m)P(m)}{\sum_m P(d/m)P(m)}, \quad (3.16)$$

where $P(d/m)$ is the probability of achieving the given data with the chosen model parameter m , and $P(m)$ is probability of the model parameter m to exist. The denominator describes the total conditional probability of observing this particular data. Clearly, if we choose the model parameters from the physically plausible set, then $P(m)$ is identically 1, and 0 if it is chosen from outside of the set. Bayes' theorem is applied in conjunction with the Markov process. The details of the sampling algorithm is not presented here since, it has been extensively applied in computational physics [55, 56], computational chemistry [57], and geophysics [58, 59, 60]. The misfit is chosen to be the ordinary Euclidean distance between the observed data and synthetic data produced by the chosen model sets weighted with the squared data error vector:

$$E^2 = \sum_{i=1}^N \frac{(d_i^{SKS} - f_i^{SKS}(\mathbf{m}^{SKS}))^2}{2(\sigma_i^{SKS})^2} + \zeta \sum_{i=1}^{N'} \frac{(d_i^{surface} - f_i^{surface}(\mathbf{m}^{surface}))^2}{2(\sigma_i^{surface})^2} + \xi \quad (3.17)$$

where σ_i , ζ , and ξ are the data error vector, a suitable normalization factor, and the coupling term, respectively. The entities associated to SKS and surface waves are labeled accordingly. Coupling term ξ may be chosen differently. To be consistent with the degree of the free terms, we choose a

quadratic coupling which is just the average of the matrix norms of

$$A = \frac{(d^{SKS} - f^{SKS}(\mathbf{m}^{SKS})) (d^{surface} - f^{surface}(\mathbf{m}^{surface}))^T}{(\sigma^{SKS}) (\sigma^{surface})}, \quad (3.18)$$

$$B = \frac{(d^{surface} - f^{surface}(\mathbf{m}^{surface})) (d^{SKS} - f^{SKS}(\mathbf{m}^{SKS}))^T}{(\sigma^{surface}) (\sigma^{SKS})} \quad (3.19)$$

i.e., $\xi = 0.5\zeta\sqrt{tr(A^T A) + tr(B^T B)}$. The complete error E^2 is written as follows

$$E^2 = \sum_{i=1}^N \frac{(d_i^{SKS} - f_i^{SKS}(\mathbf{m}^{SKS}))^2}{2(\sigma_i^{SKS})^2} + \zeta \sum_{i=1}^{N'} \frac{(d_i^{surface} - f_i^{surface}(\mathbf{m}^{surface}))^2}{2(\sigma_i^{surface})^2} + 0.5\zeta\sqrt{tr(A^T A) + tr(B^T B)} \quad (3.20)$$

with a suitable normalization constant ζ . The coupling is added in order to make the problem really “jojnt” inversion. This could be explained simply with an analogy with classical field theory. Since, E^2 is bounded from below, we can pose the problem of finding the most likely model to be a minimizing problem. The Frechet derivatives of the cost E with respect to the models m^{SKS} and $m^{surface}$ (assuming they lie in suitable function space setting) yield two different equations which would be uncoupled in the absence of the mixing term ξ . In the language of field theory this would simply imply that the optimization space is a direct sum $SKS \oplus surface$. In reality, we would want a constraint from the surface waves and therefore, the space should be a product space $SKS \otimes surface$. The extra term ξ precisely takes care of this issue. The Bayesian likelihood function $P(d/m)$ is just the Gaussian distribution (adopted in accordance with the central limit theorem):

$$P(d/m) = \exp(-E^2). \quad (3.21)$$

Once, we have constructed the associated cost functions, we may simply implement the MCMC technique to yield the desired model parameters. We do not dwell further on the methodology and proceed to the resolution tests. However, the physical setting in which we will perform the checkerboard test is real that is the station and event distributions which will be used the checkerboard test will correspond to that of the High Lava Plains of Oregon.

3.3 Resolution Tests

Before applying the technique to the real data scenario, we want to perform a checkerboard test to ensure the inversion is stable. This is a routine procedure in any inversion problem especially in

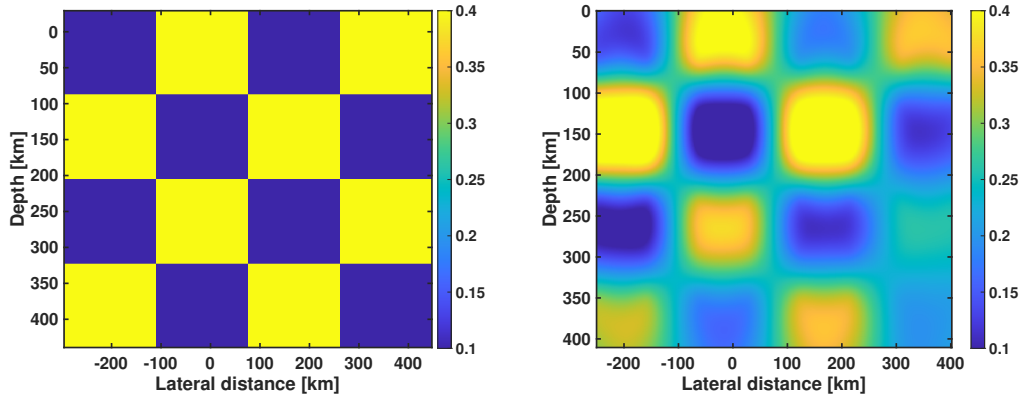


Figure 3.3: Input and output model corresponding to the checkerboard test of coupled SKS-surface wave anisotropy tomography. The coupling strength is chosen to be sufficiently high, $\zeta = 0.5$. Input model is represented in the left panel while the right panel shows the most likely recovered model.

seismic tomography problem. In addition, there are inherent challenges associated with the seismic anisotropy tomography and therefore it is important to evaluate how well our anisotropy models are resolved, and to restrict our interpretations to robust features that are well-resolved and required by the observations. In the absence of the additional constraint from surface waves, it is difficult to resolve the shallow structures i.e., structures that lie within first ~ 50 -100 km. This was explicitly depicted in [107]. However, if one adds surface waves, the resolution quality is expected to enhance, especially in the shallow parts of the model. This is primarily due to the fact that the surface waves have the property of telescopic sampling of different depths at different frequencies. The previous tests demonstrated that the HLP station configuration should allow for the resolution of anisotropic structure on length scales of ~ 50 km in most portions of the model, although there is little or no resolution of structure shallower than ~ 50 km depth. Here we build on the tests presented in [98], using the actual distribution of stations and events represented in our splitting intensity data set (rather than a synthetic, random distribution of earth-quakes over all azimuths). We first perform a checkerboard test, shown in Fig. (3.3) with input anomalies that are ~ 100 km square. Additional histograms showing probability distributions throughout the model space for this test are shown in Fig. (3.5). Because our preferred models themselves tend to exhibit variations on a spatial length scale of about 100 km (Figs.4 and 6), this is a reasonable choice for an input pattern. Because most of the variability in our recovered models is in anisotropy strength, with little variability in the fast axis orientation (and surface wave data is insensitive to variation of ϕ), in this test we hold ϕ fixed at a reasonable average value of 80° . We calculate synthetic splitting intensity values for

the input pattern and then use these values in the inversion scheme. The results of this test reveal several interesting properties of our inversion scheme. Remarkably, if we focus on the checkerboard test where contribution of surface wave is maximum ($\zeta = 0.5$, the one presented in the figure), it appears that the shallow part of the model is best resolved along with the deeper part. Therefore, the resolution depicted by the checkerboard test is better compared to the resolution produced by SKS contribution alone. This is natural to expect since, the surface wave observations are densely distributed on the left half of the line where the resolution is best. The resolution of the right part is much improved compared to pure SKS generated resolution. Of course, we remind ourselves that the resolution generated by surface waves is of higher quality than pure SKS. This is mostly due to the fact that the surface wave kernel exhibit extremely strong sensitivity at the shallow portion of the model, which is even higher than the sensitivity that SKS wave exhibit at the deeper portion. The deeper parts are resolved solely by means of the SKS waves since surface waves do not have sensitivity below ~ 200 km. Fig. 3.5 presents the posterior distributions at different cell blocks of the checkerboard. Now, we compare the checkerboard test corresponding to coupled SKS-surface wave inversion to that of pure SKS waves. Striking difference is observed at the shallow portion (i.e. for depths below 50km). While SKS provided no resolution whatsoever, coupling to the surface waves does provide a fair constraint for the shallow portions (at least the top left portion). Figure (3.4) demonstrates the checkerboard test where coupling is set to zero i.e., only SKS waves contribute.

In addition to the checkerboard test for the strength parameter, it is important to perform a checkerboard test for the orientation of the fast axis. This is simply because we want to confirm our claim of improved resolution while coupling to surface wave observations. As we have observed the resolution towards the top left portion of the model is maximum while that of the top right portion is slightly less. This is indeed confirmed by the checkerboard test for the orientation since the distribution is bimodal with the difference of heights being small. This is expected since the shallow part of the model (≤ 50 km) is dominantly constrained by the surface wave measurements. Therefore, in order to obtain a reasonably high resolution, one would require large number of surface wave measurements (density should be high). The lack of substantial number of surface wave measurements for the right part of the model indicates a lack of resolution at the shallow portion (less than or equal to 50 km). The resolution of the deeper portion remains strong since that is primarily governed by the SKS waves. Figure (3.6) demonstrates the result of the checkerboard test for the orientation parameter while figure (3.8) does so for the posterior distribution.

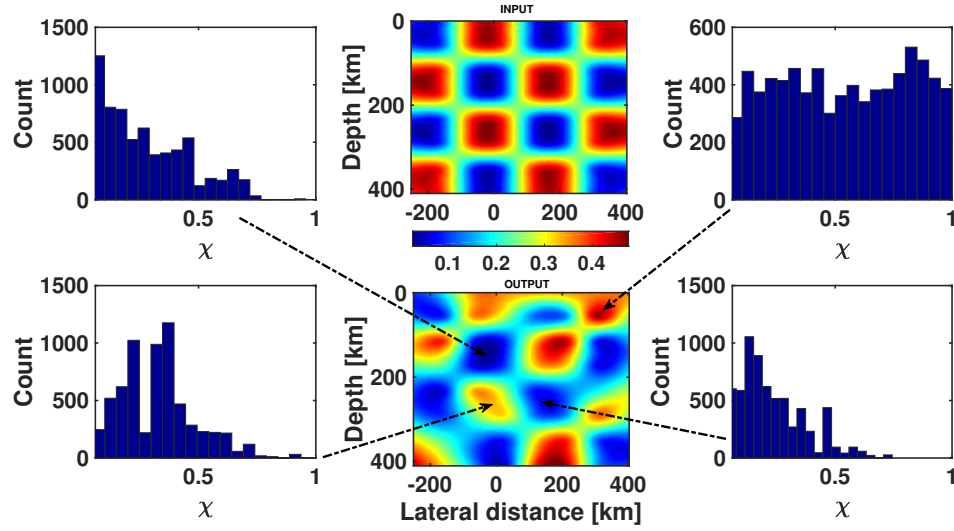


Figure 3.4: Results of checkerboard resolution test with only SKS waves. Input model is shown in the top central panel, while the bottom central panel shows the most likely recovered model, and the histograms show posterior distributions of the recovered strength of anisotropy at selected points in the model. The input pattern corresponds to a 100×100 km² checkerboard of alternating strong and weak anisotropy in a model domain of 600×400 km² with 50 km grid spacing. We only solve for strength χ ; the other parameters are held fixed with $\theta = 0$ and $\phi = 80^\circ$. This test demonstrates that the input is well resolved in portions of the model domain where the sensitivity kernels overlap. As expected, the shallow part of the model lacks such overlap and remains unresolved. Note that here we show values of the strength parameter χ , which takes values between 0 and 1, instead of % anisotropy.

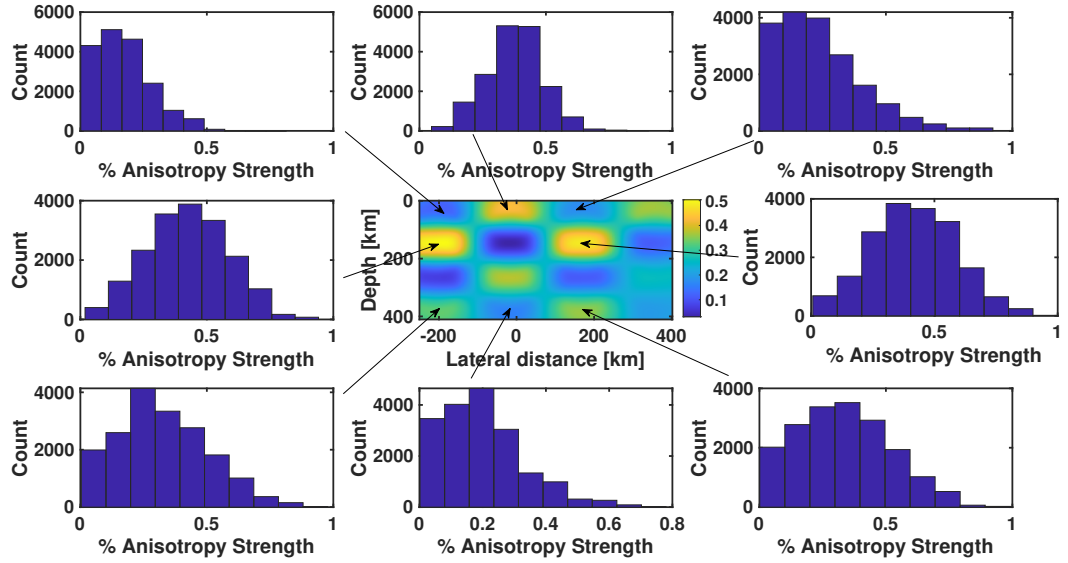


Figure 3.5: Posterior distribution corresponding to the coupled SKS-surface wave checkerboard test.

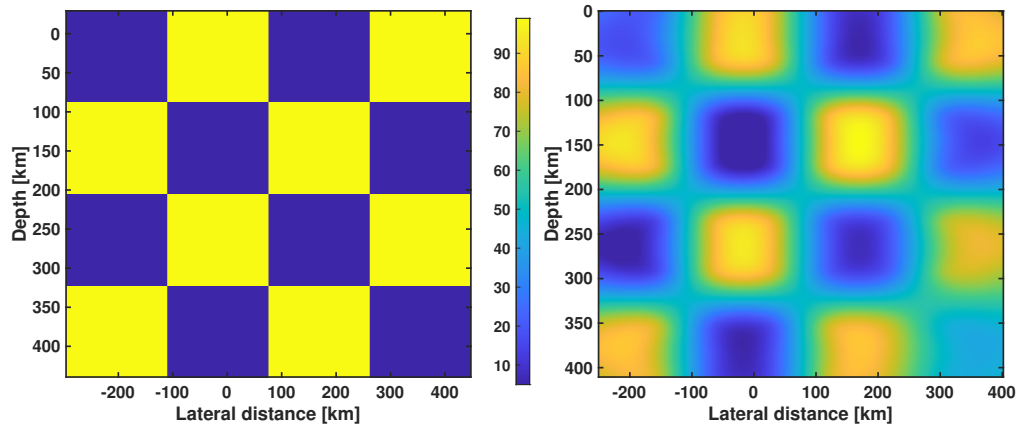


Figure 3.6: Input and output model corresponding to the orientation parameter checkerboard test of coupled SKS-surface wave anisotropy tomography. The coupling strength is chosen to be sufficiently high, $\zeta = 0.5$. Input model is represented in the left panel while the right panel shows the most likely recovered model.

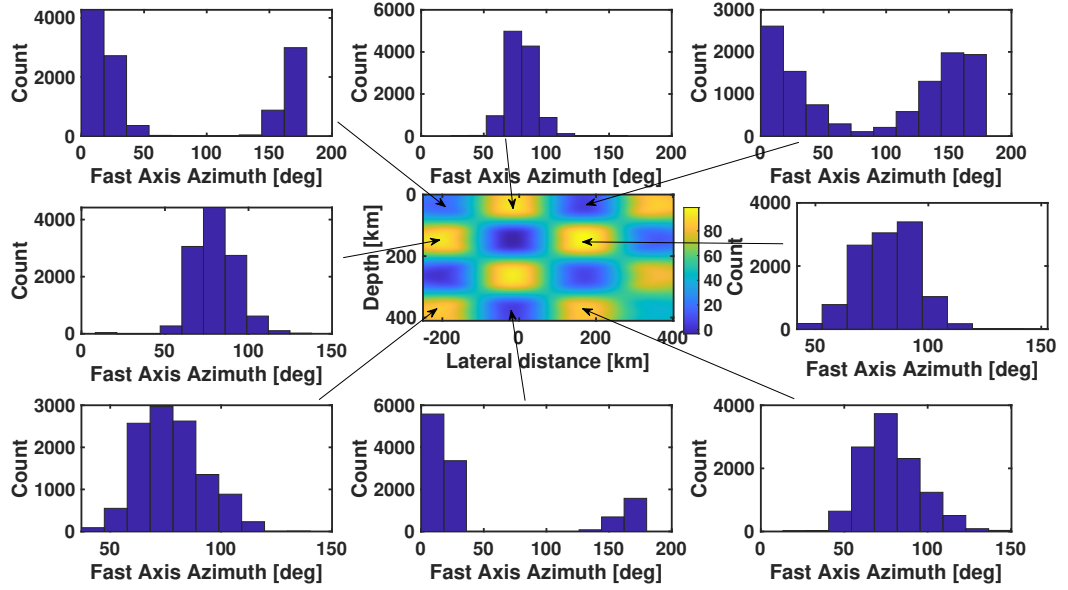


Figure 3.7: The posterior distribution of checkerboard test corresponding to the orientation parameter in the coupled SKS-surface wave inversion.

3.4 Application to real data

Once we have a strategy for a self consistent inversion, we proceed to apply the method to the real data set corresponding to HLP of Oregon. We do not present the synthetic tests here since those are similar in spirit of the ones performed in the previous chapters. Instead we will validate our model by means of more robust checkerboard test done in the previous section. After completion of the synthetic tests, we move towards applying the method to a real data set. Since in [98], we have exclusively studied the SKS splitting intensity tomography using the High Lava Plain data set, it is natural to utilize the surface wave constraints to this setting. One would certainly hope that the resolution in the shallower parts would be improved through the surface wave constraints. The High Lava Plains project and the EarthScope USArray Trans-portable Array (TA) provide a wealth of broadband seismic data in and around the HLP region. Between the HLP and TA deployments, there are 137 densely distributed broadband stations in the region, operated between 2006 and 2009. The HLP station configuration consists of two dense lines surrounded by a ‘cloud’ of stations (Fig.3). The first line spans from southwest Idaho to northwest Oregon, ending just to the west of the Cascades Range. This line follows the time-progressive track in rhyolitic volcanism (e.g. Ford

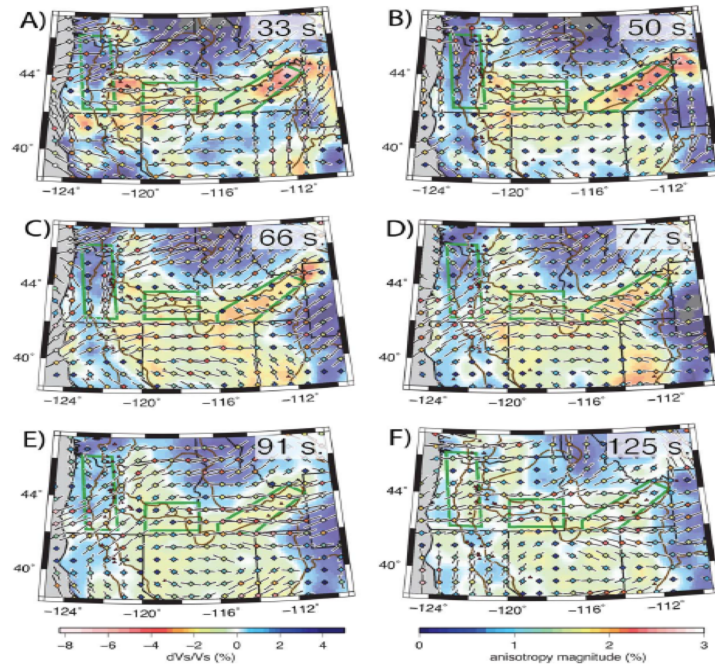


Figure 3.8: The inversion result of Wagner and Long (2013) which is used as the input data for our study. This precisely provides the information about the anisotropic contribution to the phase velocity perturbation of the Rayleigh waves.

et al., 2013). The second dense line is aligned almost N-S from Basin and Range extension (south) through HLP volcanism to the accreted terranes of Oregon’s Blue Mountains (Fig.3a). The average spacing between each station along the dense lines is approximately 10-15km, appropriate for our analysis. In this analysis, we will only use one dense line: the one that is aligned in the NW-SE direction. Similar to the pure SKS splitting intensity tomography, we applied a bandpass filter to retain energy at periods between 8 and 25 s, and visually experimented the horizontal components of the SKS waveform to retain waveforms with a high signal to noise ratio ($\text{SNR} > 2.5$). After the quality checks, we obtained 804 robust SKS splitting intensity measurements which were found to be of exceptionally high quality. Since the station and even distributions were shown in chapter two, we avoid presenting the same data. For the data corresponding to surface waves (quasi-Rayleigh wave to be precise), we used the result of Wagner and Long (2013) (see the figure 4 of their paper) for 8 distinct time periods: 33, 50, 61, 77, 58, 66, 77, 81, 100, and 125s. This is done so in order to extract the anisotropic contribution to the perturbations of the phase velocity. Using the SKS splitting data and the surface wave data (strength and the direction of anisotropy obtained by Wagner and Long (2013)), we perform the inversion for a range of suitably defined non-dimensional coupling parameter ζ . Here $\zeta = 0$ represents pure SKS part only. Therefore, for $\zeta = 0$, we retrieve the result of 2nd chapter. Notice that here we are primarily interested in the strength parameter since the orientation of the symmetry axis for horizontal scenario does not really affect the surface wave. In the next section, we describe our inversion results.

3.5 Results and discussions

Here we present the results from our 2-D inversions along the NW-SE oriented dense line of the HLP experiment (Fig.3.9). A corresponding stick diagram is presented in figure (3.12). Given the computational limitations of a model space search approach, a 2-D inversion approach is tractable (Mondal and Long, 2019). In our full joint inversion, even though we let the azimuth of the symmetry axis ϕ vary, it does not produce a substantial different inversion result since its variation is only sensed by SKS waves. However, there seems to be significant change in the strength of anisotropy in the upper portion of the model i.e., top 50-100 km. As we mentioned previously, SKS waves alone can not resolve the top 50-100 km of the model and this characteristic was rigorously proven by means of a series of resolution tests in chapter 1 and 2. Notice a few new features: the left half of the shallow portion (tp $\sim$ 50-100 km) seems to have been well resolved when the surface wave contribution is

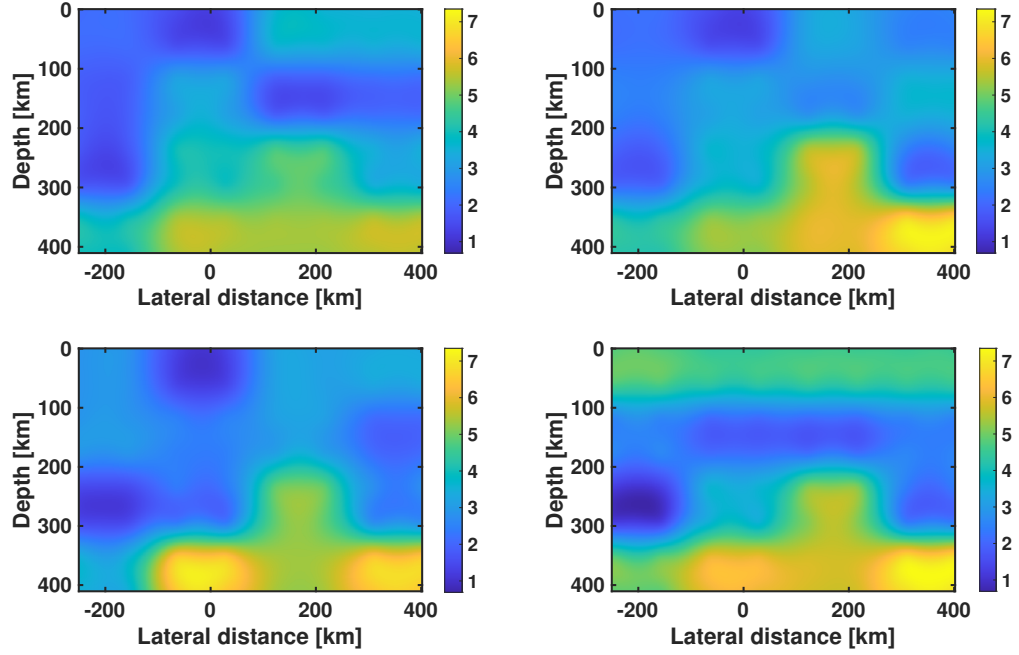


Figure 3.9: Most likely model for the NW-SE line (see chapter 2). The spatial domain vertical section is discretized by 50km50km grids. The image is obtained after smoothing the original 5050km2discretized model for visual presentation. The exact value of the coupling parameter is not important only the fact that a large ζ corresponds to higher surface waves contribution while $\zeta \equiv 0$ corresponds to no surface wave contribution. This is so because, the value of ζ can always be renormalized. From left top to bottom right in a clockwise manner: (A) $\zeta = 0.5$, (B) $\zeta = 0.1$ (C) $\zeta = 0.01$, (D) $\zeta = 0$. ζ serves as a normalization so that the relative magnitudes of the two error are consistent.

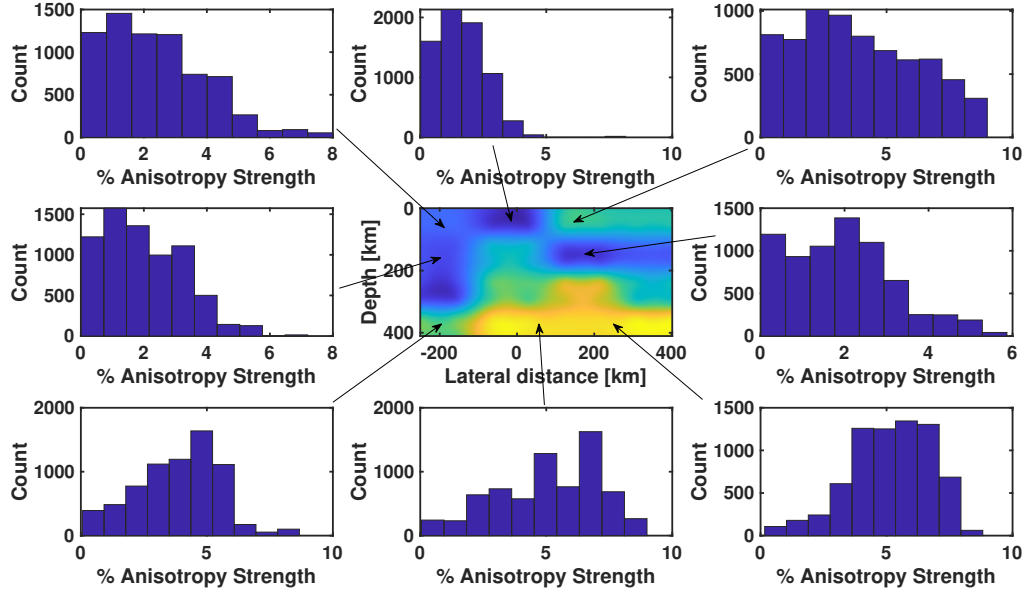


Figure 3.10: Posterior distributions of the strength of the anisotropy corresponding to the most likely recovered model for anisotropy strength along the NW-SE line, shown in Fig. 3.9. These demonstrate that the left half of shallow part of the model ($<50\text{km}$) is fairly well constrained while the right half is still poorly constrained, whereas the deeper parts, with considerable overlapping of the SKS sensitivity kernels, yield a fairly good resolution. This is so primarily because the density of measurements for surface waves is higher in the left portion.

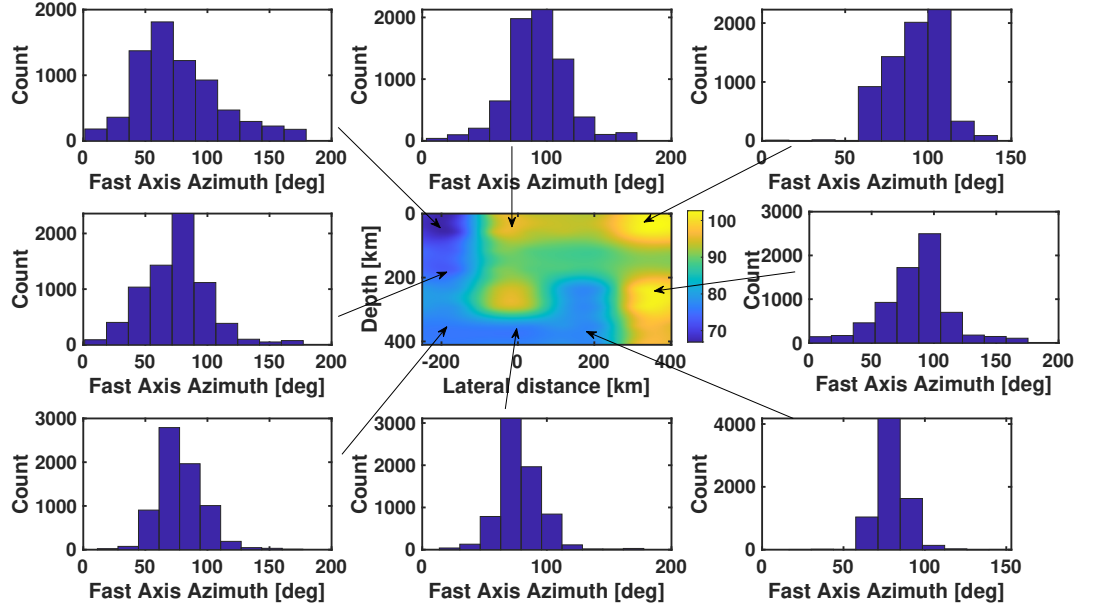


Figure 3.11: Posterior distributions of the fast axis orientation corresponding to the most likely recovered model for anisotropy strength along the NW-SE line.

strong (i.e., higher ζ value e.g., 0.5) and the magnitude of the anisotropy is low in this portion. Notice a striking difference with the case where only SKS wave is used (the last panel (D) where $\zeta = 0$). The top part in the later was completely unconstrained while the middle portion between 100 and 200 km contained low value of anisotropy strength ($\sim 1 - 2\%$). In the current situation, the strength at in this 100-200 km depth range (left half of the model) is slightly increased while its top portion assumes really low anisotropy ($\sim 1 - 2\%$). The robustness of this weak anisotropy at the crustal level is supported by the posterior distribution presented in Fig. 3.10. Once we start decreasing the surface wave strength, the model configuration starts converging (roughly) to the pure SKS case. The Deeper part of the model exhibits strong anisotropy ($\sim 6\%$) as expected. This is not surprising since at depths, the overlap of SKS kernels dominate the surface wave contribution. This magnitude is comparable to the pure SKS study of Mondal and Long (2020).

As discussed in the introduction, it is extremely important to constrain the upper mantle and crustal anisotropy. This primarily due to fact that anisotropy provides stronger constraints to understanding several important geodynamic processes which are intrinsically anisotropic in nature. There have been several studies addressing the anisotropic structures of the upper mantle. We want to compare

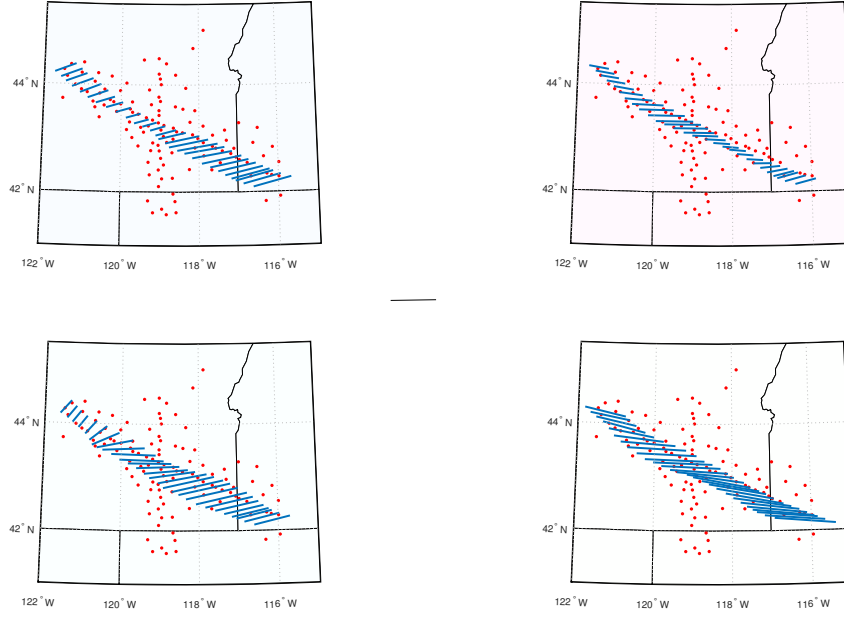


Figure 3.12: Diagram of the anisotropy strength and fast axis orientation (from our most likely models) in map view along the one dense lines at different depths. The length of the bar is scaled to anisotropy strength, while the orientation of each bar shows the local orientation of fast axis. (A) At depth 50 km: the models exhibit generally relatively weak anisotropy (less than 2%) beneath most of the study region, (B) At depth 150 km: models exhibit relatively weak anisotropy $\sim 2\%$, (C) At depth 250 km: models exhibit stronger anisotropy with maximum anisotropy reaching up to 5%. The fast axis orientation along the NW-SE array varies from 70° in the NW to $\sim 90^\circ$ in the SE (D) At depth 350 km: Both models exhibit strong anisotropy throughout, up to 7%. The fast axis orientation along the NW-SE line changes from $\sim 80^\circ$ in the NW to $\sim 100^\circ$ in the SE. The black line in the middle corresponds to 3% anisotropy.

our model with that of Yuan and Romanowicz (2010). While both of these model suggests weaker anisotropy in the shallow portion, the strength of anisotropy at the deeper level vary substantially between our model and theirs. Their model features relatively strong anisotropy in the deepest upper mantle, in that the anisotropy strength here is comparable to the anisotropy strength at shallower depths, but it is considerably weaker than in our preferred model ($\sim 2\text{-}3\%$ vs. up to $\sim 6\text{-}8\%$). This difference may reflect the fact that the model of Yuan and Romanowicz (2010) is based on a regularized inversion that may underestimate amplitudes, or it may be a consequence of their coarser lateral resolution. In addition, we also have the model provided by Wagner and Long (2013) to us. They have obtained a surface wave tomography model by relating perturbations to the phase velocity caused by the underlying anisotropy. The surface wave phase velocities, which have little or no sensitivity at depths greater than ~ 250 km, showed generally uniform fast orientations beneath the HLP with modest anisotropy strengths (values of up to $\sim 2.5\%$). Wagner and Long (2013) found that the anisotropy strengths suggested by the phase velocity maps were much smaller than needed to explain the SKS splitting delay times if all the anisotropy was distributed in the upper ~ 200 km of the mantle; furthermore, they argued that a strong contribution from crustal anisotropy (shallower than ~ 50 km) would have strongly affected the phase velocities at 33 and 40 s period, which was not observed. In the current context, however, we do observe a strongly weak contribution from the crustal part when the surface wave contribution in the inversion process is maximized. Wagner and Long (2013) left the resolution of this discrepancy to future work, and the model presented here now resolves the apparent paradox. Our preferred model for anisotropy beneath the HLP invokes strong anisotropy in the deep upper mantle, at a depth range that is beyond the sensitivity of the observations (at periods up to 143 s) used in the Wagner and Long (2013) surface wave model. The presence of deep upper mantle anisotropy can explain the apparent discrepancy between surface wave and SKS splitting constraints documented by Wagner and Long (2013).

Chapter 4

An effective theory for elastic medium with complex heterogeneity

4.1 Introduction

Homogenization of a heterogeneous medium is a classical problem in material science. There have been a tremendous amount of both theoretical and experimental studies in this sector [108, 109, 109, 110, 111, 112, 113]. However, its importance in geophysics is no less due to several reasons. One of the main reasons of course lies in seismology. As we have discussed in the previous two chapters, the resolution power of seismic tomography is limited. For example, in the context of the anisotropy tomography, we have observed that the lower bound of the resolvable length scale is of the order of 50 km. However, the size of heterogeneities present in the mantle ranges from tiny (of the order of few mm) to very large (of the order of km). Therefore, the observations and constructed models are typically characterized at a large scale primarily due to the fact that the spectra of seismic waves is limited. Obviously, there is *no* seismic waves (or associated techniques) throughout the multiverse that can resolve structures at the level of even few meters length scale. In other words, we clearly have an effective description of the system that is the low wavelength structures are integrated out in a suitable sense. When propagating through small heterogeneities, seismic waves naturally average the elastic properties of the medium. However, the calculations we have performed in the tomography section or the ones that exist in the literature do not really take into account this smoothing out effect which can lead to a different result. This is due to the fact that the tomography problem is already at the level of perturbation theory and if the effect of the homogenization (or this averaging process) is of the same order, then we clearly have a serious problem which should be addressed carefully. In addition, physics of wave propagation in a complex heterogeneous medium is fascinating by itself. The situation becomes even more interesting if one introduces randomness (spatial). In reality, Earth's subsurface is expected to possess a certain degree of randomness as well as correlated heterogeneities. Therefore, it is clear that in the current seismological context, it is quite important

to develop an effective media theory. Of course, there are plenty of already existing studies in the literature. It will be injustice to just name a few. Therefore, we give at least one example from each field where the homogenization problem is studied in detail. One of the most well known study is the study by Hashin and Strikman [108]. Their work was concerned with the determination of elastic moduli of heterogeneous medium by means of an energy technique. It was assumed that the material may be adequately described by an elastic, homogeneous, and isotropic matrix the moduli of which are known, in which particles of another elastic homogeneous and isotropic material are embedded, the moduli of which are also given. One of the important features of this study is that it is based on the static deformation theory and used the Eshelby's [114] calculation of elastic field in an ellipsoidal inclusion. In other words, there are no wave components to it. On the other hand, for a 'wave' propagation, the effective moduli is expected to be frequency dependent. Later [111] derived an effective stress-strain relationship for a representative volume of granular solids based upon the micro-mechanical methodology and a 'self-consistent' averaging technique. In addition to the analytical studies, there have also been a few numerical ones [115] to address this particular problem. Apart from these 'physical' studies, there have been numerous mathematical studies where the homogenization problem is studied from a PDE perspective. A solid foundation in the realm of analysis of PDE was developed by Evans through 'weak compactness' method [116]. Since then many similar studies have been performed. However, these mathematical studies are far beyond the realm of geophysics and we want to construct a theory which is easily tractable. There are examples of homogenization study in the context of geophysics. We note one particular study by [117] which is very relevant in the seismological context. Motivated by the recent progress in the effective field theory (EFT), renormalization group flow, and conformal field theory, we develop a self consistent theory which addresses several aspects of elastic (elastodynamic to be precise) deformations in a complex heterogeneous medium. Our results do not depend on the specified external force (stress) present as it should not. In addition, we can address the dynamical problem meaning including the frequency effect in a fairly general heterogeneous medium. Since we are using purely analytical technique, we are restricted to consider either a periodic or an infinite medium and inclusions of specific (simple) shape. This is due to the fact that we need the Green's function associated with the wave operator which can only be calculated analytically for the ideal setting described above. If one uses computational resources, then our method generalizes to medium with any shape. In addition, since the answer depends on the Fourier transform of the heterogeneity present, only simple shapes are allowed due to analytical reason. Once again, if computational resources are utilized, any general shape of inclusions may be treated as long as they have bounded Fourier transforms. The systematic

description of the methodology and result is described in the subsequent sections. First we consider a random heterogeneity (can be drawn from a ‘Gaussian’ distribution) and then specialize to ordinary heterogeneity that is without any randomness. Our result confirms the reduction of elastic constants in a purely random medium. The mathematical techniques involved here have similarities with the BRST method of quantization of the gauge theory [118, 119].

4.2 The statement of the problem

Let ρ and C^{IJKL} be the density and elasticity tensor of a physical medium which admit the following decompositions

$$\rho = \rho_0 + \zeta(x) \quad (4.1)$$

$$C^{IJKL} = C_0^{IJKL} + \eta^{IJKL}(x), \quad (4.2)$$

where (ρ_0, C_0^{IJKL}) is the homogeneous part and $(\zeta(x), \eta^{IJKL}(x))$ is the heterogeneity with a probability function $P[\eta]$. The type of randomness associated with these heterogeneous parts depends on the chosen correlation function defined as follows

$$\langle \eta(x)\eta(x') \rangle := \int [d\eta] \eta(x)\eta(x') P[\eta], \quad (4.3)$$

where $[d\eta]$ denotes the functional integration measure. The displacement field associated with the elastic waves in this medium at a frequency ω is governed by the following hyperbolic PDE

$$\rho \partial_t^2 \varphi^I - \partial_J (C^{IJKL} \partial_K \varphi_L) = 0 \quad (4.4)$$

subjected to suitable initial and boundary conditions. For our purpose we will assume that our 3 dimensional medium is either infinite or periodic 3–torus so that the boundary terms drop out. As mentioned earlier, this is simply done in order to facilitate the analytical calculations. The goal is to find an effective homogeneous elastic tensor C_{hom}^{IJKL} and density ρ_{hom} which satisfy (for frequency range of interest)

$$\rho_{hom} \partial_t^2 \varphi_{avg}^I - C_{hom}^{IJKL} \partial_J \partial_K \varphi_{avg}^L = 0, \quad (4.5)$$

where φ_{avg}^I is a suitably averaged displacement field. This is equivalent to obtaining a homogenized version of the original wave equation (4.4). We will for the moment focus on the elastic part. The major difficulty of this problem is as follows. Due to (4.3), original C^{IJKL} will never be anywhere differentiable. Therefore (4.4) is severely ill defined in the strong sense. Study of Evans [116] considers a weak formulation and through a series of estimate obtains a compactness result which yields a homogenization (in the limiting sense). Alternatively, one may invoke to invoke Ito/Stratonovich technique of stochastic integration to make sense of the equation (4.4). However such techniques are not straightforward to apply to this tensorial wave equation. We will therefore utilize a more straightforward QFT/SFT based technique. This is essentially a problem of effective field theory. This technique is in a sense equivalent to the stochastic integration technique since both are found to generate additional nonlinear interaction terms. We systematically apply the technique of effective field theory and renormalization group method to yield an effective description of the elastic coefficients as a function of a length scale involved.

4.3 Method

A random medium is a mathematical model of a complex medium. It consists of a family of media together with a probability distribution over the members of the family. Because of the random property of the medium, the associated elastic wave field also becomes random and therefore only certain ‘ensemble’ average is physically meaningful. In addition, since there are two distinct length scales involved, the physics is certainly scale dependent. In order to obtain a homogenized version of the wave equation, one needs to derive an effective field theory where the small scale parts of the wave field are integrated out. This essentially translates to the following. The length scale of the heterogeneity ranges from very small (compared to the typical elastic waves) to large (comparable to the wavelength of the typical waves). Therefore, there are complicated processes that occur while elastic waves propagate through a random medium (or simply elastostatic deformation for that matter). These include forward scattering, back scattering, multiple scattering and multi-phase generation etc. However at the level of observation, the short distance phenomena occurring due to this complex interaction (which would otherwise be absent if random heterogeneity were absent) is suppressed and only an effective physics is realized due to a finite frequency band of the observed seismic waves. The question is how to integrate out (average out or smooth out) the small scale degrees of freedom (fluctuating degrees of freedom so to speak) which respects the theory?

As mentioned in the previous paragraph, we need a ‘partition’ functional or generating functional

with respect to which the ensemble average (of entities of interest) is to be performed. Therefore, our first step is to construct such a generating functional from the solution of the wave equation (4.4). This generating functional may be expressed as an exponential of a suitable ‘action’ functional of the ‘physical’ displacement field, adjoint displacement field and two non-physical ‘Grassman’ fields. Next step would be to integrate out the small wavelength modes (the highly fluctuating modes) and obtain an ‘action’ functional of the large scale effective theory. This is essentially an averaging procedure to smooth out the short range fluctuations. The wave-number scales which will be integrated out is essentially $[\Lambda, \Lambda_0]$. We will perform all the calculations in terms of Λ and Λ_0 . For computation of the practical entities, one may simply set the values of Λ and Λ_0 according to the relevance (depending on the system property). This effective theory is essentially described by

$$\exp(iS_{eff}) = \int [d\varphi_{k>\Lambda}, d\tilde{\varphi}_{k>\Lambda}, d\bar{\psi}_{k>\Lambda}, d\psi_{k>\Lambda}] \exp(iS[\varphi, \tilde{\varphi}, \bar{\psi}, \psi]), \quad (4.6)$$

where S will be a suitably defined action functional and S_{eff} is the effective description. The equation of motion corresponding to this effective action S_{eff} will then coincide with the homogenized wave equation (4.5) by definition. One may therefore read off the homogenized coefficients directly. In addition, we derive a renormalized flow equation for the effective constants as a function of logarithmic length scale parameter l defined by $\Lambda = \Lambda_0 e^{-l}$. Here $2\pi/\Lambda_0$ may be the lower bound of the length scale of the heterogeneity and $[\frac{2\pi}{\Lambda_0}, \frac{2\pi}{\Lambda_0 e^{-l}}]$ may be the length scale which we integrate out. Notice that l is variable here and therefore one may set l according to the characteristic scales of the observable etc. For example $\Lambda_0 e^{-l}$ could very well be the typical wavelength of observed waves of interest. This ‘integrating out of large wave-number degrees of freedom’ procedure cannot be performed exactly and therefore we turn to graph expansion and restrict ourselves up to the level of 1 loop. However, we note that the series may be completed with a little more effort. 1-loop approximation is sufficient at the level of perturbations and since ‘loop’ diagrams essentially exhibit the ‘special’ characteristics of the system (due to randomness). As we shall see later in the chapter, removal of random characteristics from the perturbations yields only ‘tree’ level contributions. The plan may be summarized as follows

1. We derive a generating functional associated with this particular system
2. We separate and integrate out large wave-number/ small scale degrees of freedom to yield an effective description of the physical system. As a result, we obtain effective elastic constants.
3. Finally, we derive a renormalization group flow equations for the modified/effective elastic constants. This allows us to obtain the effective constants at all relevant length scales. We then study a

Table 4.1: Notation

Variables	Notation
Displacement vector field	φ^I
Adjoint displacement field	$\tilde{\varphi}^I$
Grassman field	ψ^I
Adjoint Grassman field	$\bar{\psi}^I$
Auxiliary fields	$J^I, \tilde{J}^I, \bar{J}^I, \mathcal{J}^I$
Partition Functional	Z
Generating functional	W
Step function	Θ
Small scale/high wave-number fields	$\varphi_>, \tilde{\varphi}_>, \bar{\psi}_>, \psi_>$
Large scale fields	$\varphi_<, \tilde{\varphi}_<, \bar{\psi}_<, \psi_<$
Action	$S[\varphi, \tilde{\varphi}, \bar{\psi}, \psi]$
Spatial coordinate	x
spatial wave number	k
Frequency	ω
Space of smooth compactly supported functions	$\mathcal{S}$
Dimension of space	N
Ordinary integral measure on $\mathbb{R}^N$	$d^N x, d^N k$
Functional integral measure	$[d\varphi], [d\tilde{\varphi}], [d\psi], [d\bar{\psi}]$
Elasticity tensor	C^{IJKL}
Density	ρ
Lamé constants	λ, μ
Random heterogeneity	η, ζ
Upper bound of the wave number	Λ_0

toy model where a possible critical phenomena arises which may have applications to the Earthquake theory.

4.4 Functional representation of 4.4: derivation of a generating functional

For practical calculations, we will avoid all the mathematical rigor. We will explain the physics as much as possible. For fully mathematically rigorous treatment of the mathematical structure arising in this study, [120, 121] are sufficient. For RG group flow [122, 123] are enough to serve the purpose (we mostly follow these). We will throughout work on $\mathbb{R}^N, N \in \mathbb{Z}^{\geq 0}$. The metric is simply denoted by $g^{IJ} \partial_I \otimes \partial_J$ which is simply identity matrix. To be covariantly consistent we denote only g_J^I by δ_J^I . The derivations will be done in a formal level. The Dirac's distribution is formally defined as

$$\delta(x) := \frac{1}{(2\pi)^N} \int e^{i\mathbf{k} \cdot x} d\mathbf{k}, \quad (4.7)$$

while the Fourier transform is denied as

$$\mathcal{F}(f)(\mathbf{k}) := \int f(x) e^{i\mathbf{k} \cdot x} dx \quad (4.8)$$

Due to its distribution property, one may naturally extend this definition of delta to the infinite dimensional setting. Note that the following elementary identity holds in finite dimension

$$\int \delta(x) d^N x = 1. \quad (4.9)$$

Now let us say that $f \in \mathcal{S}$, where $\mathcal{S}$ is a space of function equipped with suitable norm topology (for our purpose we will always assume that all the functions involved are C_0^∞ i.e., smooth with compact support). On the infinite dimensional setting one may formally define the Dirac's distribution as

$$\delta(f) := \int_{\mathcal{S}} e^{i\tilde{f}f} [d\tilde{f}] \quad (4.10)$$

where $\tilde{f} \in \mathcal{S}^* = \mathcal{S}$, $[d\tilde{f}]$ is a nonexistent (possibly) Lebesgue measure on $\mathcal{S}$. We call this the functional integration measure. This formally satisfies

$$\int_{\mathcal{S}} \delta(f) [df] = 1. \quad (4.11)$$

Now if $\mathcal{G} : \mathcal{S} \rightarrow \mathcal{S}$, then we have

$$\int_{\mathcal{S}} \delta(\mathcal{G}(f)) [d\mathcal{G}(f)] = \int_{\mathcal{S}} \delta(\mathcal{G}(f)) \frac{\delta \mathcal{G}}{\delta f} [df]. \quad (4.12)$$

We will use this basic facts. Let us now consider the main problem. We first transform the equation (4.4) to the frequency domain i.e., $\varphi \equiv \varphi(\omega, x)$

$$-\rho \omega^2 \varphi^I - \partial_J (C^{IJKL} \partial_K \varphi_L) = 0, \quad (4.13)$$

where ρ and C^{IJKL} are the density and the elasticity tensor of the medium, respectively. This equation is simply linear elliptic and therefore given the coefficients are smooth enough, has unique solution on compact manifold. Under such circumstances, one has unique time evolution given an initial data. Here we will mainly focus on the physics of propagating wave through the random medium i.e., we assume that the wave is already generated we are looking at some intermediate time interval. Now consider the case when we have random heterogeneity in the medium i.e.,

$$C^{IJKL} = C_0^{IJKL} + \eta^{IJKL}(x), \quad \rho = \rho_0 + \zeta(x) \quad (4.14)$$

Now η^{IJKL} is a nowhere differentiable function and therefore the equation (4.13) does not make sense. Therefore, we need to invoke special integration technique such as Ito or Stratonovich integration. However, even such techniques are not applicable in a straightforward sense in the current PDE setting. Therefore we will utilize the functional integration which has a direct physical significance as well. First, we assume that all the coefficients are independent of time (no visco-elasticity here, however may be important in other physically interesting scenarios such as Earthquake problem) and work in the frequency domain

$$-(\rho_0 + \zeta)\omega^2\varphi^I - \partial_J(C^{IJKL}\partial_K\varphi_L) = 0. \quad (4.15)$$

For now we assume that the background is homogeneous and therefore we densitize the e.o.m

$$-(1 + R')\omega^2\varphi^I - \partial_J(C'^{IJKL}\partial_K\varphi_L) = 0. \quad (4.16)$$

For simplicity, we will omit the dash and simply write C^{IJKL} and R for the densitized coefficients. Now in the absence of randomness involved, the solutions trace a curve $\omega \mapsto \varphi(\omega)$ in the space $\mathcal{S}$. However, while $\eta \neq 0$, the solution may theoretically span the whole space $\mathcal{S}$. Naturally there is an ‘entropy’ involved in the process. The closest analogy may be any QFT system. At the classical level the solution of field equation is simply a curve in a suitable function space. However, the quantum fluctuation introduces a type of ‘randomness’ allowing the system to explore a non measure-zero portion of the function space (theoretically all of the space). Similarly, in a statistical system, once away from the zero temperature state, the fluctuation introduces randomness and subsequently an entropy. Introduction to exploring the function space non-trivially is governed by $\hbar$ in QFT while a similar role is played by the temperature T in the statistical system. Despite few fundamental differences in physics (one has a property of interference while other does not), the mathematical foundation is identical (one system is simply related to other by a $\pi/2$ rotation on complex plane in a suitable sense, the so called wick rotation).

Now we systematically construct the partition functional and the generating functional. Let $\mathcal{F}$ be any functional of φ which actually satisfies the equation of motion. The following formal identity is satisfied by any solution of field equations

$$\mathcal{F}(\varphi_{sol}) = \int \delta(\varphi - \varphi_{sol})\mathcal{F}[\varphi][d\varphi] = \int \mathcal{F}[\varphi]\delta(\mathcal{E}(\varphi))[\det(\mathcal{E})][d\varphi] \quad (4.17)$$

where $\mathcal{E}$ is the differential operator

$$\mathcal{E} := -\omega^2 \delta^{IL} - \partial_J (C^{IJKL} \partial_K) \quad (4.18)$$

For each realization of the random perturbation η , we will have a solution φ_{sol} . Therefore, physically relevant entities must be an ‘ensemble’ average. The expectation with respect to this ensemble is given by

$$\langle \mathcal{F}(\varphi_{sol}) \rangle = \frac{\int \int \mathcal{F}[\varphi] \delta(\mathcal{E}(\varphi)) [\det(\mathcal{E})] P[\eta] [d\varphi] [d\eta]}{\int \int \delta(\mathcal{E}(\varphi)) [\det(\mathcal{E})] P[\eta] [d\eta] [d\varphi]} \quad (4.19)$$

Therefore the ‘partition functional’ may be defined as the normalization factor

$$\boxed{Z := \int \int \delta(\mathcal{E}(\varphi)) [\det(\mathcal{E})] P[\eta] [d\eta] [d\varphi]} \quad (4.20)$$

Here we note that the partition functional and generating functional are different. The former one is a formal notion while the later will be useful in practical calculations. Formally, the probability of this elastic system to be in a ‘state’ φ may naturally be written as

$$P[\varphi] := \frac{\int_{\eta} \delta(-\omega^2 \delta^{IL} - \partial_J (C^{IJKL} \partial_K \varphi_L) - R \omega^2 \varphi^I) [\det(\mathcal{E})] P(\eta) [d\eta]}{\int_{\varphi} \int_{\eta} \delta(-\omega^2 \delta^{IL} - \partial_J (C^{IJKL} \partial_K \varphi_L) - R \omega^2 \varphi^I) [\det(\mathcal{E})] P(\eta) [d\eta] [d\varphi]}$$

and naturally one has

$$\int_{\varphi} P[\varphi] [d\varphi] = 1. \quad (4.21)$$

Here the denominator may be treated as a normalization constant and identified with the ‘partition functional’ (or generating functional) of this ‘ensemble’. $P(\eta)$ is the distribution function associated with the random field η . We will assume $P(\eta) \approx e^{-\int \eta^{IJKL}(x) C R_{IJKLMNOP}(x-x') \eta^{MNOP}(x') d^N x d^N x'}$, i.e., the random perturbations η are drawn from a Gaussian or simply these perturbations are Gaussian process. In the definition above, we could keep the term involving the density perturbation R in the integration and carry out the calculations. However, since we are primarily interested in the η part i.e., the elastic part, we will for now drop this term. But, in general, our technique calculates

the effective density as well. The partition function may further be evaluated as follows

$$\begin{aligned}
Z &= \int \delta(-\omega^2 \delta^{IL} - \partial_J (C^{IJKL} \partial_K \varphi_L)) [\det(\mathcal{E})] P(\eta) [d\eta] [d\varphi] \quad (4.22) \\
&= \int \int \int e^{i \int \tilde{\varphi}_I (-\omega^2 \delta^{IL} - \partial_J (C^{IJKL} \partial_K \varphi_L)) d^N x} [\det(\mathcal{E})] P(\eta) [d\eta] [d\tilde{\varphi}] [d\varphi] \\
&= \int \int \int e^{i \int (-\omega^2 \varphi^I \tilde{\varphi}_I + C^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L) d^N x} [\det(\mathcal{E})] P(\eta) [d\eta] [d\varphi] [d\tilde{\varphi}] \\
&= \int \int \int e^{i \int (-\omega^2 \varphi^I \tilde{\varphi}_I + C^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L + \eta^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L) d^N x} [\det(\mathcal{E})] P(\eta) [d\eta] [d\varphi] [d\tilde{\varphi}],
\end{aligned}$$

where we have used (4.10), integration by parts and the fact that $\tilde{\varphi} \in \mathcal{S}$. $\tilde{\varphi}$ may be identified with the so called adjoint wave field. Alternatively one may absorb the boundary terms by adding suitable boundary conditions to the e.o.m which are then introduced in the exponential multiplied by the adjoint field and therefore cancels the extra boundary terms arising from the integration by parts. Technically this amounts to saying that the Schwartz functions are dense in the space of functions whose first derivative is square integrable (since the finite energy assumption demands that the displacement field be in that specific function space). Now how to represent the $[\det(\mathcal{E})]$ as a functional integral? We go back to the gauge theory where Fadeev-Popov ghosts are introduced to express the Jacobian [124]. In gauge theory this is typically done so at the level of gauge fixing which takes care of the redundancy present in the theory. In our context, the idea of gauge may be nonexistent, but, we remind ourselves that we had to introduce the jacobian in order to make sure that we only consider the fields φ which are solutions of the stochastic differential equation instead of the whole space. In a suitable sense, this operation is also equivalent to the removal of redundancy. Therefore, it is not really surprising to see that we need additional ‘non-physical’ ghost fields. Importantly, the Fadeev-Popov ghosts are Grassman fields and therefore satisfies certain anti-commuting property. The algebra of these grassman fields is essentially isomorphic to the exterior algebra of $\mathbb{R}^N$ i.e., $\bigoplus_{k=0}^{\infty} \bigwedge^k (\mathbb{R}^N)$. These do not have spins only the anti-commuting property. For a set of Grassman fields (basically spinors) $\{\psi_i\}_{i=1}^N$, the following are satisfied

$$\{\psi_i, \psi_j\} = 0, \int d\psi_i = 0, \int \psi_i d\psi_i = 1, \{d\psi_i, \psi_j\} = 0, \forall i, j. \quad (4.23)$$

This aforementioned Jacobian determinant appears as a gauge fixing term in quantum gauge theory, which is then replaced by a Grassman integral. This is precisely due to the following facts. Note

the following remarkable property of the Gaussian integration of the spinors

$$\int \exp \left(i \int \bar{\psi}_i(x) \mathcal{M}^{ij}(x, y) \psi_j(y) d^N x d^N y \right) [d\bar{\psi}][d\psi] = \det(\mathcal{M}^{ij}(x, y))$$

contrary to the ‘bosonic’ fields $\varphi, \tilde{\varphi}$ which satisfies (formally)

$$\int \exp \left(i \int \tilde{\varphi}_i(x) \mathcal{M}^{ij}(x, y) \varphi_j(y) d^N x d^N y \right) [d\tilde{\varphi}][d\varphi] \approx \frac{1}{\det(\mathcal{M}^{ij}(x, y))} \quad (4.24)$$

Introducing the spinors, the partition functional reads

$$\begin{aligned} Z = \int \int \int \int e^{i \int (-\omega^2 \varphi^I \tilde{\varphi}_I + C^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L + \eta^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L) d^N x} \\ e^{i \int \bar{\psi}^I (-\omega^2 \delta_{IL} - \partial_J (C_I{}^{JK}{}_{L} \partial_K)) \psi^L d^N x} P(\eta) [d\bar{\psi}][d\psi][d\eta][d\varphi][d\tilde{\varphi}]. \end{aligned} \quad (4.25)$$

Once again we assume for simplicity that the medium is isotropic. Introducing anisotropy does not cause any substantial problem, only long monotonous calculations and can be done so without much trouble. Consider the explicit representation of isotropy

$$C^{IJKL} = \lambda \delta^{IJ} \delta^{KL} + \mu (\delta^{IK} \delta^{JL} + \delta^{IL} \delta^{JK}) \quad (4.26)$$

which may be written as the background and the random perturbations (simplest case when the background is homogeneous)

$$\lambda(x) = \lambda_0 + \sqrt{g_b} \eta_b(x), \quad \mu(x) = \mu_0 + \sqrt{g_r} \eta_r(x). \quad (4.27)$$

Here $\sqrt{g_b}$ and $\sqrt{g_r}$ are the strengths of randomness associated to the two Lamé parameters. One of course can directly work with the bulk ($K := \lambda + \frac{2\mu}{3}$) and the shear modulus. Here we will stick to λ and μ . The perturbed elasticity tensor reads

$$C^{IJKL} = C_0^{IJKL} + \sqrt{g_b} \eta_b(x) \delta^{IJ} \delta^{KL} + \sqrt{g_r} \eta_r(x) (\delta^{IK} \delta^{JL} + \delta^{IL} \delta^{JK}). \quad (4.28)$$

Substituting this expression in the previous integral yields

$$\begin{aligned}
Z = \int \exp \bigg(& i \int (-\omega^2 \varphi^I \tilde{\varphi}_I + C_0^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L - \omega^2 \bar{\psi}^I \psi_I \\
& + C_0^{IJKL} \partial_J \bar{\varphi}_I \partial_K \psi_L + \sqrt{g_b} \eta_b(x) (\partial_I \tilde{\varphi}^I \partial^J \varphi_J + \partial_I \bar{\psi}^I \partial^J \psi_J) \\
& + \sqrt{g_r} \eta_r(x) (\partial_I \tilde{\varphi}^J \partial_J \varphi^I + \partial_I \tilde{\varphi}^J \partial^I \varphi_J + \partial_I \bar{\psi}^J \partial_J \psi^I + \partial_I \bar{\psi}^J \partial^I \psi_J) \bigg) d^N x \\
& P(\eta_b) [d\eta_b] P(\eta_r) [d\eta_r] [d\bar{\psi}] [d\psi] [d\varphi] [d\tilde{\varphi}].
\end{aligned}$$

Now consider the probability functionals for η_b and η_r (the simplest one Gaussian)

$$P[\eta_b] := \exp \left(- \int \eta_b(x) \zeta_b^{-1}(x - x') \eta_b(x') d^N x d^N x' \right), P[\eta_r] := \exp \left(- \int \eta_r(x) \zeta_r^{-1}(x - x') \eta_r(x') d^N x d^N x' \right). \quad (4.29)$$

where ζ_b and ζ_r are the correlation functions of η_b and η_r , respectively i.e.,

$$\zeta_b(x, x') = \langle \eta_b(x) \eta_b(x') \rangle, \zeta_r(x, x') = \langle \eta_r(x) \eta_r(x') \rangle. \quad (4.30)$$

The inverse of the correlation function is defined as follows

$$\zeta(x - x') \zeta^{-1}(x' - x'') = \delta(x - x''). \quad (4.31)$$

For a purely random medium, we have $\zeta_b(x, x') = 2\delta(x - x')$. We may of course consider Gaussian correlation as well. For generality we consider a general correlation function here. Note that if one adds a non-trivial correlation in addition to pure randomness i.e., some factor which causes order in the midst of random behaviour, then in such a competing system, one might observe interesting physics such as criticality, scale invariance, and perhaps conformal symmetry. This issue might be important in the context of earthquake since earthquake is conjectured to be a critical phenomena (a self organized one) [125, 126, 127, 128]. We will focus on these issues later. After substituting these

particular forms of correlation functions, the partition functional reads

$$\begin{aligned}
Z = & \int \exp \left(i \int (-\omega^2 \varphi^I \tilde{\varphi}_I + C_0^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L - \omega^2 \bar{\psi}^I \psi_I \right. \\
& + C_0^{IJKL} \partial_J \bar{\varphi}_I \partial_K \psi_L + \sqrt{g_b} \eta_b(x) (\partial_I \tilde{\varphi}^I \partial^J \varphi_J + \partial_I \bar{\psi}^I \partial^J \psi_J) \\
& + \sqrt{g_r} \eta_r(x) (\partial_I \tilde{\varphi}^J \partial_J \varphi^I + \partial_I \tilde{\varphi}^J \partial^I \varphi_J + \partial_I \bar{\psi}^J \partial_J \psi^I + \partial_I \bar{\psi}^J \partial^I \psi_J) \Big) d^N x \\
& \exp \left(- \int \eta_b(x) \zeta_b^{-1}(x - x') \eta_b(x') d^N x d^N x' \right) \\
& \exp \left(- \int \eta_r(x) \zeta_r^{-1}(x - x') \eta_r(x') d^N x d^N x' \right) [d\eta_b][d\eta_r][d\bar{\psi}][d\psi][d\varphi][d\tilde{\varphi}],
\end{aligned}$$

from which we need to integrate out the fluctuation part i.e.,

$$\begin{aligned}
& \int \int \exp \left(i \int (\sqrt{g_b} \eta_b(x) (\partial_I \tilde{\varphi}^I \partial^J \varphi_J + \partial_I \bar{\psi}^I \partial^J \psi_J) \right. \\
& + \sqrt{g_r} \eta_r(x) (\partial_I \tilde{\varphi}^J \partial_J \varphi^I + \partial_I \tilde{\varphi}^J \partial^I \varphi_J + \partial_I \bar{\psi}^J \partial_J \psi^I + \partial_I \bar{\psi}^J \partial^I \psi_J) \Big) d^N x \\
& \exp \left(- \int \eta_b(x) \zeta_b^{-1}(x - x') \eta_b(x') d^N x d^N x' \right) \\
& \exp \left(- \int \eta_r(x) \zeta_r^{-1}(x - x') \eta_r(x') d^N x d^N x' \right) [d\eta_b][d\eta_r].
\end{aligned} \tag{4.32}$$

Integration of this fluctuation part yields the following expression

$$\begin{aligned}
& \exp \left(-\frac{1}{4} g_b \int (\partial_I \tilde{\varphi}^I(x) \partial^J \varphi_J(x) + \partial_I \bar{\psi}^I(x) \partial^J \psi_J(x)) \zeta(x - x') \right. \\
& \left. (\partial_I \tilde{\varphi}^I(x') \partial^J \varphi_J(x') + \partial_I \bar{\psi}^I(x') \partial^J \psi_J(x')) d^N x d^N x' \right) \\
& \exp \left(-\frac{1}{4} g_r \int (\partial_I \tilde{\varphi}^J(x) \partial_J \varphi^I(x) + \partial_I \tilde{\varphi}^J(x) \partial^I \varphi_J(x) + \partial_I \bar{\psi}^J(x) \partial_J \psi^I(x) + \partial_I \bar{\psi}^J(x) \partial^I \psi_J(x)) \right. \\
& \left. \zeta_r(x - x') (\partial_I \tilde{\varphi}^J(x') \partial_J \varphi^I(x') + \partial_I \tilde{\varphi}^J(x') \partial^I \varphi_J(x') + \partial_I \bar{\psi}^J(x') \partial_J \psi^I(x') + \partial_I \bar{\psi}^J(x') \partial^I \psi_J(x')) \right. \\
& \left. d^N(x) d^N x' \right)
\end{aligned} \tag{4.33}$$

Remarkably, we note that this process yields quartic interaction terms which would otherwise be also obtainable through Ito calculus but would not be trivial like the present situation. We will separately consider the case of specific correlation functional. Let us Fourier expand the interaction Lagrangian. Formally note that the following holds

$$\zeta(x - x') = \int \zeta(k) e^{ik \cdot (x - x')} d^N k, \quad \zeta(k) = \frac{1}{(2\pi)^N} \int \zeta(x) e^{-ik \cdot x} d^N x. \tag{4.34}$$

Using this partition functional Z , we define the generating functional W as follows

$$W[J, \tilde{J}, \mathcal{J}, \tilde{\mathcal{J}}] := \int \int e^{iS_\omega[\varphi, \tilde{\varphi}, \bar{\psi}, \psi] + \int (J^I \varphi_I + \tilde{J}^I \tilde{\varphi}_I + i\bar{\psi} \cdot \mathcal{J} + i\tilde{\mathcal{J}} \cdot \psi) d^N x} [d\varphi][d\tilde{\varphi}][d\bar{\psi}][d\psi],$$

where the formal action functional is defined as

$$\begin{aligned} S[\varphi, \tilde{\varphi}] := & \int ((-\omega^2 \varphi^I \tilde{\varphi}_I + C_0^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L) - \omega^2 \bar{\psi}^I \psi_I \\ & + C_0^{IJKL} \partial_J \bar{\psi}_I \partial_K \psi_L - \frac{1}{4i} g_b \int (\partial_I \tilde{\varphi}^I(x) \partial^J \varphi_J(x) + \partial_I \bar{\psi}^I(x) \partial^J \psi_J(x)) \zeta(x - x') \\ & (\partial_I \tilde{\varphi}^I(x') \partial^J \varphi_J(x') + \partial_I \bar{\psi}^I(x') \partial^J \psi_J(x')) d^N x' \\ & - \frac{1}{4i} g_r \int (\partial_I \tilde{\varphi}^J(x) \partial_J \varphi^I(x) + \partial_I \tilde{\varphi}^J(x) \partial^I \varphi_J(x) + \partial_I \bar{\psi}^J(x) \partial_J \psi^I(x) + \partial_I \bar{\psi}^J(x) \partial^I \psi_J(x)) \\ & \zeta_r(x - x') (\partial_I \tilde{\varphi}^J(x') \partial_J \varphi^I(x') + \partial_I \tilde{\varphi}^J(x') \partial^I \varphi_J(x') + \partial_I \bar{\psi}^J(x') \partial_J \psi^I(x') + \partial_I \bar{\psi}^J(x') \partial^I \psi_J(x')) d^N x' d^N x. \end{aligned} \quad (4.35)$$

Here the adjoint fields are such that the reality condition is met. The generating functional W is constructed for practical computations such as ensemble average of entities like $\varphi \tilde{\varphi}$ etc. Here notice that the partition functional Z is physical whereas the generating functional is introduced so that we can perform a trick described below. Of course now we can split the action functional into an exactly solvable system and an interaction part following routine procedure of perturbation theory

$$S[\varphi, \tilde{\varphi}, \bar{\psi}, \psi] = S^0[\varphi, \tilde{\varphi}, \bar{\psi}, \psi] + S^I[\varphi, \tilde{\varphi}, \bar{\psi}, \psi]. \quad (4.36)$$

Utilizing this, we may now observe that the generating functional may be written as

$$\begin{aligned} W[J, \tilde{J}, \mathcal{J}, \tilde{\mathcal{J}}] = & \int \exp(iS_I) \exp\left(\int (i(-\omega^2 \varphi^I \tilde{\varphi}_I + C_0^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L) \right. \\ & \left. + i(-\omega^2 \bar{\psi}^I \psi_I + C_0^{IJKL} \partial_J \bar{\psi}_I \partial_K \psi_L) + J^I \varphi_I + i\tilde{J}^I \tilde{\varphi}_I + i\bar{\psi} \cdot \mathcal{J} + i\tilde{\mathcal{J}} \cdot \psi) d^N x\right). \end{aligned} \quad (4.37)$$

We call the functional W as the generating functional simply because, we may ‘generate’ the ensemble

average and correlation functions using it as follows (and using (4.19) and noting $\frac{\delta J(x)}{\delta J(y)} = \delta(x - y)$)

$$\begin{aligned}
\langle \varphi_M(x) \tilde{\varphi}_N(y) \rangle_{ensemble} &= \frac{1}{Z} \int \varphi_M(x) \tilde{\varphi}_N(y) \exp(iS_I) \exp\left(\int (i(-\omega^2 \varphi^I \tilde{\varphi}_I \right. \\
&\quad \left. + C_0^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L) + i(-\omega^2 \bar{\psi}^I \psi_I + C_0^{IJKL} \partial_J \bar{\psi}_I \partial_K \psi_L)) d^N x\right) \\
&= \frac{1}{W[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}]} \int \frac{\delta}{\delta J^M} \frac{\delta}{i \delta \tilde{J}^N} \exp(iS_I) \exp\left(\int (i(-\omega^2 \varphi^I \tilde{\varphi}_I + C_0^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L) \right. \\
&\quad \left. + i(-\omega^2 \bar{\psi}^I \psi_I + C_0^{IJKL} \partial_J \bar{\psi}_I \partial_K \psi_L) + J^I \varphi_I + i \tilde{J}^I \tilde{\varphi}_I + i \bar{\psi} \cdot \mathcal{J} + i \bar{\mathcal{J}} \cdot \psi) d^N x\right) |_{J=\tilde{J}=\mathcal{J}=\bar{\mathcal{J}}=0}.
\end{aligned} \tag{4.38}$$

The second line follows due to $W[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}]|_{J=\tilde{J}=\mathcal{J}=\bar{\mathcal{J}}=0} = Z$. Now in order to evaluate this integrals, we need the inverse of the linear unperturbed differential operator $-\omega^2 \delta^{IL} - C_0^{IJKL} \partial_J \partial_K$, which is nothing but the Green's function of this differential operator. Assuming periodic or asymptotically vanishing boundary condition, the Green's function or the 2-point correlation function is computed by using the following relation in Fourier domain

$$(-\omega^2 \delta^{IL} - C_0^{IJKL} \partial_J \partial_K) G_L^M(x; x') = \delta^{IM} \delta(x - x') \tag{4.39}$$

By substituting $C_0^{IJKL} = \lambda_0 \delta^{IJ} \delta^{KL} + \mu_0 (\delta^{IK} \delta^{JL} + \delta^{IL} \delta^{JK})$ and $x' = 0$, we explicitly obtain the following form of the 2-point function in the Fourier domain

$$\boxed{\omega G_J^I(k) = \frac{1}{-\omega^2 + (\lambda + 2\mu)k^2} \left(\delta_J^I + \frac{(\lambda + \mu)(k^2 \delta_J^I - k^I k_J)}{-\omega^2 + \mu k^2} \right)} \tag{4.40}$$

. This Green function is called 2-point correlation function simply because, the correlation function of φ and $\tilde{\varphi}$ in the absence of perturbations is simply G i.e.,

$$\begin{aligned}
\langle \varphi^I(x) \tilde{\varphi}_J(x') \rangle_{Z^0} &= \int \frac{i e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}}{-\omega^2 + (\lambda + 2\mu)k^2} \left(\delta_J^I + \frac{(\lambda + \mu)(k^2 \delta_J^I - k^I k_J)}{-\omega^2 + \mu k^2} \right) d^N \mathbf{k}, \\
\langle \bar{\psi}^I(x) \psi_J(x') \rangle_{Z^0} &= \int \frac{i e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}}{-\omega^2 + (\lambda + 2\mu)k^2} \left(\delta_J^I + \frac{(\lambda + \mu)(k^2 \delta_J^I - k^I k_J)}{-\omega^2 + \mu k^2} \right) d^N \mathbf{k}.
\end{aligned}$$

Notice that the dispersion shell must be the pole of the correlation function and corresponds to the P and S modes (speed $\sqrt{\lambda + 2\mu}$ and $\sqrt{\mu}$, respectively). We may in fact obtain an explicit

representation if we introduce the auxiliary fields $J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}$ as follows

$$\begin{aligned} W[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}] &= \exp(iS_I[\frac{\delta}{\delta J}, \frac{\delta}{i\delta\tilde{J}}, \frac{\delta}{i\delta\mathcal{J}}, \frac{\delta}{i\delta\bar{\mathcal{J}}}]) \exp(\int (i(-\omega^2\varphi^I\tilde{\varphi}_I + C_0^{IJKL}\partial_J\tilde{\varphi}_I\partial_K\varphi_L) \\ &\quad + i(-\omega^2\bar{\psi}^I\psi_I + C_0^{IJKL}\partial_J\bar{\psi}_I\partial_K\psi_L) + J^I\varphi_I + i\tilde{J}^I\tilde{\varphi}_I + i\bar{\psi}\cdot\mathcal{J} + i\bar{\mathcal{J}}\cdot\psi)d^Nx)) \\ &= \exp(iS_I[\frac{\delta}{\delta J}, \frac{\delta}{i\delta\tilde{J}}, \frac{\delta}{i\delta\mathcal{J}}, \frac{\delta}{i\delta\bar{\mathcal{J}}}])W^0[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}] \end{aligned}$$

A simple calculation using the definition of functional derivative $\frac{\delta J(x)}{\delta \mathcal{J}(y)} = \delta(x-y)$ confirms it. Now let us evaluate the W^0 to yield

$$\begin{aligned} W^0[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}] &= \int \exp(\int (i(-\omega^2\varphi^I\tilde{\varphi}_I + C_0^{IJKL}\partial_J\tilde{\varphi}_I\partial_K\varphi_L) \\ &\quad + i(-\omega^2\bar{\psi}^I\psi_I + C_0^{IJKL}\partial_J\bar{\psi}_I\partial_K\psi_L) + J^I\varphi_I + i\tilde{J}^I\tilde{\varphi}_I + i\bar{\psi}\cdot\mathcal{J} + i\bar{\mathcal{J}}\cdot\psi)d^Nx) \\ &= \int \exp(\int i\tilde{\varphi}_I ((G^{-1})^{IL}\varphi_L + \tilde{J}^I) d^Nx) \exp(\int i(\bar{\psi}_I(G^{-1})^{IL}\psi_L \\ &\quad + \bar{\psi}\cdot\mathcal{J} + \bar{\mathcal{J}}\cdot\psi) d^Nx) \exp(\int J^I\varphi_I d^Nx) \\ &= \int_{\varphi} \delta((G^{-1})^{IL}\varphi_L + \tilde{J}^I) \exp(\int J^I\varphi_I d^Nx) \exp(-i \int \bar{\mathcal{J}}^I G_{IJ} \mathcal{J}^J d^Nx) \\ &= \exp(- \int J^I G_{IJ} \tilde{J}^J d^Nx) \exp(-i \int \bar{\mathcal{J}}^I G_{IJ} \mathcal{J}^J d^Nx). \end{aligned} \tag{4.41}$$

$$\boxed{W^0[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}] = \exp(- \int J^I G_{IJ} \tilde{J}^J d^Nx) \exp(-i \int \bar{\mathcal{J}}^I G_{IJ} \mathcal{J}^J d^Nx)}, \tag{4.42}$$

where we have used the definition of the functional Dirac's distribution and Grassman integration.

Therefore we have the following expression for the full generating functional

$$\begin{aligned} W[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}] &= \exp(iS_I[\frac{\delta}{\delta J}, \frac{\delta}{i\delta\tilde{J}}, \frac{\delta}{i\delta\mathcal{J}}, \frac{\delta}{i\delta\bar{\mathcal{J}}}])W^0[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}] \\ &= \exp(iS_I[\frac{\delta}{\delta J}, \frac{\delta}{i\delta\tilde{J}}, \frac{\delta}{i\delta\mathcal{J}}, \frac{\delta}{i\delta\bar{\mathcal{J}}}]) \exp(- \int J^I G_{IJ} \tilde{J}^J d^Nx) \\ &\quad \exp(-i \int \bar{\mathcal{J}}^I G_{IJ} \mathcal{J}^J d^Nx). \end{aligned} \tag{4.43}$$

$$\boxed{W[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}] = \exp(iS_I[\frac{\delta}{\delta J}, \frac{\delta}{i\delta\tilde{J}}, \frac{\delta}{i\delta\mathcal{J}}, \frac{\delta}{i\delta\bar{\mathcal{J}}}]) \exp(- \int J^I G_{IJ} \tilde{J}^J d^Nx) \exp(-i \int \bar{\mathcal{J}}^I G_{IJ} \mathcal{J}^J d^Nx)}.$$

Now that we have obtained this explicit form of the generating functional, it is clear that this is much simpler form i.e., few differential operators acting on exponentials. This will facilitate the

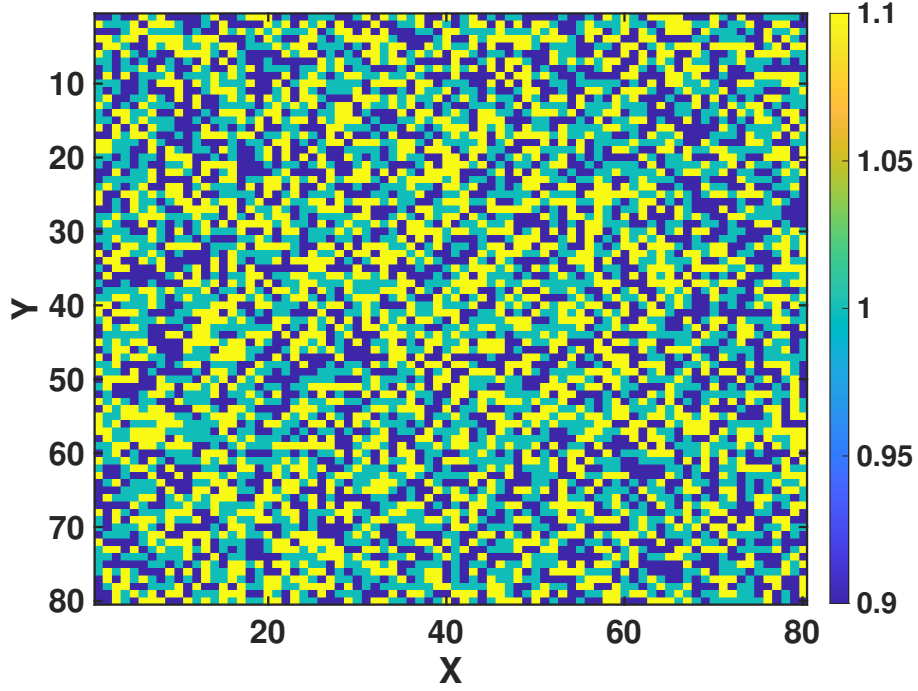


Figure 4.1: One realization of the Gaussian random medium under study on a grid 80×80 . The lame constant $\lambda = \lambda_0 + \sqrt{g_b^0} \eta(x)$ is plotted here for $\sqrt{g_b^0} = 0.1$ and $\lambda_0 = \langle \lambda \rangle = 1$.

computation of the correlation functions using graph expansion in the next section. The expectation of any functional $\mathcal{F}(\varphi, \tilde{\varphi})$ may now be written as follows

$$\langle \mathcal{F}(\varphi, \tilde{\varphi}) \rangle = \frac{1}{W[J, \tilde{J}, \mathcal{J}, \tilde{\mathcal{J}}]} \mathcal{F}\left(\frac{\delta}{\delta J}, \frac{\delta}{i\delta \tilde{J}}\right) W[J, \tilde{J}, \mathcal{J}, \tilde{\mathcal{J}}] \Big|_{J=\tilde{J}=\mathcal{J}=\tilde{\mathcal{J}}=0}. \quad (4.44)$$

Once the generating functional is derived, we may perform the necessary calculations. As mentioned earlier, we want to integrate out the degrees of freedom associated with the smaller length scales and obtain an effective theory at a larger length scale. This is executed in two steps: (a) Mode elimination by integration: in this step the small scale modes are eliminated by integrating out the large wave numbers. This is also called the course-graining procedure. (b) Re-scaling: in this step, we re-scale the associated fields suitably. The combination of the mode-elimination step with the re-scaling step defines now a mapping between the initial constants (the bare constants or non-homogenized constants) and the effective constants. Here the set of bare constants is essentially $(\lambda_0, \mu_0, g_b^0, g_r^0)$. If the effective constants are denoted as $(\lambda', \mu', g_b', g_r')$, then the RG flow $\mathcal{R}$ is essentially a semi-group $(\lambda', \mu', g_b', g_r') = \mathcal{R}(\lambda_0, \mu_0, g_b^0, g_r^0)$. The fixed points of this RG transformation

essentially gives rise to critical phenomena and as we will see towards the end if the fixed points exist then the property of the fixed points is universal. Typically in a system where two or more competing forces are present, one may expect critical behaviour. This of course include the well known Ising model where the competing factors are the randomness associated to temperature and the order generating external magnetic field. There are of course other strongly nonlinearly interacting systems where no external factor is necessary to yield critical behaviour. In these cases the non-linearity focuses and defocuses energy through interaction and such system are designated as systems with self organized criticality (SOC). Starting with the bare constants RG flow essentially provides good description of the physics at different scales via modifying the constants. In our case the constants (λ', μ') essentially becomes effective material property while viewing the system at different length scales. This analysis will explicitly give us formulas for effective material property of a random medium for small strength of randomness which is typically the case in physically interesting situations. Clearly in our system, we do not have competing forces present if we let the perturbations be purely random. However, interesting scenario arises one adds a correlation function of the type $\zeta(x, x') \sim a\delta(x - x') + be^{-(x-x')^2/l^2}$. This leads to decoherence by a and order generation by b . Depending on relative values of a and b , these two opposing effects may nullify each other to yield a critical point where one observes scale invariance (since scales are destroyed at this point). Let us now carry out the derivation of the effective action. We start with integrating the small scale modes (or highly fluctuating modes). It is natural to work in the Fourier domain since the separation of scale is trivial here. Formally we will write the field φ etc. as combinations of low wave number and high wave number in the Fourier domain as follows

$$\varphi = \varphi(k)_{>}\Theta(|k| - \Lambda) + \varphi(k)_{<}\Theta(-|k| + \Lambda), \tilde{\varphi} = \tilde{\varphi}(k)_{>}\Theta(|k| - \Lambda) + \tilde{\varphi}(k)_{<}\Theta(-|k| + \Lambda), \quad (4.45)$$

$$\psi = \psi(k)_{>}\Theta(|k| - \Lambda) + \psi(k)_{<}\Theta(-|k| + \Lambda), \bar{\psi} = \bar{\psi}(k)_{>}\Theta(|k| - \Lambda) + \bar{\psi}(k)_{<}\Theta(-|k| + \Lambda). \quad (4.46)$$

This may be slightly questionable since this partition of unity is sharp. In a more rigorous way to do this is to replace the step function by Schwartz functions i.e., one write the fields as follows

$$\varphi = \varphi(k)_{>}\chi(|k|/\Lambda) + \varphi(k)_{<}(1 - \chi(|k|/\Lambda)), \quad (4.47)$$

where $\chi(|k|/\Lambda)$ is a Schwartz function with the scale Λ i.e., $\chi(|k|/\Lambda) = 1$ for $|k| > 2\Lambda$ and $\chi(|k|/\Lambda) = 0$ for $|k| < \frac{1}{2}\Lambda$. This is simply an application of a proper filter to the displacement fields.

4.5 Mode Elimination: Course graining

Our aim is to integrate out the momenta lying in the interval $[\Lambda, \Lambda_0]$ i.e., roughly the length scale $[\frac{1}{\Lambda_0}, \frac{1}{\Lambda}]$. We can not of course integrate the large wave number modes exactly. We will therefore do it using Feynman diagrams. The action is split into three parts

$$S = S_{<\Lambda}(\varphi_{<}, \tilde{\varphi}_{<}; g_b^0, g_r^0) + S_{>\Lambda}(\varphi_{>}, \tilde{\varphi}_{>}) + S^{mix}(\varphi_{>}, \tilde{\varphi}_{>}, \varphi_{<}, \tilde{\varphi}_{<}). \quad (4.48)$$

The mixed part will contribute to the effective coupling constants. Before explicitly evaluating the action, let us first move it in the wave number domain and evaluate term by term. Now φ will represent $\varphi(\omega, k)$ and similarly for other variables.

4.5.1 Effective Action and graph expansion

Now we delve deeper into the philosophy of the mode elimination. The main points may be summarized as follows. We recall that we have split the displacement fields into higher wavenumber and lower wavenumber components. The cutoff wavenumber is described by the entity Λ . Now we treat Λ as a free parameter i.e., it can vary with the size of the heterogeneities present in the matrix. We want to integrate out all the processes whose characteristic length scale is between the length scale of the heterogeneities and that of the system we are considering (e.g., a cubic matrix of size L). This is simply because the events occurring at a length scale less than that of the size of heterogeneities present is not important from physical perspective. Recall the definition of the partition function

$$Z = \int \exp(iS[\varphi, \tilde{\varphi}, \psi, \bar{\psi}; coupling_0])[d\varphi][d\tilde{\varphi}][d\psi][d\bar{\psi}], \quad (4.49)$$

where $coupling_0$ are essentially the bare/non-homogenized elastic constants. Now we insert the splitting (4.45) into S in the above expression and integrate out the modes $(\varphi_{>}, \tilde{\varphi}_{>}, \psi_{>}, \bar{\psi}_{>})$. After integrating out these fields defined on low length scales, the partition function becomes

$$Z = \int \exp(iS_{<\Lambda}[<, <; coupling_{<}])[d\varphi_{<}][d\tilde{\varphi}_{<}][d\bar{\psi}_{<}][d\psi_{<}], \quad (4.50)$$

where the effective action $S_{<\Lambda}[\varphi_{<}, \tilde{\varphi}_{<}; coupling_{<}]$ is given by the following expression

$$\begin{aligned} \exp(iS_{<\Lambda}[<, <; coupling_{<}]) &= \exp(iS_{<\Lambda}[<, <; coupling_0]) \\ &\int \exp(iS_{>\Lambda}[>, >] + iS^{mix}[><])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}]. \end{aligned} \quad (4.51)$$

Now we clearly observe that the integrating out or averaging out procedure modifies the bare constants $coupling_0$ by adding extra terms. Previous expression after taking $\ln$ yields

$$\begin{aligned} iS_{<\Lambda}[<, <; coupling_{<}] &= iS_{<\Lambda}[<, <; coupling_0] \\ &+ \ln \left(\int \exp(iS_{>\Lambda}[>, >] + iS^{mix}[><])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}] \right). \end{aligned} \quad (4.52)$$

This may be evaluated further

$$\begin{aligned} \ln \left(\int \exp(iS_{>\Lambda}[\varphi_{>}, \tilde{\varphi}_{>}] + iS^{mix}[\varphi_{>}, \tilde{\varphi}_{>}; \varphi_{<}, \tilde{\varphi}_{<}])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}] \right) \\ = \ln \left(\frac{\int \exp(iS_{>\Lambda}[>, >] + iS^{mix}[><])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}]}{\int \exp(iS_{>\Lambda}[>, >])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}]} \right) \\ + \ln \left(\int \exp(iS_{>\Lambda}[>, >])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}] \right). \end{aligned}$$

Note here that the term $\ln \left(\int \exp(iS_{>\Lambda}[\varphi_{>}, \tilde{\varphi}_{>}])[d\varphi_{>}][d\tilde{\varphi}_{>}] \right)$ adds a constant to the effective action and as such harmless since it can always be absorbed in the normalization constant of the formal partition function. Now notice an important fact

$$\begin{aligned} \frac{\int \exp(iS_{>\Lambda}[>, >] + iS^{mix}[><])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}]}{\int \exp(iS_{>\Lambda}[\varphi_{>}, \tilde{\varphi}_{>}])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}]} \\ = \langle \exp(iS^{mix}[><]) \rangle_{Z_{>}} \end{aligned} \quad (4.53)$$

where $Z_{>}$ is defined as follows

$$Z_{>} := \int \exp(iS_{>\Lambda}[>, >])[d\varphi_{>}][d\tilde{\varphi}_{>}][d\bar{\psi}_{>}][d\bar{\psi}_{>}]. \quad (4.54)$$

We will only be interested in first few loops (which do not appear if randomness is absent). We formally expand the series to yield

$$\begin{aligned} & \langle \exp(iS^{mix}[><]) \rangle_{Z_>} \\ &= \langle 1 + iS^{mix}[><] + \frac{1}{2}(iS^{mix}[><])^2 - \dots \rangle_{Z_>} . \end{aligned} \quad (4.55)$$

Therefore, we have up to one loop

$$\begin{aligned} S_{<\Lambda}[\langle, \langle; coupling_{<}] &\approx S_{<\Lambda}[\langle, \langle; coupling_0] \\ &+ \ln \left(1 + i \langle S^{mix}[><] \rangle_{Z_>} + \frac{1}{2} \langle (iS^{mix}[><])^2 \rangle_{Z_>} \right) \\ &+ \ln \left(\int \exp(iS_{>\Lambda}[>, >]) [d\varphi_{>}] [d\tilde{\varphi}_{>}] [d\tilde{\psi}_{>}] [d\bar{\psi}_{>}] \right) \\ &\approx S_{<\Lambda}[\langle, \langle; coupling_0] + \langle S^{mix}[><] \rangle_{Z_>} \\ &- \frac{1}{2i} \langle (iS^{mix}[><])^2 \rangle_{Z_>} + \frac{1}{2i} \langle (iS^{mix}[><]) \rangle_{Z_>}^2 + \ln(Z_{>}). \end{aligned} \quad (4.56)$$

In the leading order we only consider the average with respect to the Gaussian part of the small scale action and therefore

$$\begin{aligned} S_{<\Lambda}[\langle, \langle; coupling_{<}] &\approx S_{<\Lambda}[\langle, \langle; coupling_0] + \langle S^{mix}[><] \rangle_{Z_{0>}} \\ &- \frac{1}{2i} \langle (iS^{mix}[><])^2 \rangle_{Z_{0>}} + \frac{1}{2i} \langle (iS^{mix}[><]) \rangle_{Z_{0>}}^2, \end{aligned}$$

where $Z_{0>}$ is the small scale partition function with quadratic principle part only. Notice that due to the expansion of the $\ln$, each of these terms collected properly only contributes to a connected diagram i.e., we see that $\langle S^{mix}[\varphi_{>}, \tilde{\varphi}_{>}; \varphi_{<}, \tilde{\varphi}_{<}] \rangle_{Z_{0>}}$ and $-\frac{1}{2i} \langle (iS^{mix}[\varphi_{>}, \tilde{\varphi}_{>}; \varphi_{<}, \tilde{\varphi}_{<}])^2 \rangle_{Z_{0>}}$ + $\frac{1}{2i} \langle (iS^{mix}[\varphi_{>}, \tilde{\varphi}_{>}; \varphi_{<}, \tilde{\varphi}_{<}]) \rangle_{Z_{0>}}^2$ have connected parts only at the level of Feynman diagrams. From this effective description of the associated action functional, it is now clear how the effective elastic constants will be derived. Notice that the term $S_{<\Lambda}[\langle, \langle; coupling_0]$ involves the non-homogenized constants (or simply the constants corresponding to the matrix) roughly in the following form $\lambda_0(\partial\varphi)(\partial\tilde{\varphi})$. Now, the ‘averaging out’ procedure generates extra frequency and wavenumber bound Λ (and hence length) dependent terms roughly of the form $\delta\lambda(\Lambda, w)(\partial\varphi\partial\tilde{\varphi})$ from $\langle S^{mix}[><] \rangle_{Z_{0>}}$ - $\frac{1}{2i} \langle (iS^{mix}[><])^2 \rangle_{Z_{0>}}$ + $\frac{1}{2i} \langle (iS^{mix}[><]) \rangle_{Z_{0>}}^2$ which encodes the detailed nonlinear interaction of the complex heterogeneities and the wavefield propagating through the medium. Therefore the effective constants will simply be of the type $\lambda(\omega, \Lambda) = \lambda_0 + \delta\lambda(\omega, \Lambda)$.

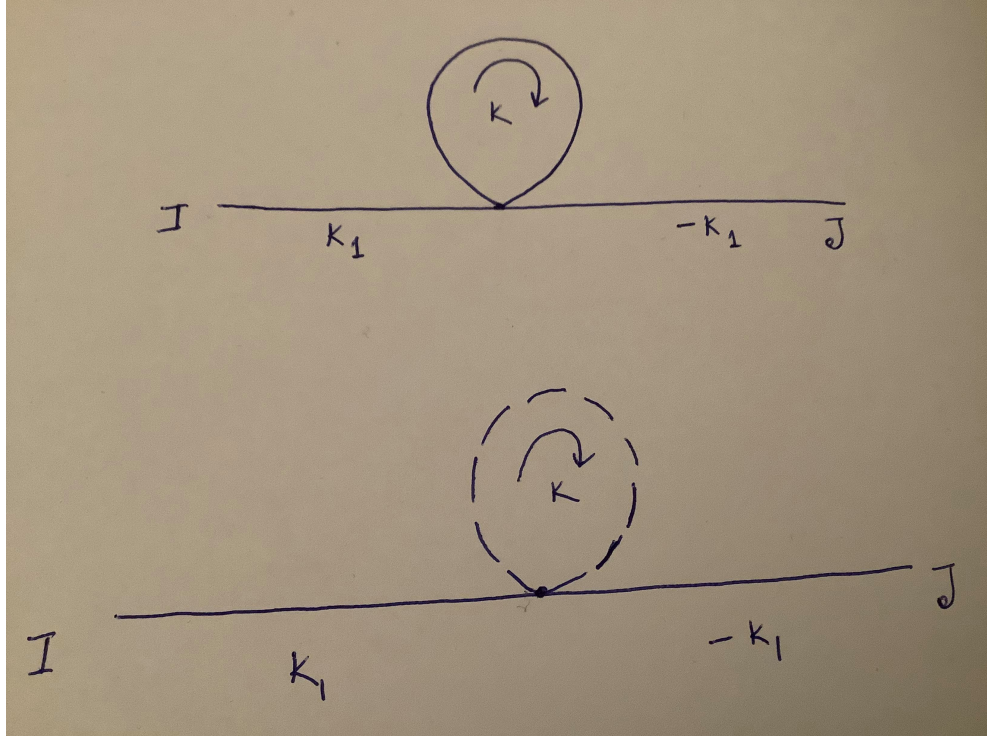


Figure 4.2: The one loop contribution to the effective coupling constants λ and μ . Notice that this loop character is purely generated because of the randomness present in the system. This is in parallel with the QFT or SFT case since the loop only appears as the ‘quantum’ correction to the classical system. The solid loop and the dashed loop represent the bosonic and grassman parts, respectively.

Note an important fact that the mixed terms with odd number of small length scale terms do not contribute at one loop level since Gaussian integral is even. The detail calculations for the exact expressions for the mixed actions (terms containing both high and low wavenumber parts) are presented in the appendix. For completeness, we include the complete perturbative series consisting of multiple loops. We refer to the diagrams (figure 4.3) for a visual illustration.

An extremely important point is that we have the loop contribution only because of the ‘randomness’ present in the system. One does *not* obtain loop contributions if the heterogeneities present in the matrix are not random. We will see when we study the special case by turning off the random behaviour of the heterogeneity later that in such cases only ‘tree’ level contribution arises. Now all we need to do is to collect all the contributions from several diagrams and sum it up. For compactness, we will not write down all the integrals here instead put them in the appendix. Since the structure of all the integrals associated are more or less same, we will just discuss the property of

$$\begin{aligned}
 (\lambda_{\text{eff}}, \mu_{\text{eff}}) = & (\lambda_0, \mu_0) + \left[\left\{ \text{---} \bigcirc \text{---} + \text{---} \bigcirc \bigcirc \text{---} + \dots \right\} \right. \\
 & + \left\{ \text{---} \bigcirc \bigcirc \text{---} + \text{---} \bigcirc \bigcirc \bigcirc \text{---} + \dots \right\} + \dots \Big] \\
 & \text{Commuting part} \\
 & + \left[\left\{ \text{---} \bigcirc \text{---} + \text{---} \bigcirc \bigcirc \text{---} + \dots \right\} \right. \\
 & + \left\{ \text{---} \bigcirc \bigcirc \text{---} + \text{---} \bigcirc \bigcirc \bigcirc \text{---} + \dots \right\} + \dots \Big] \\
 & \text{Anti-commuting part}
 \end{aligned}$$

Figure 4.3: The schematics showing how several diagrams arise to contribute to the effective constants. Ideally one should be able to sum up the whole series. But at the perturbative level we only restrict ourselves up to 1 loop.

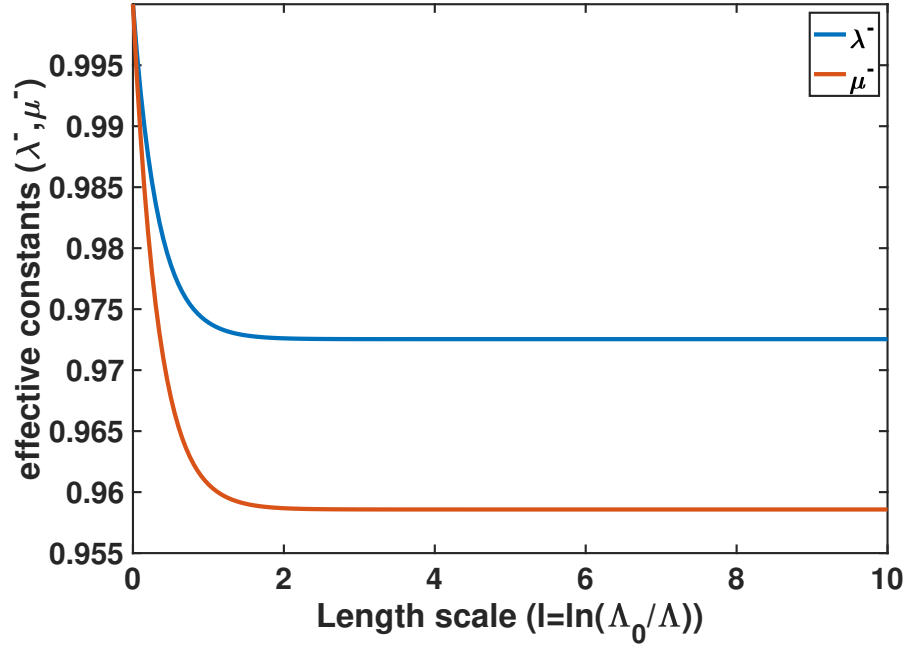


Figure 4.4: An arbitrary example: dimensionless effective coupling constants λ', μ' as a function of length scale l . Bare elastic constants $\lambda_0 = \mu_0 = 1$, strength of random heterogeneity $\sqrt{g_b^0} = \sqrt{g_r^0} = 0.1$.

one 1-loop integral. One of the loop integrals read

$$\begin{aligned}
 I &= -g_b^0 \int_{\Lambda}^{\Lambda_0} \frac{k^I k^J}{-\omega^2 + (\lambda + 2\mu)k^2} \left(\delta_{IJ} + \frac{(\lambda + \mu)(k^2 \delta_{IJ} - k_I k_J)}{-\omega^2 + \mu k^2} \right) d^N k \\
 &= -g_b^0 \int_{\Lambda}^{\Lambda_0} \frac{k^2}{-\omega^2 + (\lambda + 2\mu)k^2} d^N k.
 \end{aligned} \tag{4.57}$$

Notice a few important things that happened here. If we go back to the previous section, we remember that g_b^0 is the strength (squared) of perturbations associated to the dilation (longitudinal mode). Now physically, we would expect that this pure dilation should not effect the transverse mode. But in the integral of the first line we have the propagator contributing both to the dilation as well as transverse part due to the presence of both poles at $k = \omega/\sqrt{\lambda + 2\mu}$ and $k = \omega/\sqrt{\mu}$. However, term containing the pole $\omega/\sqrt{\mu}$ cancels out due to the fact that $k^I k^J (k^2 \delta_{IJ} - k_I k_J) = 0$. This is simply a version of the *Ward identity* in the realm of elastic wave propagation. This should guarantee that this cancellation property persists at all order of perturbation theory. This is analogous to the case of QED where the Ward identity assures that the photons do not acquire mass at all order of

perturbation theory at the quantum level. Now note the second most important thing. We have mentioned earlier that elastic theory at the nonlinear level is not well behaved, that is, it does not allow one formulate the theory at the level of very large wavenumber. This should be clear from this integral. Roughly this integral contributes to the scattering amplitude. Now, if we go all the way to the infinite wavenumber, we observe that this integral blows up. Now notice the most important part

$$\int^{\infty} \frac{k^2}{-\omega^2 + (\lambda + 2\mu)k^2} d^N k \sim \int^{\infty} k^{N-1} dk = \infty. \quad (4.58)$$

There is no way to renormalize this integral since the divergence is polynomial in nature. This is precisely a characteristics of a non-renormalizable theory. It is not surprising since the ray theory which is technically a low wavelength approximation generates caustics if the heterogeneity is complex enough. In order to alleviate this problem, we impose an upper bound on the wavenumber dictated by the lowest length scale of the heterogeneity present. We denote this upper bound by Λ_0 . Therefore the integral we want to finally evaluate is the following

$$-g_b^0 \int_{\Lambda}^{\Lambda_0} \frac{k^2}{-\omega^2 + (\lambda + 2\mu)k^2} d^N k. \quad (4.59)$$

All the calculations are presented in the appendix. Collecting all the contributions, we compute the effective elastic constants as a function of length scales Λ, Λ_0 (wavenumber scale to be precise) to be the following

$$\begin{aligned} \lambda(\Lambda, \Lambda_0) = \lambda_0 - \frac{5}{2}g_b^0 \int_{\Lambda}^{\Lambda_0} \frac{\zeta(0)k^2}{-\omega^2 + (\lambda_0 + 2\mu_0)k^2} d^N k + \frac{4g_r^0}{N(N+2)} \int_{\Lambda}^{\Lambda_0} \frac{\zeta(k)(\lambda + \mu)k^4 d^N k}{(-\omega^2 + (\lambda_0 + 2\mu_0)k^2)(-\omega^2 + \mu k^2)} \\ + O(g_b^{02}, g_r^{02}) \end{aligned}$$

$$\begin{aligned} \mu(\Lambda, \Lambda_0) = \mu_0 - \frac{3}{2}g_r^0 \int_{\Lambda}^{\Lambda_0} \frac{\zeta(0)k^2 d^N k}{-\omega^2 + (\lambda_0 + 2\mu_0)k^2} + \frac{4g_r^0}{N(N+2)} \int_{\Lambda}^{\Lambda_0} \frac{(\lambda + \mu)\zeta(k)k^4 d^N k}{(-\omega^2 + (\lambda_0 + 2\mu_0)k^2)(-\omega^2 + \mu_0 k^2)} \\ - \frac{2g_r^0}{N} \int_{\Lambda}^{\Lambda_0} \frac{\zeta(0)k^2}{-\omega^2 + (\lambda_0 + 2\mu_0)k^2} \left(1 + \frac{(\lambda + \mu)k^2}{-\omega^2 + \mu_0 k^2}\right) + O(g_b^{02}, g_r^{02}). \end{aligned}$$

Now defining $\Lambda = \Lambda_0 e^{-l}$, for a dimensionless length scale l , we may explicitly evaluate these integrals

to yield for a purely random heterogeneities, the following effective elastic constants

$$\begin{aligned}\lambda(l) = & \lambda_0 - \frac{10\pi g_b^0}{\lambda_0 + 2\mu_0} \left(\frac{1}{3} \Lambda_0^3 (1 - e^{-3l}) + \alpha^2 \Lambda_0 (1 - e^{-l}) + \frac{\alpha^3}{2} \ln \left(\frac{(\Lambda_0 - \alpha)(\Lambda_0 e^{-l} + \alpha)}{(\Lambda_0 + \alpha)(\Lambda_0 e^{-l} - \alpha)} \right) \right) \\ & + \frac{16\pi g_r^0}{15} \frac{\lambda_0 + \mu_0}{(\lambda_0 + 2\mu_0)\mu_0} \left(\frac{1}{3} \Lambda_0^3 (1 - e^{-3l}) + \beta^2 \Lambda_0 (1 - e^{-l}) + \frac{\beta^3}{2} \ln \left(\frac{(\Lambda_0 - \beta)(\Lambda_0 e^{-l} + \beta)}{(\Lambda_0 + \beta)(\Lambda_0 e^{-l} - \beta)} \right) \right) \\ & + \alpha^2 \left(\Lambda_0 (1 - e^{-l}) + \left(\frac{\alpha^2 + \beta^2}{2\beta} - \frac{\alpha^4}{2\beta(\alpha^2 - \beta^2)} \right) \ln \left(\frac{(\Lambda_0 - \beta)(\Lambda_0 e^{-l} + \beta)}{(\Lambda_0 + \beta)(\Lambda_0 e^{-l} - \beta)} \right) \right) \\ & + \frac{\alpha^3}{2(\alpha^2 - \beta^2)} \ln \left(\frac{(\Lambda_0 - \alpha)(\Lambda_0 e^{-l} + \alpha)}{(\Lambda_0 + \alpha)(\Lambda_0 e^{-l} - \alpha)} \right) \Big) + O((g_b^0)^2, (g_r^0)^2),\end{aligned}$$

$$\begin{aligned}\mu(l) = & \mu_0 - \frac{26\pi g_r^0}{3(\lambda_0 + 2\mu_0)} \left(\frac{1}{3} \Lambda_0^3 (1 - e^{-3l}) + \alpha^2 \Lambda_0 (1 - e^{-l}) + \frac{\alpha^3}{2} \ln \left(\frac{(\Lambda_0 - \alpha)(\Lambda_0 e^{-l} + \alpha)}{(\Lambda_0 + \alpha)(\Lambda_0 e^{-l} - \alpha)} \right) \right) \\ & - \frac{8\pi g_r^0(\lambda_0 + \mu_0)}{5\mu_0(\lambda_0 + 2\mu_0)} \left(\frac{1}{3} \Lambda_0^3 (1 - e^{-3l}) + \beta^2 \Lambda_0 (1 - e^{-l}) + \frac{\beta^3}{2} \ln \left(\frac{(\Lambda_0 - \beta)(\Lambda_0 e^{-l} + \beta)}{(\Lambda_0 + \beta)(\Lambda_0 e^{-l} - \beta)} \right) \right) \\ & + \alpha^2 \left(\Lambda_0 (1 - e^{-l}) + \left(\frac{\alpha^2 + \beta^2}{2\beta} - \frac{\alpha^4}{2\beta(\alpha^2 - \beta^2)} \right) \ln \left(\frac{(\Lambda_0 - \beta)(\Lambda_0 e^{-l} + \beta)}{(\Lambda_0 + \beta)(\Lambda_0 e^{-l} - \beta)} \right) \right) \\ & + \frac{\alpha^3}{2(\alpha^2 - \beta^2)} \ln \left(\frac{(\Lambda_0 - \alpha)(\Lambda_0 e^{-l} + \alpha)}{(\Lambda_0 + \alpha)(\Lambda_0 e^{-l} - \alpha)} \right) \Big) + O((g_b^0)^2, (g_r^0)^2),\end{aligned}$$

where $\alpha^2 = \frac{\omega^2}{\lambda_0 + 2\mu_0}$, $\beta^2 = \frac{\omega^2}{\mu_0}$. If we for the moment consider the static case i.e., $\omega = 0$, we have

$$\lambda(l) = \lambda_0 + \frac{16\pi g_r^0 \lambda_0 + \pi(16g_r^0 - 150g_b^0)\mu_0}{45\mu_0(\lambda_0 + 2\mu_0)} \Lambda_0^3 (1 - e^{-3l}), \quad (4.60)$$

$$\mu(l) = \mu_0 - \frac{(154\mu_0 + 24\lambda_0)\pi g_r^0}{45\mu_0(\lambda_0 + 2\mu_0)} \Lambda_0^3 (1 - e^{-3l}). \quad (4.61)$$

Here l is the length scale of interest. If one considers an infinite medium, then of course $l = \infty$ and if one is interested in a periodic lattice, then l would be the lattice size.

In practice, sometimes we want to homogenize certain heterogeneous medium which do not possess any random nature to it rather complex heterogeneity in terms of shape of the inclusions present in the matrix. As we mentioned previously, computation of the elastic field within simple shapes like ellipsoid becomes non-trivial as was depicted in Eshelby's work [114]. Since homogenization of heterogeneous medium requires such calculations, it is not really straightforward to homogenize complex heterogeneous medium where the shape of the inclusions present is arbitrary. However, using our technique, this essentially amounts to evaluating the Fourier coefficients (spatial) corresponding to the shapes present and later evaluating some scattering integrals involving these coefficients. Therefore, it is a rather straightforward special case of the current situation where a complete

random heterogeneity is treated. Of course for arbitrary shapes of the inclusions present, it is quite difficult to analytically compute the Fourier coefficients. But for such cases, one may always use a bit of computational resources to compute the Fourier coefficients. As of now our result is valid at the level of perturbation theory i.e., when the elastic constants of the inclusions differ slightly from that of the matrix. At this level, our result do match with that of Hashin and Strikman bound [108]. Since the calculations are exactly similar to the present situation we do not include all the calculations here only the results are shown in the figures (4.5, 4.6).

4.6 RG flow of effective constants

In this section, we systematically derive a system of coupled differential equations for the effective elastic constants. Given the effective constants at a particular scale, one may obtain the elastic constant at any other length scale by solving this system. This is in principle similar to the renormalization group flow for the coupling constants for statistical or quantum field theory. One might even say that our system should resemble such statistical or quantum system mathematically through the introduction of the stochastic nature. This was already discussed in the the previous sections. We proceed systematically to derive the differential equations (also know as the expression for the β - functions) for the re-scaled effective elastic constants. Choosing the usual scaling

$$k' = kb, \quad \varphi'^I(k') = \zeta_b^{-1} \varphi^I(k), \quad \tilde{\varphi}'^I(k') = \zeta_b^{-1} \tilde{\varphi}^I(k), \quad (4.62)$$

where

$$b = \frac{\Lambda_0}{\Lambda}, \quad \zeta_b = b^{\Delta_{\varphi, \tilde{\varphi}}} \sqrt{Z_b}, \quad (4.63)$$

and $\Delta_{\varphi, \tilde{\varphi}}$ is the scaling dimension of $\varphi(k)$ (or $\tilde{\varphi}(k)$; both scaling dimensions are same due to symmetry) and

$$Z_b := \frac{\text{effective coeff. of } \omega^2}{\text{coeff of } \omega^2} = 1. \quad (4.64)$$

Note that the exact same scaling is also performed for the Grassman fields and therefore we do not explicitly write them. Note an important fact that Z_b can not be obtained using dimensional analysis and only in this context where we have neglected the contribution from density perturbations, Z is

1. If write the formal action

$$S_\omega[\varphi, \tilde{\varphi}] := \int^\Lambda (\omega^2 \varphi(k) \cdot \tilde{\varphi}(-k) + (\lambda + 2\mu) k^2 \varphi(k) \cdot \tilde{\varphi}(-k) + \mu(k \cdot \varphi(k))(k \cdot \tilde{\varphi}(-k))) d^N k + \text{quartic terms} \quad (4.65)$$

the operator dimension of φ and $\tilde{\varphi}$ may be read off to yield (l denotes length scale)

$$\Delta_{\varphi, \tilde{\varphi}} = \frac{N+2}{2} \text{ since } \varphi, \tilde{\varphi} \approx l^{-\frac{N+2}{2}}. \quad (4.66)$$

Let us now go back to the large length scale (course grained) effective action in the Fourier domain to obtain a scaling for the coupling constants. Here we specifically focus on the bosonic part since the fermionic part yields exactly the same expression.

$$\begin{aligned} S_{<}[\varphi_{<}, \tilde{\varphi}_{<}] &= \int^\Lambda (\omega^2 \varphi_{<}(k) \cdot \tilde{\varphi}_{<}(-k) + (\lambda_{<} + \mu_{<}) k^2 \varphi_{<}(k) \cdot \tilde{\varphi}_{<}(-k) \\ &\quad + \mu_{<}(k \cdot \varphi_{<}(k))(k \cdot \tilde{\varphi}_{<}(-k))) d^N k + \text{grassman terms} + \text{quartic terms} \\ &= \int^{\Lambda_0} \left(Z_b \omega^2 b^{N+2} \varphi_{<}(k') \cdot \tilde{\varphi}_{<}(-k') + Z_b (\lambda_{<} + \mu_{<}) k'^2 b^{-2} b^{N+2} \varphi_{<}(k') \cdot \tilde{\varphi}_{<}(-k') \right. \\ &\quad \left. Z_b b^{-2} b^{N+2} \mu_{<}(k' \cdot \varphi_{<}(k))(k' \cdot \tilde{\varphi}_{<}(-k)) \right) b^{-N} d^N k' + \text{grassman} + \text{quartic terms} \\ &= \int^{\Lambda_0} \left(Z_b \omega^2 b^{N+2} \varphi_{<}(k') \cdot \tilde{\varphi}_{<}(-k') + Z_b (\lambda_{<} + \mu_{<}) k'^2 \varphi_{<}(k') \cdot \tilde{\varphi}_{<}(-k') \right. \\ &\quad \left. + Z_b \mu_{<}(k' \cdot \varphi_{<}(k))(k' \cdot \tilde{\varphi}_{<}(-k)) \right) d^N k' + \text{grassman} + \text{quartic terms}. \end{aligned} \quad (4.67)$$

Comparing powers of k'

$$\lambda' = Z_b \lambda_{<}, \mu' = Z_b \mu_{<} \quad (4.68)$$

We will keep the coefficients of the kinetic term unchanged which puts a constraint on Z_b . However, since we are not interested in the density part, we simply set it to 1. Therefore we have the equation for λ' and μ' as follows

$$\lambda' = Z_b \left(\lambda_0 - \frac{5}{2} g_b^0 \int_\Lambda^{\Lambda_0} \frac{\zeta(0) k^2}{-\omega^2 + (\lambda_0 + 2\mu_0) k^2} d^N k + \frac{4g_r^0}{N(N+2)} \int_\Lambda^{\Lambda_0} \frac{\zeta(k)(\lambda + \mu) k^4 d^N k}{(-\omega^2 + (\lambda_0 + 2\mu_0) k^2)(-\omega^2 + \mu k^2)} \right)$$

and

$$\mu' = Z_b(\mu_0 - \frac{3}{2}g_r^0 \int_{\Lambda}^{\Lambda_0} \frac{\zeta(0)k^2 d^N k}{-\omega^2 + (\lambda_0 + 2\mu_0)k^2} + \frac{4g_r^0}{N(N+2)} \int_{\Lambda}^{\Lambda_0} \frac{(\lambda + \mu)\zeta(k)k^4 d^N k}{(-\omega^2 + (\lambda_0 + 2\mu_0)k^2)(-\omega^2 + \mu_0 k^2)} \\ - \frac{2g_r^0}{N} \int_{\Lambda}^{\Lambda_0} \frac{\zeta(0)k^2}{-\omega^2 + (\lambda_0 + 2\mu_0)k^2} (1 + \frac{(\lambda + \mu)k^2}{-\omega^2 + \mu_0 k^2}))$$

Now let us define the logarithmic length scale $l = \ln(b)$. We neglect $O(l)$ term i.e., $\Lambda \rightarrow \Lambda_0$, $\lambda_0 \rightarrow \lambda'/Z_b, \mu_0 \rightarrow \mu'/Z_b$, and noting $Z_b = 1$. Now it's convenient to work with scale-free variables. Let us re-scale the untouched couplings

$$\omega^2 \mapsto \omega^2/\Lambda_0^2, \quad g_b' = g_b^0 \Lambda_0^N, \quad g_r' = g_r^0 \Lambda_0^N \quad (4.69)$$

which yields the following system of coupled differential equations for the effective constants

$$\partial_l \lambda' = -\frac{5\zeta(0)}{2} \frac{vol(\mathbb{S}^{N-1})Z_b g_b'}{-\omega^2 + (\lambda' + 2\mu')} + \frac{\zeta(\Lambda_0 e^{-l})}{N(N+2)} \frac{4Z_b vol(\mathbb{S}^{N-1})g_r'(\lambda' + \mu')}{(-\omega^2 + (\lambda' + 2\mu'))(-\omega^2 + \mu')} \\ \partial_l \mu' = -\frac{3\zeta(0)}{2} \frac{Z_b vol(\mathbb{S}^{N-1})g_r'}{-\omega^2 + (\lambda' + \mu')} + \frac{\zeta(k)}{N(N+2)} \frac{4Z_b vol(\mathbb{S}^{N-1})g_r'(\lambda' + \mu')}{(-\omega^2 + (\lambda' + 2\mu'))(-\omega^2 + \mu')} \\ - \frac{\zeta(0)}{N} \frac{2Z_b vol(\mathbb{S}^{N-1})g_r'}{-\omega^2 + (\lambda' + 2\mu')} (1 + \frac{(\lambda' + \mu')}{-\omega^2 + \mu'}), \\ \partial_l g_b' = O(g_b'^2, g_r'^2), \\ \partial_l g_r' = O(g_r'^2). \quad (4.70)$$

A trivial calculation shows that there are no fixed points for $N > 1$ except for the trivial ones i.e., for $g_b' = g_r' = 0$. This is expected physically since there is no competitive forces present in the system. In order to make the physics more interesting one may consider a correlation function ζ which contains two opposing factor: purely random part and a correlated part and focus on the renormalization group flow for those part only. In such case, one may hope to find a critical behaviour i.e., infinite correlation length etc. Such competing cases in a non-conservative setting (e.g., viscoelastic) may have applications to the earthquake theory since the point of failure of an earthquake is conjectured to be in a self-organized criticality class [125, 126, 127, 128].

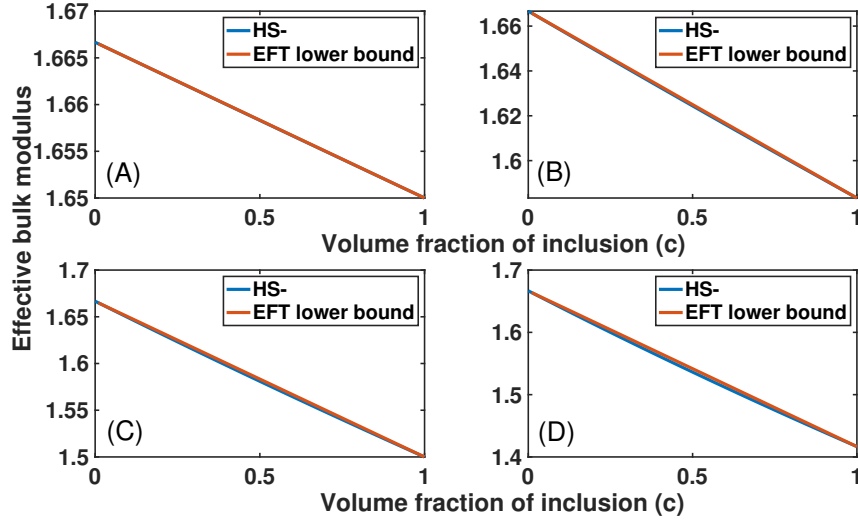


Figure 4.5: The scale-free effective calculations for the bulk modulus K with the variance of the perturbations (A) $g_b = 0.0001 = g_r$, (B) $g_b = 0.0025 = g_r$, (C) $g_b = 0.01 = g_r$, and (D) $g_b = 0.0225 = g_r$ (1%, 5%, 10%, 15%). The background constants are as follows $\lambda_0 = 1 = \mu_0$ and $K_0 = \lambda_0 + \frac{2\mu_0}{3}$.

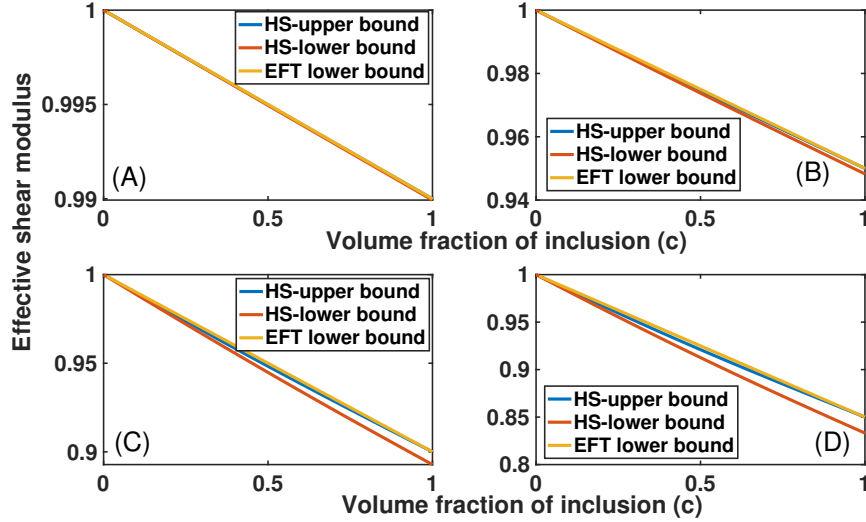


Figure 4.6: The scale-free effective calculations for the shear modulus μ with the variance of the perturbations (A) $g_b = 0.0001 = g_r$, (B) $g_b = 0.0025 = g_r$, (C) $g_b = 0.01 = g_r$, and (D) $g_b = 0.0225 = g_r$ (1%, 5%, 10%, 15%). The background constants are as follows $\lambda_0 = 1 = \mu_0$ and $K_0 = \lambda_0 + \frac{2\mu_0}{3}$.

4.7 Discussion

In this chapter we have developed a theory of homogenization and renormalization group flow of an elastic medium utilizing technique based on QFT or SFT. As mentioned in the introduction, there are few advantages to such a field theory approach. Firstly, we work at the level of a suitably defined 'action' functional (equivalently a Lagrangian) of the theory rather than at the level of equations of motion. This allows one to handle the boundary conditions in a trivial way. In addition, it provides us with a self consistent perturbation series for the effective elastic constants of a heterogeneous medium where the heterogeneity can be of any type as long as it has Fourier transform (i.e., let us say that L^1 bounded for simplicity) at least at the formal level. Secondly, our technique can handle the frequency effect which is not derivable in a straightforward way while working at the level of the equations of motion. Of course, physically realistic cases will have this dispersive effect and therefore is important. One of the main disadvantage of our technique as of now is its perturbative character i.e., it is not valid if the elastic constants of the inclusions differ substantially from that of the matrix. Our result matches with that of Hashin [108] only in the scenario when the difference is small. However, with a little more effort one may be able to complete the perturbation series and obtain convergence (even if one needs to perform re-summation trick). In addition, our technique can address several interesting physics associated with the wave propagation through the complex medium. For scattering amplitudes of wave propagation through a random medium, we refer to a classical study of Aki [129, 130, 131]. In general, there are numerous studies on this topic addressing different aspects of this rich field. Of course, one well known result is that the effective elastic wave speed reduces in a purely random medium and there are few mathematically rigorous studies conforming such fact [132]. Our analysis directly confirms such a result as seen from the figure (4.4). The physical reason for such a behaviour may be addressed by further study. In addition, we can address the issue of critical phenomena if there are heterogeneities of two different nature present in the medium which as we mentioned previously may have applications to the earthquake theory. It will also be interesting to compare our results of homogenization with the experiments. In a sense, we are just exploring the tip of the iceberg. This technique is so rich and applied to numerous branches in physics including condensed matter theory to obtain interesting physical results.

Chapter 5

Appendix

This chapter concerns with the detailed calculations behind chapter 4. In this section, we show the systematic calculations of the effective field theory. For obvious reason, we work in the Fourier domain. Notice the action functional

$$\begin{aligned}
S[\varphi, \tilde{\varphi}] := & \int ((-\omega^2 \varphi^I \tilde{\varphi}_I + C_0^{IJKL} \partial_J \tilde{\varphi}_I \partial_K \varphi_L) - \omega^2 \bar{\psi}^I \psi_I \\
& + C_0^{IJKL} \partial_J \bar{\psi}_I \partial_K \psi_L - \frac{1}{4i} g_b \int (\partial_I \tilde{\varphi}^I(x) \partial^J \varphi_J(x) + \partial_I \bar{\psi}^I(x) \partial^J \psi_J(x)) \zeta(x - x') \\
& (\partial_I \tilde{\varphi}^I(x') \partial^J \varphi_J(x') + \partial_I \bar{\psi}^I(x') \partial^J \psi_J(x')) d^N x' \\
& - \frac{1}{4i} g_r \int (\partial_I \tilde{\varphi}^J(x) \partial_J \varphi^I(x) + \partial_I \tilde{\varphi}^J(x) \partial^I \varphi_J(x) + \partial_I \bar{\psi}^J(x) \partial_J \psi^I(x) + \partial_I \bar{\psi}^J(x) \partial^I \psi_J(x)) \\
& \zeta_r(x - x') (\partial_I \tilde{\varphi}^J(x') \partial_J \varphi^I(x') + \partial_I \tilde{\varphi}^J(x') \partial^I \varphi_J(x') + \partial_I \bar{\psi}^J(x') \partial_J \psi^I(x') + \partial_I \bar{\psi}^J(x') \partial^I \psi_J(x')) d^N x' d^N x.
\end{aligned} \tag{5.1}$$

Using the wavenumber separation

$$\varphi = \varphi(k)_{>}\Theta(|k| - \Lambda) + \varphi(k)_{<}\Theta(-|k| + \Lambda), \tilde{\varphi} = \tilde{\varphi}(k)_{>}\Theta(|k| - \Lambda) + \tilde{\varphi}(k)_{<}\Theta(-|k| + \Lambda), \tag{5.2}$$

$$\psi = \psi(k)_{>}\Theta(|k| - \Lambda) + \psi(k)_{<}\Theta(-|k| + \Lambda), \bar{\psi} = \bar{\psi}(k)_{>}\Theta(|k| - \Lambda) + \bar{\psi}(k)_{<}\Theta(-|k| + \Lambda), \tag{5.3}$$

we explicitly evaluate several terms of the action functional. **1.**

$$\begin{aligned}
& - \int_x \int_{k_1, k_2} \omega^2 \varphi^I(k_1) \tilde{\varphi}(k_2) \exp(ik_1 \cdot x + ik_2 \cdot x) \\
& d^N k_1 d^N k_2 d^N x \\
& = - \int_k \omega^2 \varphi^I(k_1) \tilde{\varphi}_I(k_2) \delta(k_1 + k_2) d^N k_1 d^N k_2
\end{aligned}$$

2.

$$\int_x C_0^{IJKL} \partial_I \tilde{\varphi}_J \partial_K \varphi_L d^N x = - \int_k C_0^{IJKL} k_1^I k_2^J \delta(k_1 + k_2) \varphi_L(k_1) \tilde{\varphi}_J(k_2) d^N k_1 d^N k_2.$$

3.

$$\begin{aligned} & g_b^0 \int_x (\partial_I \tilde{\varphi}^I \partial_J \varphi^J)^2 d^N x \\ &= g_b^0 \int_k k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_I(k_1) \varphi_J(k_2) \tilde{\varphi}_K(k_3) \varphi_L(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\ & \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4 \end{aligned} \quad (5.4)$$

4.

$$\begin{aligned} & g_r^0 \int_x (\partial_I \tilde{\varphi}^J \partial_J \varphi^I \partial_K \tilde{\varphi}^L \partial_L \varphi^K) d^N x \\ &= g_r^0 \int_k k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_J(k_1) \varphi_I(k_2) \tilde{\varphi}_L(k_3) \varphi_K(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\ & \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4 \end{aligned} \quad (5.5)$$

5.

$$\begin{aligned} & 2g_r^0 \int_x (\partial_I \tilde{\varphi}^J \partial_J \varphi^I \partial_K \tilde{\varphi}^L \partial^K \varphi_L) d^N x \\ &= 2g_r^0 \int_k k_1^I k_2^J k_3^K k_{4K} \tilde{\varphi}_J(k_1) \varphi_I(k_2) \tilde{\varphi}^L(k_3) \varphi_L(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\ & \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4 \end{aligned} \quad (5.6)$$

6.

$$\begin{aligned} & g_r^0 \int_x (\partial_I \tilde{\varphi}^J \partial^I \varphi_J \partial_K \tilde{\varphi}^L \partial^K \varphi_L) d^N x \\ &= g_r^0 \int_k k_{1I} k_2^I k_{3K} k_4^K \tilde{\varphi}^J(k_1) \varphi_J(k_2) \tilde{\varphi}^L(k_3) \varphi_L(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\ & \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4. \end{aligned} \quad (5.7)$$

Now there are mixed terms where vector-Grassman fields appear together. The mixed term is simply

$$\begin{aligned}
& 2g_r^0(\partial_I \tilde{\varphi}^J \partial_J \varphi^I + \partial_I \tilde{\varphi}^J \partial^I \varphi_J)(\partial_I \bar{\psi}^J \partial_J \psi^I + \partial_I \bar{\psi}^J \partial^I \psi_J) \\
& = 2g_r^0(\partial_I \tilde{\varphi}^J \partial_J \varphi^I \partial_K \bar{\psi}^L \partial_L \psi^K + \partial_I \tilde{\varphi}^J \partial_J \varphi^I \partial_K \bar{\psi}^L \partial^K \psi_L + \partial_I \tilde{\varphi}^J \partial^I \varphi_J \partial_K \bar{\psi}^L \partial_L \psi^K \\
& \quad + \partial_I \tilde{\varphi}^J \partial^I \varphi_J \partial_K \bar{\psi}^L \partial^K \psi_L)
\end{aligned} \tag{5.8}$$

This is so because of the following fact

$$[\varphi, \psi] = [\varphi, \bar{\psi}] = [\tilde{\varphi}, \psi] = [\tilde{\varphi}, \bar{\psi}] = 0 \tag{5.9}$$

since φ and ψ live on separate spaces. In the Fourier space, once again we have the following

7.

$$\begin{aligned}
& 2g_b^0 \int_x \partial_I \tilde{\varphi}^I \partial_J \varphi^J \partial_K \bar{\psi}^K \partial_L \psi^L d^N x \\
& = 2g_b^0 \int_k k_{1I} k_{2J} k_{3K} k_{4L} \tilde{\varphi}^I(k_1) \varphi^J(k_2) \bar{\psi}^K(k_3) \psi^L(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\
& \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4
\end{aligned} \tag{5.10}$$

8.

$$\begin{aligned}
& 2g_r^0 \int_x \partial_I \tilde{\varphi}^J \partial_J \varphi^I \partial_K \bar{\psi}^L \partial_L \psi^K d^N x \\
& = \int_k k_{1I} k_{2J} k_{3K} k_{4L} \tilde{\varphi}^J(k_1) \varphi^I(k_2) \bar{\psi}^L(k_3) \psi^K(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\
& \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4
\end{aligned} \tag{5.11}$$

9.

$$\begin{aligned}
& 2g_r^0 \int_x \partial_I \tilde{\varphi}^J \partial_J \varphi^I \partial_K \bar{\psi}^L \partial^K \psi_L d^N x \\
& = \int_k k_{1I} k_{2J} k_{3K} k^{4K} \tilde{\varphi}^J(k_1) \varphi^I(k_2) \bar{\psi}^L(k_3) \psi_L(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\
& \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4
\end{aligned} \tag{5.12}$$

10.

$$\begin{aligned}
& 2g_r^0 \int_x \partial_I \tilde{\varphi}^J \partial^I \varphi_J \partial_K \bar{\psi}^L \partial_L \psi^K d^N x \\
&= 2g_r^0 \int_k k_{1I} k^{2I} k_{3K} k_{4L} \tilde{\varphi}^J(k_1) \varphi_J(k_2) \bar{\psi}^L(k_3) \psi^K(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\
& \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4
\end{aligned} \tag{5.13}$$

11.

$$\begin{aligned}
& 2g_r^0 \int_x \partial_I \tilde{\varphi}^J \partial^I \varphi_J \partial_K \bar{\psi}^L \partial^K \psi_L d^N x \\
&= 2g_r^0 \int_k k_{1I} k^{2I} k_{3K} k^{4K} \tilde{\varphi}^J(k_1) \varphi_J(k_2) \bar{\psi}^L(k_3) \psi_L(k_4) \zeta(k_3 + k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\
& \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4.
\end{aligned} \tag{5.14}$$

The appearance of $\delta(\sum_i k_i)$ will ensure the energy-momentum conservation at a vertex. The quartic terms exactly same as 1, 2, 3, 4, 5, 6 appears for the spinor fields ψ and $\bar{\psi}$. We do not write these terms since they contribute exactly same as the bosonic parts. From here one we will denote (ω, k) simply by k for convenience and $\delta(\sum_{i=1}^4 k_i)$. Now we perform the wave number separation and write it out term by term

1.

$$\begin{aligned}
& \int_k \omega^2 \varphi^I(k) \tilde{\varphi}_I(-k) d^N k = \int_k \omega^2 (\varphi^I(k)_{>\Theta(|k|-\Lambda)} + \varphi^I(k)_{<\Theta(-|k|+\Lambda)}) \\
& \quad (\tilde{\varphi}_I(k)_{>\Theta(|k|-\Lambda)} + \tilde{\varphi}_I(k)_{<\Theta(-|k|+\Lambda)}) d^N k \\
&= \int_{k=\Lambda}^{\Lambda_0} \omega^2 \varphi_{>}^I(k) \tilde{\varphi}_{I>}(-k) d^N k + \int_k^{\Lambda} \omega^2 \varphi_{<}^I(k) \tilde{\varphi}_{I<}(-k) d^N k
\end{aligned}$$

2.

$$\begin{aligned}
& \int_k C_0^{IJKL} k_I k_K \varphi_I(k) \tilde{\varphi}_J(-k) d^N k \\
&= \int_k C_0^{IJKL} k_I k_K (\varphi_J(k)_{>\Theta(|k|-\Lambda)} + \varphi_J(k)_{<\Theta(-|k|+\Lambda)}) \\
& \quad (\tilde{\varphi}_L(k)_{>\Theta(|k|-\Lambda)} + \tilde{\varphi}_L(k)_{<\Theta(-|k|+\Lambda)}) d^N k \\
&= \int_k^{\Lambda} C_0^{IJKL} k_I k_K \varphi_J(k)_{<} \tilde{\varphi}_L(k)_{<} d^N k + \int_{\Lambda}^{\Lambda_0} C_0^{IJKL} k_I k_K \varphi_J(k)_{>} \tilde{\varphi}_L(k)_{>} d^N k
\end{aligned} \tag{5.15}$$

Now we do the separation of the 4-th order terms. Term-wise

3.

$$\begin{aligned}
& g_b^0 \int_k k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_I(k_1) \varphi_J(k_2) \tilde{\varphi}_K(k_3) \varphi_L(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&= g_b^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L>}(k_4) \\
&\quad \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&\quad + g_b^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{I<}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \\
&\quad \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&\quad + g_b^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L<}(k_4) + 2\tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \\
&\quad \tilde{\varphi}_{K<}(k_3) \varphi_{L>}(k_4) + 2\tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \\
&\quad + \tilde{\varphi}_{I<}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L>}(k_4)) k_1^I k_2^J k_3^K k_4^L \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&\quad + g_b^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) + \\
&\quad + \tilde{\varphi}_{I<}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L>}(k_4) + \tilde{\varphi}_{I<}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \\
&\quad + \tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L>}(k_4) + \tilde{\varphi}_{I<}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L<}(k_4) \\
&\quad + \tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L>}(k_4) + \tilde{\varphi}_{I<}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L>}(k_4) \\
&\quad + \tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L<}(k_4)) k_1^I k_2^J k_3^K k_4^L \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4
\end{aligned} \tag{5.16}$$

We may observe the following symmetry relation under integration

$$\begin{aligned}
& \tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) k_1^I k_2^J k_3^K k_4^L \\
&= \tilde{\varphi}_{I<}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L<}(k_4) k_1^I k_2^J k_3^K k_4^L
\end{aligned} \tag{5.17}$$

and similar relations hold for other combinations. In a sense the 8 mixed terms with equal number of high and low component reduce to 4. Therefore The first mixed term yields

4.

$$g_b^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L>}(k_4) \quad (5.18)$$

$$\delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4$$

$$+ g_b^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{I<}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \quad (5.19)$$

$$\delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4$$

$$\begin{aligned} & + g_b^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L<}(k_4) + 2\tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \\ & \quad \tilde{\varphi}_{K<}(k_3) \varphi_{L>}(k_4) + 2\tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \\ & + \tilde{\varphi}_{I<}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L>}(k_4)) k_1^I k_2^J k_3^K k_4^L \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\ & + 2g_b^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) + \\ & \quad + \tilde{\varphi}_{I<}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L>}(k_4) + \tilde{\varphi}_{I<}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \\ & + \tilde{\varphi}_{I>}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L>}(k_4)) k_1^I k_2^J k_3^K k_4^L \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \end{aligned}$$

Now we focus on the second term

5.

$$\begin{aligned}
& g_r^0 \int_k k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_J(k_1) \varphi_I(k_2) \tilde{\varphi}_L(k_3) \varphi_K(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& = g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K>}(k_4) \\
& \quad \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& \quad + g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K<}(k_4) \quad (5.20) \\
& \quad \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& \quad + g_r^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{J>}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K<}(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \\
& \quad \tilde{\varphi}_{L>}(k_3) \varphi_{K>}(k_4) + \tilde{\varphi}_{J>}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K>}(k_4) + \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K<}(k_4) \\
& \quad + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K<}(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K>}(k_4)) \\
& \quad k_1^I k_2^J k_3^K k_4^L \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& \quad + g_r^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{J>}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K<}(k_4) + \\
& \quad + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K>}(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K<}(k_4) \\
& \quad + \tilde{\varphi}_{J>}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K>}(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K<}(k_4) \\
& \quad + \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K>}(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K>}(k_4) \\
& \quad + \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K<}(k_4)) k_1^I k_2^J k_3^K k_4^L \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4.
\end{aligned}$$

The third term is evaluated as

6.

$$\begin{aligned}
& 2g_r^0 \int_k k_1^I k_2^J k_3^K k_{K4} \tilde{\varphi}_J(k_1) \varphi_I(k_2) \tilde{\varphi}^L(k_3) \varphi_L(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&= 2g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_{K4} \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{>}^L(k_4) \\
&\quad \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&\quad + 2g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_{K4} \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) \quad (5.21) \\
&\quad \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&\quad + 2g_r^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{J>}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{<}^L(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \\
&\quad \tilde{\varphi}_{L>}(k_3) \varphi_{>}^L(k_4) + \tilde{\varphi}_{J>}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{>}^L(k_4) + \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) \\
&\quad + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{<}^L(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{>}^L(k_4)) \\
&\quad k_1^I k_2^J k_3^K k_{K4} \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&\quad + 2g_r^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{J>}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) + \\
&\quad + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K>}(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) \\
&\quad + \tilde{\varphi}_{J>}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K>}(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{<}^L(k_4) \\
&\quad + \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{>}^L(k_4) + \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{>}^L(k_4) \\
&\quad + \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{<}^L(k_4)) k_1^I k_2^J k_3^K k_{K4} \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4
\end{aligned}$$

The last bosonic term yields

7.

$$\begin{aligned}
& g_r^0 \int_k k_{1I} k_2^I k_{3K} k_4^K \tilde{\varphi}^J(k_1) \varphi_J(k_2) \tilde{\varphi}^L(k_3) \varphi_L(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& = g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_{I2} k_3^K k_{K4} \tilde{\varphi}_{>}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L>}^L(k_3) \varphi_{>L}^L(k_4) \\
& \quad \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& \quad + g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_{I2} k_3^K k_{K4} \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L<}^L(k_3) \varphi_{<L}^L(k_4) \quad (5.22) \\
& \quad \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& \quad + g_r^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{>}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L>}^L(k_3) \varphi_{<L}^L(k_4) + \tilde{\varphi}_{<}^J(k_1) \varphi_{J>}(k_2) \\
& \quad \tilde{\varphi}_{L>}^L(k_3) \varphi_{>L}^L(k_4) + \tilde{\varphi}_{>}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L<}^L(k_3) \varphi_{<L}^L(k_4) + \tilde{\varphi}_{<}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L<}^L(k_3) \varphi_{<L}^L(k_4) \\
& \quad + \tilde{\varphi}_{<}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L>}^L(k_3) \varphi_{<L}^L(k_4) + \tilde{\varphi}_{<}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L<}^L(k_3) \varphi_{>L}^L(k_4)) \\
& \quad k_1^I k_{I2} k_3^K k_{K4} \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& \quad + g_r^0 \int_k^{\Lambda} \int_{\Lambda}^{\Lambda_0} (\tilde{\varphi}_{>}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L<}^L(k_3) \varphi_{<L}^L(k_4) + \\
& \quad + \tilde{\varphi}_{<}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L>}^L(k_3) \varphi_{K>}^L(k_4) + \tilde{\varphi}_{<}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L<}^L(k_3) \varphi_{<L}^L(k_4) \\
& \quad + \tilde{\varphi}_{>}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L>}^L(k_3) \varphi_{K>}^L(k_4) + \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L>}^L(k_3) \varphi_{<L}^L(k_4) \\
& \quad + \tilde{\varphi}_{>}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L<}^L(k_3) \varphi_{>L}^L(k_4) + \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L<}^L(k_3) \varphi_{>L}^L(k_4) \\
& \quad + \tilde{\varphi}_{>}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L>}^L(k_3) \varphi_{<L}^L(k_4)) k_1^I k_{I2} k_3^K k_{K4} \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4.
\end{aligned}$$

Since the spinor part ($\bar{\psi}$ and ψ) produce exactly similar terms at the quartic level, we will not explicitly write down the terms. However, there are quartic mixed parts which need to be taken into consideration. However, notice that because of the symmetric nature, we will only need just one term from each mixed term in the actual calculation.

8.

$$\begin{aligned}
& 2g_b^0 \int_k k_{1I} k_{2J} k_{3K} k_{4L} \tilde{\varphi}_{<}^I(k_1) \varphi_{<}^J(k_2) \bar{\psi}_{>}^K(k_3) \psi_{>}^L(k_4) \delta\left(\sum_{i=1}^4 k_i\right) \\
& \quad d^N k_1 d^N k_2 d^N k_3 d^N k_4
\end{aligned}$$

9.

$$2g_r^0 \int_k k_{1I} k_{2J} k_{3K} k_{4L} \tilde{\varphi}_{<}^J(k_1) \varphi_{<}^I(k_2) \bar{\psi}_{>}^L(k_3) \psi_{>}^K(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4$$

10.

$$2g_r^0 \int_k k_{1I} k_{2J} k_{3K} k^{4K} \tilde{\varphi}_{<}^J(k_1) \varphi_{<}^I(k_2) \bar{\psi}_{>}^L(k_3) \psi_{L>}(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4$$

11.

$$2g_r^0 \int_k k_{1I} k^{2I} k_{3K} k_{4L} \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) \bar{\psi}_{>}^L(k_3) \psi_{>}^K(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4$$

12.

$$2g_r^0 \int_k k_{1I} k^{2I} k_{3K} k^{4K} \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) \bar{\psi}_{>}^L(k_3) \psi_{L>}(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4$$

Collecting all the terms

$$\begin{aligned} S_{<\Lambda}[\varphi_{<}, \tilde{\varphi}_{<}; C_0^{IJKL}, g_b^0, g_r^0] &= \omega^2 \int_k^\Lambda \varphi_{<}^I(k) \tilde{\varphi}_{I<}(-k) d^N k + \int_k^\Lambda C_0^{IJKL} k_I k_K \varphi_{J<}(k) \tilde{\varphi}_{L<}(k) d^N k \\ &+ g_b^0 \int_k^\Lambda k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{I<}(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\ &+ g_r^0 \int_k^\Lambda k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{K<}(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\ &+ 2g_r^0 \int_k^\Lambda k_1^I k_2^J k_3^K k_{K4} \tilde{\varphi}_{J<}(k_1) \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\ &+ g_r^0 \int_k^\Lambda k_1^I k_{I2} k_3^K k_{K4} \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \end{aligned}$$

The small scale part

$$\begin{aligned}
S_{>\Lambda}[\varphi_{>}, \tilde{\varphi}_{>}] &= \omega^2 \int_{\Lambda}^{\Lambda_0} \varphi_{>}^I(k) \tilde{\varphi}_{I>}(-k) d^N k + \int_{\Lambda}^{\Lambda_0} C_0^{IJKL} k_I k_K \varphi_{J>}(k) \tilde{\varphi}_{L>}(k) d^N k \\
&+ g_b^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K>}(k_3) \varphi_{L>}(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&+ g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{K>}(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&+ 2g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_2^J k_3^K k_{K4} \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{>}^L(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&+ g_r^0 \int_{\Lambda}^{\Lambda_0} k_1^I k_{I2} k_3^K k_{K4} \tilde{\varphi}_{>}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{>}^L(k_4) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4
\end{aligned}$$

and there are mixed part (the ones that contribute towards effective action at the level of 1-loop).

We will write the mixed part of the action integral later.

$$\begin{aligned}
-iS^{mix}[><] = & g_b^0 \int_k^\Lambda \int_\Lambda^{\Lambda_0} (\tilde{\varphi}_{I>}(k_1)\varphi_{J<}(k_2)\tilde{\varphi}_{K>}(k_3)\varphi_{L<}(k_4) + 2\tilde{\varphi}_{I>}(k_1)\varphi_{J<}(k_2) \\
& \tilde{\varphi}_{K<}(k_3)\varphi_{L>}(k_4) + 2\tilde{\varphi}_{I>}(k_1)\varphi_{J>}(k_2)\tilde{\varphi}_{K<}(k_3)\varphi_{L<}(k_4) \\
& + \tilde{\varphi}_{I<}(k_1)\varphi_{J>}(k_2)\tilde{\varphi}_{K<}(k_3)\varphi_{L>}(k_4)) k_1^I k_2^J k_3^K k_4^L \delta(\sum_{i=1}^4 k_i) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& + g_r^0 \int_k^\Lambda \int_\Lambda^{\Lambda_0} (\tilde{\varphi}_{J>}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{K<}(k_4) + \tilde{\varphi}_{J<}(k_1)\varphi_{I<}(k_2) \\
& \tilde{\varphi}_{L>}(k_3)\varphi_{K>}(k_4) + \tilde{\varphi}_{J>}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{K>}(k_4) + \tilde{\varphi}_{J>}(k_1)\varphi_{I>}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{K<}(k_4) \\
& + \tilde{\varphi}_{J<}(k_1)\varphi_{I>}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{K<}(k_4) + \tilde{\varphi}_{J<}(k_1)\varphi_{I>}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{K>}(k_4)) \\
& k_1^I k_2^J k_3^K k_4^L \delta(\sum_{i=1}^4 k_i) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& + 2g_r^0 \int_k^\Lambda \int_\Lambda^{\Lambda_0} (\tilde{\varphi}_{J>}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{<}^L(k_4) + \tilde{\varphi}_{J<}(k_1)\varphi_{I<}(k_2) \\
& \tilde{\varphi}_{L>}(k_3)\varphi_{>}^L(k_4) + \tilde{\varphi}_{J>}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{>}^L(k_4) + \tilde{\varphi}_{J>}(k_1)\varphi_{I>}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{<}^L(k_4) \\
& + \tilde{\varphi}_{J<}(k_1)\varphi_{I>}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{<}^L(k_4) + \tilde{\varphi}_{J<}(k_1)\varphi_{I>}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{>}^L(k_4)) \\
& k_1^I k_2^J k_3^K k_{K4} \delta(\sum_{i=1}^4 k_i) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& + g_r^0 \int_k^\Lambda \int_\Lambda^{\Lambda_0} (\tilde{\varphi}_{>}^J(k_1)\varphi_{J<}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{<}^L(k_4) + \tilde{\varphi}_{<}^J(k_1)\varphi_{J<}(k_2) \\
& \tilde{\varphi}_{L>}(k_3)\varphi_{>}^L(k_4) + \tilde{\varphi}_{>}^J(k_1)\varphi_{J<}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{>}^L(k_4) + \tilde{\varphi}_{>}^J(k_1)\varphi_{J>}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{<}^L(k_4) \\
& + \tilde{\varphi}_{<}^J(k_1)\varphi_{J>}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{<}^L(k_4) + \tilde{\varphi}_{<}^J(k_1)\varphi_{J>}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{>}^L(k_4)) \\
& k_1^I k_{I2} k_3^K k_{K4} \delta(\sum_{i=1}^4 k_i) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& + 2g_b^0 \int_k k_{1I} k_{2J} k_{3K} k_{4L} \tilde{\varphi}_{<}^I(k_1)\varphi_{<}^J(k_2)\bar{\psi}_{>}^K(k_3)\psi_{>}^L(k_4)\delta(\sum_{i=1}^4 k_i) \\
& d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& + 2g_r^0 \int_k k_{1I} k_{2J} k_{3K} k_{4L} \tilde{\varphi}_{<}^J(k_1)\varphi_{<}^I(k_2)\bar{\psi}_{>}^L(k_3)\psi_{>}^K(k_4)\delta(\sum_{i=1}^4 k_i) \\
& d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& + 2g_r^0 \int_k k_{1I} k_{2J} k_{3K} k^{4K} \tilde{\varphi}_{<}^J(k_1)\varphi_{<}^I(k_2)\bar{\psi}_{>}^L(k_3)\psi_{L>}(k_4)\delta(\sum_{i=1}^4 k_i) \\
& d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& + 2g_r^0 \int_k k_{1I} k^{2I} k_{3K} k_{4L} \tilde{\varphi}_{<}^J(k_1)\varphi_{J<}(k_2)\bar{\psi}_{>}^L(k_3)\psi_{>}^K(k_4)\delta(\sum_{i=1}^4 k_i) \\
& d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
& + 2g_r^0 \int_k k_{1I} k^{2I} k_{3K} k^{4K} \tilde{\varphi}_{<}^J(k_1)\varphi_{J<}(k_2)\bar{\psi}_{>}^L(k_3)\psi_{L>}(k_4)\delta(\sum_{i=1}^4 k_i)
\end{aligned}$$

5.0.1 Graph expansion for the quadratic part

(N.B:- There is a factor $\pm i\epsilon$ in the denominator of the propagator; this is necessary in order for the integral to converge).

Now consider the quadratic part of the action $S_>$ and find the expectation $\langle S^{mix}[><] \rangle_{Z_{0>}}$, $Z_{0>} = \int e^{-S_>^{quad}} [d\varphi_>][d\tilde{\varphi}_>][d\bar{\psi}_>][d\bar{\psi}_>]$. We will perform this operation term by term. As an example consider the expectation value of the term $g_b^0 \int_k^\Lambda \int_{\Lambda_0}^\Lambda \delta(\sum_{i=1}^4 k_i) k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4)$. We may compute the expectation value explicitly in the Fourier domain using (4.43)

$$\begin{aligned} & \langle g_b^0 \int_k^\Lambda \int_{\Lambda_0}^\Lambda \delta(\sum_{i=1}^4 k_i) k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{I>}(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \rangle_{Z_{0>}} \\ &= \frac{g_b^0}{W[J, \tilde{J}, \mathcal{J}, \bar{\mathcal{J}}]} \int_k^\Lambda \int_{\Lambda_0}^\Lambda \delta(\sum_{i=1}^4 k_i) k_1^I k_2^J k_3^K k_4^L \tilde{\varphi}_{K<}(k_3) \varphi_{L<}(k_4) \frac{\delta}{J^J(k_2)} \frac{\delta}{i\tilde{J}^I(k_1)} \\ & \quad \exp(-\int J^I(k) G_{IJ}(k) \tilde{J}^J(-k) d^N k) \exp(-i \int \bar{\mathcal{J}}^I(k) G_{IJ}(k) \mathcal{J}^J(-k) d^N k) \\ & \quad \Big|_{J=\tilde{J}=\mathcal{J}=\bar{\mathcal{J}}=0} \end{aligned}$$

Even if we can perform the functional integration at each step, it is most convenient to write down the propagator rules and use them. The rules read

$$\begin{aligned} & \langle \tilde{\varphi}_M(k_1) \varphi_N(k_2) \rangle = -iG_{MN}(|k_1|) \delta(k_1 + k_2), \langle \tilde{\varphi}_M(k_1) \tilde{\varphi}_N(k_2) \rangle = 0, \\ & \langle \varphi_M(k_1) \varphi_N(k_2) \rangle = 0, \langle \bar{\psi}_M(k_1) \psi_N(k_2) \rangle = -iG_{MN}(|k_1|) \delta(k_1 + k_2), \\ & \langle \varphi_M(k_1) \bar{\psi}_M(k_1) \rangle = 0, \langle \varphi_M(k_1) \psi_M(k_1) \rangle = 0, \\ & \langle \tilde{\varphi}_M(k_1) \bar{\psi}_M(k_1) \rangle = 0, \langle \tilde{\varphi}_M(k_1) \psi_M(k_1) \rangle = 0. \end{aligned} \tag{5.24}$$

This rule is essentially quite similar in spirit with that of Yang-Mills theory quantization. Now it is actually straightforward to compute the expectation term by term. We denote the expectation with respect to $Z_{0>}$ simply by $>$. We number the terms according to their sequential appearance

1. $\mathbf{g}_b^0$

$$\begin{aligned}
& 4g_b^0 < \tilde{\varphi}_{I>}(k_1)\varphi_{J<}(k_2)\tilde{\varphi}_{K<}(k_3)\varphi_{L>}(k_4)k_1^I k_2^J k_3^K k_4^L >> \\
& = -4g_b^0 \int_k^\Lambda \varphi_{J<}(k_2)\tilde{\varphi}_{K<}(k_3)k_2^J k_3^K \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} k^I k^L G_{IL}(|k|) d^N k \\
& = -4g_b^0 \int_k^\Lambda \varphi_{J<}(k_2)\tilde{\varphi}_{K<}(k_3)k_2^J k_3^K \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(0)k^2}{-\omega^2 + (\lambda + 2\mu)k^2} d^N k
\end{aligned} \tag{5.25}$$

2.2 $\mathbf{g_r^0}$

$$\begin{aligned}
& g_r^0 < \tilde{\varphi}_{J<}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{K>}(k_4)k_1^I k_2^J k_3^K k_4^L >> \\
& = -g_r^0 \int_k^\Lambda \tilde{\varphi}_{J<}(k_1)\varphi_{I<}(k_2)k_1^I k_2^J \int_\Lambda^{\Lambda_0} \frac{\zeta(0)k^2 d^N k}{-\omega^2 + (\lambda + 2\mu)k^2}
\end{aligned} \tag{5.26}$$

2.3 $\mathbf{g_r^0}$

$$\begin{aligned}
& g_r^0 < \tilde{\varphi}_{J>}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{K>}(k_4)k_1^I k_2^J k_3^K k_4^L >> \\
& = -\frac{g_r^0}{N} \int_k^\Lambda \varphi_{L<}^L(k_2)\tilde{\varphi}_{L<}(k_3)(k_2 \cdot k_3)\delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(k)k^2}{-\omega^2 + (\lambda + 2\mu)k^2} \left(1 + \frac{(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}\right) \\
& \quad + \frac{g_r^0}{N(N+2)} \int_k^\Lambda (k_2^L k_3^I \tilde{\varphi}_{L<}(k_3)\varphi_{I<}(k_2) + k_2^I k_3^L \tilde{\varphi}_{L<}(k_3)\varphi_{I<}(k_2) \\
& \quad + (k_2 \cdot k_3)\varphi_{I<}(k_2)\tilde{\varphi}^I(k_3)) \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(k)(\lambda + \mu)k^4 d^N k}{(-\omega^2 + (\lambda + 2\mu)k^2)(-\omega^2 + \mu k^2)}
\end{aligned} \tag{5.27}$$

2.4 $\mathbf{g_r^0}$

$$\begin{aligned}
& g_r^0 < \tilde{\varphi}_{J>}(k_1)\varphi_{I>}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{K<}(k_4)k_1^I k_2^J k_3^K k_4^L >> \\
& = -g_r^0 \int_k^\Lambda \tilde{\varphi}_{J<}(k_1)\varphi_{I<}(k_2)k_1^I k_2^J \int_\Lambda^{\Lambda_0} \frac{\zeta(0)k^2 d^N k}{-\omega^2 + (\lambda + 2\mu)k^2}
\end{aligned} \tag{5.28}$$

2.5 $\mathbf{g_r^0}$

$$\begin{aligned}
& g_r^0 < \tilde{\varphi}_{J<}(k_1)\varphi_{I>}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{K<}(k_4)k_1^I k_2^J k_3^K k_4^L >> \\
& = -\frac{g_r^0}{N} \int_k^\Lambda \varphi_{L<}^L(k_2)\tilde{\varphi}_{L<}(k_3)(k_2 \cdot k_3)\delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(k)k^2}{-\omega^2 + (\lambda + 2\mu)k^2} \left(1 + \frac{(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}\right) \\
& \quad + \frac{g_r^0}{N(N+2)} \int_k^\Lambda (k_2^L k_3^I \tilde{\varphi}_{L<}(k_3)\varphi_{I<}(k_2) + k_2^I k_3^L \tilde{\varphi}_{L<}(k_3)\varphi_{I<}(k_2) \\
& \quad + (k_2 \cdot k_3)\varphi_{I<}(k_2)\tilde{\varphi}^I(k_3)) \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{(\lambda + \mu)\zeta(k)k^4 d^N k}{(-\omega^2 + (\lambda + 2\mu)k^2)(-\omega^2 + \mu k^2)}
\end{aligned} \tag{5.29}$$

3.1 $\mathbf{g_r^0}$

$$2g_r^0 < \tilde{\varphi}_{J>}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{<}^L(k_4)k_1^I k_2^J k_3^K k_{K4} >> = 0 \quad (5.30)$$

3.2 $\mathbf{g_r^0}$

$$\begin{aligned} & 2g_r^0 < \tilde{\varphi}_{J<}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L>}(k_3)\varphi_{>}^L(k_4)k_1^I k_2^J k_3^K k_{K4} >> \\ & = -2g_r^0 \int_k^\Lambda \tilde{\varphi}_J(k_1)\varphi_I(k_2)k_1^I k_2^J \delta(k_1 + k_2) \int_\Lambda^{\Lambda_0} \frac{\zeta(0)k^2}{-\omega^2 + (\lambda + 2\mu)k^2} \left(N + \frac{(N-1)(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}\right) d^N k \end{aligned} \quad (5.31)$$

3.3 $\mathbf{g_r^0}$

$$\begin{aligned} & 2g_r^0 < \tilde{\varphi}_{J>}(k_1)\varphi_{I<}(k_2)\tilde{\varphi}_{L<}(k_3)\varphi_{>}^L(k_4)k_1^I k_2^J k_3^K k_{K4} >> \\ & = -2g_r^0 \int_k^\Lambda \varphi_{I<}(k_2)\tilde{\varphi}_{L<}(k_3)\delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} k_1^I k_2^J k_3 \cdot k_1 G_J^L(|k_1|) \\ & = -2g_r^0 \int_k^\Lambda \varphi_{I<}(k_2)\tilde{\varphi}_{L<}(k_3)\delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{k^I k_2^J k_3^M k_M}{-\omega^2 + (\lambda + 2\mu)k^2} \left(\delta_J^L + \frac{(\lambda + \mu)k^2 \delta_J^L - k^L k_J}{-\omega^2 + \mu k^2}\right) d^N k \end{aligned} \quad (5.32)$$

Now once again we use the fact of the ‘Gaussian-shell’ integration and so we split $k \otimes k$

$$k^I k_M = \frac{k^2}{N} \delta_M^I + (k^I k_M - \frac{k^2}{N} \delta_M^I) \quad (5.33)$$

where the trace-less part does not contribute to the integral yielding

$$\begin{aligned} & -2g_r^0 \int_k^\Lambda \varphi_{I<}(k_2)\tilde{\varphi}_{L<}(k_3)k_2^J k_3^I \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{k^2}{-\omega^2 + (\lambda + 2\mu)k^2} \left(1 + \frac{(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}\right) \\ & + 2g_r^0 \int_k^\Lambda \varphi_{I<}(k_2)\tilde{\varphi}_{L<}(k_3)k_2^J k_3^M \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{(\lambda + \mu)k^I k_M k^L k_J d^N k}{(-\omega^2 + (\lambda + 2\mu)k^2)(-\omega^2 + \mu k^2)} \end{aligned}$$

Now split $k \otimes k \otimes k \otimes k$ as follows

$$k^I k_M k^L k_J = \frac{k^4}{N(N+2)} (\delta_M^I \delta_J^L + \delta_J^I \delta_M^L + g^{IL} g_{MJ}) + \text{trace} - \text{free part} \quad (5.34)$$

$$\begin{aligned}
\delta_M^I \delta_J^L \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) k_2^J k_3^M &= k_2^L k_3^I \tilde{\varphi}_{L<}(k_3) \varphi_{I<}(k_2), \\
\delta_J^I \delta_M^L \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) k_2^J k_3^M &= k_2^I k_3^L \tilde{\varphi}_{L<}(k_3) \varphi_{I<}(k_2), \\
g^{IL} g_{MJ} \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) k_2^J k_3^M &= (k_2 \cdot k_3) \varphi_{I<}(k_2) \tilde{\varphi}^I(k_3).
\end{aligned} \tag{5.35}$$

Once again, the trace-free part does not contribute to the integral. The final expression becomes

$$\begin{aligned}
& -\frac{2g_r^0}{N} \int_k^\Lambda \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) k_2^L k_3^I \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(k) k^2}{-\omega^2 + (\lambda + 2\mu) k^2} \left(1 + \frac{(\lambda + \mu) k^2}{-\omega^2 + \mu k^2}\right) \\
& + \frac{2g_r^0}{N(N+2)} \int_k^\Lambda \left((k_2^L k_3^I \tilde{\varphi}_{L<}(k_3) \varphi_{I<}(k_2) + k_2^I k_3^L \tilde{\varphi}_{L<}(k_3) \varphi_{I<}(k_2) \right. \\
& \left. + (k_2 \cdot k_3) \varphi_{I<}(k_2) \tilde{\varphi}^I(k_3) \right) \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{(\lambda + \mu) \zeta(k) k^4 d^N k}{(-\omega^2 + (\lambda + 2\mu) k^2)(-\omega^2 + \mu k^2)}
\end{aligned}$$

3.4 $\mathbf{g_r^0}$

$$\begin{aligned}
& 2g_r^0 < \tilde{\varphi}_{J>}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) k_1^I k_2^J k_3^K k_{K4} > > \\
& = -2g_r^0 \int_k^\Lambda \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) (k_3 \cdot k_4) \delta(k_3 + k_4) \int_\Lambda^{\Lambda_0} \frac{\zeta(0) k^2}{-\omega^2 + (\lambda + 2\mu) k^2} d^N k
\end{aligned} \tag{5.36}$$

3.5 $\mathbf{g_r^0}$

$$\begin{aligned}
& 2g_r^0 < \tilde{\varphi}_{J<}(k_1) \varphi_{I>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{<}^L(k_4) k_1^I k_2^J k_3^K k_{K4} > > \\
& = -\frac{2g_r^0}{N} \int_k^\Lambda \varphi_{I<}(k_2) \tilde{\varphi}_{L<}(k_3) k_2^L k_3^I \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(k) k^2}{-\omega^2 + (\lambda + 2\mu) k^2} \left(1 + \frac{(\lambda + \mu) k^2}{-\omega^2 + \mu k^2}\right) \\
& + \frac{2g_r^0}{N(N+2)} \int_k^\Lambda \left((k_2^L k_3^I \tilde{\varphi}_{L<}(k_3) \varphi_{I<}(k_2) + k_2^I k_3^L \tilde{\varphi}_{L<}(k_3) \varphi_{I<}(k_2) \right. \\
& \left. + (k_2 \cdot k_3) \varphi_{I<}(k_2) \tilde{\varphi}^I(k_3) \right) \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{(\lambda + \mu) \zeta(k) k^4 d^N k}{(-\omega^2 + (\lambda + 2\mu) k^2)(-\omega^2 + \mu k^2)}
\end{aligned} \tag{5.37}$$

4.2 $\mathbf{g_r^0}$

$$\begin{aligned}
& g_r^0 < \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{>}^L(k_4) (k_1 \cdot k_2) (k_3 \cdot k_4) > > \\
& = -g_r^0 \int_k^\Lambda \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) (k_1 \cdot k_2) \delta(k_1 + k_2) \int_\Lambda^{\Lambda_0} \frac{\zeta(0) k^2}{-\omega^2 + (\lambda + 2\mu) k^2} (N \\
& + \frac{(N-1)(\lambda + \mu) k^2}{-\omega^2 + \mu k^2}) d^N k
\end{aligned} \tag{5.38}$$

4.3 $\mathbf{g_r^0}$

$$\begin{aligned}
& g_r^0 < \tilde{\varphi}_{>}^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{>}^L(k_4) (k_1 \cdot k_2) (k_3 \cdot k_4) > > \quad (5.39) \\
& = -g_r^0 \int_k^\Lambda \varphi_{J<}(k_2) \tilde{\varphi}_{L<}(k_3) \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{k^A k^B k_{2A} k_{3B}}{-\omega^2 + (\lambda + 2\mu)k^2} (g^{JL} \\
& \quad + (\lambda + \mu) \frac{k^2 g^{JL} - k^L k^J}{-\omega^2 + \mu k^2}) d^N k \\
& = -\frac{g_r^0}{N} \int_k^\Lambda \varphi_{<}^L(k_2) \tilde{\varphi}_{L<}(k_3) (k_2 \cdot k_3) \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(k) k^2}{-\omega^2 + (\lambda + 2\mu)k^2} (1 + \frac{(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}) \\
& \quad + \frac{g_r^0}{N(N+2)} \int_k^\Lambda (\varphi_{<}^L(k_2) \tilde{\varphi}_{L<}(k_3) (k_2 \cdot k_3) + k_2 \cdot \tilde{\varphi}(k_3) k_3 \cdot \varphi(k_2) + k_3 \cdot \tilde{\varphi}(k_3) k_2 \cdot \varphi(k_2) \\
& \quad \delta(k_2 + k_3)) \int_\Lambda^{\Lambda_0} \frac{\zeta(k) (\lambda + \mu) k^4 d^N k}{(-\omega^2 + (\lambda + 2\mu)k^2)(-\omega^2 + \mu k^2)}
\end{aligned}$$

4.4 $\mathbf{g_r^0}$

$$\begin{aligned}
& g_r^0 < \tilde{\varphi}_{>}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L<}(k_3) \varphi_{<}^L(k_4) (k_1 \cdot k_2) (k_3 \cdot k_4) > > \quad (5.40) \\
& = -g_r^0 \int_k^\Lambda \tilde{\varphi}_{<}^J(k_1) \varphi_{J<}(k_2) (k_1 \cdot k_2) \delta(k_1 + k_2) \int_\Lambda^{\Lambda_0} \frac{\zeta(0) k^2}{-\omega^2 + (\lambda + 2\mu)k^2} (N \\
& \quad + \frac{(N-1)(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}) d^N k
\end{aligned}$$

4.5 $\mathbf{g_r^0}$

$$\begin{aligned}
& g_r^0 < \tilde{\varphi}_{<}^J(k_1) \varphi_{J>}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{<}^L(k_4) (k_1 \cdot k_2) (k_3 \cdot k_4) > > \quad (5.41) \\
& = -\frac{g_r^0}{N} \int_k^\Lambda \varphi_{<}^L(k_2) \tilde{\varphi}_{L<}(k_3) (k_2 \cdot k_3) \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(k) k^2}{-\omega^2 + (\lambda + 2\mu)k^2} (1 + \frac{(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}) \\
& \quad + \frac{g_0}{N(N+2)} \int_k^\Lambda (\varphi_{<}^L(k_2) \tilde{\varphi}_{L<}(k_3) (k_2 \cdot k_3) + k_2 \cdot \tilde{\varphi}(k_3) k_3 \cdot \varphi(k_2) + k_3 \cdot \tilde{\varphi}(k_3) k_2 \cdot \varphi(k_2) \\
& \quad \delta(k_2 + k_3)) \int_\Lambda^{\Lambda_0} \frac{(\lambda + \mu) \zeta(k) k^4 d^N k}{(-\omega^2 + (\lambda + 2\mu)k^2)(-\omega^2 + \mu k^2)}.
\end{aligned}$$

Now the mixed spinor contributions

5.1

$$\begin{aligned}
& 2g_b^0 < \tilde{\varphi}_{<}^I(k_1) \varphi_{<}^J(k_2) \bar{\psi}_{>}^K(k_3) \psi_{>}^L(k_4) k_{1I} k_{2J} k_{3K} k_{4L} > > \quad (5.42) \\
& = -2g_b^0 \int_k^\Lambda \varphi_{J<}(k_2) \tilde{\varphi}_{K<}(k_3) k_2^J k_3^K \delta(k_2 + k_3) \int_\Lambda^{\Lambda_0} \frac{\zeta(0) k^2}{-\omega^2 + (\lambda + 2\mu)k^2} d^N k
\end{aligned}$$

5.2

$$\begin{aligned}
& 2g_r^0 < \tilde{\varphi}_{<}^J(k_1)\varphi_{<}^I(k_2)\bar{\psi}_{>}^L(k_3)\psi_{>}^K(k_4)k_{1I}k_{2J}k_{3K}k_{4L} >> \\
& = -2g_r^0 \int_k^\Lambda \tilde{\varphi}_{J<}(k_1)\varphi_{I<}(k_2)k_1^I k_2^J \int_\Lambda^{\Lambda_0} \frac{\zeta(0)k^2 d^N k}{-\omega^2 + (\lambda + 2\mu)k^2}
\end{aligned} \tag{5.43}$$

5.3

$$\begin{aligned}
& 2g_r^0 < \tilde{\varphi}_{<}^J(k_1)\varphi_{<}^I(k_2)\bar{\psi}_{>}^L(k_3)\psi_{L>}(k_4)k_{1I}k_{2J}k_{3K}k_{4K} >> \\
& = -2g_r^0 \int_k^\Lambda \tilde{\varphi}_J(k_1)\varphi_I(k_2)k_1^I k_2^J \delta(k_1 + k_2) \int_\Lambda^{\Lambda_0} \frac{\zeta(0)k^2}{-\omega^2 + (\lambda + 2\mu)k^2} (N \\
& \quad + \frac{(N-1)(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}) d^N k
\end{aligned} \tag{5.44}$$

5.4

$$\begin{aligned}
& 2g_r^0 < \tilde{\varphi}_{<}^J(k_1)\varphi_{J<}(k_2)\bar{\psi}_{>}^L(k_3)\psi_{>}^K(k_4)k_{1I}k_{2I}k_{3K}k_{4L} >> \\
& = -2g_r^0 \int_k^\Lambda \tilde{\varphi}_{L<}(k_3)\varphi_{<}^L(k_4)(k_3 \cdot k_4)\delta(k_3 + k_4) \int_\Lambda^{\Lambda_0} \frac{\zeta(0)k^2}{-\omega^2 + (\lambda + 2\mu)k^2} d^N k
\end{aligned} \tag{5.45}$$

5.5

$$\begin{aligned}
& 2g_r^0 < \tilde{\varphi}_{<}^J(k_1)\varphi_{J<}(k_2)\bar{\psi}_{>}^L(k_3)\psi_{L>}(k_4)(k_1 \cdot k_2)(k_3 \cdot k_4) >> \\
& = -2g_r^0 \int_k^\Lambda \tilde{\varphi}_J(k_1)\varphi_I(k_2)k_1^I k_2^J \delta(k_1 + k_2) \int_\Lambda^{\Lambda_0} \frac{\zeta(0)k^2}{-\omega^2 + (\lambda + 2\mu)k^2} (N \\
& \quad + \frac{(N-1)(\lambda + \mu)k^2}{-\omega^2 + \mu k^2}) d^N k
\end{aligned} \tag{5.46}$$

and similarly other terms for the fermionic part alone.

5.0.2 Quartic part

Now consider the 4th order contributions i.e., the connected part produced by the term $-\frac{1}{2} < (S^{mix}[\varphi_{>}, \tilde{\varphi}_{>}; \varphi_{<}, \tilde{\varphi}_{<}])^2 >_{Z_{0>}} + \frac{1}{2} (< (S^{mix}[\varphi_{>}, \tilde{\varphi}_{>}; \varphi_{<}, \tilde{\varphi}_{<}]) >_{Z_{0>}})^2$. Once again, we need only

compute few terms (~ 100) and rest will follow from symmetry

$$\begin{aligned}
(g_r^0)^2 &< \int_k^\Lambda \int_\Lambda^{\Lambda_0} (\tilde{\varphi}_<^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_{L>}(k_3) \varphi_{>}^L(k_4)) \delta\left(\sum_{i=1}^4 k_i\right) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&\quad (\tilde{\varphi}_<^J(k'_1) \varphi_{J<}(k'_2) \tilde{\varphi}_{L>}(k'_3) \varphi_{>}^L(k'_4)) \delta\left(\sum_{i=1}^4 k'_i\right) (k_1 \cdot k_2) (k_3 \cdot k_4) (k'_1 \cdot k'_2) (k'_3 \cdot k'_4) \\
&\quad d^N k'_1 d^N k'_2 d^N k'_3 d^N k'_4 > 0 \\
&= \frac{(g_r^0)^2}{\exp(-i \int \tilde{\varphi}_>^I G_{IJ}^- \varphi_{>}^J) [d\varphi_{>}] [d\tilde{\varphi}_{>}]} \int_k^\Lambda \int_\Lambda^{\Lambda_0} \delta\left(\sum_{i=1}^4 k_i\right) \delta\left(\sum_{i=1}^4 k'_i\right) (k_1 \cdot k_2) (k_3 \cdot k_4) \\
&\quad (k'_1 \cdot k'_2) (k'_3 \cdot k'_4) \tilde{\varphi}_<^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_<^L(k'_1) \varphi_{L<}(k'_2) \frac{\delta}{\delta \tilde{J}^J[k_3]} \frac{\delta}{\delta J_J[k_4]} \frac{\delta}{\delta \tilde{J}^L[k'_3]} \frac{\delta}{\delta J_L[k'_4]} \\
&\quad \frac{\exp\left(\int J^I(k) G_{IJ}(|k|) \tilde{J}^J(-k) d^N k\right)}{\det(-G^{-1})} \Big|_{J=\tilde{J}=0} \\
&= (g_r^0)^2 \int_k^\Lambda \int_\Lambda^{\Lambda_0} \delta\left(\sum_{i=1}^4 k_i\right) \delta\left(\sum_{i=1}^4 k'_i\right) (k_1 \cdot k_2) (k_3 \cdot k_4) (k'_1 \cdot k'_2) (k'_3 \cdot k'_4) \\
&\quad \tilde{\varphi}_<^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_<^L(k'_1) \varphi_{L<}(k'_2) G_{LJ}(-k'_3) G^{LJ}(k'_4) \delta(k'_3 + k_4) \delta(k_3 + k'_4) d^N k_1 d^N k_2 d^N k_3 d^N k_4 \\
&\quad d^N k'_1 d^N k'_2 d^N k'_3 d^N k'_4 + \text{disconnected part} \\
&= (g_r^0)^2 \int_k^\Lambda \int_\Lambda^{\Lambda_0} \delta\left(\sum_{i=1}^4 k_i\right) \delta(k'_1 + k'_2 - k_3 - k_4) (k_1 \cdot k_2) (k_3 \cdot k_4) (k'_1 \cdot k'_2) (k'_3 \cdot k'_4) \\
&\quad \tilde{\varphi}_<^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_<^L(k'_1) \varphi_{L<}(k'_2) G_{LK}(k_4) G^{LK}(-k_3) d^N k_1 d^N k_2 d^N k_3 d^N k_4 d^N k'_1 d^N k'_2.
\end{aligned}$$

Now if we consider $k_3 + k_4 = k$ then

$$\begin{aligned}
&(g_r^0)^2 \int_k^\Lambda \int_\Lambda^{\Lambda_0} \tilde{\varphi}_<^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_<^L(k'_1) \varphi_{L<}(k'_2) (k_1 \cdot k_2) (k'_1 \cdot k'_2) \quad (5.47) \\
&\quad ((k - k_4) \cdot k_4)^2 G_{LK}(k_4) G^{LK}(k_4 - k) \delta(k_1 + k_2 + k) \delta(k'_1 + k'_2 - k) \\
&\quad d^N k_1 d^N k_2 d^N k'_1 d^N k'_2 d^N k d^N k_4 \\
&= (g_r^0)^2 \int_k^\Lambda d^N k_1 d^N k_2 d^N k'_1 d^N k'_2 \tilde{\varphi}_<^J(k_1) \varphi_{J<}(k_2) \tilde{\varphi}_<^L(k'_1) \varphi_{L<}(k'_2) \delta(k_1 + k_2 + k'_1 + k'_2) \\
&\quad (k_1 \cdot k_2) (k'_1 \cdot k'_2) \int_\Lambda^{\Lambda_0} \frac{((k'_1 + k'_2 - k_4) \cdot k_4)^2}{-\omega^2 + (\lambda + 2\mu)k_4^2} \left(g_{LM} + \frac{(\lambda + \mu)(k_4^2 g^{LM} - k_4^L k_4^M)}{-\omega^2 + \mu k_4^2} \right) \\
&\quad \frac{1}{-\omega^2 + (\lambda + 2\mu)(k_4 - k'_1 - k'_2)^2} \\
&\quad \left(g_{LM} + \frac{(\lambda + \mu)((k_4 - k'_1 - k'_2)^2 g_{LM} - (k_4^M - k_1^M - k_2^M)(k_4^M - k_1^M - k_2^M))}{-\omega^2 + \mu(k_4 - k'_1 - k'_2)^2} \right) d^N k_4.
\end{aligned}$$

Now as usual we choose the upper bound of the momentum k to be Λ_0 . Now consider the term

$$\begin{aligned}
(g_r^0)^2 &< \int_k^\Lambda \int_\Lambda^{\Lambda_0} \tilde{\varphi}_>^J(k_1) (\varphi_{J<}(k_2) \tilde{\varphi}_>^L(k_3) \varphi_{L<}(k_4) \tilde{\varphi}_>^J(k'_1) \varphi_{J<}(k'_2) \tilde{\varphi}_>^L(k'_3) \varphi_{L<}(k'_4) \\
&\quad \delta(\sum_i k_i) \delta(\sum_i k'_i) (k_1 \cdot k_2) (k_3 \cdot k_4) (k'_1 \cdot k'_2) (k'_3 \cdot k'_4) >_0 > \\
&= (g_r^0)^2 \int_k^\Lambda \int_\Lambda^{\Lambda_0} \varphi_{J<}(k_2) \varphi_{L<}(k_4) \varphi_{J<}(k'_2) \varphi_{L<}(k'_4) \\
&\quad \delta(\sum_i k_i) \delta(\sum_i k'_i) (k_1 \cdot k_2) (k_3 \cdot k_4) (k'_1 \cdot k'_2) (k'_3 \cdot k'_4) > \\
&\quad \frac{\delta}{\delta \tilde{J}^J[k_1]} \frac{\delta}{\delta \tilde{J}^J[k_3]} \frac{\delta}{\delta J^L[k'_2]} \frac{\delta}{\delta J^L[k'_4]} \frac{\exp\left(\int J^I(k) G_{IJ}(|k|) \tilde{J}^J(-k) d^N k\right)}{\det(-G^{-1})} \Big|_{J=\tilde{J}=0} \\
&= (g_r^0)^2 \int d^N k_2 d^N k'_1 d^N k'_3 d^N k_4 \delta(k'_1 + k'_3 + k_2 + k_4) \int_\Lambda^{\Lambda_0} (k_1 \cdot k_2) (k_1 \cdot k_4) (k'_1 \cdot k_1) (k'_3 \cdot k_1) \\
&\quad \frac{1}{[-\omega^2 + (\lambda + 2\mu)k_1^2]^2} \left\{ \left(\varphi_{J<}(k_2) \varphi_{L<}(k_4) \tilde{\varphi}_<^L(k'_1) \tilde{\varphi}_<^J(k'_3) + \frac{\lambda + 2\mu}{-\omega^2 + \mu k_1^2} \right. \right. \\
&\quad \left. \left(k_1^2 \varphi_{J<}(k_2) \varphi_{L<}(k_4) \tilde{\varphi}_<^L(k'_1) \tilde{\varphi}_<^J(k'_3) - k_{1J} \varphi_{<}^J(k_2) k_{1N} \tilde{\varphi}_<^N(k'_3) \tilde{\varphi}_<^L(k'_1) \varphi_{L<}(k_4) \right) \right. \\
&\quad + \frac{\lambda + \mu}{-\omega^2 + \mu k_1^2} \left(k_1^2 \varphi_{J<}(k_2) \varphi_{L<}(k_4) \tilde{\varphi}_<^L(k'_1) \tilde{\varphi}_<^J(k'_3) - k_{1J} \varphi_{<}^J(k_2) k_{1N} \tilde{\varphi}_<^N(k'_3) \tilde{\varphi}_<^L(k'_1) \varphi_{L<}(k_4) \right) \\
&\quad + \frac{\lambda + \mu}{-\omega^2 + \mu k_1^2} \left(k_1^2 \varphi_{J<}(k_2) \varphi_{L<}(k_4) \tilde{\varphi}_<^L(k'_1) \tilde{\varphi}_<^J(k'_3) - k_{1L} \varphi_{<}^L(k_2) k_{1M} \tilde{\varphi}_<^M(k'_3) \tilde{\varphi}_<^J(k'_1) \varphi_{J<}(k_2) \right) \\
&\quad + \left[\frac{\lambda + \mu}{-\omega^2 + \mu k_1^2} \right]^2 \left(k_1^4 \varphi_{J<}(k_2) \varphi_{L<}(k_4) \tilde{\varphi}_<^L(k'_1) \tilde{\varphi}_<^J(k'_3) - k_1^2 \varphi_{L<}(k_4) \tilde{\varphi}_<^L(k'_1) [k_1 \cdot \varphi_{<}(k_2)] \right. \\
&\quad \left. [k_1 \cdot \tilde{\varphi}_{<}(k'_3)] - k_1^2 \varphi_{L<}(k_2) \tilde{\varphi}_<^L(k'_3) [k_1 \cdot \varphi_{<}(k_4)] [k_1 \cdot \tilde{\varphi}_{<}(k'_1)] + [k_1 \cdot \varphi_{<}(k_2)] [k_1 \cdot \tilde{\varphi}_{<}(k'_3)] \right. \\
&\quad \left. \left. [k_1 \cdot \varphi_{<}(k_4)] [k_1 \cdot \tilde{\varphi}_{<}(k'_1)] \right) \right\} d^N k_1.
\end{aligned}$$

In the previous integral, because the integration is in the shell $k \in [-\Lambda_0, -\Lambda] \cup [\Lambda, \Lambda_0]$, odd powers of k_1^i integrals vanishes. In other words the integrals contributing must have the trace structure only

$$\begin{aligned}
[k_1 \cdot \varphi_{<}(k_2)] [k_1 \cdot \tilde{\varphi}_{<}(k'_3)] &= k_1^L \varphi_{L<}(k_2) k_1^M \tilde{\varphi}_{M<}(k'_3) \\
&= \frac{1}{N} k_1^2 \varphi_{L<}(k_2) \tilde{\varphi}_{M<}(k'_3) g^{LM} \\
&= \frac{1}{N} k_1^2 \varphi_{<}^L(k_2) \tilde{\varphi}_{L<}(k'_3).
\end{aligned} \tag{5.48}$$

Now for the term $[k_1 \cdot \varphi_{<}(k_2)] [k_1 \cdot \tilde{\varphi}_{<}(k'_3)] [k_1 \cdot \varphi_{<}(k_4)] [k_1 \cdot \tilde{\varphi}_{<}(k'_1)]$, we have $k_1^A k_1^B k_1^C k_1^D$. We only need its trace part. Therefore we write

$$k_1^A k_1^B k_1^C k_1^D = \frac{k_1^4}{N(N+2)} (g^{AB} g^{CD} + g^{AC} g^{BD} + g^{AD} g^{BC}) + \text{traceless part}. \tag{5.49}$$

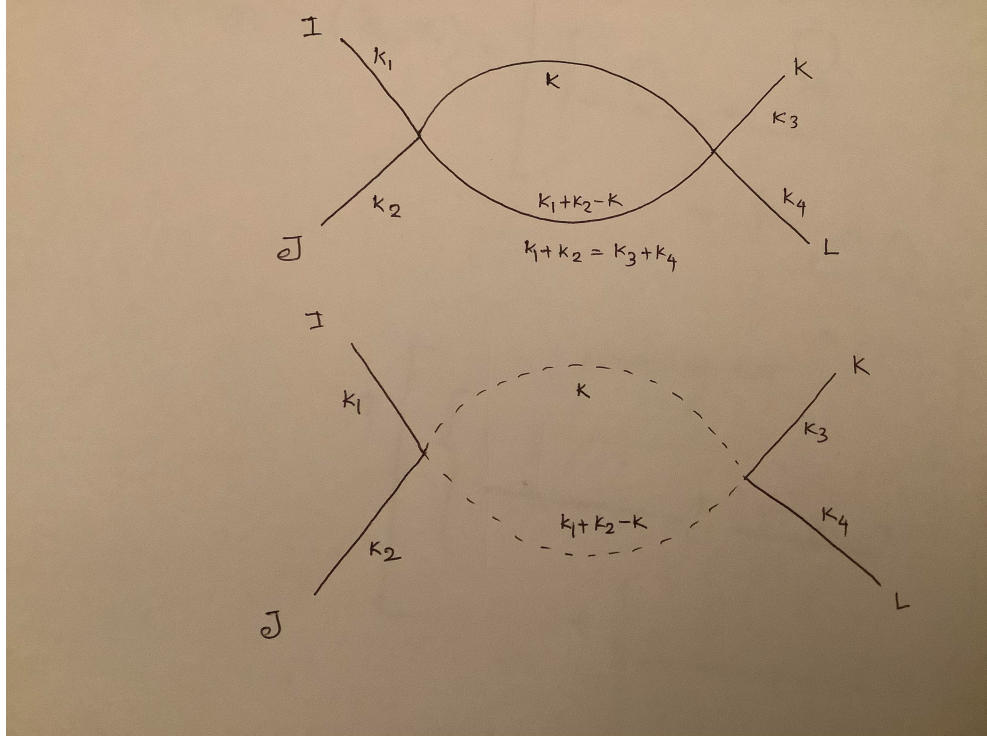


Figure 5.1: The one loop contribution to the quartic coupling constants g_b and g_r . The top and bottom diagrams represent the commuting and anti-commuting parts, respectively.

Therefore the last term becomes

$$\begin{aligned}
 & [k_1 \cdot \varphi_{<}(k_2)][k_1 \cdot \tilde{\varphi}_{<}(k'_3)][k_1 \cdot \varphi_{<}(k_4)][k_1 \cdot \tilde{\varphi}_{<}(k'_1)] \\
 &= \frac{k_1^4}{N(N+2)} \left(\varphi_{<}^M(k_2) \tilde{\varphi}_{M<}(k'_3) \varphi_{<}^L(k_4) \tilde{\varphi}_{L<}(k'_1) + \varphi_{<}^M(k_2) \tilde{\varphi}_{L<}(k'_3) \varphi_{M<}(k_4) \tilde{\varphi}_{<}^L(k'_1) \right. \\
 & \quad \left. + \varphi_{<}^M(k_2) \tilde{\varphi}_{L<}(k'_3) \varphi_{<}^L(k_4) \tilde{\varphi}_{M<}(k'_1) \right)
 \end{aligned} \tag{5.50}$$

and similarly for the following

$$\begin{aligned}
 (k_1 \cdot k_2)(k_1 \cdot k_4)(k'_1 \cdot k_1)(k'_3 \cdot k_1) &= \frac{k_1^4}{N(N+2)} \left((k_2 \cdot k_4)(k'_1 \cdot k'_3) \right. \\
 & \quad \left. + (k_2 \cdot k'_1)(k_4 \cdot k'_3) + (k_2 \cdot k'_3)(k_4 \cdot k'_1) \right) + \text{traceless part}
 \end{aligned} \tag{5.51}$$

Now when traceless-traceless parts collide in the integral as product, they survive. Let us write the

following

$$k_1^A k_1^B k_1^C k_1^D = \frac{k_1^4}{N(N+2)} (g^{AB} g^{CD} + g^{AC} g^{BD} + g^{AD} g^{BC}) \quad (5.52)$$

$$+ \left(k_1^A k_1^B k_1^C k_1^D - \frac{k_1^4}{N(N+2)} (g^{AB} g^{CD} + g^{AC} g^{BD} + g^{AD} g^{BC}) \right).$$

The traceless parts collide in the integral to yield

$$\begin{aligned} & \left(k_1^A k_1^B k_1^C k_1^D - \frac{k_1^4}{N(N+2)} (g^{AB} g^{CD} + g^{AC} g^{BD} + g^{AD} g^{BC}) \right) \quad (5.53) \\ & \left(k_1^L k_1^M k_1^N k_1^P - \frac{k_1^4}{N(N+2)} (g^{LM} g^{NP} + g^{LN} g^{MP} + g^{LP} g^{MN}) \right) \\ & = k_1^A k_1^B k_1^C k_1^D k_1^L k_1^M k_1^N k_1^P - \frac{k_1^4}{N(N+2)} (g^{AB} g^{CD} + g^{AC} g^{BD} + g^{AD} g^{BC}) k_1^L k_1^M k_1^N k_1^P \\ & \quad - k_1^A k_1^B k_1^C k_1^D \frac{k_1^4}{N(N+2)} (g^{LM} g^{NP} + g^{LN} g^{MP} + g^{LP} g^{MN}) \\ & \quad + \frac{k_1^8}{N^2(N+2)^2} (g^{AB} g^{CD} + g^{AC} g^{BD} + g^{AD} g^{BC}) (g^{LM} g^{NP} + g^{LN} g^{MP} + g^{LP} g^{MN}). \end{aligned}$$

Once again we only need to worry about the trace part where k_1^8 appears and therefore we only need to worry about the cross-trace part. Figure (5.1) depicts the diagram for this quartic contribution.

Bibliography

- [1] Adam M Dziewonski and Don L Anderson. Preliminary reference earth model. *Physics of the earth and planetary interiors*, 25(4):297–356, 1981.
- [2] Zhen Liu, Jeffrey Park, and Shun-ichiro Karato. Seismological detection of low-velocity anomalies surrounding the mantle transition zone in japan subduction zone. *Geophysical Research Letters*, 43(6):2480–2487, 2016.
- [3] Yang Shen, Cecily J Wolfe, and Sean C Solomon. Seismological evidence for a mid-mantle discontinuity beneath hawaii and iceland. *Earth and Planetary Science Letters*, 214(1-2):143–151, 2003.
- [4] Kyoko N Matsukage, Yu Nishihara, and Shun-ichiro Karato. Seismological signature of chemical differentiation of earth’s upper mantle. *Journal of Geophysical Research: Solid Earth*, 110(B12), 2005.
- [5] Shuqing Zhang and Shun-ichiro Karato. Lattice preferred orientation of olivine aggregates deformed in simple shear. *Nature*, 375(6534):774–777, 1995.
- [6] Shun-ichiro Karato, Haemyeong Jung, Ikuo Katayama, and Philip Skemer. Geodynamic significance of seismic anisotropy of the upper mantle: New insights from laboratory studies. *Annu. Rev. Earth Planet. Sci.*, 36:59–95, 2008.
- [7] Martin L Smith and FA Dahlen. The azimuthal dependence of love and rayleigh wave propagation in a slightly anisotropic medium. *Journal of Geophysical Research*, 78(17):3321–3333, 1973.
- [8] Sébastien Chevrot. Finite-frequency vectorial tomography: a new method for high-resolution imaging of upper mantle anisotropy. *Geophysical Journal International*, 165(2):641–657, 2006.
- [9] Maureen D Long, V Maarten, and Robert D Van Der Hilst. Wave-equation shear wave splitting tomography. *Geophysical Journal International*, 172(1):311–330, 2008.

- [10] Anne Sieminski, Hanneke Paulssen, Jeannot Trampert, and Jeroen Tromp. Finite-frequency SKS splitting: measurement and sensitivity kernels. *Bulletin of the Seismological Society of America*, 98(4):1797–1810, 2008.
- [11] LP Vinnik, GL Kosarev, and LI Makeeva. Lithosphere anisotropy from the observation of sks and sks waves. *Doklady Akademii Nauk SSSR*, 278(6):1335–1339, 1984.
- [12] Martha Kane Savage and Paul G Silver. Mantle deformation and tectonics: constraints from seismic anisotropy in the western united states. *Physics of the Earth and Planetary Interiors*, 78(3-4):207–227, 1993.
- [13] Paul G Silver. Seismic anisotropy beneath the continents: Probing the depths of geology. *Annual review of earth and planetary sciences*, 24(1):385–432, 1996.
- [14] MK Savage. Seismic anisotropy and mantle deformation: what have we learned from shear wave splitting? *Reviews of Geophysics*, 37(1):65–106, 1999.
- [15] Maureen D Long and Thorsten W Becker. Mantle dynamics and seismic anisotropy. *Earth and Planetary Science Letters*, 297(3-4):341–354, 2010.
- [16] Clinton P Conrad and Mark D Behn. Constraints on lithosphere net rotation and asthenospheric viscosity from global mantle flow models and seismic anisotropy. *Geochemistry, Geophysics, Geosystems*, 11(5), 2010.
- [17] Maureen D Long and Paul G Silver. The subduction zone flow field from seismic anisotropy: A global view. *science*, 319(5861):315–318, 2008.
- [18] Maureen D Long and Paul G Silver. Mantle flow in subduction systems: The slab flow field and implications for mantle dynamics. *Journal of Geophysical Research: Solid Earth*, 114(B10), 2009.
- [19] M Faccenda and FA Capitanio. Development of mantle seismic anisotropy during subduction-induced 3-d flow. *Geophysical Research Letters*, 39(11), 2012.
- [20] Maureen D Long and Erin A Wirth. Mantle flow in subduction systems: The mantle wedge flow field and implications for wedge processes. *Journal of Geophysical Research: Solid Earth*, 118(2):583–606, 2013.
- [21] Cecily J Wolfe and Sean C Solomon. Shear-wave splitting and implications for mantle flow beneath the melt region of the east pacific rise. *Science*, 280(5367):1230–1232, 1998.

- [22] J-M Kendall, GW Stuart, CJ Ebinger, ID Bastow, and D Keir. Magma-assisted rifting in Ethiopia. *Nature*, 433(7022):146, 2005.
- [23] Miles Bodmer, Douglas R Toomey, Emilie E Hooft, John Nábělek, and Jochen Braunmiller. Seismic anisotropy beneath the juan de fuca plate system: Evidence for heterogeneous mantle flow. *Geology*, 43(12):1095–1098, 2015.
- [24] Sutatcha Hongsresawat, Mark P Panning, Raymond M Russo, David A Foster, Vadim Monteiller, and Sébastien Chevrot. Usarray shear wave splitting shows seismic anisotropy from both lithosphere and asthenosphere. *Geology*, 43(8):667–670, 2015.
- [25] Maureen D Long, Kenneth G Jackson, and John F McNamara. SKS splitting beneath transportable array stations in eastern north America and the signature of past lithospheric deformation. *Geochemistry, Geophysics, Geosystems*, 17(1):2–15, 2016.
- [26] Sanne Cottaar and Barbara Romanowicz. Observations of changing anisotropy across the southern margin of the African LLSVP. *Geophysical Journal International*, 195(2):1184–1195, 2013.
- [27] Heather A Ford and Maureen D Long. A regional test of global models for flow, rheology, and seismic anisotropy at the base of the mantle. *Physics of the Earth and Planetary Interiors*, 245:71–75, 2015.
- [28] Neala Creasy, Maureen D Long, and Heather A Ford. Deformation in the lowermost mantle beneath australia from observations and models of seismic anisotropy. *Journal of Geophysical Research: Solid Earth*, 122(7):5243–5267, 2017.
- [29] Maureen D Long and Paul G Silver. Shear wave splitting and mantle anisotropy: measurements, interpretations, and new directions. *Surveys in Geophysics*, 30(4-5):407–461, 2009.
- [30] Paul G Silver and Maureen D Long. The non-commutivity of shear wave splitting operators at low frequencies and implications for anisotropy tomography. *Geophysical Journal International*, 184(3):1415–1427, 2011.
- [31] David L Abt and Karen M Fischer. Resolving three-dimensional anisotropic structure with shear wave splitting tomography. *Geophysical Journal International*, 173(3):859–886, 2008.
- [32] Vadim Monteiller and Sébastien Chevrot. How to make robust splitting measurements for single-station analysis and three-dimensional imaging of seismic anisotropy. *Geophysical Journal International*, 182(1):311–328, 2010.

- [33] Sébastien Chevrot. Multichannel analysis of shear wave splitting. *Journal of Geophysical Research: Solid Earth*, 105(B9):21579–21590, 2000.
- [34] Maureen D Long and Rob D van der Hilst. Upper mantle anisotropy beneath Japan from shear wave splitting. *Physics of the Earth and Planetary Interiors*, 151(3-4):206–222, 2005.
- [35] Noémie Favier and Sébastien Chevrot. Sensitivity kernels for shear wave splitting in transverse isotropic media. *Geophysical Journal International*, 153(1):213–228, 2003.
- [36] Noémie Favier, Sébastien Chevrot, and Dimitri Komatitsch. Near-field influence on shear wave splitting and traveltimes sensitivity kernels. *Geophysical Journal International*, 156(3):467–482, 2004.
- [37] Frank J Calixto, Danielle Robinson, Eric Sandvol, Suzanne Kay, David Abt, Karen Fischer, Ben Heit, Xiaohui Yuan, Diana Comte, and Patricia Alvarado. Shear wave splitting and shear wave splitting tomography of the southern Puna plateau. *Geophysical Journal International*, 199(2):688–699, 2014.
- [38] Vadim Monteiller and Sébastien Chevrot. High-resolution imaging of the deep anisotropic structure of the San Andreas Fault system beneath southern California. *Geophysical Journal International*, 186(2):418–446, 2011.
- [39] James Wookey. Direct probabilistic inversion of shear wave data for seismic anisotropy. *Geophysical Journal International*, 189(2):1025–1037, 2012.
- [40] Karl F Graff. *Wave motion in elastic solids*. Courier Corporation, 2012.
- [41] Keiiti Aki and Paul G Richards. *Quantitative seismology*. 2002.
- [42] Shmuel Agmon. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Communications on Pure and Applied Mathematics*, 15(2):119–147, 1962.
- [43] Henrik Schmidt and Gerard Tango. Efficient global matrix approach to the computation of synthetic seismograms. *Geophysical Journal of the Royal Astronomical Society*, 84(2):331–359, 1986.
- [44] Ghislain R Franssens. Calculation of the elasto-dynamic green’s function in layered media by means of a modified propagator matrix method. *Geophysical Journal International*, 75(3):669–691, 1983.

- [45] Freeman Gilbert and George E Backus. Propagator matrices in elastic wave and vibration problems. *Geophysics*, 31(2):326–332, 1966.
- [46] Fabio Cammarano, Saskia Goes, Pierre Vacher, and Domenico Giardini. Inferring upper-mantle temperatures from seismic velocities. *Physics of the Earth and Planetary Interiors*, 138(3-4):197–222, 2003.
- [47] BLN Kennett, ER Engdahl, and R Buland. Constraints on seismic velocities in the earth from traveltimes. *Geophysical Journal International*, 122(1):108–124, 1995.
- [48] Thomas Mensch and Patrick Rasolofosaon. Elastic-wave velocities in anisotropic media of arbitrary symmetry? generalization of thomsen’s parameters ε , δ and γ . *Geophysical Journal International*, 128(1):43–64, 1997.
- [49] Jules Thomas Browaeys and Sébastien Sébastien Chevrot. Decomposition of the elastic tensor and geophysical applications. *Geophysical Journal International*, 159(2):667–678, 2004.
- [50] A Reuss. Berechnung der fließgrenze von mischkristallen auf grund der plastizitätsbedingung für einkristalle. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 9(1):49–58, 1929.
- [51] Thorsten W Becker, Sebastien Chevrot, Vera Schulte-Pelkum, and Donna K Blackman. Statistical properties of seismic anisotropy predicted by upper mantle geodynamic models. *Journal of Geophysical Research: Solid Earth*, 111(B8), 2006.
- [52] Sébastien Chevrot and Robert D Van Der Hilst. On the effects of a dipping axis of symmetry on shear wave splitting measurements in a transversely isotropic medium. *Geophysical Journal International*, 152(2):497–505, 2003.
- [53] Nicolas Bourbaki. *Elements of Mathematics: General Topology. Part 1,[chapters 1-4]*. Hermann, 1966.
- [54] Jun S Liu. *Monte Carlo strategies in scientific computing*. Springer Science & Business Media, 2008.
- [55] F Arias de Saavedra and MH Kalos. Bilinear diffusion quantum monte carlo methods. *Physical Review E*, 67(2):026708, 2003.
- [56] WMC Foulkes, L Mitas, RJ Needs, and G Rajagopal. Quantum monte carlo simulations of solids. *Reviews of Modern Physics*, 73(1):33, 2001.

- [57] Brian L Hammond, William A Lester, and Peter James Reynolds. *Monte Carlo methods in ab initio quantum chemistry*, volume 1. World Scientific, 1994.
- [58] Klaus Mosegaard and Malcolm Sambridge. Monte carlo analysis of inverse problems. *Inverse problems*, 18(3):R29, 2002.
- [59] Jun Korenaga and Shun-Ichiro Karato. A new analysis of experimental data on olivine rheology. *Journal of Geophysical Research: Solid Earth*, 113(B2), 2008.
- [60] Erin A Wirth, Maureen D Long, and John C Moriarty. A Markov chain Monte Carlo with Gibbs sampling approach to anisotropic receiver function forward modeling. *Geophysical Journal International*, 208(1):10–23, 2016.
- [61] Maureen D Long, Haiying Gao, Amanda Klaus, Lara S Wagner, Matthew J Fouch, David E James, and Eugene Humphreys. Shear wave splitting and the pattern of mantle flow beneath eastern Oregon. *Earth and Planetary Science Letters*, 288(3-4):359–369, 2009.
- [62] Lara S Wagner and Maureen D Long. Distinctive upper mantle anisotropy beneath the High Lava Plains and Eastern Snake River Plain, Pacific Northwest, USA. *Geochemistry, Geophysics, Geosystems*, 14(10):4647–4666, 2013.
- [63] Bradford J Foley and Maureen D Long. Upper and mid-mantle anisotropy beneath the Tonga slab. *Geophysical Research Letters*, 38(2), 2011.
- [64] Yu-Pin Lin, Li Zhao, and Shu-Huei Hung. Full-wave effects on shear wave splitting. *Geophysical Research Letters*, 41(3):799–804, 2014.
- [65] Caroline M Eakin, Maureen D Long, Lara S Wagner, Susan L Beck, and Hernando Tavera. Upper mantle anisotropy beneath Peru from SKS splitting: Constraints on flat slab dynamics and interaction with the Nazca ridge. *Earth and Planetary Science Letters*, 412:152–162, 2015.
- [66] O Ola, AW Frederiksen, T Bollmann, S van der Lee, F Darbyshire, E Wolin, J Revenaugh, C Stein, S Stein, and M Wyssession. Anisotropic zonation in the lithosphere of central north america: Influence of a strong cratonic lithosphere on the mid-continent rift. *Tectonophysics*, 683:367–381, 2016.
- [67] Xiaotao Yang, Gary L Pavlis, Michael W Hamburger, Stephen Marshak, Hersh Gilbert, John Rupp, Timothy H Larson, Chen Chen, and N Seth Carpenter. Detailed crustal thickness variations beneath the illinois basin area: Implications for crustal evolution of the midcontinent. *Journal of Geophysical Research: Solid Earth*, 122(8):6323–6345, 2017.

- [68] Cecilia A Nyamwandha and Christine A Powell. Seismic anisotropy beneath the Mississippi Embayment and the New Madrid seismic zone: A study of shear wave splitting. *Journal of Geophysical Research: Solid Earth*, 121(11):8239–8253, 2016.
- [69] Emily Hopper, Karen M Fischer, Lara S Wagner, and Robert B Hawman. Reconstructing the end of the Appalachian orogeny. *Geology*, 45(1):15–18, 2017.
- [70] John C Aragon, Maureen D Long, and Margaret H Benoit. Lateral variations in sks splitting across the MAGIC array, central appalachians. *Geochemistry, Geophysics, Geosystems*, 18(11):4136–4155, 2017.
- [71] Xiaoran Chen, Yiran Li, and Vadim Levin. Shear wave splitting beneath eastern north American continent: Evidence for a multi-layered and laterally variable anisotropic structure. *Geochemistry, Geophysics, Geosystems*, 2018.
- [72] Eugene D Humphreys and David D Coblenz. North American dynamics and western US tectonics. *Reviews of Geophysics*, 45(3):doi:10.1029/2005RG000181, 2007.
- [73] Brennan T Jordan, Anita L Grunder, Robert A Duncan, and Alan L Deino. Geochronology of age-progressive volcanism of the Oregon High Lava Plains: Implications for the plume interpretation of Yellowstone. *Journal of Geophysical Research: Solid Earth*, 109(B10, doi: 10.1029/2003JB002776), 2004.
- [74] Christy B Till, Timothy L Grove, Richard W Carlson, Julie M Donnelly-Nolan, Matthew J Fouch, Lara S Wagner, and William K Hart. Depths and temperatures of < 10.5 Ma mantle melting and the lithosphere-asthenosphere boundary below southern Oregon and northern California. *Geochemistry, Geophysics, Geosystems*, 14(4):864–879, doi: 10.1002/ggge.20070, 2013.
- [75] Mark T Ford, Anita L Grunder, and Robert A Duncan. Bimodal volcanism of the High Lava Plains and northwestern basin and range of oregon: Distribution and tectonic implications of age-progressive rhyolites. *Geochemistry, Geophysics, Geosystems*, 14(8):2836–2857, doi: 10.1002/ggge.20175, 2013.
- [76] Maureen D Long, Christy B Till, Kelsey A Druken, Richard W Carlson, Lara S Wagner, Matthew J Fouch, David E James, Timothy L Grove, Nicholas Schmerr, and Chris Kincaid. Mantle dynamics beneath the Pacific Northwest and the generation of voluminous back-arc volcanism. *Geochemistry, Geophysics, Geosystems*, 13(8, doi:10.1029/2012GC004189), 2012.

- [77] Victor E Camp. Plume-modified mantle flow in the northern Basin and Range and southern Cascadia back-arc region since ca. 12 ma. *Geology*, 47(8):695–699, 2019.
- [78] Robert B Smith, Michael Jordan, Bernhard Steinberger, Christine M Puskas, Jamie Farrell, Gregory P Waite, Stephan Husen, Wu-Lung Chang, and Richard O’Connell. Geodynamics of the Yellowstone hotspot and mantle plume: Seismic and GPS imaging, kinematics, and mantle flow. *Journal of Volcanology and Geothermal Research*, 188(1-3):26–56, 2009.
- [79] Victor E Camp and Martin E Ross. Mantle dynamics and genesis of mafic magmatism in the intermontane Pacific Northwest. *Journal of Geophysical Research: Solid Earth*, 109(B8, doi:10.1029/2003JB002838), 2004.
- [80] David E James, Matthew J Fouch, Richard W Carlson, and Jeffrey B Roth. Slab fragmentation, edge flow and the origin of the Yellowstone hotspot track. *Earth and Planetary Science Letters*, 311(1-2):124–135, 2011.
- [81] Tiffany Leonard and Lijun Liu. The role of a mantle plume in the formation of Yellowstone volcanism. *Geophysical Research Letters*, 43(3):1132–1139, 2016.
- [82] Tristin C Hales, David L Abt, ED Humphreys, and Joshua J Roering. A lithospheric instability origin for Columbia River flood basalts and Wallowa Mountains uplift in northeast Oregon. *Nature*, 438(7069):842, 2005.
- [83] Victor E Camp and Barry B Hanan. A plume-triggered delamination origin for the Columbia River basalt group. *Geosphere*, 4(3):480–495, 2008.
- [84] B Tikoff, B Benford, and S Giorgis. Lithospheric control on the initiation of the Yellowstone hotspot: Chronic reactivation of lithospheric scars. *International Geology Review*, 50(3):305–324, 2008.
- [85] Maureen D Long. The Cascadia Paradox: Mantle flow and slab fragmentation in the Cascadia subduction system. *Journal of Geodynamics*, 102:151–170, 2016.
- [86] Mei Xue and Richard M Allen. Origin of the Newberry hotspot track: Evidence from shear-wave splitting. *Earth and Planetary Science Letters*, 244(1-2):315–322, 2006.
- [87] Kelly H Liu, Ahmed Elsheikh, Awad Lemnifi, Uranbaigal Purevsuren, Melissa Ray, Hesham Refayee, Bin B Yang, Youqiang Yu, and Stephen S Gao. A uniform database of teleseismic

- p shear wave splitting measurements for the western and central United States.
- Geochemistry, Geophysics, Geosystems*
- , 15(5):2075–2085, doi: 10.1002/2014GC005267, 2014.
- [88] Brandon Schmandt and Eugene Humphreys. Complex subduction and small-scale convection revealed by body-wave tomography of the western united states upper mantle. *Earth and Planetary Science Letters*, 297(3-4):435–445, 2010.
 - [89] William B Hawley and Richard M Allen. The fragmented death of the Farallon plate. *Geophysical Research Letters*, 46(13):7386–7394, doi: 10.1029/2019GL083437, 2019.
 - [90] Lara S Wagner, Matthew J Fouch, David E James, and Sara Hanson-Hedgecock. Crust and upper mantle structure beneath the Pacific Northwest from joint inversions of ambient noise and earthquake data. *Geochemistry, Geophysics, Geosystems*, 13(12):doi: 10.1029/2012GC004353, 2012.
 - [91] Lara S Wagner, Matthew J Fouch, David E James, and Maureen D Long. The role of hydrous phases in the formation of trench parallel anisotropy: Evidence from Rayleigh waves in Cascadia. *Geophysical Research Letters*, 40(11):2642–2646, doi: 10.1002/grl.50525, 2013.
 - [92] Kevin C Eagar, Matthew J Fouch, David E James, and Richard W Carlson. Crustal structure beneath the High Lava Plains of eastern Oregon and surrounding regions from receiver function analysis. *Journal of Geophysical Research: Solid Earth*, 116(B2, doi:10.1029/2010JB007795), 2011.
 - [93] Chin-Wu Chen, David E James, Matthew J Fouch, and Lara S Wagner. Lithospheric structure beneath the High Lava Plains, Oregon, imaged by scattered teleseismic waves. *Geochemistry, Geophysics, Geosystems*, 14(11):4835–4848, doi: 10.1029/2010JB007795, 2013.
 - [94] Huaiyu Yuan and Barbara Romanowicz. Depth dependent azimuthal anisotropy in the western US upper mantle. *Earth and Planetary Science Letters*, 300(3-4):385–394, 2010.
 - [95] Kelly H Liu and Stephen S Gao. Estimation of the depth of anisotropy using spatial coherency of shear-wave splitting parameters. *Bulletin of the Seismological Society of America*, 101(5):2153–2161, doi:10.1785/0120100258, 2011.
 - [96] Paul G Silver and Martha K Savage. The interpretation of shear-wave splitting parameters in the presence of two anisotropic layers. *Geophysical Journal International*, 119(3):949–963, 1994.

- [97] MJ Bezada, M Faccenda, and DR Toomey. Representing anisotropic subduction zones with isotropic velocity models: A characterization of the problem and some steps on a possible path forward. *Geochemistry, Geophysics, Geosystems*, 17(8):3164–3189, doi:10.1002/2016GC006507, 2016.
- [98] Puskar Mondal and Maureen D Long. A model space search approach to finite-frequency SKS splitting intensity tomography in a reduced parameter space. *Geophysical Journal International*, 217(1):238–256, 2019.
- [99] Gavin P Hayes, Ginevra L Moore, Daniel E Portner, Mike Hearne, Hanna Flamme, Maria Furtney, and Gregory M Smoczyk. Slab2, a comprehensive subduction zone geometry model. *Science*, 362(6410):58–61, 2018.
- [100] David Ruelle. *Statistical mechanics: Rigorous results*. World Scientific, 1999.
- [101] Miriam Christina Reiss and Georg Rumpker. Splitracer: Matlab code and GUI for semiautomated analysis and interpretation of teleseismic shear-wave splitting. *Seismological Research Letters*, 88(2A):392–409, 2017.
- [102] KA Druken, MD Long, and C Kincaid. Patterns in seismic anisotropy driven by roll-back subduction beneath the High Lava Plains. *Geophysical Research Letters*, 38(13, doi:10.1029/2011GL047541), 2011.
- [103] Quan Zhou, Jiashun Hu, Lijun Liu, Thomas Chaparro, Dave R Stegman, and Manuele Faccenda. Western US seismic anisotropy revealing complex mantle dynamics. *Earth and Planetary Science Letters*, 500:156–167, 2018.
- [104] Shun-ichiro Karato and Patrick Wu. Rheology of the upper mantle: A synthesis. *Science*, 260(5109):771–778, 1993.
- [105] Tomonori Miyazaki, Kenta Sueyoshi, and Takehiko Hiraga. Olivine crystals align during diffusion creep of Earth’s upper mantle. *Nature*, 502(7471):321, 2013.
- [106] Fan-Chi Lin, Michael H Ritzwoller, Yingjie Yang, Morgan P Moschetti, and Matthew J Fouch. Complex and variable crustal and uppermost mantle seismic anisotropy in the western United States. *Nature Geoscience*, 4(1):55, doi:10.1038/ngeo1036, 2011.
- [107] Puskar Mondal and Maureen D Long. Strong seismic anisotropy in the deep upper mantle beneath the cascadia backarc: Constraints from probabilistic finite-frequency sks splitting intensity tomography. *Earth and Planetary Science Letters*, 539:116172, 2020.

- [108] Zvi Hashin. The elastic moduli of heterogeneous materials. Technical report, HARVARD UNIV CAMBRIDGE MA, 1960.
- [109] ZSHTR Hashin and S Shtrikman. A variational approach to the theory of the elastic behaviour of polycrystals. *Journal of the Mechanics and Physics of Solids*, 10(4):343–352, 1962.
- [110] Zvi Hashin and Shmuel Shtrikman. A variational approach to the theory of the elastic behaviour of multiphase materials. *Journal of the Mechanics and Physics of Solids*, 11(2):127–140, 1963.
- [111] Anil Misra and Ching S Chang. Effective elastic moduli of heterogeneous granular solids. *International Journal of Solids and Structures*, 30(18):2547–2566, 1993.
- [112] Edward J Garboczi and James G Berryman. Elastic moduli of a material containing composite inclusions: effective medium theory and finite element computations. *Mechanics of materials*, 33(8):455–470, 2001.
- [113] J Peter Watt and Louis Peselnick. Clarification of the hashin-shtrikman bounds on the effective elastic moduli of polycrystals with hexagonal, trigonal, and tetragonal symmetries. *Journal of Applied Physics*, 51(3):1525–1531, 1980.
- [114] John Douglas Eshelby. The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proceedings of the royal society of London. Series A. Mathematical and physical sciences*, 241(1226):376–396, 1957.
- [115] Erik H Saenger. Numerical methods to determine effective elastic properties. *International Journal of Engineering Science*, 46(6):598–605, 2008.
- [116] Lawrence C Evans. *Weak convergence methods for nonlinear partial differential equations*. Number 74. American Mathematical Soc., 1990.
- [117] Paul Cupillard and Yann Capdeville. Non-periodic homogenization of 3-d elastic media for the seismic wave equation. *Geophysical Journal International*, 213(2):983–1001, 2018.
- [118] C Duval, J Elhadad, MJ Gotay, J Śniatycki, and GM Tuynman. Quantization and bosonic brst theory. *Annals of Physics*, 206(1):1–26, 1991.
- [119] Glenn Barnich, Friedemann Brandt, and Marc Henneaux. Local brst cohomology in gauge theories. *Physics Reports*, 338(5):439–569, 2000.

- [120] James Glimm and Arthur Jaffe. *Quantum physics: a functional integral point of view*. Springer Science & Business Media, 2012.
- [121] Barry Simon. *Functional integration and quantum physics*, volume 86. Academic press, 1979.
- [122] Kevin Costello. *Renormalization and effective field theory*. Number 170. American Mathematical Soc., 2011.
- [123] Peter Kopietz, Lorenz Bartosch, and Florian Schütz. *Introduction to the functional renormalization group*, volume 798. Springer, 2010.
- [124] Ludwig Dmitrievich Faddeev. Faddeev-popov ghosts. *Scholarpedia*, 4(4):7389, 2009.
- [125] Anne Sornette and Didier Sornette. Self-organized criticality and earthquakes. *EPL (Europhysics Letters)*, 9(3):197, 1989.
- [126] Filippo Caruso, Alessandro Pluchino, Vito Latora, Sergio Vinciguerra, and Andrea Rapisarda. Analysis of self-organized criticality in the olami-feder-christensen model and in real earthquakes. *Physical Review E*, 75(5):055101, 2007.
- [127] Zeev Olami, Hans Jacob S Feder, and Kim Christensen. Self-organized criticality in a continuous, nonconservative cellular automaton modeling earthquakes. *Physical review letters*, 68(8):1244, 1992.
- [128] Per Bak. Self-organized criticality. *Physica A: Statistical Mechanics and its Applications*, 163(1):403–409, 1990.
- [129] R-S Wu and Keiiti Aki. Scattering characteristics of elastic waves by an elastic heterogeneity. *Geophysics*, 50(4):582–595, 1985.
- [130] Yuehua Zeng, Feng Su, and Keiiti Aki. Scattering wave energy propagation in a random isotropic scattering medium: 1. theory. *Journal of Geophysical Research: Solid Earth*, 96(B1):607–619, 1991.
- [131] RS Wu and K Aki. Elastic wave scattering by a random medium and the small-scale inhomogeneities in the lithosphere. *Journal of Geophysical Research: Solid Earth*, 90(B12):10261–10273, 1985.
- [132] Frank C Karal Jr and Joseph B Keller. Elastic, electromagnetic, and other waves in a random medium. *Journal of Mathematical Physics*, 5(4):537–547, 1964.